Florentin Smarandache

(author and editor)

Collected Papers

Neutrosophic Theories and Applications

Volume XV

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(author and editor)

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Volume XV

Neutrosophic Theories and Applications

Peer-Reviewers:

Maissam Jdid
Faculty of Science, Damascus University, SYRIA
maissam.jdid66@damascusuniversity.edu.sy

Maikel Leyva-Vázquez
Universidad de Guayaquil, Guayas, ECUADOR
maikel.leyvav@ug.edu.ec

Giorgio Nordo
University of Messina, Sant'Agata, Messina, ITALY
giorgio.nordo@unime.it

Erick González Caballero
Asociación Latinoamericana de Ciencias Neutrosóficas, Havana, CUBA
erickgc@yandex.com

Surapati Pramanik
Nandalal Ghosh B.T. College, Panpur, West Bengal, INDIA
sura_pati@yahoo.co.in

Mohammad Hamidi
University of Payame Noor, Tehran, IRAN
m.hamidi@pnu.ac.ir

Victor Christianto
Malang Institute of Agriculture (IPM), Malang, INDONESIA
victorchristianto@gmail.com

Takaaki Fujita
Independent Researcher, Tokyo, JAPAN
takaaki.fujita060@gmail.com

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Division of Mathematics and Sciences
University of New Mexico
705 Gurley Ave., Gallup Campus
NM 87301, United States of America

University of Guayaquil
Av. Kennedy and Av. Delta
"Dr. Salvador Allende" University Campus
Guayaquil 090514, Ecuador

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Introductory Note

This fifteenth volume of **Collected Papers** is an extensive work, comprising 77 articles that span almost 1000 pages. The papers, originally published in various scientific journals, cover the broad and interdisciplinary fields of neutrosophics and other areas of study, including mathematics, fuzzy sets, intuitionistic fuzzy sets, picture fuzzy sets, information fusion, robotics, statistics, and extenics. The volume features articles authored by Florentin Smarandache, either alone or in collaboration with 123 co-authors. A complete list of these co-authors is provided: Shumaila Abbas, Raja Abdullah Abdulfatah, Sabah S. Abdulrahman, Mohammad Abobala, Mohamed Abosalih, Sunday Adesina Adebisi, Lorenzo Affè, Abuobida Ahal, Ashik Ahirwar, Abuobida Alfahal, Yaser Ahmad Alhasan, Hanan Alohali, B. Ananda Priya, S. Angammal, N. Angel, T. Ashika, Faiza Asif, Muhammad Aslam, Aws A. Azzidan, Abdelaziz Babiker, Esam Bader, Assia Bakali, Ranulfo Paiva Barbosa, Noel Batista-Hernández, Sania Batool, Romualdas Bausys, Mahmut Can Bozyıgt, Said Broumi, Erick González Caballero, T. Chalapathi, Victor Christianto, Bing Cui, Tugrul Daim, Bijan Davvaz, Muhammet Deveci, Meghana Dirisala, Seyyed Ahmad Edalatpanah, Ahmed El-Massry, Tawfik Elmasry, A. Elrawy, Takaaki Fujita, N. Ramila Gandhi, Harish Garg, Mona Gharib, P. Gnanachandra, İlgin Gökasar, Hasan Gökbaş, Hannah Grace, S. Karunya Helen Gunaseeli, Mohammad Hamidi, Iftikhar Ul Haq, D. Harish Babu, Masooma Raza Hashmi, Farnaz Sheikh Hassani, Jesús Rafael Hechavarría-Hernández, K. Hemabala, A. Ibrahim, Sadia Iqbal, Mehtap Isik, Saeid Jafari, Maissam Jdid, Ilanthenral Kandasamy, Vasantha Kandasamy, Muthusamy Karthika, Huda E. Khalid, Neha Andaleeb Khalid, Rehan Ahmad Khan Sherwani, B. Srinivasa Kumar, K. Kumaraswamy Naidu, Anamika Kumari, Mukesh Kushwaha, Yingcang Ma, Madelaine Al Tahan, Nivetha Martin, Mohamed Talea, Mona Mohamed, Swaminathan Mohanaselvi, Alaa A. Mohammed, D. Nagarajan, Muhammad Ali Najjar, Giorgio Nordo, Murat Olgun, Dragan Pamucar, P. Pandiammal, Antonios Paraskevas, Mani Parimala, Aurimas Petrovas, S. Rajeswari, Muhammad Riaz, Rafael Rojas-Gualdron, Yaser Saber, Muhammad Saeed, Rajesh Kumar Saini, A. A. Salama, Muhammad Saleem, S. Sandhiya, Tanzeela Shaheen, Riya Eliza Shaju, Rehan Ahmad Khan Sherwani, Rajesh Singh, Ali Sorourkhah, B. Srinivasa Kumar, C. Sugapriya, Ayat A. Temraz, Hugo Terashima-Marin, Mahima Thakur, S. S. Thakur, A. Theeba, Shobh Nath Tiwari, Hamza Ghazanfar Toor, Selçuk Topal, Mehmet Ünver, Maikel Leyva Vázquez, Satya Bhushan Verma, Mohammad Kaif Wajid, Mohammad Saif Wajid, Mohd Anas Wajid, Tomasz Witczak, Subhash Kumar Yadav, Aws A. Zaidan, Kazimieras Edmundas Zavadskas, Dan Zhang, Hengdong Zhu.

Keywords

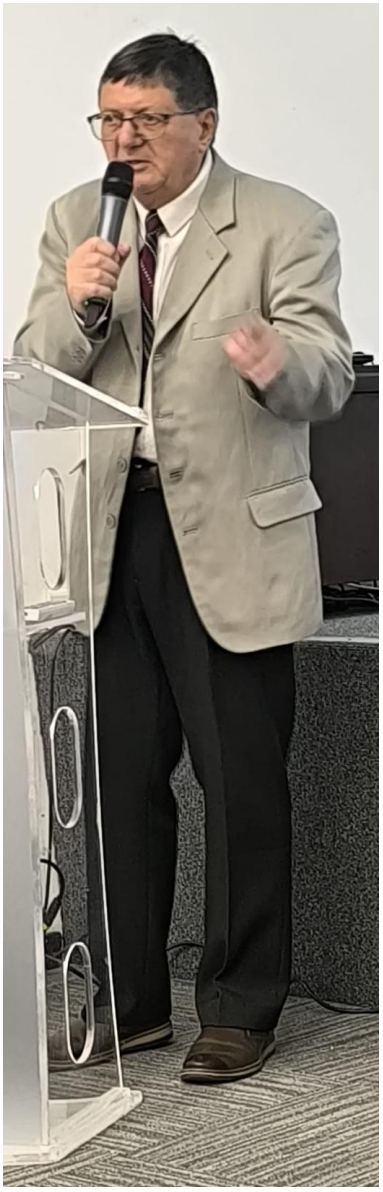
Neutrosophic Set, Neutrosophic Logic, Neutrosophic Theory, Fuzzy Sets, Intuitionistic Fuzzy Sets, Picture Fuzzy Sets, Uncertainty, Decision-Making, Information Fusion, Multi-Criteria Decision-Making (MCDM), Complex Neutrosophic Sets, Plithogenic Sets, NeutroGeometry, Neutrosophic Rings, Neutrosophic Probability, Neutrosophic Statistics, Neutrosophic Graphs, Neutrosophic Numbers, Algebraic Structures, Single-Valued Neutrosophic Sets, Neutrosophic Algebra, Extenics, Machine Learning, Data Science, Image Processing, Medical Diagnosis, Optimization, Risk Analysis, Social Science, Engineering, Artificial Intelligence, Robotics, Neutrosophic Soft Sets, Neutrosophic Measure, Neutrosophic Integral, Interval-Valued Neutrosophic Sets, Q-Rung Orthopair Neutrosophic Sets, Rough Sets, Pura Vida Neutrosophic Algebra, Tropical Algebra, Fuzzy Cognitive Maps, Triangular Neutrosophic Numbers, Neutrosophic Topology, Neutrosophic Groups, Neutrosophic Semigroups, Neutrosophic Vector Spaces, Neutrosophic Ideal Theory, Neutrosophic Lattices, Neutrosophic Modules, Neutrosophic Normed Spaces, Neutrosophic Calculus, Neutrosophic Differential Equations, Neutrosophic Control Theory, Neutrosophic Cybernetics, Neutrosophic Systems, Neutrosophic Field Theory, Neutrosophic Physics, Neutrosophic Chemistry, Neutrosophic Biology, Neutrosophic Psychology, Neutrosophic Sociology, Neutrosophic Economics, Neutrosophic Linguistics, Neutrosophic Philosophy, Neutrosophic Paradoxism, Neutrosophic Algorithms, Neutrosophic Data Mining, Neutrosophic Clustering, Neutrosophic Classification, Neutrosophic Regression, Neutrosophic Neural Networks, Neutrosophic Expert Systems, Neutrosophic Control Systems, Neutrosophic Security, Neutrosophic Cryptography, Neutrosophic Networks, Neutrosophic Sensor Networks, Neutrosophic Quantum Computing, Neutrosophic Information Retrieval, Neutrosophic Knowledge Representation, Neutrosophic Reasoning, Neutrosophic Logic Programming, Neutrosophic Game Theory.

FLORENTIN SMARANDACHE is an emeritus professor of Mathematics at the University of New Mexico, United States. He got his MSc in Mathematics and Computer Science from the University of Craiova, Romania, PhD in Mathematics from the State University of Kishinev, Moldova, and Postdoctoral in Applied Mathematics from Okayama University of Sciences, Japan, and The Guangdong University of Technology, Guangzhou, China.

He is the founder of neutrosophy (generalization of dialectics), neutrosophic set, logic, probability and statistics since 1995 and has published hundreds of papers and books on neutrosophic physics, superluminal and instantaneous physics, unmatter, quantum paradoxes, absolute theory of relativity, paradoxism, outerart, neutrosophy as a new branch of philosophy, Law of Included Multiple-Middle, multispace and multistructure, hypersoft set, IndetermSoft Set and IndetermHyperSoft Set, SuperHyperGraph, SuperHyperTopology, SuperHyperAlgebra, Neutrosophic SuperHyperAlgebra, degree of dependence and independence between neutrosophic components, refined neutrosophic set, neutrosophic over-under-off-set, plithogenic set / logic / probability / statistics, neutrosophic triplet and duplet structures, quadruple neutrosophic structures, extension of algebraic structures to NeutroAlgebra and AntiAlgebra, NeutroGeometry & AntiGeometry, Dezert-Smarandache Theory and so on.

In addition, he published many books of poetry, dramas, children' stories, translations, essays, a novel, folklore collections, traveling memories, and art albums.

http://fs.unm.edu/FlorentinSmarandache.htm



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Neutrosophic Theories and Applications

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Egypt	**6**
Greece	**1**
India	**40**
Indonesia	**1**
Iran	**5**
Iraq	**3**
Italy	**2**
Japan	**1**
Jordan	**1**
Lithuania	**3**
Morocco	**4**
Nigeria	**1**
Oman	**1**
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Poland	**1**
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Serbia	**2**
Syria	**2**
Turkey	**9**
United Arab Emirates	**1**
United Kingdom	**2**
United States	**1**

Introduction to the MultiNeutrosophic Set

Florentin Smarandache

Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract: In the real word, in most cases, everything (an attribute, event, proposition, theory, idea, person, object, action, etc.) is evaluated in general by many *sources* (called *experts*), not only one. The more sources evaluate a subject, the better accurate result (after fusioning all evaluations). That's why, in this paper, we straightforwardly extend the Refined Neutrosophic Set to the MultiNeutrosophic Set, and we show that the last two are isomorphic. A MultiNeutrosophic Set is a Neutrosophic Set whose all elements' degrees of truth/indeterminacy/falsehood are evaluated by many (Multi) sources.

Afterwards, we introduce a total order on the set of n-plets of the form (p, r, s), we build the operators on the (p, r, s)-plets, and show several applications of the MultiNeutrosophic Sets.

Several particular cases of the MultiNeutrosophic Sets are presented: such as MultiFuzzy Set, MultiIntuitionistic Fuzzy Set, MultiPicture Fuzzy Set, and other Multi(Fuzzy Extension) Set.

1. General Definition of the Neutrosophic Set (or Subset Neutrosophic Set - *SNS*)

Let $\mathcal{U}$ be a universe of discourse and a subset *N* of it.

Then:

$N = \{x, (T, I, F), x \in U\}$

is called a *Neutrosophic Set*, where *T, I, F* are subsets of [0, 1], and they are called respectively degrees of Truth (*T*), Indeterminacy (*I*), and Falsehood (*F*) of the element *x* with respect to the set *A*. No other restrictions on *T*, *I,* and *F*. Of course, it implies that:

$0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3.$

The most used (particular cases are):

i) If *T, I, F* are all single-values (numbers) from [0, 1], then one has a *Singe-Valued Neutrosophic Set* (SVNS);

ii) If *T, I, F* are intervals included in [0, 1], then one has an *Interval-Valued Neutrosophic Set* (IVNS).

2. The MultiNeutrosophic Set

In the real word, in most cases, everything: an attribute, event, proposition, theory, idea, person, object, action, etc., is evaluated in general by many *sources* (called *experts*), let's denote them by $S_1, S_2, \ldots, S_n$, where the number of sources $n \geq 2$ (to ensure the MultiSource). The more sources evaluate a subject, the better accurate result (after fusioning all evaluations).

Therefore, let's assume the degree of truth (or membership) of the generic element *x* with respect to the set *N* is evaluated by *p* sources of information, that give the following results,

respectively $T_1, T_2, \ldots, T_p$;

and the degree of indeterminacy (neither truth/membership, nor falsehood/nonmembership) of the element *x* with respect to the set *N* is evaluated by *r* sources of information, that give the following results, respectively: $I_1, I_2, \ldots, I_r$;

while the degree of falsehood (or nonmembership) of the element *x* with respect to the set *N* is also evaluated by *s* sources of information that give the following results, respectively: $F_1, F_2, \ldots, F_s$;

where all $T_1,\ T_2, \dots, T_p,\ I_1,\ I_2, \dots, I_r,\ F_1,\ F_2, \dots, F_s$ are subsets of [0,1], with *p, r, s* integers ≥ 0, and at least one of *p, r, s* is ≥ 2 (in order to ensure the multiplicity of at least one of: truth, indeterminacy, or falsehood), with $p + r + s = n \geq 2$.

All n sources may be either totally independent two by two, or partially independent and partially dependent, or totally dependent - according to the need of each specific application.

In the situation where there is some dependence between sources, we understand that either they communicate with each other and share information (influencing each other), or the same source may evaluate two or three of the components: truth, indeterminacy, falsehood of the same element.

3. General Definition of MultiNeutrosophic Set (or Subset MultiNeutrosophic Set - *SMNS*)

Let $\mathcal{U}$ be a universe of discourse and *M* a subset of it. Then, a *MultiNeutrosophic Set* is:

$M = \{x, x(T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s), x \in \mathcal{U},$

where p, r, s are integers ≥ 0, $p + r + s = n \geq 2$,

and at least one of *p, r, s* is ≥ 2, in order to ensure the existence of multiplicity of at least one neutrosophic component: truth/membership, indeterminacy, or falsehood/nonmembership;

all subsets $T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s \subseteq [0, 1]$;

$0 \leq \sum_{j=1}^{p} inf T_j + \sum_{k=1}^{r} inf I_k + \sum_{l=1}^{s} inf F_l \leq \sum_{j=1}^{p} sup T_j + \sum_{k=1}^{r} sup I_k + \sum_{l=1}^{s} sup F_l \leq n\}$.

No other restrictions apply on these neutrosophic multicomponents.

$T_1, T_2, \dots, T_p$ are multiplicities of the truth, each one provided by a different source of information (expert).

Similarly, $I_1, I_2, \dots, I_r$ are multiplicities of the indeterminacy, each one provided by a different source.

And $F_1, F_2, \dots, F_s$ are multiplicities of the falsehood, each one provided by a different source.

The **Degree of MultiTruth** (MultiMembership), also called *MultiDegree of Truth*, of the element *x* with respect to the set *M* are $T_1, T_2, \dots, T_p$;

the **Degree of MultiIndeterminacy** (MultiNeutrality), also called *MultiDegree of Indeterminacy*, of the element *x* with respect to the set *M* are $I_1, I_2, \dots, I_r$;

and the **Degree of MultiFalsehood** (MultiNonmembership), also called *MultiDegree of Falsehood*, of element *x* with respect to the set *M* are $F_1, F_2, \dots, F_s$.

All these $p + r + s = n \geq 2$ are assigned by *n* sources (experts) that may be:

— either totally independent;

— or partially independent and partially dependent;

— or totally dependent;

according or as needed to each specific application.

A generic element *x* with regard to the MultiNeutrosophic Set *A* has the form:

$x(\underbrace{T_1, T_2, \dots, T_p;}_{\text{multi-truth}}\ \underbrace{I_1, I_2, \dots, I_r;}_{\text{multi-indeterminacy}}\ \underbrace{F_1, F_2, \dots, F_s}_{\text{multi-falsehood}})$

In many particular cases $p = r = s$, and a source (expert) assigns all three degrees of truth, indetermincay, and falsehood (T_j, I_j, F_j) for the same element.

4. Particular Cases of MultiNeutrosophic Set (MNS)

Upon the types of sets that the neutrosophic components are, one has:

i. *Single-Valued MultiNeutrosophic Set (SVMNS),*

when all neutrosophic components

$T_j,\ 1 \leq j \leq p;\ T_j,\ I_k, 1 \leq k \leq r$, and $F_l, 1 \leq l \leq s$,

are single-values (numbers), such that all $T_j, I_k, F_l \in [0, 1]$.

ii. *Interval-Valued MultiNeutrosophic Set (IVMNS),*

when all neutrosophic components

$T_j,\ 1 \le j \le p;\ T_j,\ I_k, 1 \le k \le r$, and $F_l, 1 \le l \le s$,
are interval-values, such that all $T_j, I_k, F_l \subseteq [0,1]$.

5. Particular Cases of Single-Valued MultiNeutrosophic Set (*SVMNS*)

a. *MultiFuzzy Set*, by setting $p \ge 2$, and $r = s = 0$, into the above SVMNS Definition.

b. *MultiIntuitionistic Set*, by setting $r = 0$, p and $s \ge 1$, with $p + s \ge 3$, and $0 \le T_j + F_l \le 1$, for all $j \in \{1, 2, \dots p\}, l \in \{1, 2, \dots s\}$, into the SVMNS Definition.

c. *MultiPythagorean Fuzzy Set*, by letting $r = 0$, and $p, s \ge 3$, with $p + s \ge 3$, and $0 \le T_j^2 + F_l^2 \le 1$, for $j \in \{1, 2, \dots, p\}$, $l \in \{1, 2, \dots, s\}$, into the SVMNS Definition.

d. *MultiFermatean Fuzzy Set*, by letting $r = 0$, and $p, s \ge 1$, with $p + s \ge 3$, and $0 \le T_j^3 + F_l^3 \le 1$, for $j \in \{1, 2, \dots, p\}$, $l \in \{1, 2, \dots, s\}$, into SVMNS Definition.

e. *Multi q-Rung Orthopair Fuzzy Set*, by letting $r = 0$, and $p, s \ge 1$, with $p + s \ge 3$, and $0 \le T_j^q + F_l^q \le 1$, with $q \ge 1$, for $j \in \{1, 2, \dots, p\}$, $l \in \{1, 2, \dots, s\}$, into SVMNS Definition.

f. *MultiPicture Fuzzy Set*, by letting $p, r, s \ge 1$, with $p + r + s \ge 4$, and $0 \le T_j + N_k + F_e \le 1$, for $j \in \{1, 2, \dots, p\}$, $k \in \{1, 2, \dots, r\}$, $l \in \{1, 2, \dots, s\}$, where N_k is considered neutral (as in neutrosophic set is ideterminacy) into SVMNS Definition.

g. *MultiSpherical Set*, by setting $p, r, s \ge 1$, with $p + r + s \ge 4$, and $0 \le T_j^2 + I_k^2 + F_e^2 \le 1$, and $R_{jkl} = \sqrt{1 - T_j^2 - I_k^2 - F_e^2}$, for all $j \in \{1, 2, \dots, p\}$, $k \in \{1, 2, \dots, r\}$, $l \in \{1, 2, \dots, s\}$, into the SVMNS Definition.

6. Particular Cases of Interval-Valued MultiNeutrosophic Set (*IVMNS*)

In an identical way we get the *Particular Cases of Interval-Valued MultiNeutrosophic Set*, as being Interval-Valued (fuzzy and fuzzy-extension) sets, replacing the single-valued components by interval components and using the operations of intervals:

For any $[a, b],\ [c, d] \subseteq [0, 1]$, where $a \le b$ and $c \le d$, one has:

$[a, b] + [c + d] = [min\{a + c, 1\}, min\{b + d, 1\}]$

$[a, b]^n = [min\{a^n, 1\}, min\{b^n, 1\}]$

$1 - [a, b] = [1 - b, 1 - a]$

$[a, b] - [c, d] = [max\{a - d, 0\}, max\{b - c, 0\}]$.

7. Application of Single-Valued MultiNeutrosophic Set

Let $M = \{A, B, C, D\}$ be a group of students.

Their performance in science is evaluated by several professors (= sources of information, experts).

Let's assume that three professors $P_1,\ P_2,\ P_3$ evaluate the degrees of positive knowledge (truth) of the students, and:

– Professor P_1 assigns the value T_1 respectively to all the students;

– Professor P_2 assigns the value T_2 respectively to all the students;

– Professor P_3 assigns the value T_3 respectively to all the students, as follows:

A($T_1 = 0.8,\ T_2 = 0.6,\ T_3 = 0.7$),

B($T_1 = 0.6,\ T_2 = 0.9,\ T_3 = 0.5$),

C($T_1 = 0.4,\ T_2 = 0.4,\ T_3 = 0.6$),

D($T_1 = 0.7,\ T_2 = 0.0,\ T_3 = 0.4$).

But two professors Q_1 and Q_2 are not very sure of the students' performances and assign indeterminate degrees (I_1 and I_2 respectively) to the students:

$A\ (I_1 = 0.2, I_2 = 0.3)$,

$B\ (I_1 = 0.5, I_2 = 0.4)$,

$C\ (I_1 = 0.1, I_2 = 0.0)$,

$D\ (I_1 = 0.3, I_2 = 0.1)$.

Further on, four professor $R_1, R_2, R_3,$ and R_4, dissatisfied with the students' performance, assign negative evaluations (falsehood degrees), $F_1, F_2, F_3,$ and F_4 respectively:

$A\ (F_1 = 0.7, F_2 = 0.4, F_3 = 0.5, F_4 = 0.4)$,
$B\ (F_1 = 0.6, F_2 = 0.3, F_3 = 0.5, F_4 = 0.1)$,
$C\ (F_1 = 0.2, F_2 = 0.1, F_3 = 0.2, F_4 = 0.3)$,
$D\ (F_1 = 0.5, F_2 = 0.2, F_3 = 0.1, F_4 = 0.2)$.

The students have been evaluated by $3 + 2 + 4$ sources of information. In the case that all sources were independent two by two, one has 9 sources. But, if there was some dependence (i.e. the same professor assigning, for example, not only the truth, but also the indeterminacy and/or the falsehood, the number of independent sources is < 9).

The more sources evaluate a subject, the better accurate result.

We got the following single-valued MultiNeutrosophic Set, where each element has the form:

$x(\{T_1, T_2, T_3\}, \{I_1, I_2\}, \{F_1, F_2, F_3, F_4\})$.

$M = \{A(\{0.8, 0.6, 0.7\}, \{0.2, 0.3\}, \{0.7, 0.4, 0.5, 0.4\}),$

$B(\{0.6, 0.9, 0.3\}, \{0.5, 0.4\}, \{0.6, 0.3, 0.5, 0.1\}),$

$C(\{0.4, 0.4, 0.6\}, \{0.1, 0.0\}, \{0.2, 0.1, 0.2, 0.3\}),$

$D(\{0.7, 0.0, 0.4\}, \{0.3, 0.1\}, \{0.5, 0.2, 0.1, 0.2\})$.

7.1. Remark on previous Application

The Single-Valued MultiNeutrosophic Set (SVMNS) coincides in form with the particular case of the Subset Neutrosophic Set (SNS) by taking the neutrosophic components as discrete subsets of the form $\{a_1, a_2, \dots, a_m\} \subset [0,1], m \geq 1$.

For example, considering the student A, his degree of truth (membership) is $T(A) = \{0..8, 0.6, 0.7\}$, his degree of indeterminacy-membership is $I(A) = \{0.2, 0.3\}$, and his degree of falsehood (nonmembership) is $F(A) = \{0.7, 0.4, 0.5, 0.4\}$, from the point of view of Subset Neutrosophic Set.

i. The first distinction is that in the case of Subset Neutrosophic Set, *only one source (expert)* provides information about let's say the student degree $T(A) = \{0.8, 0.6, 0.7\}$, while in the case of Single-Value Multi Neutrosophic Set, three sources provide information on $T(A)$, i.e. one source evaluates the student A degree of truth as 0.8, the second one as 0.6, and the third one as 0.7. The more experts evaluating, the better accuracy, whence the SVMNS better evaluates than the SNS. Similarly for the degree of indeterminacy $I(A)$, and the degree of falsehood $F(A)$.

ii. The second distinction is in applying the neutrosophic operators, since in general the operators for the Subset Neutrosophic Sets are different from the operators for the Single-Valued MultiNeutrosophic Set (we'll see it below on Section 13).

8. Ranking of n-valued MultiNeutrosophic types of the same (*p, r, s*)-form

$(T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s)$,

where p, r, s are integers ≥ 0, and $p + r + s = n \geq 2$, and at least one of $p, r, s \geq 2$ to be sure that we have multiplicity for at least one neutrosophic component (either truth, or indeterminacy, or falsehood).

The first research in n-ranking neutrosophic triplets was done in 2023 by V. Lakshmana Gomathi Nayagam, and Bharanidharan R. [3], using the dictionary ranking.

We propose an easier n-ranking, but this is rather an approximation. Let's compute the following.

1. Average Positivity:

$$\frac{\sum_{j=1}^{p} T_j + \sum_{k=1}^{r}(1 - I_k) + \sum_{e=1}^{s}(1 - F_e)}{p + r + s}$$

2. Average (Truth-Falsehood):

$$\frac{\sum_{j=1}^{p} T_j - \sum_{e=1}^{s} F_e}{p+s}$$

3. Average Truth

$$\frac{\sum_{j=1}^{p} T_j}{p}$$

Let's compare $(T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s) \equiv N$
with $(T_1', T_2', \dots, T_p';\ \ I_1', I_2', \dots, I_r';\ \ F_1', F_2', \dots, F_s') \equiv N'$.

If their *Average Positivity* is the same, one gets (1):

$$\sum_{j=1}^{p} T_j - \sum_{k=1}^{r} I_k - \sum_{e=1}^{s} F_e = \sum_{j=1}^{p} T_j' - \sum_{k=1}^{r} I_k' - \sum_{e=1}^{s} F_e'$$

If their *Average (Truth-Falsehood)* is the same, one gets (2):

$$\sum_{j=1}^{p} T_j - \sum_{e=1}^{s} F_e = \sum_{j=1}^{p} T_j' - \sum_{e=1}^{s} F_e'$$

whence, by combining (1) and (2), one gets (3):

$$\sum_{k=1}^{r} I_k = \sum_{k=1}^{r} I_k'$$

If their *Average Truth* is the same, one gets (4):

$$\sum_{j=1}^{p} T_j = \sum_{j=1}^{p} T_j'$$

Then, from (2) and (4), one gets:

$$\sum_{e=1}^{s} F_e = \sum_{e=1}^{s} F_e'$$

Therefore $N = N'$ means that their corresponding averages of truths, indeterminacies, and falsehoods respectively are equal:

$$\frac{1}{p}\sum_{j=1}^{p} T_j = \frac{1}{p}\sum_{j=1}^{p} T_j'$$

$$\frac{1}{r}\sum_{k=1}^{r} I_k = \frac{1}{r}\sum_{k=1}^{r} I_k'$$

$$\frac{1}{s}\sum_{e=1}^{s} F_e = \frac{1}{s}\sum_{e=1}^{s} F_e'$$

9. Ranking *n*-valued MultiNeutrosophic tuples of different (*p, r, s*)–forms

Let's consider two n-valued multi neutrosophic tuples of the forms (p_1, r_1, s_1) and respectively (p_2, r_2, s_2), where $p_1, r_1, s_1, p_2, r_2, s_2$ are integers ≥ 0, and $p_1 + r_1 + s_1 = n_1 \geq 2$, and at least one of p_1, r_1, s_1 $is \geq 2$ to be sure that we have multiplicity for at least one neutrusophic component (either truth, or indeterminacy, or falsehood); similarly $p_2 + r_2 + s_2 \geq 2$, and at least one of $p_2, r_2, s_2 \geq 2$. Let's take the following Single-Valued Multi Neutrosophic Tulpes (SVMNT):

$SVMNT = (T_1, T_2, \dots, T_{p_1};\ \ I_1, I_2, \dots, I_{r_1};\ \ F_1, F_2, \dots, F_{s_1})$ of (p_1, r_1, s_1)-form, and

$SVMNT' = (T_1', T_2', \dots, T_{p_2}'; I_1', I_2', \dots, I_{r_2}';\ \ F_1', F_2', \dots, F_{s_2}')$ of (p_2, r_2, s_2)-form.

We make the classical averages of truth (T_a), indeterminancies (I_a) and falsehood (F_a), respectively for:

$SVMNT = \ (T_a, I_a, F_a)$

and the averages of truths (T_a'), indeterminancies (I_a'), and falsehood (F_a') respectively for:

$SVMNT' = (T_a', I_a', F_a')$.

And then we apply the Score (S), Accuracy (A), and Certainty (C) Functions, as for the single valued neutrosophic set:

1. *Compute the Score Function* (average of positiveness)

$$S(T_a, I_a, F_a) = \frac{(T_a + (1 - I_a) + (1 - F_a)}{3}$$

$$S(T_a', I_a', F_a') = \frac{T_a' + 1 - I_a' + F_a')}{3}$$

(i) if $S(T_a, I_a, F_a) > S(T_a', I_a', F_a')$, then $SVMNT > SVMNT'$;

(ii) if $S(T_a, I_a, F_a) < S(T_a', I_a', F_a')$, then $SVMNT < SVMNT'$;

(iii) and if $S(T_a, I_a, F_a) = S(T_a', I_a', F_a')$,

then go to the second step.

2. *Compute the Accuracy Function* (difference between the truth and falsehood)

$$A(T_a, I_a, F_a) = T_a - F_a$$

$$A(T_a', I_a', F_a') = T_a' - F_a'$$

(i) if $A(T_a, I_a, F_a) > A(T_a', I_a', F_a')$, then $SVMNT > SVMNT'$;

(ii) if $A(T_a, I_a, F_a) < A(T_a', I_a', F_a')$, then $SVMNT < SVMNT'$;

(iii) and if $A(T_a, I_a, F_a) = A(T_a', I_a', F_a')$,

then go to the third step.

3. *Compute the Certainty Function* (truth)

$$C(T_a, I_a, F_a) = T_a$$

$$C(T_a', I_a', F_a') = T_a'$$

(i) if $C(T_a, I_a, F_a) > C(T_a', I_a', F_a')$, then $SVMNT > SVMNT'$;

(ii) if $C(T_a, I_a, F_a) < C(T_a', I_a', F_a')$, then $SVMNT < SVMNT'$;

(iii) if $C(T_a, I_a, F_a) = C(T_a', I_a', F_a')$, then $SVMNT$ and $SVMNT'$ are multi-neutrosophically equal, i.e. $T_a = T_a'$, $I_a = I_a'$, $F_a = F_a'$, or their corresponding truth, indeterminancy, and falsehood averages are equal.

10. Example 1

Example where all sources providing information have equal weights.

Assume the student George is evaluated by several professors from his university with respect to his skills in science:

George$(\{0.8, 0.9, 0.3\}, \{0.2\}, \{0.6, 0.7\})$

While the student John is evaluated with respect to the same scientific skills by some of the previous professors and by other professors from the same university:

John$(\{0.7, 1.0, 0.6, 0.5\}, \{01., 0.4\}, \{0.2, 0.8, 0.7\})$

Which student does better than the others?

Let's compute the averages.

John$\left(\frac{0.7+1.0+0.6+0.5}{4}, \frac{0.1+0.4}{2}, \frac{0.2+0.8+0.7}{3}\right) \simeq$ John(0.70, 0.25, 0.57).

George$\left(\frac{0.8+0.9+0.3}{3}, \frac{0.2}{1}, \frac{0.6+0.7}{2}\right) \simeq$ George(0.67, 0.20, 0.65).

The Score Function:

S(George) = $\frac{0.67+(1-0.20)\_(1-0.65)}{3} \simeq 0.61$.

S(John) = $\frac{0.70+(1-0.25)\_(1-0.57)}{3} \simeq 0.63$.

John has better scientific skills than George, since S(John) $\simeq 0.63 > 0.61 \approx$S(George).

This may be explained from the fact that if more or less sources evaluate the same element x of a given set, we make the average of evaluations.

In cases some sources have a greater weight in evaluation than others, one uses the *weighted averages*, indexed as T_{wa}, I_{ua}, F_{va} and $T'_{w'a'}$, $I'_{u'a'}$, $F'_{v'a'}$, respectively.

Because the sources may be independent or partially independent, the sum of weights should not necessarily be equal to 1. As such, one has:

$$T_{wa} = \frac{w_1T_1 + w_2T_2 + \cdots + w_{p_1}T_{p_1}}{w_1 + w_2 + \cdots + w_{p_1}}$$

where $w_1, w_2, \ldots, w_{p_1} \in [0,1]$, while the sum $w_1 + w_2 + \cdots + w_{p_1}$ may be < 1, or $= 1$, or > 1;

$$I_{ua} = \frac{u_1I_1 + u_2I_2 + \cdots + u_{r_1}I_{r_1}}{u_1 + u_2 + \cdots + u_{r_1}}$$

where $u_1, u_2, \ldots, u_{r_1} \in [0,1]$, while the sum $u_1 + u_2 + \cdots + u_{r_1}$ may be < 1, or $= 1$, or > 1;

$$F_{va} = \frac{v_1F_1 + v_2F_2 + \cdots + v_{s_1}F_{s_1}}{v_1 + v_2 + \cdots + v_{s_1}}$$

where $v_1, v_2, \ldots, v_{s_1} \in [0,1]$, while the sum $v_1 + v_2 + \cdots + v_s$ may be < 1, or $= 1$, or > 1.

Similarly,

$$T'_{w'a} = \frac{w'_1T'_1 + w'_2T'_2 + \cdots + w'_{p_2}T'_{p_2}}{w'_1 + w'_2 + \cdots + w'_{p_2}}$$

where $w'_1, w'_2, \ldots, w'_{p_2} \in [0,1]$, while the sum $w'_1 + w'_2 + \cdots + w'_{p_1}$ may be < 1, or $= 1$, or > 1;

$$I'_{u'a} = \frac{u'_1I'_1 + u'_2I'_2 + \cdots + u'_{r_2}I'_{r_2}}{u_1 + u_2 + \cdots + u_{r_1}}$$

where $u'_1, u'_2, \ldots, u'_{r_2} \in [0,1]$, while the sum $u'_1 + u'_2 + \cdots + u'_{r_1}$ may be < 1, or $= 1$, or > 1;

$$F'_{v'a} = \frac{v'_1F'_1 + v'_2F'_2 + \cdots + v'_{s_2}F'_{s_2}}{v'_1 + v'_2 + \cdots + v'_{s_2}}$$

where $v'_1, v'_2, \ldots, v'_{s_2} \in [0,1]$, while the sum $v'_1 + v'_2 + \cdots + v'_s$ may be < 1, or $= 1$, or > 1.

And, similarly, one applies the Score, Accuracy, and Certainty Functions on these weighted averages to rank them.

$$S(T_{wa}, I_{ua}, F_{va}) = \frac{T_{wa}+(1-I_{ua})+(1-F_{va})}{3}$$
$$A(T_{wa}, I_{ua}, F_{va}) = T_{wa} - F_{va}$$
$$C(T_{wa}, I_{ua}, F_{va}) = T_{wa}$$

$$S(T'_{w'a}, I'_{u'a}, F'_{v'a}) = \frac{T'_{w'a}+(1-I'_{u'a})+(1-F'_{v'a})}{3}$$
$$A(T'_{w'a}, I'_{u'a}, F'_{v'a}) = T'_{w'a} - F'_{v'a}$$
$$C(T'_{w'a}, I'_{u'a}, F'_{v'a}) = T'_{w'a}$$

11. Example 2

Let's retake the *Example 1*:
George$(\{0.8, 0.9, 0.3\}, \{0.2\}, \{0.6, 0.7\})$,
and John$(\{0.7, 1.0, 0.6, 0.5\}, \{0.1, 0.4\}, \{0.2, 0.8, 0.7\})$,
and assume the six evaluators of George have the following corresponding weights respectively:
0.6, 0.7, 0.4; 0.3; 0.8, 0.7;
while the nine evaluators of John have the following corresponding weights respectively:
0.7, 0.2, 0.5, 0.1; 0.8, 0.3; 0.9, 0.4, 0.6.
Let's compute the weighted averages.
For George:

$$T_{wa} = \frac{0.8 \cdot (0.6) + 0.9 \cdot (0.7) + 0.3 \cdot (0.4)}{0.6 + 0.7 + 0.4} \simeq 0.72$$

$$I_{ua} = \frac{0.2 \cdot (0.3)}{0.3} = 0.20$$

$$F_{va} = \frac{0.6 \cdot (0.8) + 0.7 \cdot (0.7)}{0.8 + 0.7} \simeq 0.65.$$

We got George (0.72, 0.20, 0.65).
For John:

$$T'_{w'a} = \frac{0.7 \cdot (0.7) + 1.0 \cdot (0.2) + 0.6 \cdot (0.5) + 0.5 \cdot (0.1)}{0.7 + 0.2 + 0.5 + 0.1} \simeq 0.69$$

$$I'_{u'a} = \frac{0.1 \cdot (0.8) + 0.4 \cdot (0.3)}{0.8 + 0.3} \simeq 0.18$$

$$F'_{v'a} = \frac{0.2 \cdot (0.9) + 0.8 \cdot (0.4) + 0.7 \cdot (0.6)}{0.9 + 0.4 + 0.6} \simeq 0.48$$

We got John (0.69, 0.18, 0.48).
Compute the score functions in order to rank them.
S(George) = S(0.72, 0.20, 0.65) = $\frac{0.72+(1-0.20)+(1-0.65}{3} \simeq 0.62$.
S(John) = S(0.69, 0.20, 0.65) = $\frac{0.69+(1-0.20)+(1-0.65)}{3} \simeq 0.61$.
Therefore, now George is better, because S(George) = 0.62 $>$ 0.61 = S(John).

12. Isomorphism between Subset Refined Neutrosophic Set (*SRNS*) and Subset MultiNeutrosophic Set (*SMNS*)

The Subset Refined Neutrosophic Set was first introduced by Smarandache [4] in 2013.

12.1. Definition of Subset Refined Neutrosophic Set (SRNS)

Let $\mathcal{U}$ be a universe of discourse, and a set $R \subset \mathcal{U}$.
Then a Subset Refined Neutrosophic R is defined as follows:
$R = \{x, x(T, I, F), x \in \mathcal{U}\}$,
where T is refined/split into p sub-truths,
I is refined/split into r sub-indeterminacies,
$I = \langle I_1, I_2, \dots, I_r \rangle, I_k \subseteq [0, 1],\ 1 \leq k \leq r$,
and F is refined/split into s sub-falsehoods,
$F = \langle F_1, F_2, \dots, F_s \rangle, F_l \subseteq [0, 1],\ 1 \leq l \leq s$,
where $p, r, s \geq 0$ are integers, and $p + r + s = n \geq 2$, and at least one of p, r, s is ≥ 2 in order to ensure the existence of refinement (splitting).

Similarly, in particular cases, $p = r = s$, meaning that each component *T, I, F* is refined/split into the same member of sub-components.

The isomorphism is obvious:

$$\varphi: SMNS \to SRNS$$
$$\varphi(T_j) = T_j,\ 1 \le j \le p,$$
$$\varphi(I_k) = I_k,\ 1 \le k \le r,$$
$$\varphi(F_l) = F_l,\ 1 \le l \le s.$$

But while T_j, I_k, F_l from SMNS are duplicates (or multi-truth, multi-indeterminacy, multi-falsehood respectively), the corresponding T_j, I_k, F_l from SRNS are parts (or sub-truth, sub-indeterminacy, sub-falsehood respectively).

13. Operators on Multi (and Refined) Neutrosophic Sets/Logic

i. The case when the neutrosophic tuples have the same (p, r, s)-format.

Let $\vee_N$, $\wedge_N$, $\neg_N$, $\to_N$, $\leftrightarrow_N$ be the neutrosophic union, intersection, complement (negation), inclusion (implication), equality (equivalence) respectively.

While $\vee_F$, $\wedge_F$, $\neg_F$, $\to_F$, $\leftrightarrow_F$ the fuzzy operators respectively, where $\vee_F$ and $\wedge_F$ are t-conorm and t-norm respectively, afterwards fuzzy negation, fuzzy implication, and fuzzy equivalence respectively.

Also, by notation, one considers:

$(T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s) \equiv \left(T_j, 1 \le j \le p;\ \ I_k, 1 \le k \le r; F_l, 1 \le l \le s\ \right).$

The operations will be a straightforward extension from the (1, 1, 1)-format (*T, I, F*) to the (p, r, s)-format.

Multi/Refined Neutrosophic Union

$$\left(T_1, T_2, \dots, T_{p_1};\ \ I_1, I_2, \dots, I_{r_1};\ \ F_1, F_2, \dots, F_s\right) \vee_N \left(T'_1, T'_2, \dots, T'_p; I'_1, I'_2, \dots, I'_r;\ \ F'_1, F'_2, \dots, F'_s\right)$$
$$= \left(T_1 \vee_F T'_1, T_2 \vee_F T'_2\ \ , \dots, T_p \vee_F T'_p;\ \ I_1 \wedge_F I'_2, \dots, I_r \wedge_F I'_r; F_1 \wedge_F F'_1, F_2 \wedge_F F'_2, \dots, F_s \wedge_F F'_s\right)$$

Shortly, we may write:

$$\left(T_j, 1 \le j \le p; I_k, 1 \le k \le r; F_l, 1 \le l \le s\right) \vee_N \left(T'_j, 1 \le j \le p; I'_k, 1 \le k \le r; F'_l, 1 \le l \le s\right)$$
$$= \left(T_j \vee_F T'_j, 1 \le j \le p; I_k \wedge_F I'_k, 1 \le k \le r; F_l \wedge_F F'_l, 1 \le l \le s\right)$$

Multi/Refined Neutrosophic Intersection

$$\left(T_j, 1 \le j \le p; I_k, 1 \le k \le r; F_l, 1 \le l \le s\right) \wedge_N \left(T'_j, 1 \le j \le p; I'_k, 1 \le k \le r; F'_l, 1 \le l \le s\right)$$
$$= \left(T_j \wedge_F T'_j, 1 \le j \le p; I_k \vee_F I'_k, 1 \le k \le r; F_l \vee_F F'_l, 1 \le l \le s\right)$$

Multi/Refined Neutrosophic Negation

$$\neg_N\left(T_j, 1 \le j \le p; 1 \le k \le r; F_l, 1 \le l \le s\right) = \left(F_l, 1 \le l \le s; 1 - I_k, 1 \le k \le r; T_j, 1 \le j \le p\right)$$

Multi/Refined Neutrosophic Implication and Equivalence

Let $A = \left(T_j, 1 \le j \le p; I_k, 1 \le k \le r; F_l, 1 \le l \le s\right)$
and $A' = \left(T'_j, 1 \le j \le p; I'_k, 1 \le k \le r; F'_l, 1 \le l \le s\right).$
Then $A \to_N A'$ means $(\neg_N A) \vee_N A'$
and $A \leftrightarrow_N A'$ means $\left[[A \to_N A']\text{and}[A' \to_N A]\right]$.

ii. The case when the neutrosophic tuples have different (p, r, s)-formats.

Let $B_1 = \left(T_j, 1 \le j \le p_1;\ \ I_k, 1 \le k \le r_1;\ \ F_l, 1 \le l \le s_1\right)$ of (p_1, r_1, s_1)-format,
and $B_2 = \left(T'_j, 1 \le j \le p_2;\ \ I'_k, 1 \le k \le r_2;\ \ F'_l, 1 \le l \le s_2\right)$ of (p_2, r_2, s_2)-format.
We compute the weight average of each neutrosophic component of both tuples, and we get:
$B_1 = (T_{wa}, I_{ua}, F_{va})$,
and $B_2 = (T'_{w\prime a}, I'_{u\prime a}, F'_{v\prime a})$,

which have the (1, 1, 1)-form, let's simplify their notation under the form:

$B_1 = (T, I, F)$

and $B_2 = (T', I', F')$.

and one applies the well-known and most used neutrosophic operators:

$$(T, I, F) \vee_N (T', I', F') = (T \vee_F T', I \wedge_F I', F \wedge_F F')$$

$$(T, I, F) \wedge_N (T', I', F') = (T \wedge_F T', I \vee_F I', F \vee_F F')$$

$$\neg(T, I, F) \vee_N (F, 1 - I, T)$$

$(T, I, F) \to_N (T', I', F')$ is the same as $(F, 1 - I, T) \vee_N (T', I', F')$, or $(F \vee_F T', (1 - I) \wedge_F I', T \wedge_F F')$

$(T, I, F) \leftrightarrow_N (T', I', F')$ is the same as $(T, I, F) \to_N (T', I', F')$ and $(T', I', F') \to_N (T, I, F)$

or $(F \vee_F T', (1 - I) \wedge_F I', T \wedge_F F')$ neutrosophic and $(F' \vee_F T, (1 - I_1') \wedge_F I, T' \wedge_F F)$,

or $([(F \vee_F T') \wedge_F (F' \vee_F T)], [(1 - I) \wedge_F I'] \vee_F [(1 - I_1') \wedge_F I], [(T \wedge_F F') \wedge_F (T' \wedge_F F)]$).

13.1. Weight Averaging and Neutrosophic Operators

The (weight) averaging and the neutrosophic operators for (*p, r, s*)-tuples, in general, do not commute.

13.2. Counter-Example

Let's consider the (2,3,2)-tuples:

A = $(\{0.2, 0.3\}, \{0.1, 0.4, 0.5\}, \{0.6, 0.9\})$

and B = $(\{0.8, 0.4\}, \{0.6, 0.0, 0.3\}, \{0.5, 0.6\})$

i. Union, then Averaging Union:

$$A \vee_N B =$$

$$= \left(\left\{max\{0.2, 0.8\}, max\{0.3, 0.4\}, \left\{min\{0.1, 0.6\}\right\}, min\{0.4, 0.0\}, min\{0.5, 0.3\}, \left\{min\{0.6, 0.5\}\right\}, min\{0.9, 0.6\}\right\}\right)$$

$$= (\{0.8, 0.4\}, \{0.1, 0.0, 0.3\}, \{0.5, 0.6\})$$

Averaging:

$$A \vee_N B = \left(\frac{0.8 + 0.4}{2}, \frac{0.1 + 0.0 + 0.3}{3}, \frac{0.5 + 0.6}{2}\right) \simeq (0.60, 0.13, 0.55)$$

ii. Reversely: Averaging, then Union.

Averaging:

$$A = \left(\frac{0.2 + 0.3}{2}, \frac{0.1 + 0.4 + 0.5}{3}, \frac{0.6 + 0.9}{2}\right) \simeq (0.25, 0.33, 0.75)$$

$$B = \left(\frac{0.8 + 0.4}{2}, \frac{0.6 + 0.0 + 0.3}{3}, \frac{0.5 + 0.6}{2}\right) \simeq (0.60, 0.30, 0.55)$$

Union:

$$A \vee_N B = (max\{0.25, 0.60\}, min\{0.33, 0.30\}, min\{0.75, 0.53\})$$

$$= (0.60, 0.30, 0.55) \neq$$

$(0.60, 0.13, 0.55)$.

Conclusion: The MultiNeutrosophic Set was introduced now for the first time. It is a neutrosophic set whose elements' degrees of truth / indeterminacy / falsehood are evaluated by many sources to get a better accurate result. The ranking of the n-tuples of the form (p, r, s) and their operators were also built on.

References

1. Florentin Smarandache, A Unifying Field in Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics (sixth edition online), 1998 - 2007, ProQuest Information & Learning, Ann Arbor, Michigan, USA, https://fs.unm.edu/eBook-Neutrosophics6.pdf
2. F. Smarandache, The Scor The Score, Accuracy, and Certainty Functions determine a Total Order on the Set of Neutrosophic Triplets (T, I, F), Neutrosophic Sets and Systems, Vol. 38, 1-15, 2020, https://fs.unm.edu/NSS/TheScoreAccuracyAndCertainty1.pdf
3. V. Lakshmana Gomathi Nayagam, and Bharanidharan R., A Total Ordering on n - Valued Refined Neutrosophic Sets using Dictionary Ranking based on Total ordering on n - Valued Neutrosophic Tuplets , Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 379-396, https://fs.unm.edu/NSS/TotalOrderingRefined23.pdf
4. Smarandache, Florentin; n-Valued Refined Neutrosophic Logic and Its Applications to Physics, Progress in Physics, Vol. 4, 143 - 136, 2013, https://fs.unm.edu/RefinedNeutrosophicSet.pdf .

Neutrosophic TwoFold Algebra

Florentin Smarandache

Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract

We introduce for the first time the concepts of Neutrosophic TwoFold Algebra with its corresponding Neutrosophic TwoFold Law, and their derivatives of Fuzzy (and all fuzzy-extensions) Two Fold Algebras and Laws. A practical application and a numerical example are also presented.

Keywords: Classical Algebra, Hybrid Algebra, Neutrosophic Algebra, Neutrosophic TwoFold Algebra, Neutrosophic TwoFold Law, Multiple Concentrations, Immiscible Liquids, Spaces Composed of Heterogeneous Subspaces.

1 | Introduction

Applications of the TwoFold Algebra in general occur in chemistry for the mixtures of liquids of various concentrations, and of immiscible liquids as well, and in any field whose space (set) is made up of heterogeneous subspaces (subsets) of elements.

2 | Neutrosophic TwoFold Algebra

This is called a TwoFold Algebra because it has two types of algebras:

(i) The first algebra is with respect to the elements x belonging to a set A (classical type algebra);

(ii) and the second algebra is with respect to the neutrosophic components (t, i, f) of the elements.

This is a hybrid structure, because a classical algebraic operation is inter-related with a fuzzy (or fuzzy-extensions) operation.

In the following we use, as fuzzy-extensions, the neutrosophic set / logic / probability.

3 | Definition of Neutrosophic TwoFold Algebra

Let U be a universe of disclosure and a non-empty neutrosophic set $A \subset U$,

$$A_{(T,I,F)} = \{x(T_A(x), I_A(x), F_A(x)), (T_A(x), I_A(x), F_A(x)) \in [0,1]^3, x \in U\},$$

where $T_A(x), I_A(x), F_A(x)$ are degrees of truth-membership, indeterminacy-membership and falsehood-membership of the generic element x with respect to the set A.

4 | Definition of Neutrosophic TwoFold Law

In consequence, we define the Neutrosophic TwoFold Law Δ as follows:

$$\Delta: A_{(T,I,F)} \times A_{(T,I,F)} \to A_{(T,I,F)}$$

$$x_{1(t_1,i_1,f_1)} \Delta x_{2(t_2,i_2,f_2)} = (x_1 \# x_2)_{(t_1,i_1,f_1)*(t_2,i_2,f_2)}$$

The law Δ is formed by two *sub-laws* $\#$ and $*$ respectively, that may be totally dependent, partially dependent and partially independent, or totally independent of each other, in function of the application they are used to.

5 | The Most General Form of Neutrosophic TwoFold Law

It is defined as below:

$$g: A_{(T,I,F)} \times A_{(T,I,F)} \to A_{(T,I,F)}$$

$$g(x_1,t_1,i_1,f_1,x_2,t_2,i_2,f_2) = h(x_1,t_1,i_1,f_1,x_2,t_2,i_2,f_2)_{(T(x_1,t_1,i_1,f_1,x_2,t_2,i_2,f_2),I(x_1,t_1,i_1,f_1,x_2,t_2,i_2,f_2),F(x_1,t_1,i_1,f_1,x_2,t_2,i_2,f_2))}$$

where all five functions g, h, T, I, F are functions of eight variables $x_1,t_1,i_1,f_1,x_2,t_2,i_2,f_2$.

6 | Practical Application of General Neutrosophic TwoFold Algebra

Applications of the TwoFold Algebra in general occur in chemistry for the mixtures of liquids of various concentrations, and of immiscible liquids as well, and in any field whose space (set) is made up of heterogeneous subspaces (subsets) of elements.

Let's consider: 2 liters of concentrated liquid of three substances that do not mixt with each other (immiscible): $2_{(0.6,0.1,0.3)} = (1.2, 0.2, 0.6)$, because 0.6 of 2 = 60% of 2 = 0.6 × 2 = 1.2 liters, 0.1 of 2 = 0.1 × 2 = 0.2 liters, and 0.3 of 2 = 0.3 × 2 = 0.6 liters, with 1.2 + 0.2 + 0.6 = 2.

Therefore, the space (quantity) of 2 liters is formed of three heterogeneous sub-spaces (sub-quantities) of 1.2, 0.2, and respectively 0.6 liters. Similarly, the space (quantity) of 3 liters of concentrated liquid is formed of three heterogeneous sub-spaces (sub-quantities) of 1.5, 1.2, and 0.3 liters, $3_{(0.5,0.4,0.1)} = (1.5, 1.2, 0.3)$, where 1.5 + 1.2 + 0.3 = 3.

Therefore, both liquids, mixed together, give:

$$(x_1 + x_2)_{\left(\frac{x_1t_1+x_2t_2}{x_1+x_2}, \frac{x_1i_1+x_2i_2}{x_1+x_2}, \frac{x_1f_1+x_2f_2}{x_1+x_2}\right)}$$

where $x_1 = 2$ with $(t_1, i_1, f_1) = (0.6, 0.1, 0.3)$, and $x_2 = 3$ with $(t_2, i_2, f_2) = (0.5, 0.4, 0.1)$, whence $x_1 + x_2 = 2 + 3 = 5$ liters of concentrations: and $\left(\frac{x_1t_1+x_2t_2}{x_1+x_2}, \frac{x_1i_1+x_2i_2}{x_1+x_2}, \frac{x_1f_1+x_2f_2}{x_1+x_2}\right) = (2.7, 1.4, 0.9)$, therefore $5_{(2.7,1.4,0.9)}$, where $2.7+1.4+0.9=5.$

The space of 5 liters is formed by three heterogeneous sub-spaces of 2.7, 1.4, and 0.9 liters.

7 | Fuzzy and (Fuzzy-Extensions) TwoFold Algebra

Applications of the TwoFold Algebra in general occur in chemistry for the mixtures of liquids of various concentrations, and of immiscible liquids as well, and in any field whose space (set) is made up of heterogeneous subspaces (subsets) of elements.

The Neutrosophic TwoFold Algebra may be adjusted to any fuzzy-extensions theory. Let *U* be a universe of discourse, and let *A* be a non-empty set included in *U*.

(i) Fuzzy TwoFold Algebra

Let $A_{(T)} = \{x(T_A), T_A \in [0,1]; x \in U\}$, where T_A is the degree of truth-membership of the generic element x with respect to the set A.

The Fuzzy TwoFold Law is then defined as:

$\Delta: A_{(T)} \times A_{(T)} \to A_{(T)}$

$x_{1(t_1)} \Delta\, x_{2(t_2)} = (x_1 \# x_2)_{(t_1 * t_2)}$

Similarly, the Fuzzy TwoFold Law Δ is formed by two (totally dependent, partially dependent and partially independent, or totally independent) sub-laws $\#$ and $*$.

(ii) Intuitionistic Fuzzy TwoFold Algebra

$A_{(T,F)} = \{x(T_A(x), F_A(x)), (T_A(x), F_A(x)) \in [0,1]^2, x \in U\}$, where $T_A(x)$ and $F_A(x)$ are degrees of truth-membership and falsehood-nonmembership of the generic element x with respect to the set A.

The Intuitionistic TwoFold Law is defined as:

$\Delta: A_{(T,F)} \times A_{(T,F)} \to A_{(T,F)}$

$x_{1(t_1,f_1)} \Delta\, x_{2(t_2,f_2)} = (x_1 \# x_2)_{(t_1,f_1)*(t_2,f_2)}$

(iii) Similarly, for {any fuzzy-extensions [1] } TwoFold Algebra and Law such as:

Inconsistent Intuitionistic Fuzzy Set (or Picture Fuzzy Set, or Ternary Fuzzy Set) TwoFold Algebra and Law;

Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type) TwoFold Algebra and Law;

q-Rung Orthopair Fuzzy TwoFold Algebra and Law;

Spherical Fuzzy TwoFold Algebra and Law;

n-HyperSpherical Fuzzy TwoFold Algebra and Law;

Refined Neutrosophic and MultiNeutrosophic TwoFold Algebra and Law;

Plithogenic TwoFold Algebra and Law.

As future research it would be to develop some of the above new types of TwoFold Algebras and Laws, with their real-life applications.

8 | Single-Valued, Interval-Valued, and Subset-Value Fuzzy (and fuzzy-extensions) TwoFold Algebras and Laws

All previous (fuzzy or fuzzy-extensions) TwoFold Algebras and Laws are Single-Valued, meaning that their components (degrees of membership, indeterminacy, nonmembership, etc.) are single numbers included in [0, 1].

But, if the components are interval-valued, or most generally subset-valued, of [0, 1], they are called Interval-Valued (Fuzzy or fuzzy-extensions) TwoFold Algebras and Laws and respectively Subset-Valued (Fuzzy or fuzzy-extensions) TwoFold Algebras and Laws.

9 | Numerical Example of Neutrosophic TwoFold Algebra and Law

Let U be a neutrosophic universal set and A be a non-empty subset of U.

Let $A_{(T,I,F)} = \{x_j(t_j, i_j, f_j), x_j \in U, t_j, i_j, f_j \in [0,1]\}$ be a neutrosophic set endowed with the Neutrosophic TwoFold Law defined as follows: $x_1(t_1, i_1, f_1)\, \Delta\, x_1(t_2, i_2, f_2) = (x_1 \cdot x_2)[(t_1, i_1, f_1) * (t_2, i_2, f_2)]$

where each sub-law is defined respectively as:

$$\cdot : A^2 \to A$$

and

$$*: \{(t, i, f); t, i, f \in [0,1]\}^2 \to \{(t, i, f); t, i, f \in [0,1]\}.$$

Let's assume the classical integer multiplication *modulo 3* as first classical sub-operation (sub-law), $x_1 \cdot x_2 = (x_1 \cdot x_2)(mod\ 3)$ and the neutrosophic sub-law be

$$(t_1, i_1, f_i) * (t_2, i_2, f_2) = \left(\frac{t_1+t_2}{2}, \frac{i_1+i_2}{2}, \frac{f_1+f_2}{2}\right)$$

or the averages of *t, i, f* neutrosophic components respectively.

Therefore, $x_1(t_1, i_1, f_i)\, \Delta\, x_1(t_2, i_2, f_2) = (x_1 \cdot x_2)(mod\ 3)_{\left(\frac{t_1+t_2}{2}, \frac{i_1+i_2}{2}, \frac{f_1+f_2}{2}\right)}$

Let $A = \{0(0.2,0.4,0.6), 1(0.8,0.2,0.0), 2(0.4,0.8,0.6)\}$ be a neutrosophic set.

Table 1. of the Neutrosophic TwoFold Law Δ.

Δ	0(0.2,0.4,0.6)	1(0.8,0.2,0.0)	2(0.4,0.8,0.6)
0(0.2,0.4,0.6)	0(0.2,0.4,0.6)	0(0.5,0.3,0.3)	0(0.3,0.6,0.6)
1(0.8,0.2,0.0)	0(0.5,0.3,0.3)	1(0.8,0.2,0.0)	2(0.6,0.5,0.3)
2(0.4,0.8,0.6)	0(0.3,0.6,0.6)	2(0.6,0.5,0.3)	1(0.4,0.8,0.6)

Let $A = \{0(0.2,0.4,0.6), 1(0.8,0.2,0.0), 2(0.4,0.8,0.6)\}$ be a neutrosophic set.

The Neutrosophic TwoFold Law Δ is partially inner defined, since for example

$1(0.8,0.2,0.0)\, \Delta\, 1(0.8,0.2,0.0) = 1(0.8,0.2,0.0) \in A$

and partially outer-defined, because for example

$2(0.4,0.8,0.6)\, \Delta\, 1(0.8,0.2,0.8) = 2(0.6,0.5,0.3) \notin A$, but $2_{(0.6,0.5,0.3)} \in U \backslash A$.

Anti-commutative:

because (according to Table 1) for any two elements

$a_1(t_1, i_1, f_i)$ and $a_2(t_2, i_2, f_2) \in A$, with $a_1 \neq a_2$, one has:

$a_1(t_1, i_1, f_i)\, \Delta\, a_2(t_2, i_2, f_2) = a_2(t_2, i_2, f_2)\, \Delta\, a_1(t_1, i_1, f_i) \notin A$

Anti-neutral element

Since there is no neutral element (see the above Table 1).

Anti-inverse element

And no element has an inverse, since there is no neutral element.

We got an unusual hybrid structure!

10 | Conclusions

In this paper we have founded a new type of hybrid algebra, called Neutrosophic TwoFold Algebra and its corresponding Neutrosophic TwoFold Law, by combining two algebras:

the first algebra is a classical algebra of the elements, and the second algebra is an algebra of the elements' components (the degrees of membership / indeterminacy / nonmembership).

Then we extended it to all Fuzzy (and fuzzy-extensions) sets. The TwoFold Algebras have applications in any field whose space is formed by heterogeneous sub-spaces.

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Author Contributaion

All authors contributed equally to this work.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] Florentin Smarandache, Neutrosophic Set is a Generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, and n-HyperSpherical Fuzzy Set, while NeutroSophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision (revisited), Journal of New Theory, 29 (2019) 01-35; arXiv, Cornell University, New York City, NY, USA, pp. 1-50, 17-29 November 2019, https://arxiv.org/ftp/arxiv/papers/1911/1911.07333.pdf ; The University of New Mexico, Albuquerque, USA, Digital Repository, pp. 1-50, https://digitalrepository.unm.edu/math_fsp/21.

Theory of Distances in NeutroGeometry

Erick González Caballero

Asociación Latinoamericana de Ciencias Neutrosóficas, Havana, CUBA

Florentin Smarandache

Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract. NeutroGeometry is one of the most recent approaches to geometry. In NeutroGeometry mod-els, the main condition is to satisfy an axiom, definition, property, operator and so on, that is neither entirely true nor entirely false. When one of these concepts is not satisfied at all it is called AntiGeometry. One of the problems that this new theory has had is the scarcity of models. Another open problem is the definition of angle and distance measurements within the framework of NeutroGeometry. This paper aims to introduce a general theory of distance measures in any NeutroGeometry. We also present an algorithm for distance measurement in real-life problems.

Keywords: NeutroGeometry, path, rectifiable path, single-valued neutrosophic set, Taxicab geometry, Chinese checker metric

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1 Introduction

NeutroGeometries have recently emerged as new proposals by Professor F. Smarandache within the extensive history of geometry. NeutroGeometries are heirs of the neutrosophic vision of the world founded and created by F. Smarandache himself [1, 2]. Neutrosophy is the branch of philosophy that addresses the existence of neutrality, the erroneous, unknown, contradictory, paradoxical, incoherent, inconsistent and neutral, among other concepts of this type [3].

Neutrosophy as a new branch of philosophy has had a favorable impact on some branches of knowledge, such as the emergence of neutrosophic sets as a generalization of fuzzy sets, intuitionistic fuzzy sets and interval-valued fuzzy sets. Also, those logic has led to neutrosophic logic. On the other hand, applications of neutrosophy have appeared in decision-making, digital image processing and statistics, to solve problems in psychology, sociology, economics, physics and mathematics.

NeutroAlgebras appeared before NeutroGeometries, where the degree of indeterminacy is part of some algebraic structure models, besides the degrees of truthfulness and falsehood [4]. However, we must highlight that the study of NeutroGeometries has as a precedent mixed geometries or Smarandachean geometries several decades ago. The definition of a mixed geometry is the acceptance of geometries where at least one axiom is Smarandachely negated in one of the following ways: (a) It is fulfilled by one part of the elements of the structure and is not fulfilled by the rest of them, (b) It is not fulfilled in one way by one part of the elements and is not fulfilled by another part of the elements, but in a different way [5].

That is to say, when an axiom is Smarandachely negated, a geometric structure can exist where the axiom is fulfilled for some elements of the space and not fulfilled for others. For example, a plane that is defined where the fifth Euclidean postulate is satisfied by some straight lines and other lines do not satisfy it. Also, we can have that Euclid's fifth postulate is never fulfilled, but in two different ways: on the one hand, with elements that satisfy the axiom of multiple parallels, and on the other hand with lines and points where there is no parallelism.

This initial historical Smarandache's idea of denying concepts within geometric structure led him to the notion of also including indeterminacy as part of the structure. Thus, NeutroGeometry is understood as the geometric structure where there may be some degree of indeterminacy besides the degrees of truthfulness and falsehood, that is, at least one concept within the structure is partially satisfied by the elements of the structure with some degree of indeterminacy. When one of the concepts, definitions, operations or axioms, among others, is not fulfilled in any case, it is called AntiGeometry. A Smarandache

Geometry is a NeutroGeometry when one axiom is partially false and partially true (and it may also be partially indeterminate), or an AntiGeometry when at least one axiom is totally false.

These more modern geometric approaches recover the primeval emergence of geometry as the branch of mathematics dedicated to the study of real-life physical objects. This first objective had become over time an unreal abstract approximation of physical reality. However, it is obvious that in many cases it is not possible to travel a path through the shortest distance. This is because, in the geometric spaces of the real world, there is indeterminacy, vagueness, uncertainty and so on. This has been Smarandache's main motivation for the creation of NeutroGeometries.

Today, this new theory lacks some points that would make it more solid. Some of them are the scarcity of geometric models where this theory is manifested. Hence, this paper aims to contribute to the solidification of NeutroGeometry introducing a theory of distance within the geometric structures in the neutrosophic framework. However, this is not a simple task. There are multiple circumstances recognized by some authors within the geographical scope, where a region can change in time and space [6]. In the so-called vague regions, one of the elements to consider is geometry [7]. In them, uncertainty is associated with each point, line and region that is studied, where indeterminacy can be due to multiple causes and is not easy to model.

The purpose of vague regions is the use of spatial databases within the geographical scope, such as geographical information systems (GIS), and their application in the modeling of geographical conditions that are uncertain in time and space [8]. They use probabilities, fuzzy sets and supervaluationism to model uncertainty or vagueness [9]. The elements that are components of geography such as forests, plains and mountains cannot be defined exactly in their dimensions, and in everyday life people refer to them vaguely. Neither the relationships between these elements escape the vagueness of natural language for referring to these geographical elements. It is common to use vague terms such as “near”, “far” and “small area” rather than precise distance or area measures such as 1 km, 10 km and 3 km^2.

In this paper, however, we propose other objectives, although the motivations are the same. We work directly with geometric properties to solve real problems. Nevertheless, we treat geometric structures beyond the Euclidean terrestrial geometry of GIS. Specifically, we deal with the problem of distances by trajectories in the so-called NeutroGeometries from any geometry applied to any region of the universe.

In classical geometry, the distance between two points is calculated in a segment contained in the geodesic line, where the distance is the shortest one. In NeutroGeometry this is not necessarily true. It depends on the person's knowledge and the obstacles that may be encountered along the way. It is a subjective and cognitive distance, in the sense that it depends on each person - on their knowledge, the historical knowledge about the region and the obstacles that objectively interrupt the path - to determine the shortest path to follow between one point and another. The indeterminacy may be due to the presence of unwanted phenomena such as a conflictive zone, there is a territory dominated by wild animals, there is a river with impassable parts, there are mountains that are difficult to climb and many more examples that we can think of.

It also depends on the orientation; to walk from point A to point B is not always equivalent to walking in the opposite direction. For example, if there is a hill, it is not the same to walk uphill as downhill for the same track. Transiting a river is not the same as navigating with the current or against it.

Also, if we are traveling through a river, it is not efficient to travel a section by boat, then go to land and later continue by boat, only to continue along a straight line. It is better to navigate the river, even if this means deviating at times from the path that coincides with the geodesic line.

They are reasons that motivate this article to propose a general solution to find the shortest distance from one point to another regardless of the geometry in question, whether it is Euclidean or non-Euclidean. This is an important element to consider once a geometric model is established. The other point is the definition of angular measurements, but depending on the geometry in question it is possible to maintain the predefined measurements of angles. However, the problem of finding distances is critical in situations where there are indeterminate or uncertain regions, where the real characteristics of the terrain to be covered may not be known or have degrees of uncertainty.

This paper is divided into section 2 where the preliminary concepts necessary to understand the contribution of this paper to NeutroGeometry theory are explained. Section 3 contains the elements of the theoretical proposal that is the objective of this article. These results are argued with some examples of geometries such as the well-known Taxicab geometry or Chinese checker metric that are cases of the herein proposed theory. Section 4 consists of conclusions.

2. Basic Notions of NeutroGeometry

Smarandache added the prefixes Neutro- and Anti- to certain mathematical structures’ names [1, 2,

10-12]. The first of them indicates that the concept presents degrees of truth, falsehood and indeterminacy, and the second one denotes that the concept is not fulfilled in all cases. For example, an Algebra is made up of a set of elements with one or more operations among them that satisfy at least one axiom. NeutroAlgebra satisfies these conditions with a degree of truthfulness, a degree of indeterminacy and a degree of falsity. On the other hand, AntiAlgebra does not satisfy at least one of these conditions at all. So, NeutroAlgebra satisfies its conditions in a triad (t, i, f) of truthfulness, indeterminacy and falseness, where $(t, i, f) \neq (1, 0, 0)$ in at least one of them. All the conditions of classical algebra are satisfied in a triad of $(1, 0, 0)$. AntiAlgebra fulfills that any of its conditions are never satisfied corresponding to the triad $(0, 0, 1)$.

Similarly, NeutroGeometry is made up of elements, generally they are the concepts of "point", "line", "plane", "space" and "hyperspace". There are relationships between them, for example, a point is contained in a line and a line is contained in a plane, and so on. They must satisfy certain axioms. These axioms usually either satisfy or contradict the Euclidean axioms, specifically Euclid's five postulates, with emphasis placed on the fifth postulate.

It is not possible to talk about NeutroGeometry without referring to mixed geometries or Smarandache geometries that emerged several decades ago:

"A Smarandache geometry is a geometry which has at least one Smarandache denied axiom, i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalidated, or only invalidated but in multiple distinct ways and a Smarandache n-manifold is an n-manifold that supports a Smarandache geometry." [5]

However, a mixed geometry only takes into account the negation of the concept in a certain sense. NeutroGeometry may deny some components of classical geometry, but it also takes into account situations where there is indeterminacy due to indistinguishability, error, ignorance, paradoxes, contradictions, vagueness, inconsistency and so on.

A mixed geometry may contain some elements that satisfy Euclid's fifth postulate of parallelism, while other elements do not satisfy it. Moreover, a mixed geometry can contradict the fifth postulate, on the one hand, because more than one parallel to a given line passes through an external point and on the other hand because of the absence of parallelism for other elements.

There are real-life situations that cannot be modeled using a mixed geometry. An example illustrated in Smarandache's writings is a river with unknown areas. Due to the lengths of the river compared to the terrestrial measures, we can utilize a geometric Euclidean model. However, it is unrealistic to consider a straight line for measuring the shortest distance between two points within the river. This is because the straight line that joins both points can pass through an unknown area, so we are not sure whether this area is passable or not. Then this can no longer be the shortest line to travel through both points. It is necessary to deviate in some way along another path avoiding this region or otherwise risk exploring an unknown place, see Figure 1.

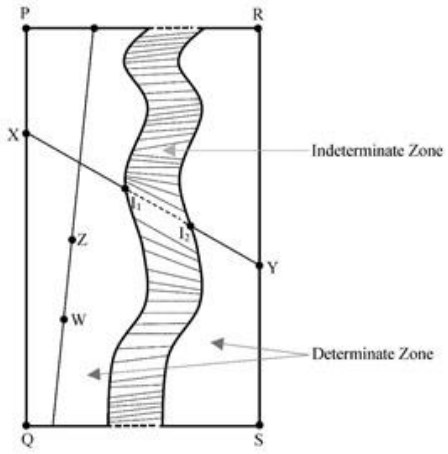

Figure 1: NeutroGeometry example. River with an Indeterminate Zone [1].

In the next section, we propose a theory of distances where there is indeterminacy. Nevertheless, we can consult some references where this topic is discussed, [13-16].

3. Distances in NeutroGeometries

This section contains the details of the proposed theory of distances within the framework of NeutroGeometry.

The first point to consider is that NeutroGeometry must be characterized by the geometry from which it is defined. This can be either Euclidean, hyperbolic, elliptical or mixed geometry. Such a starting geometric structure must have a distance function defined between any two points of the geometric structure. We denote as (G, d_G) this initial geometric space with such a distance. To simplify the results, in this paper we will limit the study to the plane, therefore the geometric structures will only contain points and lines, and the plane is generally defined as a 2-dimensional surface on Euclidean space.

Thus, the two-dimensional coordinate system XOY, where O is the origin together with the distance function forms a metric space. Let us call it (X, d_X) and recall that a metric space is defined from a set of elements and a distance function among them [17].

The other important definition is a *path*, which is a parametric function p: $[0,1] \to X$. That is, it is a function defined on the coordinate system. The paths will be assumed to be rectifiable. In other words, considering the partitions $\mathrm{T} = [t_0 = 0, t_1, t_2, \cdots, t_{n-1}, t_n = 1]$ there is always:

$$L(p) = sup_T\{\sum_{i=1}^{n} d(p(t_{i-1}), p(t_i))\} \quad (1)$$

$L(p)$ is called the *length of* p in (G, d_G), where $\sup_{\mathrm{T}}$ is the supremum over the set of finite partitions T. What is more, let us suppose that we are in the presence of a *path metric space* that is, given $x_1, x_2 \in X$, then:

$$d_G(x_1, x_2) = inf\{L(p): p \text{ is a rectifiable path from } x_1 \text{ to } x_2\} \quad (2)$$

Where inf is the abbreviation of infimum.

Let us also consider other definitions of paths. For example, each path is oriented when for all partitions $\mathrm{T} = [t_0 = 0, t_1, t_2, \cdots, t_{n-1}, t_n = 1]$ we obtain paths that go from x_1 to x_2, however, we can define $-p(t_i) = p(t_{n-i})$ for obtaining the same path in the opposite direction.

Let us call *subpath* when the path is obtained from p such that for $p(0) = x_1$ and $p(1) = x_2$ the subpath is taken as $q(0) = p(u)$ or $q(1) = p(v)$ for $u > 0$ or $v < 1$. That is, geometrically the subpath is a path contained in the trajectory of the path.

Definition 1. Suppose that besides $L(p)$ we define a *passability function* $\varepsilon(p) \in [0,1]$ for each rectifiable path p. $\varepsilon(p)$ satisfies the following conditions:

i. If $p = \emptyset$ or one of the points $p(t_i)$ is an ideal point, then $\varepsilon(p) = 0$. Let us recall that the ideal points in models of non-Euclidean geometries do not belong to the models, but they are on the border. E.g., the Poincaré Disk model contains the interior points of the unit circle except for the border, thus the circumference of radius 1 is the set of ideal points. So, the distance from one point to an ideal point is infinite.

ii. When $p \equiv x \in X$, that is, the path is formed by a single point, we have $\varepsilon(p) = 1$.

iii. $\forall q$ subpath of p it is fulfilled $\varepsilon(p) \leq \varepsilon(q)$.

Definition 1 does not require that $\varepsilon(p) = \varepsilon(-p)$.

As a consequence of Definition 1, we have the following properties:

1. p, q and r are three paths such that $p(1) = q(0)$. Let us denote by $r = p \cup q$ the operation of obtaining the union of the two paths where $r\left(\frac{t}{2}\right) = p(t)$ and $r\left(\frac{t+1}{2}\right) = q(t)$. Therefore, we have $r(0) = p(0)$, $r(1) = q(1)$, $r\left(\frac{1}{2}\right) = p(1) = q(0)$. In this case $\varepsilon(r) \leq min(\varepsilon(p), \varepsilon(q))$ according to Definition 1 condition (iii).

2. Let p, q and r be three paths equally oriented, such that for certain intervals $\mathrm{I}_1, \mathrm{I}_2 \subset \mathrm{I} = [0,1]$ $r(I) \equiv p(\mathrm{I}_1) \equiv q(\mathrm{I}_2)$. This operation is denoted by $r = p \cap q$. So $max(\varepsilon(p), \varepsilon(q)) \leq \varepsilon(r)$, according to Definition 1 condition (iii) as well.

Definition 2. Given (G, d_G) is a metric space ([17]) in a geometric structure. The *NeutroGeometric distance* between x_1 and x_2 for $x_1, x_2 \in X$ is defined as:

$$d_{NG}(x_1, x_2) = inf\left\{\frac{L(p)}{\varepsilon(p)}: p \text{ is a rectifiable path from } x_1 \text{ to } x_2\right\} \quad (3)$$

Where $\varepsilon(p)$ is the passability function of p.

Some properties of $d_{NG}(x_1, x_2)$ are the following:

1. Fixing p, when the passability function increases, the distance $d_{NG}(x_1, x_2)$ decreases.

2. When $\forall p\ \varepsilon(p) = 1$ then we have $d_{NG}(x_1, x_2) = d_G(x_1, x_2)$. This happens when there is no indeterminacy or obstacle between x_1 and x_2, which is the hypothesis assumed in classical geometries.

3. In general, $d_{NG}(x_1, x_2)$ is not necessarily symmetrical, thus $d_{NG}(x_1, x_2) \neq d_{NG}(x_2, x_1)$.

4. $\forall x \in X\ d_{NG}(x, x) = 0$, since if p is the path that goes from x to itself then since definition we have $\varepsilon(p) = 1$ and $L(p) = 0$.

5. In some recent models, there are cases where $d_G(x_1, x_2) = \infty$ when an ideal point is contained in the path. Then $d_{NG}(x_1, x_2) = \infty$, since from Equation 3 we have $d_{NG}(x_1, x_2) = \frac{\infty}{0} = \infty \cdot \infty = \infty$.

6. When $\varepsilon(p) = 0$, that is, when there is total certainty of the impassibility along the path p, then we have $\frac{L(p)}{\varepsilon(p)} = \infty$ either for $L(p) < \infty$ or $L(p) = \infty$.

$L(p)$ is defined from the characteristics of geometry (G, d_G). However, for the geometric space (NG, d_{NG}) we must define the passability function $\varepsilon(p)$.

For example, for each path it can be determined $\varepsilon(p)$ as the probability that there is transitibility along the path p. Also $\varepsilon(p)$ can be the function of the possibility of traveling along the path p, where this is a possibility measure developed by Dubois and Prade in the field of fuzzy logic [18, 19]. Equivalently, if a multi-agent system is modeled, $\varepsilon(p)$ can be interpreted as the permission that exists to travel the path in the scope of the deontic logic. $\varepsilon(p)$ can be a truth value in fuzzy logic [20], can be a numerical function obtained from the ordered pair of both membership and non-membership values within the intuitionistic fuzzy logic [21], can be an element of an interval-valued fuzzy set [22, 23] or may be a function of a neutrosophic number [24, 25].

In the latter case when there is a neutrosophic valuation formed by the triple $n = (t, i, f)$ we will need to convert n into a real value using the following formula of the Score function [26]:

$$\mathcal{S}(n) = \frac{2+t-i-f}{3} \qquad (4)$$

For the neutrosophic order, defined as $n_1 \preccurlyeq n_2$ if and only if $t_1 \leq t_2, i_2 \leq i_1$ and $f_2 \leq f_1$, where $n_1 = (t_1, i_1, f_1)$ and $n_2 = (t_2, i_2, f_2)$; then we have $(0,1,1)$ is the smallest value of n and $(1,0,0)$ is the maximum value. Therefore, the score of $(1,0,0)$ corresponds to $\mathcal{S}\big((1,0,0)\big) = 1$ and the score of $(0,1,1)$ corresponds to $\mathcal{S}\big((0,1,1)\big) = 0$.

When we are using intervals, they can be des-neutrosophied converting them into a real single value using [27]:

$$\lambda([a, b]) = \frac{a+b}{2} \qquad (5)$$

Also, we can adapt the calculation of the infimum to intervals, and as a final result we obtain an interval containing indeterminacy that can be converted into the so-called neutrosophic number.

Note that $d_{NG}(x_1, x_2)$ is increasing for the length of the path but is decreasing for the passability function of the path. Therefore, the longer the path is, the greater the value of the distance in NeutroGeometry is, both because the length increases and also because the passability function could decrease.

The theory presented so far is a generalization of already known geometries. For example, Minkowski's Taxicab geometry and the distance function that bears this name [28, 29]:

$$\mathrm{d_T(A, B)} = |\mathrm{x_A} - \mathrm{x_B}| + |\mathrm{y_A} - \mathrm{y_B}| \qquad (6)$$

Where (x_A, y_A) are the coordinates of A and (x_B, y_B) are the coordinates of B in the Euclidean Cartesian plane.

This geometry is based on Euclidean geometry (G, d_G), where the points, lines and measurements between angles are the usual ones. Only the distance function between two points is changed to the one represented in Equation 6.

It is assumed that the only valid trajectories are those that go in a straight line in angles of 0, $\frac{\pi}{2}$, π or $\frac{3\pi}{2}$ concerning the x-axis. That is, the only possible paths follow trajectories parallel to one of the two coordinate axes. This geometry models the movement of a car in a city where the blocks are square and have the same dimensions. The taxicab can only turn right or left $\frac{\pi}{2}$ radians when it arrives at the corner of the block, see Figure 2.

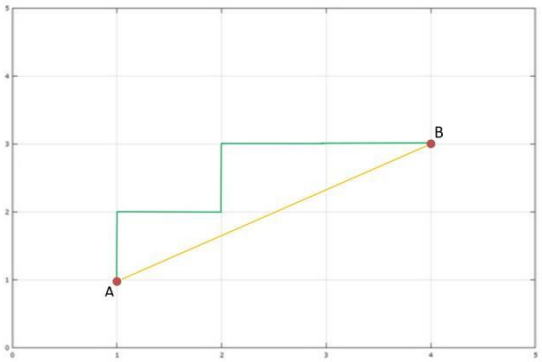

Figure 2: Trajectory from point A to B using paths in the form of broken lines (green lines) in Taxicab geometry. Observe the Euclidean straight line in yellow.

Therefore, paths that contain other types of straight lines are not allowed, thus the passability function is defined by:

$$\varepsilon(p) = \begin{cases} 1, if\ p \text{ is a path containing segments always parallels to one axis} \\ 0, \text{otherwise} \end{cases}$$

$$\text{So, } \frac{L(p)}{\varepsilon(p)} = \begin{cases} L(p), if\ p \text{ is a path containing segments always parallel to one axis} \\ \infty, \text{otherwise} \end{cases}$$

It is easy to check that the previous condition is satisfied only when $d_{NG}(A,B) = d_T(A,B)$. In Figure 2 the path represented with yellow line is therefore not allowed $\frac{L(p)}{\varepsilon(p)} = \infty$. The path of broken lines in green has a length equal to 5 and it can be proven that this is the distance between the two points for this geometry.

This same idea can be extended for other distances inspired by the Taxicab geometry. For example, when we have the following distance:

$$d_C(\mathrm{A},\mathrm{B}) = \max(|\mathrm{x_A} - \mathrm{x_B}|, |\mathrm{y_A} - \mathrm{y_B}|) + (\sqrt{2} - 1)\min(|\mathrm{x_A} - \mathrm{x_B}|, |\mathrm{y_A} - \mathrm{y_B}|) \qquad (7)$$

This is known as the *Chinese checker metric*. So, in this geometry it is only allowed movements in straight diagonal lines of $\frac{\pi}{4}$ radians in addition to those already defined in Taxicab geometry [30]. See Figure 3.

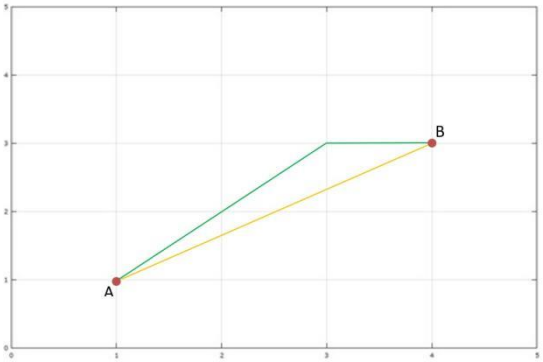

Figure 3: Trajectory from point A to B using paths in the form of lines corresponding to the Chinese checker metric.

In general, it is known that in the cities there are streets with blocks of different geometric shapes, they can be roundabouts, triangular, irregular, as well as square or rectangular, among others. Hence, our theory of distances is more exact, since each of these paths is measurable regardless of the type of shape it presents, see Figure 4.

Figure 4: The streets of a city (in black) have different shapes.

Let us observe that Taxicab or Chinese Checker geometry cannot be applied to measure distances in Figure 4. However, in this city, it is possible to apply Equation 3 because the paths to travel are rectifiable curves in Euclidean geometry as circular and rectilinear paths.

Also, in the case of real traffic, there are additional restrictions. E.g., although ideally we can turn down some streets, this is not always allowed, so we have to comply with traffic laws. These types of real-life situations are modeled thanks to the non-symmetry of the defined distance NG.

Apart from this, an interesting question is why we choose neutrosophy to model the function $\varepsilon(p)$. The neutrosophic theory offers greater possibilities for representing the different types of situations that may arise. In addition, it allows greater accuracy that is achieved at the cost of greater indeterminacy to represent the transitibility from one point to another through a path.

In general, we propose the following operations on the passability functions when there are unions or intersections of paths.

1. $\varepsilon(p \cup q) = min(\varepsilon(p), \varepsilon(q))$,
2. $\varepsilon(p \cap q) = max(\varepsilon(p), \varepsilon(q))$,

A clear limitation is that Equation 3 may be impractical since it requires analyzing all possible paths. In daily life, people make decisions about which path to take according to a good enough decision-making rather than an optimal one. Then, the amount of calculation of possible trajectories is reduced which would be too cumbersome.

The calculation of the path that minimizes the distance between two points in Equation 2 is simplified when we know the type of geometry we are dealing with. So, the path is identified with the geodesic and its equation. For example, in the Euclidean plane, the straight line between the points is the shortest path, and it is also very simple to calculate its algebraic equation. This is not necessarily the case in real life, where we need to know the terrain to get from one point to another throughout the shortest trajectory.

In daily life, human beings use Equation 3 more than Equation 2. Using the theory explained so far is an arduous task to select each path and calculate the shortest distance. In reality, decision-making in daily life is not based on selecting the optimal one, but rather on a "good enough" solution. From the distance defined in Equation 3, it is possible to define an approximate distance based on a finite subset of paths instead of all possible paths.

This leads to the following algorithm shown in Table 1.

Algorithm for deciding on the shortest path

1. Given a region R of the plane, two points A and B are defined as contained in R, and we want to measure the approximate minimum distance to go from A to B.
2. Let $E = \{e_1, e_2, \cdots, e_m\}$ be a group of locals, specialists, geographers, among others who know the terrain to be measured.
3. Γ denotes the subset of possible paths p from A to B, such that it has finite cardinality.
4. Each of the respondents in E is asked based on a scale of 0-10 to rate each $p \in \Gamma$ in terms of:
   (a) Feasibility to go from A to B passing through p.
   (b) Indeterminacy (due to ignorance, indifference, among others) to go from A to B passing through p.
   (c) Impossibility to go from A to B (passing through p).
5. The values of (t, i, f) are obtained in the following way:

$$t = \frac{sum(assessments\ in\ (a))}{10 \cdot m},\ i = \frac{sum(assessments\ in\ (b))}{10 \cdot m},\ f = \frac{sum(assessments\ in\ (c))}{10 \cdot m}$$

6. It is calculated $\varepsilon(p)$ according to Equation 4 for each of the possible values of (t, i, f). It is also calculated $L'(p)$ as a sufficiently approximate value of the actual length $L(p)$ of the path p. I.e., given $\epsilon > 0$ a prefixed allowed error then $|L'(p) - L(p)| \leq \epsilon$.

7. The approximate distance NG between both points is calculated by Equation 8.

$$d'_{NG}(A,B) = inf\left\{\frac{L'(p)}{\varepsilon(p)} : p \in \Gamma\right\} \quad (8)$$

Table 1: Algorithm for calculating the shortest approximate distance in NG based on neutrosophic numbers.

In this way, the calculation of a distance is linked to the collective knowledge about the paths to travel, rather than through an inherent single function for each geometry.

Example 1. Suppose we are in a boat, and we want to navigate a river whose bed has irregularities so that the left bank has calm waters and the right bank has turbulent waters in a certain section. However, we wish to go from point A to point B on the right side of the river. Furthermore, the path of the river itself is sinuous, see Figure 5.

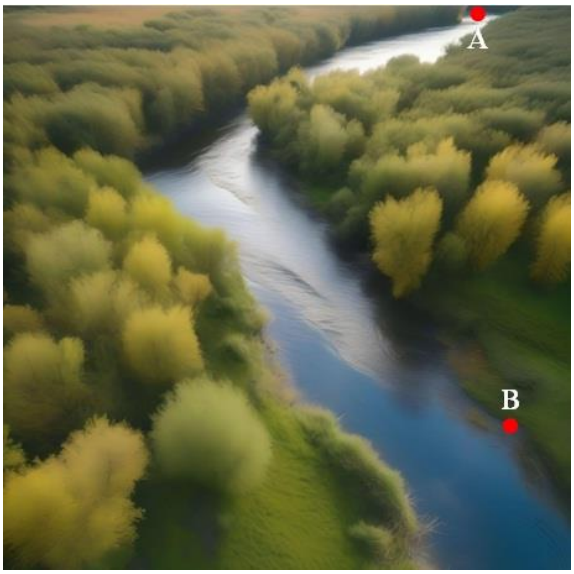

Figure 5: Picture of the river of the example. We need to go from point A to point B. This picture was generated by an AI tool.

That is why the shortest path that passes on the right is impassable and hence $\varepsilon(p_1) = \mathcal{S}\big((0,1,1)\big) = 0$. The path that goes through the center has some areas with a percentage of danger, let us say $\varepsilon(p_2) = \mathcal{S}\big((0.5,0.1,0.4)\big) = 0.66667$. While the path that goes left and then turns right is the longest one and it is completely safe, therefore $\varepsilon(p_3) = \mathcal{S}\big((1,0,0)\big) = 1$, see Figure 6.

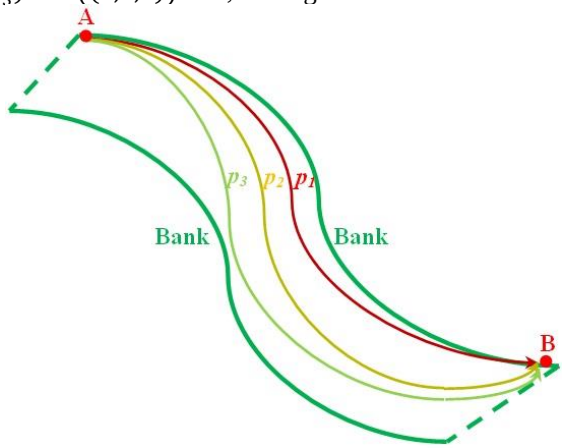

Figure 6: Map of the river showing the three paths, p_1, p_2 and p_3.

Let us suppose that the Euclidean length of each path from A to B is viz., $l(p_1) = 1.5\ Km$, $l(p_2) = 1.9\ Km$ and

$l(p_3) = 2.1\ Km$. To go through the Euclidean straight line, we would have to navigate sections of the river, then go overland in a swampy area, then navigate another section, and so on, which is not practical.

The NG lengths of the three paths in the example are, $l_{NG}(p_1) = \frac{1.5}{0}\ Km = \infty\ Km$, $l_{NG}(p_2) = \frac{1.9}{0.66667}\ Km = 2.85\ Km$, while $l_{NG}(p_3) = \frac{2.1}{1}\ Km = 2.1\ Km$, therefore we prefer the path p_3 to make the crossing, even though it is the longest one according to Euclidean geometry.

That is why the distance NG between A and B is at most 2.1 Km. Note that we have selected a finite number of paths because otherwise, we would have to search for the length of an infinite number of them.

Furthermore, if we have a path p_4 that begins at a point C before A and then continues through p_1, according to our definition of $\varepsilon(\cdot)$ we have $\varepsilon(p_4) \leq \varepsilon(p_1) = 0$, therefore $l_{NG}(p_1) = \infty\ Km$ as well. That is, any path that contains p_1 as a subpath will have infinite length in this geometry.

Another topic of interest is the way that path lengths are calculated. In this paper, we have proposed using the Euclidean arc length, which is a method that can be adapted to other geometries always respecting the definitions of geodesics and lengths, see Figure 7.

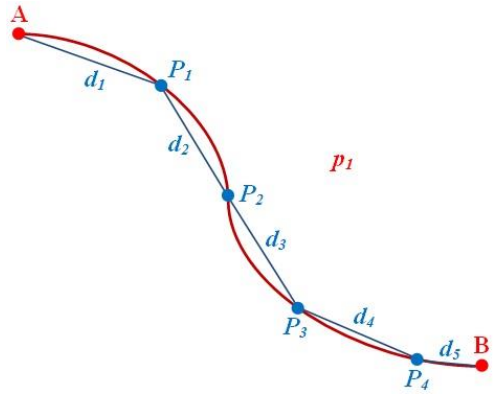

Figure 7: Arc length of the path p_1.

Figure 7 exemplifies the calculus of length of p_1. We select the points $A = P_0, P_1, P_2, P_3, P_4$ and $B = P_5$, all of them are on the curve. Then, we calculate the distances, $d_1 = d_E(A, P_1)$, $d_2 = d_E(P_1, P_2)$, $d_3 = d_E(P_2, P_3)$, $d_4 = d_E(P_3, P_4)$ and $d_5 = d_E(P_4, B)$, where $d_E(\cdot,\cdot)$ is the Euclidean metric. So, the length of p_1 is approximated by $d = \sum_{i=1}^{5} d_i$. For more accuracy, we must use more points on the curve.

In any case, in real life, there are cartographic methods to calculate the length of a path, including satellite spatial vision. Also in certain situations, we can walk along a path at a more or less constant speed and take the transit time from one point to another. Hence, we can approximately calculate the distance (length of the path) by the formula $distance = speed \cdot (time_B - time_A)$.

Conclusion

In this paper, we propose for the first time a theory to measure distance in NeutroGeometry. NeutroGeometry is assumed to be defined from a geometric structure that can be Euclidean or non-Euclidean. Using the distance measure in that base geometric structure, we introduced a distance defined in the NeutroGeometry structure, which depends on a passability function and the length of the path that joins both points. We specify some axioms that the passability function must fulfill. We also verify that the new measure generalizes the cases of classical geometries, as well as other geometries such as Taxicab and Chinese checker metrics. It is more convenient that the passability function depends on neutrosophy and not on other uncertainty models such as fuzzy sets or intuitionistic fuzzy sets models, because a neutrosophic set allows a greater number of possible states of knowledge to be expressed more accurately. Additionally, we propose an algorithm to calculate the approximate distance between two points in the NeutroGeometry plane based on a survey applied to a group of locals, geographers and experts, among others. The theory presented so far, and the proposed algorithm allows us to solve the problem of calculating the distance when there is uncertainty or indeterminacy in a two-dimensional region. These results bring us closer to solving real-life problems than utilizing classical geometries.

References

[1]. Smarandache, F. (2021). NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries. Neutrosophic Sets and Systems, 46, 456-477, http://fs.unm.edu/NSS/NeutroGeometryAntiGeometry31.pdf
[2]. Smarandache, F. (2023). Real Examples of NeutroGeometry & AntiGeometry, 55, 568-575, https://fs.unm.edu/NSS/ExamplesNeutroGeometryAntiGeometry35.pdf
[3]. Latreche, A., Barkat, O., Milles, S., and Ismail, F. (2020). Single valued neutrosophic mappings defined by single valued neutrosophic relations with applications. Neutrosophic Sets and Systems, 1, 203-220.
[4]. Smarandache, F. (2020). NeutroAlgebra is a Generalization of Partial Algebra. International Journal of Neutrosophic Science, 2(1), 8-17.
[5]. Chimienti, S. P. and Bencze, M. (1998). Smarandache paradoxist geometry. Smarandache Notions Journal, 9(1-3), 42.
[6]. Pfoser, D. and Tryfona, N. (2001). Capturing Fuzziness and Uncertainty of Spatiotemporal Objects. In: Advances in Databases and Information Systems: 5th East European Conference, ADBIS 2001 Vilnius, Lithuania, September 25-28, 2001 Proceedings 5 (pp. 112-126). Springer Berlin-Heidelberg.
[7]. Gonzalez-Perez, C., Pereira-Fariña, M., Martín-Rodilla, P., and Tobalina-Pulido, L. (2023). Dealing with Vagueness in Archaeological Discourses. In Discourse and Argumentation in Archaeology: Conceptual and Computational Approaches (pp. 137-157). Cham: Springer International Publishing.
[8]. Jana, S. and Mahanta, J. (2023). Boundary of a fuzzy set and its application in GIS: a review. Artificial Intelligence Review, 56(7), 6477-6507.
[9]. Kulik, L. (2001). A Geometric Theory of Vague Boundaries Based on Supervaluation. In: Spatial Information Theory Foundations of Geographic Information Science, COSIT 2001, Morro Bay, CA, USA (pp. 44-59) Berlin: Springer.
[10]. Singh, P. K. (2021). Anti-geometry and neutrogeometry characterization of non-Euclidean data. Journal of Neutrosophic and Fuzzy Systems, 1(1), 24-33.
[11]. Singh, P. K. (2022). NeutroAlgebra and NeutroGeometry for Dealing the Heteroclinic Patterns. In Theory and Applications of NeutroAlgebras as Generalizations of Classical Algebras (pp. 90-102). IGI Global.
[12]. Granados, C. (2023). A Review of AntiGeometry and NeutroGeometry and Their Application to Real Life. NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World, 1-16; https://www.igi-global.com/book/neutrogeometry-neutroalgebra-superhyperalgebra-today-world/292031 and https://www.igi-global.com/book/neutrogeometry-neutroalgebra-superhyperalgebra-today-world/292031
[13]. González-Caballero, E. (2023). Introduction to the Finite NeutroGeometries: The Mixed Projective-Affine Geometry. In NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World (pp. 52-80). IGI Global.
[14]. González-Caballero, E. (2023). "NeutroGeometry Laboratory": The New Software Dedicated to Finite NeutroGeometries. In NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World (pp. 131-155). IGI Global.
[15]. González-Caballero, E. (2023). A Constructive Introduction to Finite Mixed Projective-Affine-Hyperbolic Planes. In NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World (pp. 156-186). IGI Global.
[16]. Gonzalez-Caballero, E. (2023). Applications of NeutroGeometry and AntiGeometry in Real World. International Journal of Neutrosophic Science, 21(1), 14-33.
[17]. O'Searcoid, M. (2006). Metric spaces. Springer Science & Business Media.
[18]. Dubois, D., Godo, L., and Prade, H. (2023). An elementary belief function logic. Journal of Applied Non-Classical Logics, 33(3-4), 582-605.
[19]. Liu, C., and Martin, R. (2024). Inferential models and possibility measures. In Handbook of Bayesian, Fiducial, and Frequentist Inference (pp. 344-363). Chapman and Hall/CRC.
[20]. Zimmermann, H. J. (2010). Fuzzy set theory. Wiley interdisciplinary reviews: computational statistics, 2(3), 317-332.
[21]. Atanassov, K. T. (2012). On intuitionistic fuzzy sets theory (Vol. 283). Springer.
[22]. Gehrke, M., Walker, C., and Walker, E. (1996). Some comments on interval valued fuzzy sets. structure, 1(2), 751-759.
[23]. Lee, K. M., Lee, K. M., and Cios, K. J. (2001). Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets. In Computing and information technologies: exploring emerging technologies (pp. 433-439).

[24]. Luo, M., Zhang, G., and Wu, L. (2022). A novel distance between single valued neutrosophic sets and its application in pattern recognition. Soft Computing, 26(21), 11129-11137.
[25]. Smarandache, F. (2024) Foundation of Appurtenance and Inclusion Equations for Constructing the Operations of Neutrosophic Numbers Needed in Neutrosophic Statistics. Neutrosophic Systems with Applications, 15, 16-32.
[26]. Smarandache, F. (2020) The Score, Accuracy, and Certainty Functions determine a Total Order on the Set of Neutrosophic Triplets (T,I,F). Neutrosophic Sets and Systems, 38, 1-14.
[27]. Mandal, K. (2020). On Deneutrosophication. Neutrosophic Sets and Systems, 38, 409-423.
[28]. Krause, E.F. (1986) Taxicab Geometry. An Adventure in Non-Euclidean Geometry. New York: Dover Publications.
[29]. Thompson, K. P. (2011). The nature of length, area, and volume in taxicab geometry. International Electronic Journal of Geometry, 4(2), 193-207.
[30]. Chen, G. (1992). Lines and circles in taxicab geometry (Doctoral dissertation, Central Missouri State University).

NeutroGeometry and Fractal Geometry

Erick González-Caballero Asociación Latinoamericana de Ciencias Neutrosóficas, Havana, CUBA
Maikel Y. Leyva-Vázquez Universidad de Guayaquil (UG), Ciudadela Universitaria, Guayaquil, Guayas, ECUADOR
Noel Batista-Hernández Universidad Bolivariana del Ecuador (UBE), Km 5.5 Vía Durán, Guayas, ECUADOR
Florentin Smarandache Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract. Geometries are structures with certain elements like points, lines, planes, and spaces, among others, that satisfy certain definitions, axioms, properties, and theorems for the total of the elements. NeutroGeometries are geometric structures that meet at least one of these concepts only partially, never for 100% or 0% of the elements. Until now, NeutroGeometries have been developed from the ideas of classical geometries such as Euclidean, Hyperbolic, Elliptic, Mixed (Smarandache) geometries, among others where axiomatization is the basis of their construction. This paper aims to discuss some ideas about the relationship between NeutroGeometries and fractal geometry. This relation is not necessarily obvious; it is mainly established because fractals are structures used to model deterministic chaotic phenomena. The fractal dimension is a numerical value used to measure the complexity of the figure and the maps that represent chaotic phenomena. The more complex the phenomenon, the more unpredictable it becomes and therefore the more uncertain and indeterminate. This indeterminacy is essentially ontological since it deals mostly with natural phenomena. This relationship is proposed in this article for associating the concepts of NeutroGeometry that present degrees of uncertainty or indeterminacy and fractal geometries that model phenomena where unpredictability exists. This idea is reinforced in some works where a direct relationship between entropy and the fractal dimension is demonstrated.

Keywords: NeutroConcept, NeutroGeometry, indeterminacy, unpredictability, fractal geometry, entropy, chaos, fractal dimension.

1 Introduction

Neutrosophy is the branch of philosophy that studies neutrality, understanding in this concept the neutral, the indeterminate, the erroneous, the contradictory, the imprecise, the paradoxical, the unknown, and the inconsistent, among other concepts. This philosophical theory introduced by Professor F. Smarandache has expanded to various parts of knowledge. Especially mathematics has been enriched with concepts such as Neutrosophic Sets in the form of Single-Valued Neutrosophic Sets or Interval-Valued Neutrosophic Sets. These generalize fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets, and Pythagorean fuzzy sets, among many others.

One of the fields in which Neutrosophy has been presented is Geometry. Several decades ago Smarandache introduced a movement called Paradoxist, where mixed geometric structures were studied. Specifically, the following definition was used [1]:

"An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalidated, or only invalidated but in multiple distinct ways. A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom."

In this type of geometry, emphasis is placed on the non-satisfaction of some axiom. Above all, the controversial Euclid's fifth postulate about the existence of a single parallel that passes through a point outside a straight line. However, within the framework of Neutrosophy, Smarandache defined NeutroGeometry much more recently as part of a series of concepts coming from neutrosophic logic. He defined what he called a NeutroConcept, compared to Concept and AntiConcept. Each of these categories was associated with a triple of values (T, I, F) that represents the triple of Truthfulness, Indeterminacy, and Falseness, respectively, of satisfaction of the concept [2].

A Geometry satisfies each axiom, definition, property, theorem, and proposition, among others, for all elements, that is, with a true degree of satisfaction $(1,0,0)$. The AntiGeometry satisfies it for none in any Concept that is $(0,0,1)$. The elements of the NeutroGeometry satisfy it for the rest of the intermediate values of (T, I, F). The idea of the existence of indeterminacy within the geometric structure is important, although it is enough that a certain degree of truth exists in NeutroGeometry when $I = 0$ [2-4].

The Mixed Geometries or Smarandache Geometries are included within the concept of NeutroGeometry. However, NeutroGeometries have greater logical implications, they are logic applied to Geometry rather than pure geometry modified in its classical axioms as occurs within Mixed Geometries [5].

There is a relatively large amount of research within the field of Mixed Geometries, e.g. the works of H. Iseri who contributed to the Smarandache Manifolds, where geometric structures are composed of "disks" in the shape of equilateral triangles, such that the length of their edges is taken as the unit of measurement within the structure. In these manifolds, all parallelism properties belonging to elliptic, Euclidean, or hyperbolic geometries appear within the same structure [6]. We also owe the creation of a simple geometric model in [7], although it is mathematically poorly founded and developed. More recently, other authors have worked in this line of research such as Carlos Granados [8]. Other works within the field of Smarandache Geometries can be consulted related to finite geometry [9-11] and some real-life cases where Mixed Geometries are satisfied [12]. This last article also outlines some examples of NeutroGeometry that is not Mixed Geometry. We can also consult the development from Euclidean geometry to Smarandache geometry in [13].

However, a step taken where results are presented in the field of NeutroGeometries beyond the Smarandache Geometries appears in the article [14]. In this article, the author proposes a distance formula within the framework of NeutroGeometry. The mathematical simplicity and at the same time the unreality of the use of distances in classical geometries, whether Euclidean or classical non-Euclidean, becomes evident. This unreality is because the world that geometry tries to model is not homogeneous, and therefore it is not possible to travel from one point to another by the shortest classical distance, since the path may be crossed by obstacles or indeterminacies.

The aforementioned article deals with paths represented in a coordinate system on a two-dimensional surface. The proposed distance formula is valid for any geometry, viz., Euclidean, hyperbolic, elliptic, or mixed. However, the paths proposed to connect one point to another must be rectifiable, that is, there must be a way to measure them. A measure of uncertainty or indeterminacy based on the opinion or experience of a group of people is also proposed and a decision-making algorithm is recommended to solve the problem of moving from one point to another in a practical way.

That is why the idea of distance in this case is associated with a situation of uncertainty or indetermination with a high epistemological component. So, moving from one place to another will depend on people's knowledge of the best path and their cognitive uncertainty or indeterminacy. Another important idea that does not appear, at least explicitly, in the aforementioned works is the relationship of geometry with the dynamic changes of a system. The latter is because these are classical geometries; more emphasis is placed on static realities that do not change over time.

Taking into account all explained so far, it remains to ask if the concept of NeutroGeometry has any relationship with geometry as atypical as fractal since all authors have focused on the idea of associating it with classical geometries. Fractal geometry focuses on the study of the geometric object called fractal which was coined in this way by Benoit Mandelbrot from the Latin word *fractus*, meaning broken or fractured [15]. These objects have been widely studied. Among their characteristics are that they are usually not rectifiable and self-similar. This means that they usually have infinite length even though they occupy a limited space and also have shapes that repeat in the same or similar way at different scales. The visualization of these objects is reminiscent of shapes within nature that cannot be described with the help of classical geometries; however, fractals are usually constructed with simple recurring equations. This is supported by computational development that allows the plotting of these shapes with an adequate resolution.

Associated with the concept of a fractal is the fractal dimension. This is a numerical value of dimension which in fractals are non-integer values. In classical curves or geometric figures, these values are integers, for example, points have dimension 0, rectifiable lines and curves have dimension 1, surfaces have dimension 2, and so on. The fractal dimension has application to study curves that are natural, for example, the crack in a concrete surface, the coastline of a country, and the blood vessel networks of the human body, among many others.

The relationship between fractal forms and the representation of chaotic dynamic systems is known and has been well-studied. These are deterministic phenomena that produce completely different results from small changes in the initial conditions. They are non-random phenomena, since when the same initial value is entered, the results are the same, but on the other hand, a very small change in the initial value gives completely different results. Despite this, they are phenomena that are described with the help of curves included within a limited space.

This paper aims to reflect on and discuss the relationship that may exist between NeutroGeometries and Fractal Geometry. We maintain the thesis that there is a relationship although this has its subtleties. Chaotic phenomena are provided with a high degree of complexity, which is manifested in the fractal dimension of the so-called maps of their graphic representation. In some cases, a direct correlation occurs between the fractal dimension and entropy, for example in the study of the spatial growth of cities. Therefore, it is not difficult to realize that the complexity of natural phenomena implies greater unpredictability and therefore greater uncertainty and indeterminacy. This type of indeterminacy has a primarily ontological nature.

The article is divided into a preliminary section where the most important concepts of NeutroGeometry and fractal geometry are explained. In the following section, we develop the ideas that allow us to link both concepts.

The last section contains the conclusions.

2 Preliminaries

This section contains the basic notions of NeutroGeometry, AntiGeometry, fractal geometry, and fractal dimension.

2.1 NeutroGeometry and AntiGeometry

According to neutrosophic logic, a proposition has a valuation (T, I, F), where T means the degree of truthfulness, I is the degree of indeterminacy and F is the degree of falseness. A Concept is satisfied with valuation according to the triple $(T, I, F) = (1,0,0)$, an AntiConcept is satisfied with the valuation $(T, I, F) = (0,0,1)$, and a NeutroConcept is satisfied for the rest of the possible values of the triple.

Geometry is defined as a geometric structure composed only of Concepts, NeutroGeometry is a geometric structure with at least one NeutroConcept and no AntiConcept, and AntiGeometry is a geometric structure that contains at least one AntiConcept [2, 4].

Specifically, NeutroGeometry is a geometric structure composed of elements such as points, lines, planes, spaces, hyperspaces, and so on, where some definition, axiom, theorem, and property, among others, is a NeutroConcept. That is, it fulfills that there is a NeutroAxiom or a NeutroProperty or a NeutroDefinition, and so on. The intuitive idea to define NeutroGeometry is that the physical world in which we live is not homogeneous and contains indeterminations [2, 4]. For example, although the straight line is the shortest one between two points in a city, it is not always possible to follow the straight path because there may be construction elements such as buildings that can prevent our passage. In this way, the shortest path would be a broken line following the traffic.

It is also clear that Mixed Geometries or Smarandache Geometries are NeutroGeometries. In them, the Smarandachely denied axioms are partially fulfilled. An example that can be consulted is the model shown in Figure 1.

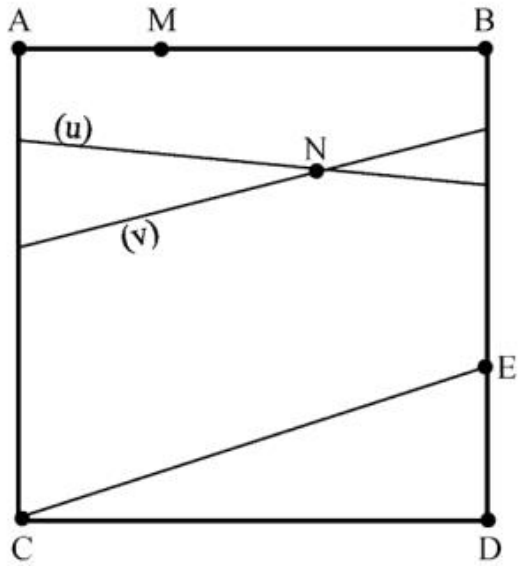

Figure 1: Bhattacharya's model of Smarandache Geometry. See [7].

Figure 1 shows a model made up of a rectangle where the points are the Euclidean points contained in the interior of the rectangle and those on the border. The lines go from one point on the edge AC to another point on the edge BD. If the line CE is taken, it does not have any parallel that passes through any point inside the segment DE. It has at least two parallel lines (u and v) that pass through point N and a single parallel that passes through point M. This model is one of the first to appear in a mixed geometry.

Figure 2 shows another model where a NeutroGeometry is not a Smarandache Geometry.

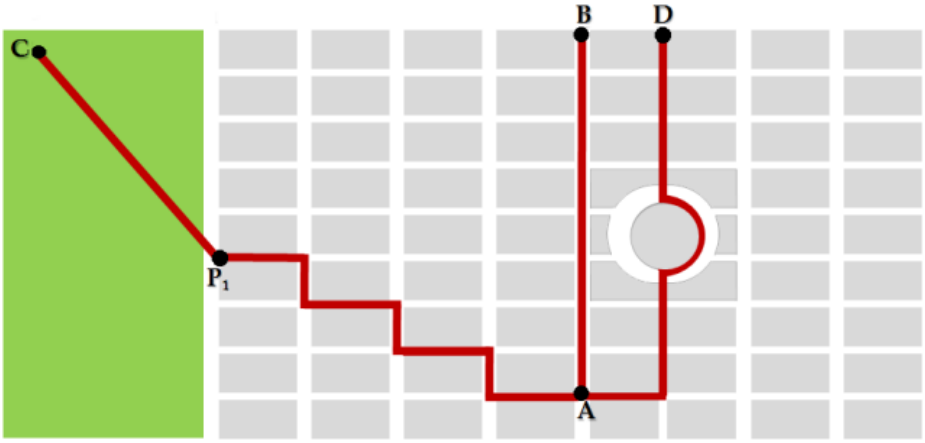

Figure 2: Euclidean model of an urban area showing a park (green area), blocks (gray rectangles), roundabout (gray circle), and paths (red lines). Source: [12].

Formally, the elements in Figure 2 correspond to Euclidean geometry; among the proposed paths, only AB and P_1C can be joined in straight Euclidean lines. There is no geometry defined in the literature where AD is a line in the sense of geodesic. The line AC presents a valid path for Taxicab Geometry in the section AP_1 and the Euclidean geometry in the section P_1C.

Specifically, to travel on the path AC shown in the figure, the shortest line (geodesic) is to follow the path marked in red. However, this path is not valid in both Euclidean geometry and Taxicab Geometry. This is why the property being a geodesic is a NeutroProperty in this example. This is the shortest path in real life, but it is not a line in Euclidean and Taxicab geometries, only partially.

These drawbacks show the rigidity of geometries to provide us with solutions to real-life problems because they are based on axioms that prioritize cognitive simplicity over applicability to daily life problems. However, the search for a more general solution is obtained by sacrificing the simplicity of the classical models. An example of this is the distance proposal for NeutroGeometry as explained below [14].

Definition 1 ([14])**.** Given (G, d_G) is a metric space in a geometric structure. The *NeutroGeometric distance* between x_1 and x_2 for $x_1, x_2 \in X$ is defined as:

$$d_{NG}(x_1, x_2) = inf\left\{\frac{L(p)}{\varepsilon(p)} : p \text{ is a rectifiable path from } x_1 \text{ to } x_2\right\} \qquad (1)$$

Where $\varepsilon(p)$ is the passability function of p.

This formula explains that if we want to calculate the neutrosophic distance that joins the points x_1 and x_2 in a plane corresponding to a certain geometry (Euclidean, hyperbolic, elliptic, mixed, taxicab, another one) having a coordinate system and a metric space associated with the base geometry, then the length of all the rectifiable paths that join both points are divided by a function called passability that measures the uncertainty or indeterminacy that joins both points along the chosen path.

This formula is a decision-making method where the distance is sought taking into account what is the shortest path in the classical sense and also the real possibility of passing through this path taking into account the knowledge of the locals or those people who know the place. It is therefore an uncertainty and indeterminacy that are epistemological.

The example that appears in the article is the following:

Example 1 ([14])**.** Suppose we are in a boat and we want to navigate a river whose bed has irregularities so that the left bank has calm waters and the right bank has turbulent waters in a certain section. However, we wish to go from point A to point B on the right side of the river. Furthermore, the path of the river itself is sinuous, see Figure 3.

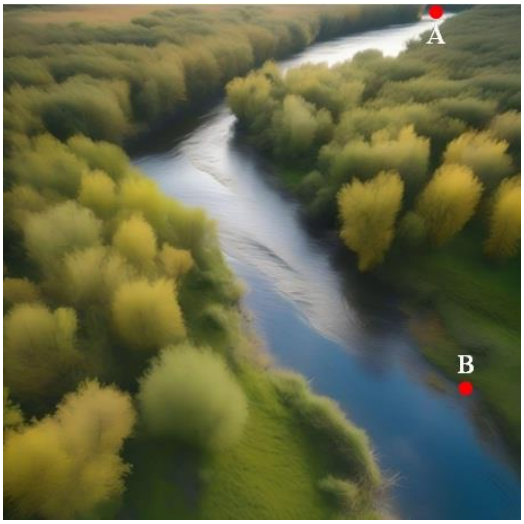

Figure 3: Picture of the river of the example. We need to go from point A to point B. This image was generated by an AI tool. Source: [14].

That is why the shortest path that passes on the right is impassable and hence $\varepsilon(p_1) = 0$. The path that goes through the center has some areas with a percentage of danger, let us say $\varepsilon(p_2) = 0.66667$. While the path that goes left and then turns right is the longest, and it is completely safe, therefore $\varepsilon(p_3) = 1$, see Figure 4.

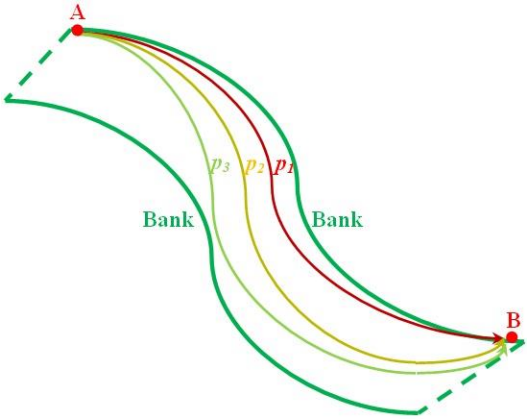

Figure 4: Map of the river showing the three paths, p_1, p_2, and p_3. Source: [14].

Let us suppose that the Euclidean length of each path from A to B is viz., $l(p_1) = 1.5\ km$, $l(p_2) = 1.9\ km$ and $l(p_3) = 2.1\ km$. To go on the shortest path in the Euclidean distance, i.e. a straight line we would have to navigate sections of the river, then go overland in a swampy area, then navigate another section, and so on, which is not practical.

The NeutroGeometric lengths of the three paths in the example are, $l_{NG}(p_1) = \frac{1.5}{0}\ km = \infty\ km$, $l_{NG}(p_2) = \frac{1.9}{0.66667}\ km = 2.85\ km$, while $l_{NG}(p_3) = \frac{2.1}{1}\ km = 2.1km$, therefore we prefer the path p_3 to make the crossing, even though it is the longest according to Euclidean geometry.

2.2 Fractal Geometry

Fractal Geometry is due to the mathematician Benoît Mandelbrot who introduced it in the mid-1970s [15]. Fractals are geometric objects that have irregularity as a characteristic. They are usually curves or surfaces that are not differentiable at any point and also have infinite lengths (they are not rectifiable). Another characteristic is that they are self-similar, that is, when the scale of a portion of the object is increased, a figure equal or similar to the entire object is obtained, and this occurs at any scale. This self-similarity can be exact as in classical fractals, approximate as in nature where the parts are approximately equal to the whole, and statistical self-similarity where some self-similarity characteristics are preserved.

A classic example of a fractal is the Koch curve; Figure 5 shows the construction of the so-called Koch snowflake.

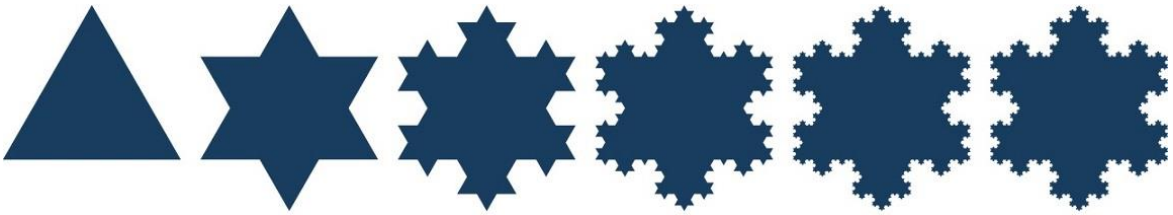

Figure 5: Koch snowflakes fractal. Source: http://www.pixabay.com.

To construct this fractal, an equilateral triangle is taken as seen in the figure. From each of its edges, the segment that is in the middle third is eliminated and replaced by two edges of an equilateral triangle. For the new edges, the process is repeated infinitely. From this process, a bounded object is obtained, but one that has infinite length and is also self-similar.

These objects cannot be described with elements of classical geometry and are reminiscent of elements of nature such as broccoli, cauliflower, and trees, among others. For example, see Figure 6 with the details of a broccoli and Figure 7 contains an artificially generated tree using code in the Python language.

Figure 6: Broccoli showing self-similarity properties. Source: http://www.pixabay.com.

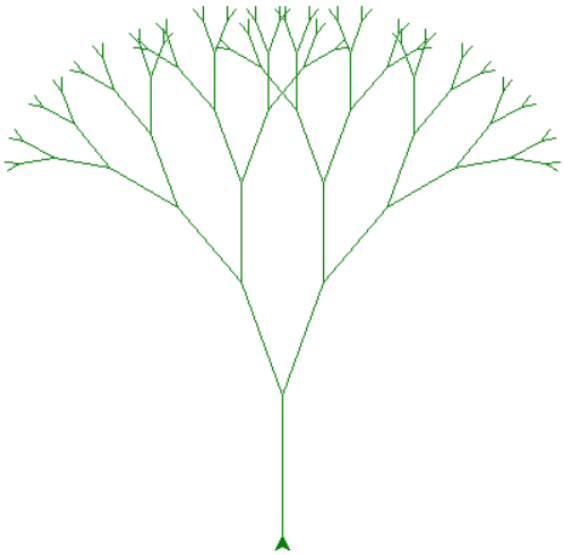

Figure 7: Tree generated using Python code.

An important concept within this theory is the fractal dimension. While classical objects have an integer topological dimension, the fractal dimension can be real, not integer. For example, the Koch curve has a fractal dimension of $d_f = \frac{\ln(4)}{\ln(3)} \approx 1.2619$. This number means that the curve has more complexity than a smooth curve in the two-dimensional space such as a straight line in Euclidean geometry, however, it is not so complex as to occupy a complete two-dimensional space, for example, a square that has topological dimension 2.

The number above is obtained from the general formula:

$$d_f = \frac{\ln m}{\ln r} \qquad (2)$$

Where r is the scale factor, while m is the minimum number of copies of this square needed to cover the object. For example, in the Koch snowflake fractal, 4 squares with a scale factor of 3 are needed to cover the entire figure [16, 17]. This is used when the figure satisfies the self-similarity property.

There is more than one definition for the fractal dimension. A simple one is the Box Dimension, which is calculated from Equation 3 [17].

$$d_B = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \qquad (3)$$

If the limit exists, where $N(\varepsilon)$ is the minimum D-dimensional ε side cubes needed to cover the D-dimensional figure.

This dimension is used not only in fractals but also in real objects, a classic example is the box dimension of the coasts of Great Britain [18]. There are phenomena where its fractal dimension is used as a measurement, since classical measurements such as length or area, among others, do not make sense for these objects. Among these cases is the spatial growth of cities over time or the crack in the concrete or wall, see Figures 8 and 9 [19].

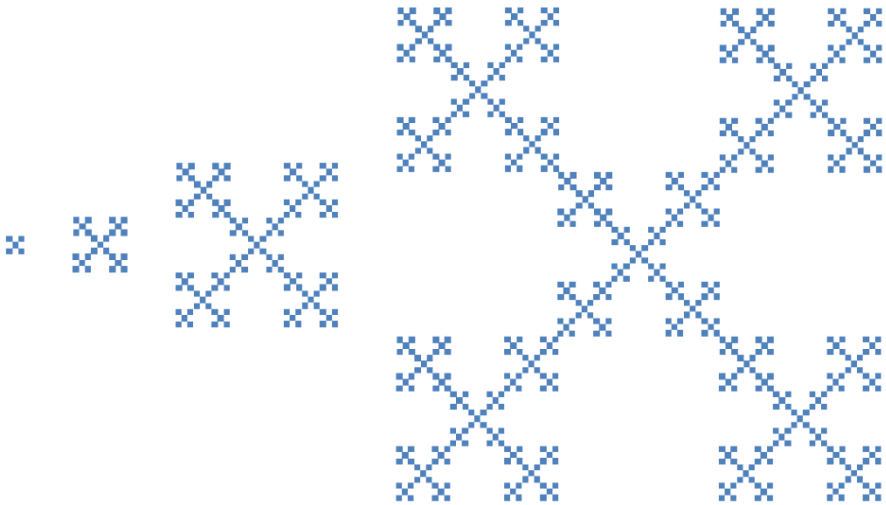

Figure 8: Self-similar fractal used to model the spatial growth of a city. Source: The authors.

Figure 9: Crack in the wall. It is a figure that cannot be represented by classical geometries. Source: http://www.pixabay.com.

3 NeutroGeometry and Fractal Geometry

Perhaps when the idea of creating a theory of NeutroGeometry arose, the possibility of associating this concept with fractal geometry was not thought of. However, there are common points between these two geometries. The NeutroGeometries arose from the Paradoxist movement where only two concepts opposed to each other that co-existed in the same structure were taken into account, mainly the concept of axiom, and within the axioms the fifth postulate of Euclid and its variants of non-Euclidean geometries. However, fractal geometry is not represented axiomatically; therefore it does not make sense to talk about Mixed or Smarandache fractal geometries in this context.

From a formal point of view, both theories can be combined with the self-similarity property. There are curves in the plane that are not self-similar, we have as an example any known smooth curve, a very simple one is the circumference. At the opposite extreme, we have curves that have exact self-similarity such as classic fractals like the one seen in Figure 5. It is also known that there are intermediate curves, where the self-similarity is partial as occurs in several fractals, or even on surfaces like mountains, self-similarity is said to be statistical, since only some parts are somewhat similar to the whole. That is why self-similarity is a Concept that has an AntiConcept (not self-similarity) and a NeutroConcept (approximate or statistical self-similarity).

As for the measure of the complexity of the fractal figure, which is the fractal dimension, which is taken due to the lack of others such as length or area, there is no single measure either. The Box Dimension is just one of them and does not always coincide with the other definitions of fractal dimension.

There are:

$d_T, d_{HB}, d_{MB}, d_E, d_C$,
Where:
d_T: is the topological dimension that is always an integer,
d_{HB}: is the Haussdorff-Besicovitch dimension,
d_{MB}: is the Minkowski-Bouligand or box-counting dimension that had been explained previously,
d_P: is the packaging dimension,
d_C: is the dimension of the Euclidean space that contains the figure and is an integer.

Therefore, if d_f is the fractal dimension of the object, then $d_f \in [d_T, d_C]$ or more specifically for fractal objects we have $d_f \in [min\{d_{HB}, d_{MB}, d_P\}, max\{d_{HB}, d_{MB}, d_P\}]$. That is, if there is a measurement called "fractal dimension" of a fractal object, it must be contained in this last interval. However, in general, it is fulfilled $d_f \in [d_T, d_C]$. Also, any fractal dimension is always calculated approximately.

This is reinforced by the definition of multifractal dimension, because some objects have a self-similarity that does not hold equally for all scales and it is necessary to define a series of fractal dimensions instead of a single numerical value, to describe the object [17].

However, the relationship between NeutroGeometry and fractality is especially noticeable in the representation of deterministic chaos through fractal theory. A system is chaotic when it exhibits aperiodic behavior and is sensitive to initial conditions; therefore it is not predictable [17]. That is, despite it is not random, for small changes in the initial conditions we obtain large changes in the results. We are going to illustrate this idea with the help of the logistic equation shown in (4).

$$x_{n+1} = r x_n (1 - x_n) \qquad (4)$$

This equation represents the growth of a population dynamically over time, note that it is a recurring equation.

The behavior of this equation depends on the parameter r. It has been studied that:

For $r < 1$ it is fulfilled $x_n \to 0$ when $n \to \infty$,
For $1 < r < 3$ it is fulfilled $x_n \to a \neq 0$ when $n \to \infty$, see Figure 10.
For $r > 3$ the graph oscillates between two values (Figure 11), four values (Figure 12), and so on until reaching a chaotic state (Figure 13).

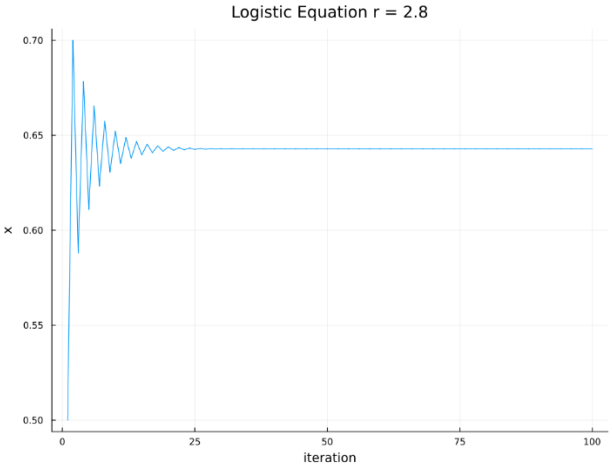

Figure 10: Convergence of the logistic equation for $r = 2.8$. See [16, 17].

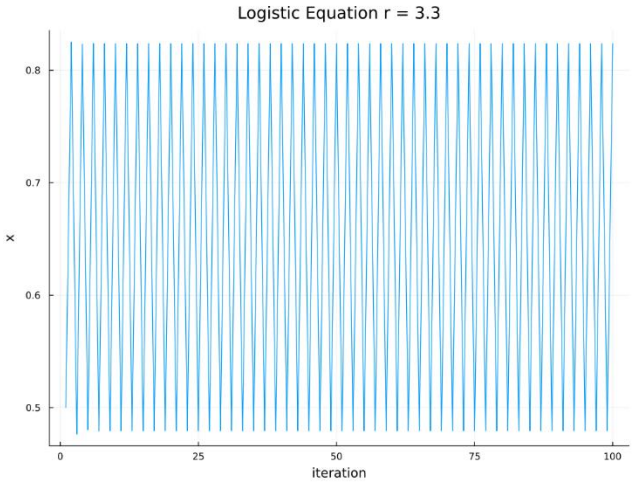

Figure 11: Convergence of the logistic equation for $r = 3.3$. See [16, 17].

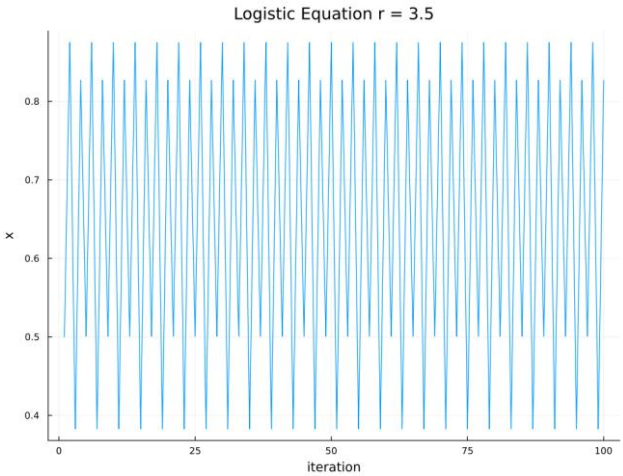

Figure 12: Convergence of the logistic equation for $r = 3.5$. See [16, 17].

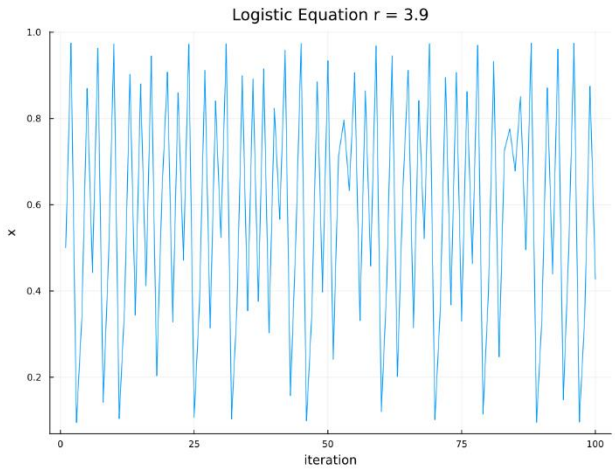

Figure 13: Convergence of the logistic equation for $r = 3.9$. See [16, 17].

To represent the Logistic Map, each value of the parameter r is plotted against the convergence points of Equation 4 for this parameter. This is represented in Figure 14.

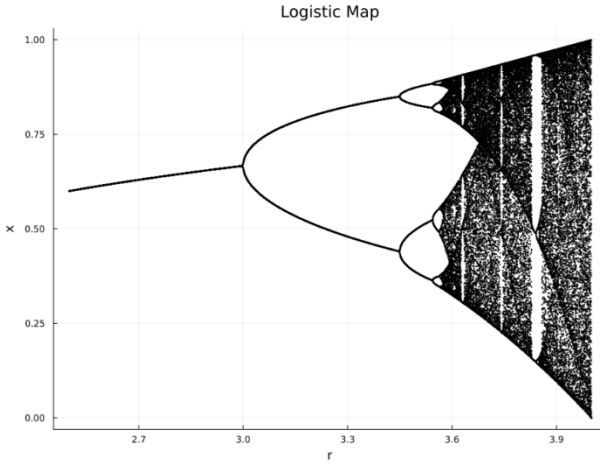

Figure 14: Logistic Map. Plot of parameter vs. point(s) of convergence. See [16, 17].

Figure 14 represents the Logistics Map. It is not difficult to notice that it presents some characteristics of approximate self-similarity. This map requires multifractal dimension theory to calculate its complexity.

We can also notice that for $r = 2.8$ its result is completely predictable, however, for some value $r > 3$ the value begins to bifurcate, obtaining indeterminate prediction results. The larger the r value becomes, the greater the indeterminacy and unpredictability of the system, see Figures 10-12 and Figure 14. This is manifested to a greater degree for certain values close to $r = 4$ which is where the system becomes chaotic, Figures 13 and 14. This is a classic example of the relationship between fractal geometry and the behavior of nonlinear dynamical systems.

This could also be an example where Neutrosophy that studies indeterminacy corresponds totally with the idea of fractal geometry and nonlinear dynamic systems and chaos [20].

This last idea is reinforced by some studies where the fractal dimension is associated with entropy. Entropy is the measure of order of the system [19]. In some phenomena such as the spatial growth of cities or the agglomeration of cities, a positive correlation has been found between the fractal dimension and entropy. This means that in these cases the complexity of urban spatial growth can be measured both with the help of entropy and with the fractal dimension. We can also intuitively link these concepts when it becomes evident that indeterminacy, which is an important component of Neutrosophy, is typical of complex systems. A complex system is understood to be one where very different results are obtained for small variations in the initial conditions. So we can appreciate the relationship between NeutroGeometry[21] and Fractal Geometry[22] when the indeterminacy is intrinsic to the system or what is the same when it is an ontological indetermination [23, 24].

Conclusion

NeutroGeometry is a logical approach to geometric systems based on neutrosophic logic. A NeutroGeometry contains at least one NeutroConcept and never an AntiConcept. However, this idea originally intended for classical geometric systems, such as Euclidean geometry, classical non-Euclidean geometries, or mixed Smarandache geometries, did not take into account other less orthodox geometries such as fractal geometry. Fractal geometry does not respond to any specific axiom, although it is composed of objects called fractals that are immersed in a Euclidean line, plane, space, or hyperspace and are composed of points or segments of points. The objective of this article was to reflect and discuss the relationship that may exist between the ideas of NeutroGeometry, Neutrosophy, and fractal geometry or the phenomena in which it is applied such as nonlinear or chaotic systems. On the one hand, self-similarity properties are NeutroProperties in some cases, when it is approximate or statistical. On the other hand, if it is considered that there is an "intrinsic" fractal dimension of an object, this can only be determined accurately when it is understood as a number within an indeterminate interval since there are several definitions of fractal dimension that do not always coincide each other.

Perhaps the most interesting relation is in the application of fractal geometry to model nonlinear dynamic systems or chaotic systems. These systems can be indeterminate or unpredictable, which falls within the field of study of Neutrosophy. In turn, the maps of some of these systems, such as the logistic map, present self-similarity properties typical of fractal objects. Although this self-similarity may not be exact.

References

[1] Mao, L. (2006). Combinatorial Speculations and the Combinatorial Conjecture for Mathematics. In Smarandache Geometries & Map Theory with Applications (I) (pp. 1-23). Chinese Branch Xiquan House.

[2] Smarandache, F. (2021). NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries. Neutrosophic Sets and Systems, 46, 456-476.

[3] Singh, P. K. (2021). Anti-geometry and neutrogeometry characterization of non-Euclidean data. Journal of Neutrosophic and Fuzzy Systems, 1, 24-33.

[4] Smarandache, F. (2023). Real Examples of NeutroGeometry & AntiGeometry. Neutrosophic Sets and Systems, 55, 568-575.

[5] Pătrașcu, I. (2023). Smarandache Geometries (or Hybrids). Craiova: Infinite Study.

[6] Iseri, H. (2002). Smarandache manifolds. Craiova: Infinite Study.

[7] Bhattacharya, S. (2004). A model of the Smarandache geometries. Journal of Recreational Mathematics, 33, 166.

[8] Granados, C. (2023). A Review of AntiGeometry and NeutroGeometry and Their Application to Real Life. NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World, 1-16.

[9] González-Caballero, E. (2023). Introduction to the Finite NeutroGeometries: The Mixed Projective-Affine Geometry. In NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World (pp. 52-80). IGI Global.

[10] González-Caballero, E. (2023). "NeutroGeometry Laboratory": The New Software Dedicated to Finite NeutroGeometries. In NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World (pp. 131-155). IGI Global.

[11] González-Caballero, E. (2023). A Constructive Introduction to Finite Mixed Projective-Affine-Hyperbolic Planes. In NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World (pp. 156-186). IGI Global.

[12] Gonzalez-Caballero, E. (2023). Applications of NeutroGeometry and AntiGeometry in Real World. International Journal of Neutrosophic Science, 21, 14-33.

[13] Gonzalez-Caballero, E. (2024). A Brief Journey from Euclidian to Smarandachean Geometry. London: LAP LAMBERT Academic Publishing. ISBN: 978-620-7-65460-4.

[14] González-Caballero, E. and Smarandache, F. (2024). Theory of Distances in NeutroGeometry. Neutrosophic Sets and Systems, 67, 179-189.

[15] Falconer, K. (2014). Fractal geometry: mathematical foundations and applications. John Wiley & Sons.

[16] Talanquer, V. (1996). Fractus, fracta, fractal. Fractales de Laberintos a Espejos (In Spanish). Mexico D.F.: FCE-Fondo de Cultura Económica.

[17] Strogatz, S.H. (1994) Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering. Cambridge: Westview Press.

[18] Husain, A., Gumma, S., Sajid, M., Reddy, J., and Alresheedi, M. T. (2024). A novel fractal interpolation function algorithm for fractal dimension estimation and coastline geometry reconstruction: a case study of the coastline of Kingdom of Saudi Arabia. The European Physical Journal B: Condensed Matter and Complex Systems, 97, 1-16.

[19] Chen, Y., Wang, J. and Feng, J. (2017). Understanding the Fractal Dimensions of Urban Forms through Spatial Entropy. Entropy, 19, 600.

[20] Al-Kaff, M. O., El-Metwally, H. A., Elsadany, A. E. A., and Elabbasy, E. M. (2024). Exploring chaos and bifurcation in a discrete prey–predator based on coupled logistic map. Scientific Reports, 14, 16118.

[21] Smarandache, F. (2022). La Neutro-Geometría y la Anti-Geometría como Alternativas y Generalizaciones de las Geometrías no Euclidianas. Neutrosophic Computing and Machine Learning, 20, 91-104

[22] Sendker, F. L., Lo, Y. K., Heimerl, T., Bohn, S., Persson, L. J., Mais, C. N., ... & Hochberg, G. K. (2024). Emergence of fractal geometries in the evolution of a metabolic enzyme. Nature, 1-7.

[23] Nabeeh, N. (2023). Assessment and contrast the sustainable growth of various road transport systems using intelligent neutrosophic multi-criteria decision-making model. Sustainable Machine Intelligence Journal, 2, 2-1.
[24] Vázquez, M. L., Ricardo, J. E., & Smarandache, F. (2024). Enhancing Set-Theoretic Research Methods with Neutrosophic Sets. HyperSoft Set Methods in Engineering, 2, 96-103.

Using Neutrosophic Trait Measures to Analyze Impostor Syndrome in College Students after COVID-19 Pandemic with Machine Learning

Riya Eliza Shaju, Meghana Dirisala, Muhammad Ali Najjar, Ilanthenral Kandasamy, Vasantha Kandasamy

School of Computer Science and Engineering (SCOPE), Vellore Institute of Technology, Katpadi, Vellore,Tamil Nadu, INDIA

Florentin Smarandache

Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract. Impostor syndrome or Impostor phenomenon is a belief that a person thinks their success is due to luck or external factors, not their abilities. This psychological trait is present in certain groups like women. In this paper, we propose a neutrosophic trait measure to represent the psychological concept of the trait-anti trait using refined neutrosophic sets. This study analysed a group of 200 undergraduate students for impostor syndrome, perfectionism, introversion and self-esteem: after the COVID pandemic break in 2021. Data labelling was carried out using these neutrosophic trait measures. Machine learning models like Support Vector Machine(SVM), K-nearest neighbour (K-NN), and random forest were used to model the data; SVM provided the best accuracy of 92.15%.

Keywords: Neutrosophic psychology; Impostor syndrome; Neutrosophic trait measure; SVM, KNN; Random forest

1. Introduction

Impostor Syndrome or Impostor Phenomenon is a person's internal conviction that their success has happened due to pure chance, an external error, or hard work but not their capabilities or intellect. Individuals with impostor syndrome fear that others will someday learn that they are fakes [1]. They feel they do not belong in their working or academic environment

despite qualifications, accomplishments, and achievements [2]. Impostor syndrome is appropriate to ethnic, racial, and gender groups [3], especially in creative arts-based careers where success is not readily quantifiable [4]. Impostor syndrome was also thought to be prevalent in women and varies with gender. In 1978, psychologists Pauline Rose Clance and Suzanne Imes presented the concept of an Impostor phenomenon. Their research examined graduate and undergraduate women who were relatively successful but felt "overvalued" by their counterparts or supervisors and further felt like imposters [5]. Furthermore, this research established that the impostor phenomenon happened more in females than males. It is attributed to how different genders synthesize their success. Men believe success comes from within, while women consider that success comes from outside. To complicate matters, it emerges that the more educated and skilled a woman is, the more she doubts her abilities [6]. Due to this, women with IS often work more than others to finally achieve the status they fear they have never earned. [5] coined the term Impostor Phenomenon (IP) to describe a sensation of internal intellectual phoniness that seemed especially widespread among a select group of female high-achievers. They were terrified of being labelled as "impostors" who did not belong "among all these brilliant, intelligent individuals." Rather than skill or competence, many ascribed their success to chance, hard effort, acquaintance with well-connected people, being at the correct place at the perfect time, fates, or individual characteristics such as charisma and the capability to interact effectively.

The Clance IP Scale was created to assess the idea that people are successful by superficial standards but have a false sense of personal ineptitude. The scale evaluates phenomena such as self-doubt and achievement by coincidence [7]. The study assesses the predictive accuracy of several machine learning methods. They proved that approaches, like ensemble learning, are better than simple machine learning algorithms [8]. According to a recent systematic analysis of the literature published in 2020, the prevalence of IS ranges from 9 to 82% [9]. According to [10], the IS prevalence among medical students and trainees was 22 -60%, and 33-44%, respectively. 15% of women dentistry students in the United States reported IS, while 57.8 % of youngsters in Saudi Arabia showed symptoms of Impostor Syndrome [11] [12].

The effect of impostorism on a specific leadership behaviour component of task delegation was analyzed in [13]. The population included 190 managers of various industries, with a prevalence of 74.6%. [3] examined the connection between the impostor phenomenon and racial discrimination in over one hundred and fifty African-American university students aged 18 to 19 through a cross-sectional survey. Respondents with more significant levels of IS also conveyed more survivor guilt feelings. [14] analyzed the prevalence of Impostor syndrome in final-year nursing students in a cross-sectional survey from Australia, the UK and New Zealand. The population selection included over 200 nursing students, of which 45.1% had mild IS,

33.4% were classified as repeatedly having IS feelings, and 8.3% were depicted as frequently experiencing intense IS experiences. A positive weak correlation between IS and preparedness for practice was found. [15] analyzed the connection between IS, perceived prejudice, and mental-health issues among minority trainees. The population sample was 322 College students with a mean age of 21 years (70% women) through a cross-sectional survey.

Perfectionism was described as "insisting of oneself or others a higher quality of performance than is needed by the circumstances". David Burn (1980) [16] produced a perfectionism scale, one of the first instruments to quantify perfectionism. He defined them as one whose ideals are high above reach or reason, who strains unremittingly and compulsively toward unattainable goals. Here, perfectionism is defined as a person's self-description of his or her performance style as perfectionistic, and most psychiatrists would likely agree with this assessment. The multidimensional perfectionism scale [17] is a test used to assess the character trait of perfectionism. Impostor syndrome in the classroom was analyzed in [18] to evaluate the influences of gender, level, grade, GPA, and individual characteristics on impostor syndrome among high school students. The study was conducted on 104 English honours students in grades 9-12, and there were no gender-based differences in impostorism. Impostor Syndrome was analyzed in [19] on 506 college students of mean age 21yrs (79% women) through a cross-sectional survey.

Women were notably more inclined to convey impostor feelings than men. Perfectionism, test anxiety and mental health were mainly related to IS, but low self-esteem was not. In [20] investigated gender disparities in anxiety of success and failure and IS on 104 marketing managers, of which response rate was 92.9% and mean age was 35yrs and 49% were women through a cross-sectional survey. Among male and female managers, significant positive correlations were observed between fear of failure and IS.

In [21], the authors evaluated emotional fatigue and work satisfaction among faculty with IS on 16 and 310 academic faculties for two studies, respectively. Study 1 had 63% women, and Study 2 had 59% women through a cross-sectional survey. Women and men vary in their coping techniques for managing impostor syndrome. [22] analyzed the impostor syndrome among Austrian doctoral students and evaluated gender differences in the impostor syndrome of nearly 631 students. Females had more fear of success and failure and lesser self-esteem than men. Faculties reported higher levels of impostor syndrome and research self-efficacy than non-faculty members. In [23], the authors explored the presence and connection between IS and burnout syndrome in internal medicinal residents.

[2] examined whether IS is a homogeneous construct or whether different types of persons with impostor syndrome can be distinguished based on related characteristics with 242 professionals in administration positions. No association was found between impostor syndrome and gender.

The authors have tried correlating well-being to impostor syndrome and gender role orientation in [24]. The population sample was 379 college students, which was a cross-sectional survey. People with high impostor syndrome scored less in well-being and self-acceptance. Significant differences were found in Impostor Syndrome by gender role orientation. [6] sampled the opinions of five doctors of various disciplines of medicine and their experiences with impostor syndrome. [25] identifies the prevalence of impostor syndrome among computer science students by conducting a cross-sectional survey on 203 college students. Additionally, it validates that the women students had more elevated levels of impostor feelings than the men.

[26] studied IS among first-generation and continuing-generation university students, with a population sample was 388 college students. After researching the relationship between IS level and perfectionism in these populations, only socially stipulated perfectionism was discovered to be mainly related to impostor syndrome among college students.

However, none of these studies thought there could be indeterminacy while making conclusions about these studies. As we see, sometimes, we may be unable to distinguish the presence or absence of impostor syndrome. In those cases, neutrosophic models can be used. Further, all results or conclusions may involve a certain amount of uncertainty in that situation. Neutrosophic will play a significant role.

Neutrosophy was introduced as a generalization to fuzzy theory. It handles the neutralities/indeterminacy present in the real-world scenario [27–31]. Neutrosophy has been recently used in psychology and will be very beneficial in analyzing impostor syndrome problems. Smarandache presented refined neutrosophic set (RNS) in [32] and further evolved into Double Valued NS (DVNS) [33], Triple Refined Indeterminate NS (TRINS) [34], and Multi Refined NS (MRNS) [35]. The indeterminate Likert scaling was defined using TRINS. Neutrosophy and neutrosophic psychology has been used to study several psychological problems [36–38].

This paper proposes the neutrosophic measure for Imposter syndrome based on neutrosophic traits and psychology. This is implemented using data collection using a specifically designed questionnaire after the data preprocessing and applying appropriate machine learning algorithms.

This paper is organized as follows: Section one is an introductory, extensive literature survey regarding impostor syndrome in young adults, and neutrosophy is given. Section two covers basic concepts of neutrosophy, psychology, indeterminate Likert scaling and Impostor syndrome. Neutrosophic measures for Impostor syndrome are introduced in section three.

Section four provides the dataset description along with the methodology and deals with the working of the proposed model, including calculating impostor syndrome scores. Section five deals with data analysis along with the machine learning models used. Results and discussion are given in section six. The last section discusses the limitations and concludes the study.

2. **Basic Concepts**

This subsection deals with the basic concepts of Neutrosophy and Neutrosophic psychology. The neutrosophic psychological framework is based on Freud's theory of conscious/unconcious memory and preconscious memory, it also includes one more state known as the aconscious state. Based on the neutrosophic theory, the extended neutrosophic psychology is denoted by $(< A >), < NeutA >, < antiA >)$.

Refined Neutrosophic sets are used to capture this, and several applications of neutrosophic psychology can be found in [32].

Definition 2.1. The refined neutrosophic set is described such that the truth T is split into several kinds of truths: $T_1, T_2, \ldots, T_p$, Indeterminate I into different indeterminacies: $I_1, I_2, \ldots, I_r$ and False F into different falsehoods: $F_1, F_2, \ldots, F_s$, where all $0 < p, r, s \leq 1$ are integers, and $p + r + s = n$.

Definition 2.2. [34] Consider X a space of objects with elements in X denoted by x. A TRINS A in X is represented by truth $T_A(x)$, indeterminacy leaning towards truth $IT_A(x)$, indeterminacy $I_A(x)$, indeterminacy leaning towards falsity $IF_A(x)$, and falsity $F_A(x)$ membership functions. For each component $x \in X$, there are

$$0 \leq T_A(x) + IT_A(x) + I_A(x) + IF_A(x) + F_A(x) \leq 5 \tag{1}$$

Thus, a TRINS A can be described by

$$A = \langle x, T_A(x), IT_A(x), INT_A(x), IAT_A(x), AT_A(x) \rangle : x \in X \tag{2}$$

and is characterized by trait $T_A(x)$, indeterminacy trait $IT_A(x)$, neutral trait $INT_A(x)$, indeterminate anti-trait $IAT_A(x)$ and anti trait $AT_A(x)$ membership functions.

Detailed examples and working of TRINS can be obtained from [34, 35, 39].

3. **Neutrosophic trait measures**

According to neutrosophic psychology, there are several trait and anti-trait pairs. Some of the most common trait and anti-trait pairs related to the Impostor syndrome are
Extraversion-Introversion
Perfectionism-Imperfectionism

Self esteem- Self non-esteem
Honesty - Dishonesty

Let us, for example, consider the extraversion-introversion, trait-anti-trait pair. A person can be an extrovert in some situations, at the same time, he/she can also be an introvert while interacting with some other people. There are also chances that s/he is an ambivert. To accurately capture this kind of personality trait, we have developed a questionnaire. We now proceed to describe the questionnaire.

The questionnaire is framed to cover all these traits and anti-traits pairs. We propose in the following section neutrosophic impostor syndrome measures. Instead of using a complete indeterminate Likert scaling-based questionnaire, we have framed the questionnaire differently since using the indeterminate Likert scaling questionnaire might require more professional support.

We have asked them to answer based on a 5-point Likert scaling to make the evaluation easier. Their answers were combined/aggregated into suitable neutrosophic measures.

We use the concept of TRINS defined by [34]. We extend the concept of TRINS values to psychology and propose a novel architecture for generating neutrosophic values from the questionnaire.

Here, we introduce the concept of neutrosophic trait measures for a trait, which is later used to define neutrosophic impostor syndrome and neutrosophic perfectionist measures.

Definition 3.1. Consider X to be a collection of all trait- anti pair with elements of X denoted by x. A neutrosophic trait S is based on a 5-tuple refined neutrosophic set. It is denoted as a neutrosophic set by

$$S = \langle x, Tr_S(x), ITr_S(x), NTr_S(x), IATr_S(x), ATr_S(x)\rangle : x \in X \tag{3}$$

where $Tr_S(x)$ denotes the degree of presence of trait S, which is based on the truth membership of TRINS, $ITr_S(x)$ denotes the degree of presence of indeterminate trait S, which is based on the truth leaning towards indeterminacy membership of TRINS, $NTr_S(x)$ denotes the degree of presence of neutral trait, which is based on the indeterminate membership of TRINS, $IATr_S(x)$ denotes the degree of presence of indeterminate anti-trait S, which is based on the false leaning towards indeterminacy membership of TRINS, $ATr_S(x)$ denotes the degree of presence of anti-trait S which is based on the false membership of TRINS.

Next, we define the three functions, namely, accuracy function, score function and certainty function.

Definition 3.2. The accuracy function a, defined over the Neutrosophic trait S of x as

$$a(S(x)) = Tr_S(x) - ATr_S(x) \tag{4}$$

Definition 3.3. The score function s, defined over the Neutrosophic trait S of x as

$$s(S(x)) = (Tr_S(x) + ITr_S(x) + (1 - NTr_S(x)) + (1 - IATr_S(x)) + (1 - ATr_S(x)))/5 \tag{5}$$

Definition 3.4. The certainty function c, defined over the Neutrosophic trait S of x as

$$c(S(x)) = Tr_S(x). \tag{6}$$

Neutrosophic Impostor syndrome measure calculations

We propose the Neutrosophic Impostor Syndrome IS as Neutrosophic Trait measure IS in X as given above in 3, it is characterized by the degree of Impostor syndrome using the trait and anti-trait membership values. The Neutrosophic Impostor syndrome is given by

$$IS = \langle x, Tr_{IS}(x), IrT_{IS}(x), NTr_{IS}(x), IATr_{IS}(x), ATr_{IS}(x)\rangle : x \in X \tag{7}$$

It is important to note that the anti-trait of Imposter syndrome is Peter's principle or the Dunning-Kruger effect. Similarly, the neutrosophic perfectionist NP is given by

$$NP = \langle x, Tr_{NP}(x), ITr_{NP}(x), INTr_{NP}(x), IATr_{NP}(x), ATr_{NP}(x)\rangle : x \in X \tag{8}$$

4. System Architecture

Figure 1 provides the proposed framework's architecture. We prepared a separate questionnaire, performed data collection, and pre-processed the data. Neutrosophic trait scores were calculated for each data point, and the data was labelled. Data analysis using exploratory data analysis was done, and then machine learning algorithms were implemented. Discussion and conclusions inferred from EDA and machine learning models are presented in the last module.

4.1. *Questionnaire Design*

We wanted to cover a set of interconnections between impostor syndrome, perfectionism, self-esteem and post-covid confidence levels in students after college reopened. Generally, impostor syndrome questionnaires , are based on the 5-point Likert scale, expect the summation of answers to provide the results.

Our questionnaire was designed innovatively with 29 items to cover impostor characteristics, perfectionism, self-esteem, emotional quotient and introversion-extroversion characteristics. Certain questionnaire sections were framed intentionally, so the respondents had to answer similar concept-based questions at least twice. Some questions were asked in the reversed

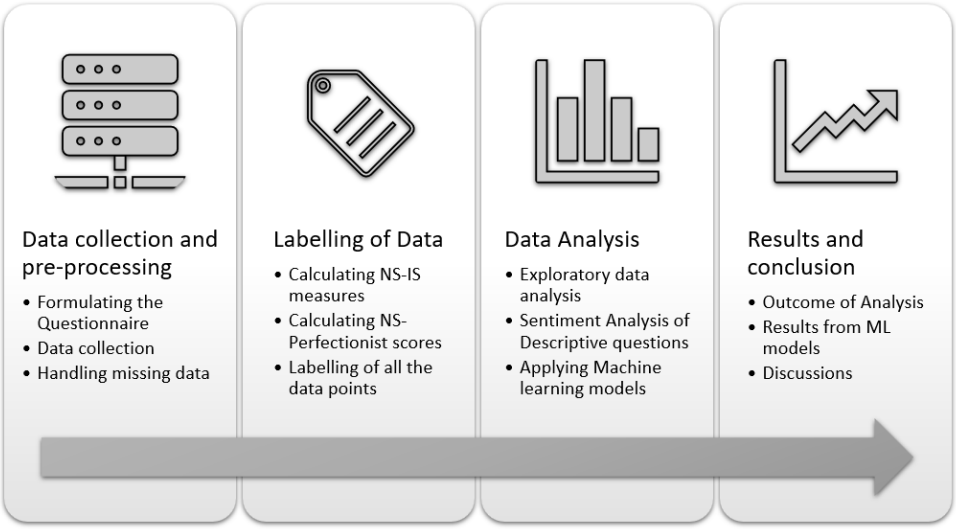

Data collection and pre-processing
• Formulating the Questionnaire
• Data collection
• Handling missing data
Labelling of Data
• Calculating NS-IS measures
• Calculating NS-Perfectionist scores
• Labelling of all the data points
Data Analysis
• Exploratory data analysis
• Sentiment Analysis of Descriptive questions
• Applying Machine learning models
Results and conclusion
• Outcome of Analysis
• Results from ML models
• Discussions

FIGURE 1. The overall system architecture

direction so that a respondent might capture a more genuine picture of themselves in one way or the other.

The survey conducted was anonymous. Questions were on a 5-point Likert scale, from "Strongly Agree" to "Strongly Disagree". The subjects were asked about their age and gender. All the questions were masked and worded positively to hide the direct intentions of the purpose of the questions. The complete questionnaire is provided in Appendix A. Certain salient features are discussed here to highlight its uniqueness.

Impostor Syndrome related: In the questionnaire, 13 items were focused on Impostor Syndrome; they are items 3-5, 8-14, 19-20 and 23 in Appendix 7.

Consider items Q4 and Q8

Q4: Many times, you feel crushed by constructive criticism, seeing it as evidence of your "ineptness"?

Q8: In rare cases, you feel crushed by constructive criticism, seeing it as evidence of your "ineptness"?

Item 8 is a reversed question of item 4. If a person has already made up his/her mind that the SA option is the default, or if they are trying to hide things, there are chances that they might exhibit impostor syndrome in the case of reversed questions.

Similarly, consider Q3, Q11 and Q20.

Q3. Do you chalk your success up to fates, luck or error?

Q11. You blame your luck for success rather than hard work.

Q20. Most of your success has been a stroke of luck.

All these questions deal with the luck factor.

Perfectionism related: Generally, impostor syndrome is related to perfection and self-esteem issues. To capture the same, questions related to perfection were asked. Items 6, 7, 15, 17 and 18 are related to perfectionism.
Self-esteem related: Two Likert scale questions were asked regarding their self-image in the house. The second part of the questionnaire consists of descriptive questions on whether the subjects' self-esteem (the question says confidence, though) in their college life had changed after the pandemic and their response to a generalized gender-based self-esteem statement.

IQ-EQ and introversion-extroversion related: Items 21 and 22 are related to IQ and EQ in self-judgment. The introversion-extroversion aspect is dealt with in items 25-27. It also consisted of a quantitative question about how comfortable they can be without being judged in different social settings like home, college, on social media, an online alter-ego and with friends.

4.2. *Methodology*

Data Collection: The dataset has been obtained by conducting a survey consisting of a detailed pen-and-paper questionnaire of 29 items, of which 22 were 5-point Likert scale-based questions and two written questions, covering aspects to measure the participants' levels of experiencing Impostor syndrome, perfectionism and gender. The respondents were 200 university students from Vellore, India.

The survey was conducted in the post-pandemic environment, with in-person classroom attendance. The students who were wiling voluteeraily participated in the study. Their ages ranged from 19 to 22. There was no monetary reward or added incentive for partaking. The study complied with the ethical research regulation of the college from which the respondents were recruited. The collected data was then manually entered into a CSV file by us.

Data pre-processing: Cleaning the manually entered data is an important data pre-processing step:

(1) Many issues were there in data entry; this was cleaned by hardcoding and replacing the anomalies found.
(2) Dropped serial number column for smoother working of machine learning models.
(3) As mentioned earlier, the questionnaire has inverted specific questions. To ensure that each answer's extremes indicated the same phenomenon, the answers (SD $\rightarrow$ SA) have been inverted for some questions (8, 12, 18) to facilitate this.
(4) Filled 'NA' values with 'N' (Neutral) to preserve the item's weight.

Weightage to questions: Of the thirteen items under consideration for Impostor syndrome, items 4, 8, 11-13, 19 & 20 were given a weightage of 2, and others were given a weightage of 1.

4.3. *Labelling of Data*

Neutrosophic Impostor Syndrome measures

The impostor syndrome is calculated from all three functions (accuracy, score and certainty), and a counter variable called result is used. It is initialized to 0.

• Neutrosophic impostor syndrome accuracy cutoff: If $a(IS) \geq 0$, result = result + 1 else result = result.

• Neutrosophic impostor syndrome score cutoff: If $s(IS) \geq 0.53$, result = result + 1 else result = result.

• Neutrosophic impostor syndrome certainty cutoff: If $c(IS) \geq 0.1$, result = result + 1 else result = result.

If the result ≥ 2, the data point is labelled Yes for impostor syndrome.

Similarly, the neutrosophic perfectionist accuracy, score and certainty are calculated, but the labelling is done directly only from item 15. The calculated neutrosophic measures are used for machine learning modelling.

Introversion score calculation: These questions (items 25, 26 and 27) were asked to see how introverted or extroverted the students were. Based on their responses to the questions, an extrovert-introvert score was calculated, which ranged from 0 to 10. If the score was less than or equal to five, the person was labelled an extrovert, while if the score was greater than five, they were labelled introverted.

The first item (25) was asked to show where the students could be themselves without being judged by their surroundings or peers. The next question was used to observe how comfortable the subjects felt at certain places in their daily life, like home, college, friends, workplace, online, and their alter-ego. Students were expected to answer this question in percentages. The following question was targeted to find where they had to pretend to be somebody else they were not. Depending on their responses to all three questions, the score calculation has been divided into six categories: nowhere, home, online and alter-ego, friends, college and workplace. The score variable is altered according to their choice for each question. A detailed score updation is given below: In the first question, if the student chose nowhere, it indicates that they cannot truly be themselves anywhere and are heavily inclined towards being an introvert. The score is then increased by ten. Suppose the student responded that they are comfortable at home. They pretend to be someone they are not at home while being comfortable (home >=50); 1 point is added to the score. If they chose that they felt judged at home but also gave a percentage of being comfortable at home >=70, it could be drawn that they assume they

TABLE 1. Summary of Dataset after labelling

Title	Overall Value	Female	Male	Undisclosed
Total people	200	41	131	28
number with IS	114	26	68	20
% of people with IS	57%	63.41%	51.91%	71.42%
number of people with perfectionism	57	18	33	6
% of people with perfectionism	28.50%	43.90%	25.19%	21.42%
number of people with IS having perfectionism	36	10	33	3
% of people with IS having perfectionism	31.50%	27.77%	91.66%	8.33%

feel comfortable at a place despite knowing they are judged there. 2 points are then added to the score. If this is not the case, two is removed from the score.

If the student chose online or alter-ego, they pretended to be someone they are not online and answered that they were comfortable(>=60), then; if they were comfortable having an alter-ego (%>=50), then three points are added to the score since it indicates that he/she might have that personality that they think is pretension. However, they cannot express it because they are introverted, and the online community gives a sense of safety to express themselves, else Three is subtracted from the score since they are not comfortable having an online alter-ego, indicating extroverts. Else, if they feel they can be themselves online without being judged and are comfortable in alter-ego(>60%), then the score is increased by 4, else if they were not that comfortable($\leq$ 60%), the score is increased by 1.

If the student chose with friends, they pretended to be someone else in front of their friends and be comfortable around them >60%, and if they feel they could be themselves, the score is increased by two else if they are comfortable being themselves at college >50.

5. **Data analysis and Machine learning module**

After the labelling of the data, exploratory data analysis was performed on the data; the results are tabulated in Table 1, which is discussed in detail in the next section. Out of the 200 participants, there were 41 females, 131 males and 28 people who chose not to disclose their gender. In total, 114 people had impostor syndrome, which is close to 57%.

Random forest: Random forest is an ML algorithm that uses multiple decision trees to make predictions. It is an extension of the bagging method that uses both bagging and feature randomness to create an uncorrelated forest of decision trees. Random Forest is commonly used for both classification and regression problems. It is a flexible and easy-to-use algorithm

that usually produces excellent results, even without hyper-parameter tuning. The dataset was split into two parts: 75% for training and 25% for testing.

Support Vector Machine Support Vector Machine (SVM) is a supervised ML algorithm for classification and regression analysis. It is used to find the best boundary between two data classes by maximizing the margin between them. SVMs are often used in image classification, text classification, and bioinformatics. SVM was implemented for the dataset. The dataset split was a 70:30 ratio. We used a linear kernel to implement the model to avoid overfitting data.

K-Nearest Neighbours (K-NN) It is a simple ML algorithm founded on a supervised learning technique. It uses the likeness between the new case/data and available cases. It classifies a new data point based on the similarity of the dimensional features for each data point. For this model, the dataset was split into an 80:20 ratio.

Sentiment Analysis of descriptive questions

Calculating SVNS values: Expression of sentiment is very complex, but the VADER package is defined to understand online language closely. It has use cases to encompass utf-8 encoded emojis, emoticons (:D, :P, XD), slang words(sux), slang words modified (kinda, friggin), use of all caps(GOOD), use of exclamation points (good!!), and usage of typical negations (not good).

VADER contains inbuilt pre-defined objects like SentimentIntensityAnalyzer() and polarity_scores(). SentimentIntensityAnalyzer() takes input in string format and returns four values: positive (pos), negative(neg), neutral(neu) and compound(comp); where $0 \geq$ pos, neg, neu, comp ≤ 1. The scores denote how much positivity, negativity and neutrality lie in the sentence and the variable "compound" is calculated by normalizing the three scores. The closer the value of the compound is to "1", the higher the positivity of the sentence. If the compound value is ≥ 0.05, it is considered positive; if compound ≤ -0.05, it is negative, or else it is neutral. Since we are focusing on a gender-based study, we asked two questions to the subjects; one based on their views about self-image issues in men and fear of failure in women and the next question describes self-confidence pre and post covid.

6. **Results and Discussions**

6.1. *Results from EDA*

Of the 200 participants involved in the study, 114 had imposter syndrome. Of the 41 females, 26 had imposter syndrome; out of 131 males, 68 had imposter syndrome. There was a significant difference in the occurrence of Imposter syndrome between females and males, with 63.41% and 51.91%, respectively. It is visible and validated that women are likelier to

TABLE 2. Results from Machine Learning models

Models	Random forest	SVM	KNN
Train: Test Ratio	75:25	70:30	80:20
Accuracy (%)	83.7	92.15	82.35
Precision(weighted avg)	95.2	90.9	85.7
Recall(weighted avg)	76.9	90.9	85.7

have imposter syndrome. While 28.5% of the total respondents were perfectionists, 43.90% were perfectionists among the females. And only 25.19% of males were perfectionists.

It is interesting to note that 31.5% of people with Imposter syndrome are also perfectionists. However, there is a vast difference between females and males. Only 27.7% of women with imposter syndrome are perfectionists. In contrast, almost 91.66% of men with imposter syndrome are perfectionists. The correlation between perfectionism and imposter syndrome is observed here.

6.2. *Results from Machine Learning Models*

Three machine learning models, namely Random forest, SVM and KNN, were implemented using Python. The results are tabulated in Table 2.

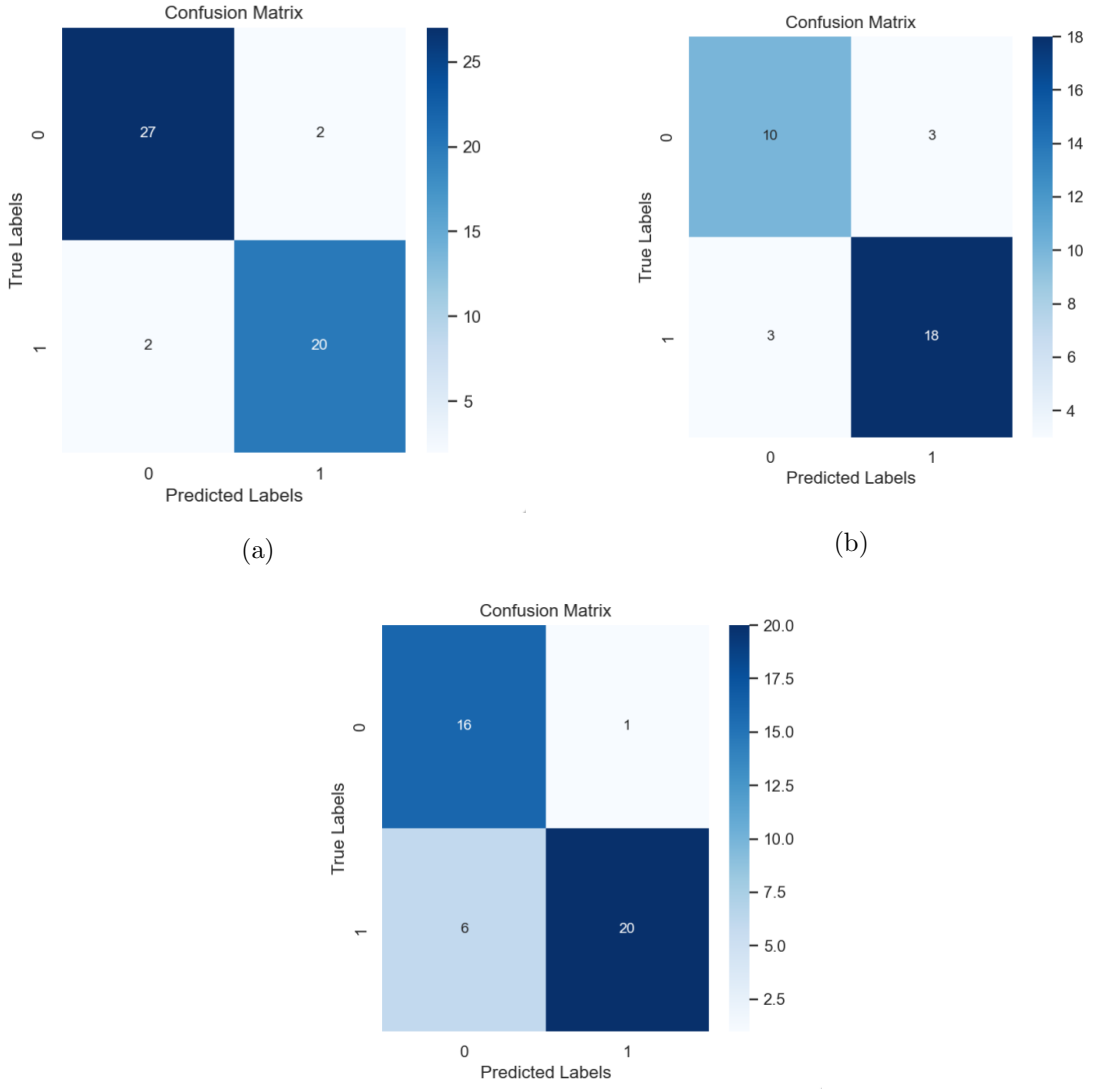
Random forest: The features used in the model include the student's age, their responses to questions, sentiment scores for two descriptive questions, and labels for perfectionist and impostor syndrome. The random forest model had a high accuracy of 83.7%, which means it is suitable for solving problems with many features. Figure 2a gives the confusion matrix for the random forest model.

SVM: SVM was implemented for the dataset. The dataset was split into a 70:30 ratio. We used a linear kernel to implement the model to avoid overfitting data. The best accuracy was obtained with the SVM model; we attained an accuracy of 92.15%. Figure 2b gives the confusion matrix for the SVM model.

KNN: KNN is a supervising machine learning model that stores all the data and classifies a new data point based on the similarity of the dimensional features for each data point. For this model, the dataset was split into an 80:20 ratio. We achieved an accuracy of 80.20%. The confusion matrix for KNN model is given in Figure 2c.

6.3. *Results from sentiment Analysis*

The data set was split based on gender and sentiment analysis was conducted for each long answer question. For the first question, men had a compound score of 0.1491 (positive) and women had 0.2689(positive); meaning women agreed that they face self-image issues more than



(a) (b)

(c)

FIGURE 2. Confusion Matrix: (a) Support Vector Model; (b) K-NN ; (c) Random Forest

men. While analysing the second question, men had a compound score of 0.2545 (positive) and women had 0.2580(positive); meaning men and women equally believe that covid has affected their self-confidence.

7. Conclusions

Neutrosophic trait measures were introduced in this paper based on refined neutrosophic sets. A group of 200 students participated in the study; it was conducted and labelled using Neutrosophic trait measures. Impostor syndrome was analysed along with perfectionism, self-esteem issues and introversion. The ratio of students with Impostor syndrome was the same

across the genders; no gender-based difference was found. After labelling the data, machine learning models like SVM, KNN and Random forest were implemented. SVM performed the best of the three models.

Limitation and Future Study: The study's primary limitation is based on the fact that the number of female participants is less here since the number of female students in STEM fields is lesser than that of boys in STEM. For future research, gender-oriented studies can be taken up with more participants. With more detailed data collection based on (Q16 and Q25), gender-related studies can be conducted to predict how Impostor syndrome affects the genders in STEM fields. The socio-economic background of the student should have been considered. Similarly, the educational background, like first-generation learners, can be considered in future studies.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Sample Questionnaire

(1) Gender:
(2) Age:
(3) Do you chalk your success up to fates, luck or error?
(4) Many times, you feel crushed by constructive criticism, seeing it as evidence of your "ineptness"?
(5) Do you believe "If I can do it, anybody can"?
(6) Do you agonize over the smallest flaws in your work?
(7) Do you believe that everything you do must be completely perfect?
(8) In rare cases you feel crushed by constructive criticism, seeing it evidence of your "ineptness"?
(9) Do you feel incompetent despite attaining success?
(10) You compare your abilities to people around you and think that others may be more intelligent than you.
(11) You blame your luck for success rather than hard work.
(12) Do you think shortcut to success makes you smarter
(13) You have doubts about your abilities despite people around you trusting you.
(14) Do you often fear not meeting other people's expectations?
(15) I am a perfectionist
(16) Do you feel like a non-valuable member of the family if you don't participate in domestic work?

(17) 'I have to be good at a particular activity to enjoy it' (As in, if you picked up a new hobby like painting, the only way you feel good about doing it is if you are using perfect techniques and doing it the "right" way)

(18) I believe that means is more important than the ends

(19) When people compliment you, you think you are not as accomplished as they think

(20) Most of your success has been a stroke of luck

(21) You have an above-average IQ score

(22) Your emotional quotient is better than your general IQ score

(23) You downplay compliments from others.

(24) Do you feel like an unimportant family member if you don't involve in decision-making process?

(25) Which places do you think you can be yourself without being judged?
Home College With friends Work Online Nowhere

(26) How comfortable are you with being yourself at (please give a percentage)
Home College With friends Work Online Alter Online Egos

(27) What places do you think you pretend to be someone you are not?
Home College With friends Work Online Nowhere

(28) At least 70% of individuals have dissatisfaction in their lives. Women mostly face self-image issues and in men, it is driven by the fear of not being successful or letting people down. Do you agree with this? And in your experience, how have you seen variances to the mentioned scenarios?

(29) As the campus has reopened after conducting classes, examinations, and project reviews online over the past 2 years, do you feel that this might bring up changes in confidence levels, due to the adjustments which may have arisen doubts in self-esteem from the initial change of method?

References

1. John Kolligian Jr and Robert J Sternberg. Perceived fraudulence in young adults: Is there an'imposter syndrome'? *Journal of personality assessment*, 56(2):308–326, 1991.
2. Mona Leonhardt, Myriam N Bechtoldt, and Sonja Rohrmann. All impostors aren't alike–differentiating the impostor phenomenon. *Frontiers in psychology*, 8:1505, 2017.
3. Donte L Bernard, Quiera M Lige, Henry A Willis, Effua E Sosoo, and Enrique W Neblett. Impostor phenomenon and mental health: The influence of racial discrimination and gender. *Journal of counseling psychology*, 64(2):155, 2017.
4. Anna Parkman. The imposter phenomenon in higher education: Incidence and impact. *Journal of Higher Education Theory and Practice*, 16(1):51, 2016.
5. Pauline Rose Clance and Suzanne Ament Imes. The imposter phenomenon in high achieving women: Dynamics and therapeutic intervention. *Psychotherapy: Theory, research & practice*, 15(3):241, 1978.

6. Georgia E Hodes, Madeline L Pfau, Marylene Leboeuf, Sam A Golden, Daniel J Christoffel, Dana Bregman, Nicole Rebusi, Mitra Heshmati, Hossein Aleyasin, Brandon L Warren, et al. Individual differences in the peripheral immune system promote resilience versus susceptibility to social stress. *Proceedings of the National Academy of Sciences*, 111(45):16136–16141, 2014.
7. Brian F French, Sarah C Ullrich-French, and Deborah Follman. The psychometric properties of the clance impostor scale. *Personality and Individual Differences*, 44(5):1270–1278, 2008.
8. Md Khan, Nafiul Alam, M Miah, Md Shahjalal, Talha Bin Sarwar, Md Rokon, et al. Predicting young imposter syndrome using ensemble learning. *Complexity*, 2022, 2022.
9. Dena M Bravata, Sharon A Watts, Autumn L Keefer, Divya K Madhusudhan, Katie T Taylor, Dani M Clark, Ross S Nelson, Kevin O Cokley, and Heather K Hagg. Prevalence, predictors, and treatment of impostor syndrome: a systematic review. *Journal of General Internal Medicine*, 35(4):1252–1275, 2020.
10. Michael Gottlieb, Arlene Chung, Nicole Battaglioli, Stefanie S Sebok-Syer, and Annahieta Kalantari. Impostor syndrome among physicians and physicians in training: a scoping review. *Medical education*, 54(2):116–124, 2020.
11. Alison M Holliday, Galina Gheihman, Cynthia Cooper, Amy Sullivan, Hiroe Ohyama, David E Leaf, and Rebecca Karp Leaf. High prevalence of imposterism among female harvard medical and dental students. *Journal of general internal medicine*, 35(8):2499–2501, 2020.
12. Saad Alrayyes, Umar F Dar, Musab Alrayes, Ahmed Alghutayghit, and Nouf Alrayyes. Burnout and imposter syndrome among saudi young adults: The strings in the puppet show of psychological morbidity. *Saudi Medical Journal*, 41(2):189, 2020.
13. Myriam N Bechtoldt. Wanted: Self-doubting employees—managers scoring positively on impostorism favor insecure employees in task delegation. *Personality and Individual Differences*, 86:482–486, 2015.
14. Martin Christensen, Aimee Aubeeluck, Diana Fergusson, Judy Craft, Jessica Knight, Lisa Wirihana, and Ed Stupple. Do student nurses experience imposter phenomenon? an international comparison of final year undergraduate nursing students readiness for registration. *Journal of advanced nursing*, 72(11):2784–2793, 2016.
15. Kevin Cokley, Leann Smith, Donte Bernard, Ashley Hurst, Stacey Jackson, Steven Stone, Olufunke Awosogba, Chastity Saucer, Marlon Bailey, and Davia Roberts. Impostor feelings as a moderator and mediator of the relationship between perceived discrimination and mental health among racial/ethnic minority college students. *Journal of Counseling Psychology*, 64(2):141, 2017.
16. Murray W Enns and Brian J Cox. The nature and assessment of perfectionism: A critical analysis. 2002.
17. Paul L Hewitt, Gordon L Flett, Wendy Turnbull-Donovan, and Samuel F Mikail. The multidimensional perfectionism scale: Reliability, validity, and psychometric properties in psychiatric samples. *Psychological Assessment: A Journal of Consulting and Clinical Psychology*, 3(3):464, 1991.
18. Barbara Heflin Cromwell. *The impostor phenomenon in the classroom: Personality and cognitive correlates*. Old Dominion University, 1989.
19. Claire E Cusack, Jennifer L Hughes, and Nadi Nuhu. Connecting gender and mental health to imposter phenomenon feelings. *Psi Chi Journal of Psychological Research*, 18(2), 2013.
20. Sharon Fried-Buchalter. Fear of success, fear of failure, and the imposter phenomenon among male and female marketing managers. *Sex roles*, 37(11):847–859, 1997.
21. Holly M Hutchins, Lisa M Penney, and Lisa W Sublett. What imposters risk at work: Exploring imposter phenomenon, stress coping, and job outcomes. *Human Resource Development Quarterly*, 29(1):31–48, 2018.
22. Gregor Jöstl, Evelyn Bergsmann, Marko Lüftenegger, Barbara Schober, and Christiane Spiel. When will they blow my cover? *Zeitschrift für Psychologie*, 2015.
23. Jenny Legassie, Elaine M Zibrowski, and Mark A Goldszmidt. Measuring resident well-being: impostorism and burnout syndrome in residency. *Journal of general internal medicine*, 23(7):1090–1094, 2008.

24. Aysa N September, Michael McCarrey, Anna Baranowsky, Chantal Parent, and Dwayne Schindler. The relation between well-being, impostor feelings, and gender role orientation among canadian university students. *The Journal of social psychology*, 141(2):218–232, 2001.
25. Adam Rosenstein, Aishma Raghu, and Leo Porter. Identifying the prevalence of the impostor phenomenon among computer science students. In *Proceedings of the 51st ACM Technical Symposium on Computer Science Education*, pages 30–36, 2020.
26. Chelsey L Holden, Lindsay E Wright, Angel M Herring, and Pat L Sims. Imposter syndrome among first-and continuing-generation college students: The roles of perfectionism and stress. *Journal of College Student Retention: Research, Theory & Practice*, page 15210251211019379, 2021.
27. Maissam Jdid, Florentin Smarandache, and Said Broumi. Inspection assignment form for product quality control using neutrosophic logic. *Neutrosophic Systems with Applications*, 1:4–13, 2023.
28. Ahmed Sleem, Nehal Mostafa, and Ibrahim Elhenawy. Neutrosophic critic mcdm methodology for ranking factors and needs of customers in product's target demographic in virtual reality metaverse. *Neutrosophic Systems with Applications*, 2:55–65, 2023.
29. Jagan Obbineni, Ilanthenral Kandasamy, WB Vasantha, and Florentin Smarandache. Combining swot analysis and neutrosophic cognitive maps for multi-criteria decision making: a case study of organic agriculture in india. *Soft Computing*, pages 1–22, 2023.
30. Ibrahim M Hezam. An intelligent decision support model for optimal selection of machine tool under uncertainty: Recent trends. *Neutrosophic Systems with Applications*, 3:35–44, 2023.
31. Kritika Mishra, Ilanthenral Kandasamy, Vasantha Kandasamy WB, and Florentin Smarandache. A novel framework using neutrosophy for integrated speech and text sentiment analysis. *Symmetry*, 12(10):1715, 2020.
32. Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite Study*, 4:143–146, 2013.
33. Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.
34. Ilanthenral Kandasamy and Florentin Smarandache. Triple refined indeterminate neutrosophic sets for personality classification. In *2016 IEEE Symposium Series on Computational Intelligence (SSCI)*, pages 1–8. IEEE, 2016.
35. Ilanthenral Kandasamy, WB Vasantha, Jagan M Obbineni, and Florentin Smarandache. Sentiment analysis of tweets using refined neutrosophic sets. *Computers in Industry*, 115:103180, 2020.
36. Florentin Smarandache. *Neutropsychic Personality: A mathematical approach to psychology*. Infinite Study, 2018.
37. Carmen Verónica Valenzuela Chicaiza, Olga Germania Arciniegas Paspuel, Paola Yesenia, Carrera Cuesta, and Sary Del Rocío Álvarez Hernández. Neutrosophic psychology for emotional intelligence analysis in students of the autonomous university of los andes, ecuador. *Neutrosophic Sets and Systems*, 34:1–8, 2020.
38. Iván Pimienta Concepción, Elizabeth Mayorga Aldaz, Lenin Gabriel Flores, and Erick González Caballero. *Neutrosophic Scale to Measure Psychopathic Personalities Based on Triple Refined Indeterminate Neutrosophic Sets*, volume 37. Infinite Study, 2020.
39. Ilanthenral Kandasamy, WB Kandasamy, Jagan M Obbineni, and Florentin Smarandache. Indeterminate likert scale: feedback based on neutrosophy, its distance measures and clustering algorithm. *Soft Computing*, 24(10):7459–7468, 2020.

Complex Fermatean Neutrosophic Graph and Application To Decision Making

Said Broumi[1,2*], Swaminathan Mohanaselvi[3], Tomasz Witczak[4], Mohamed Talea[1], Assia Bakali[5], and Florentin Smarandache[6]

[1] Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco
[2]Regional Center for the Professions of Education and Training(C.R.M.E.F), Casablanca-settat , Morocco
[3] Department of Mathematics, SRM Institute of Science and Technology Kattankulathur-603203, India
[4]Institute of Mathematics, Faculty of Science and Technology, University of Silesia, Bankowa 14, 40-007 Katowice, Poland
[5] Ecole Royale Navale-Boulevard Sour Jdid, Morocco
[6] Department of Mathematics,University of New Mexico,705 Gurley Avenue, Gallup, NM 87301, USA

Broumi, Said, et al.. "Complex Fermatean Neutro Graphs with Application." *Decision Making: Applications in Management and Engineering*, 2023, p. 1-28. doi:10.31181/dmame24022023b.

Abstract: *A new growing area of the neutrosophic set theory called complex neutrosophic sets (CNS) provides useful tools for dealing with uncertainty in complex valued physical variables that are observed in the actual world. A CNS takes values for the truth, indeterminacy and falsity membership functions in the complex plane's unit circle. In this research, a novel concept of complex fermatean neutrosophic graph (CFNG) is established and various basic graphical ideas such as the order, size, degree and total degree of a vertex of it are introduced. Also, set theoretical operations such as complement, union, join, ring-sum and cartesian product of CFNG are studied. Further, the concept of a regular graph under a complex fermatean neutrosophic environment is introduced. Finally, we make use of the proposed CFNG in solving a multi-criteria decision-making problem in which the graphical structure of attributes is uncertain. This study also demonstrates the application of a CFNG in the educational system to evaluate a lecturer's research productivity.*

Key words: *Complex neutrosophic sets, Complex fermatean Neutrosophic sets, Neutrosophic graphs, Complex Fermatean neutrosophic graph.*

1. Introduction

The idea of fuzzy sets (FS) was initially proposed by (Zadeh, 1965). It is very good at coping with the uncertainty that exists in a real-world decision-making. FS considers an element's membership degree (α) alone and does not allow us to independently select non-membership degree (β). In this sense, fuzzy sets are still somewhat classical. However, this fact may be considered as a disadvantage. Thus, (Atanassov, 1986) proposed intuitionistic fuzzy sets [IFS] which facilitate the decision maker (DM) to assign α and β independently with the condition $0 \le \alpha + \beta \le 1$. By relaxing the restriction of IFS to $0 \le \alpha^2 + \beta^2 \le 1$, Yager (2013, 2014) proposed Pythagorean fuzzy sets [PFS]. Senapati and Yager (2019, 2019a) proposed Fermatean fuzzy set (FFS) by further relaxing the restriction of PFS $0 \le \alpha^3 + \beta^3 \le 1$ to handle a wider range of uncertainty. A comparison between the space of FS, IFS, PFS, FFS, and PS is shown in Table 1.

Table 1. Comparison between FS, IFS, PFS, FFS, and PS

Extension	Condition	Example	Explanation
FS	$0 \le \alpha \le 1$	$A = \{(a, 0.1), (b, 0.5)\}$	Doesn't consider β
IFS	$0 \le \alpha + \beta \le 1$	$A = \{(a, 0.3, 0.5), (b, 0.6, 0.3)\}$ $B = \{(a, 0.5, 0.7), (b, 0.7, 0.4)\}$	Condition for IFS satisfies for A but fails for B as $\alpha + \beta > 1$
PFS	$0 \le \alpha^2 + \beta^2 \le 1$	$B = \{(a, 0.4, 0.8), (b, 0.5, 0.6)\}$ $C = \{(a, 0.7, 0.8), (b, 0.9, 0.6)\}$	Condition for PFS satisfies for B but fails for C as $\alpha^2 + \beta^2 > 1$
FFS	$0 \le \alpha^3 + \beta^3 \le 1$	$C = \{(a, 0.7, 0.8), (b, 0.9, 0.6)\}$	Condition for FFS satisfies for C
NS	$0 \le \alpha + \beta + \gamma \le 3$	$D = \{(a, 0.7, 0.6, 0.9), (b, 0.6, 0.8, 0.7)\}$	Includes γ

One can add that there also other solutions of this kind. For example, Ibrahim et al. (2021) introduced (3,2)-fuzzy sets while Shanmathi and Nirmala (2022) studied (4,2)-fuzzy sets. In both cases, appropriate topological spaces were studied too. As for the neutrosophic sets (NS), they were initiated by (Smarandache (1999). They incorporate the indeterminacy membership degree (γ) along with the truth membership degree (α) and falsity membership degree (β). Details will be described later in the paper. Another aspect of our work is related to complex numbers and functions. Many physical quantities in practical problems such as wave function in quantum mechanics are complex-valued. Buckley (1989) first initiated the formal definition of complex fuzzy numbers (CFN) and their types. Later on, Ramot et al. (2002) presented the concept of complex fuzzy sets (CFS) by extending

FS in the real field to the complex field. The membership degree of CFS is of the form $\alpha.e^{i\omega}$ where $i=\sqrt{-1}$ whose amplitude term α lies in [0,1] and phase term ω lies in $[0,2\pi]$. Many researchers have focused their attention on CFS over the past few years. Yazdanbakhsh and Dick (2018) reviewed CFS and highlighted potential future research topics in it. Many researchers investigated and worked on CFS and its extensions. Alkouri and Salleh (2012) generalized the concept of CFS to IFS and proposed a complex intuitionistic fuzzy set (CIFS) whose membership and non-membership functions are complex valued. Ullah et al. (2019) presented the concept of complex Pythagorean fuzzy set (CPyFS). Chinnadurai et al. (2021) presented complex Fermatean fuzzy set (CFFS) in rectangular coordinates and its applications in decision-making problems. Ali and Smarandache (2017) proposed complex neutrosophic sets (CNS). Extensions of CFS under various fuzzy environment are shown in Table 2.

Table 2. Complex fuzzy sets under various fuzzy environment

References	Extensions
(Ramot et al., 2002)	CFS
(Alkouri and Salleh, 2012)	CIFS
(Ullah et al., 2019).	CPyFS
(Chinnadurai et al., 2021)	CFFS
(Ali and Smarandache, 2017)	CNS

Our paper deals with graph theory too (but in a context of uncertainty). Therefore, we may say that fuzzy graphs (FG) are developed to model uncertainties in graphical network and their extensions are explored by many researchers (Saumya and Hegde, 2022). Thirunavukarasu et al. (2016) extended FG to complex fuzzy graphs (CFG) and proposed energy of CFG. Yaqoob et al. (2018) studied about complex neutrosophic graphs (CNG). Also, Yaqoob et al. (2019) suggested the complex intuitionistic fuzzy graphs (CIFG) using some basic operations. Akram and Naz (2019) proposed complex Pythagorean fuzzy graphs (CPFG). Shoaib et al. (2022) introduced the concept of picture fuzzy graphs and their properties. Also, Shoaib et al. (2022a) initiated the notion of complex spherical fuzzy graph. Pythagorean neutrosophic fuzzy graphs (PFNG) was suggested by Ajay and Chellamani (2020). Antony and Jansi (2021) proposed the fermatean neutrosophic sets (FNS) and Broumi et al. (2022) explored the fermatean neutrosophic graphs (FNG) with their applications. Motivated by these works we initiated the concept, complex fermatean neutrosophic fuzzy sets (CFNS) and complex fermatean neutrosophic fuzzy graph (CFNG). Butt et al. (2022) established a complex dombi fuzzy graph and their properties. In the single valued neutrosophic environment, regular graphs are explored by Samina et al. (2017). Uncertain decision-making problems can be easily organized and modeled using CFNG theory.

An overview of the motivation for this paper is provided below:

1. A complex Fermatean neutrosophic set (CFNS) is capable of handling the situation well when faced with imprecise and intuitive knowledge in an uncertain decision-making process.
2. Due to the CFNS's phase term, there is no information loss, and are incredibly effective at making decisions and have a wider range of applications.

The primary contributions of this work are as follows:

1. First and foremost, we initiated the new idea of CFNGs.
2. We studied the properties of CFNGs.

3. We introduced the order, degree, size and total degree of CFNGs.
4. We introduced the operations of CFNG such as complement, union, join, ring sum and cartesian product of CFNG.
5. Also studied about the regularity of CFNG and its properties.
6. Discussed the application of CFNG in MCDM problems using the proposed arithmetic and geometric operators.

This paper is arranged as follows. In part 2, we presented the preliminary findings relating to CNG. Some important properties of CFNGs such as the order, size, primary operations, degree, total degree and regularity are introduced in section 3. In section 4, an application of CFNG in educational system is discussed. The conclusion and forthcoming work are discussed in section 5.

2. Preliminaries

This section provides an introduction to few fundamental definitions of CNS to further explain the innovative idea of CFNS.

2.1. Some important definitions

Definition 2.1 (Smarandache, 1999): A neutrosophic set (NS) S on a non-empty universe X is described by $T_S, I_S, F_S : X \to]0^-, 0^+[$. That is

$$S = \left\{\left(u, T_S(u), I_S(u), F_S(u)\right) : u \in X\right\}$$

with $0^- \le T_S(u) + I_S(u) + F_S(u) \le 3^+$. Here $T_S(u), I_S(u)$ and $F_S(u)$ are truth, indeterminacy and falsity membership values respectively and are real standard or non-standard subsets of $]0^-, 0^+[$. It is worth to mention that the definition above is very general. In particular, it allows to combine neutrosophic sets with some ideas taken from the world of non-standard analysis. However, the vast majority of researchers limit their interest to single-valued neutrosophic sets. In this case T, I and F are just real numbers from [0, 1]. This approach is more practical. Also interval-valued neutrosophic sets are studied.

Definition 2.2 (Ajay and Chellamani, 2020): A Pythagorean neutrosophic set (PNS) S defined on X is represented as

$$S = \left\{\left(u, T_S(u), I_S(u), F_S(u)\right) : u \in X\right\}$$

with $0^- \le T_S^2(u) + I_S^2(u) + F_S^2(u) \le 2^+$ and $0 \le T_S^2(u) + F_S^2(u) \le 1;\ 0 \le I_S^2(u) \le 1$. Here $T_S(u), F_S(u)$ are dependent components and $I_S(u)$ is an independent component.

Definition 2.3 (Antony and Jansi, 2021): A Fermatean neutrosophic set (FNS) S defined on X is represented as

$$S = \left\{\left(u, T_S(u), I_S(u), F_S(u)\right) : u \in X\right\}$$

with $0^- \le T_S^3(u) + I_S^3(u) + F_S^3(u) \le 2^+$ and $0 \le T_S^3(u) + F_S^3(u) \le 1;\ 0 \le I_S^3(u) \le 1$.

Definition 2.4 (Ali and Smarandache, 2017): A complex neutrosophic set (CNS) S is described on X is represented as

$$S=\left\{\left(\mathrm{u},\mathrm{T}_S\left(u\right),\mathrm{I}_S\left(u\right),\mathrm{F}_S\left(u\right)\right):u\in X\right\}$$

where $\mathrm{T}_S\left(u\right),\mathrm{I}_S\left(u\right),\mathrm{F}_S\left(u\right)$ are complex-valued functions and are of the form $\mathrm{T}_S\left(u\right)=r_S\left(u\right).e^{i\,\omega_S(\mathrm{u})}$; $\mathrm{F}_S\left(u\right)=k_S\left(u\right).e^{i\,\rho_S(\mathrm{u})}$; $\mathrm{I}_S\left(u\right)=t_S\left(u\right).e^{i\,\theta_S(\mathrm{u})}$; where $r_S\left(u\right),\omega_S(u),k_S\left(u\right),\rho_S(u),t_S\left(u\right),\theta_S(u)\in[0,1]$ such that $0^-\le r_S\left(u\right)+k_S\left(u\right)+t_S\left(u\right)\le 3^+$. The amplitude terms of CNS are $r_S\left(u\right),k_S\left(u\right),t_S\left(u\right)$ and the phase terms are $\omega_S(u),\rho_S(u),\theta_S(u)$. The triplet $\delta=\left(r.e^{i2\pi\omega_S},t.e^{i2\pi\theta_S},k.e^{i2\pi\rho_S}\right)$ is referred to as a complex Neutrosophic number.

Definition 2.5 (Yaqoob et al, 2018; Khan et al., 2022): A mapping $Q=\left(\mathrm{T}_Q,\mathrm{I}_Q,\mathrm{F}_Q\right):X\times X\to[0,1]$ is said to be a neutrosophic relation on X for $\mathrm{T}_Q\left(u,v\right),\mathrm{I}_Q\left(u,v\right),\mathrm{F}_Q\left(u,v\right)\in[0,1]$ where $u,v\in X$.

Definition 2.6 (Yaqoob et al., 2018): A neutrosophic graph (NG) on X is a pair $G=(P,Q)$ where P and Q are NS defined on vertex set V and edge set $E\subseteq V\times V$ respectively such that

$$\begin{cases}T_Q\left(u,v\right)\le\min\left\{T_P\left(u\right),T_P\left(v\right)\right\}\\ \mathrm{I}_Q\left(u,v\right)\le\min\left\{I_P\left(u\right),\mathrm{I}_P\left(v\right)\right\}\\ F_Q\left(u,v\right)\le\max\left\{F_P\left(u\right),\mathrm{F}_P\left(v\right)\right\}\end{cases}$$

Such that $0\le T_Q\left(u,v\right)+\mathrm{I}_Q\left(u,v\right)+F_Q\left(u,v\right)\le 3$ for all $u,v\in V$.

Definition 2.7 (Ajay and Chellamani, 2020): A Pythagorean neutrosophic graph (PyNG) on X is a pair $G=(P,Q)$ where P and Q are PyNS defined on vertex set V and edge set $E\subseteq V\times V$ respectively such that

$$\begin{cases}T_Q\left(u,v\right)\le\min\left\{T_P\left(u\right),T_P\left(v\right)\right\}\\ \mathrm{I}_Q\left(u,v\right)\le\min\left\{I_P\left(u\right),\mathrm{I}_P\left(v\right)\right\}\\ F_Q\left(u,v\right)\le\max\left\{F_P\left(u\right),\mathrm{F}_P\left(v\right)\right\}\end{cases}$$

Such that $0\le T_Q^2\left(u,v\right)+I_Q^2\left(u,v\right)+F_Q^2\left(u,v\right)\le 2$ for all $u,v\in V$.

Definition 2.8 (Antony and Jansi, 2021): A Fermatean neutrosophic graph (FNG) on X is a pair $G=(P,Q)$ where P and Q are FNS defined on vertex set V and edge set $E\subseteq V\times V$ respectively such that

$$\begin{cases}T_Q\left(u,v\right)\le\min\left\{T_P\left(u\right),T_P\left(v\right)\right\}\\ \mathrm{I}_Q\left(u,v\right)\le\min\left\{I_P\left(u\right),\mathrm{I}_P\left(v\right)\right\}\\ F_Q\left(u,v\right)\le\max\left\{F_P\left(u\right),\mathrm{F}_P\left(v\right)\right\}\end{cases}$$

Such that $0\le T_Q^3\left(u,v\right)+I_Q^3\left(u,v\right)+F_Q^3\left(u,v\right)\le 2$ for all $u,v\in V$.

Definition 2.9 (Yaqoob et al., 2018): A complex neutrosophic graph (CNG) on X is a pair $G=(P,Q)$ where P and Q are CNS defined on vertex set V and edge set $E\subseteq V\times V$ respectively such that

$$\begin{cases} r_Q(u,v).e^{i\omega_Q(u,v)} \le \min\{r_P(u), r_P(v)\}.e^{i\min\{\omega_P(u),\omega_P(v)\}} \\ t_Q(u,v).e^{i\theta_Q(u,v)} \le \min\{t_P(u), t_P(v)\}.e^{i\min\{\theta_P(u),\theta_P(v)\}} \\ k_Q(u,v).e^{i\rho_Q(u,v)} \le \max\{k_P(u), k_P(v)\}.e^{i\min\{\rho_P(u),\rho_P(v)\}} \end{cases}$$

Where Q is complex neutrosophic relation on P.

3. Complex Fermatean neutrosophic graphs

This section explores the concept of the Complex Fermatean Neutrosophic Set (CFNS) and the Complex Fermatean Neutrosophic Graph (CFNG) as well as their fundamental operations.

3.1. Proposed CFNS and some important definitions

Definition 3.1: A complex Pythagorean neutrosophic set (CPNS) S on X is given by

$$S = \{(u, T_S(u), I_S(u), F_S(u)) : u \in X\}$$

where $T_S(u) = r_S(u).e^{i\omega_S(u)}$; $F_S(u) = k_S(u).e^{i\rho_S(u)}$; $I_S(u) = t_S(u).e^{i\theta_S(u)}$; with $0^- \le r_S^2(u) + t_S^2(u) + k_S^2(u) \le 2^+$ and $0^- \le \omega_S^2(u) + \theta_S^2(u) + \rho_S^2(u) \le 2\pi$.

Definition 3.2: A complex Fermatean neutrosophic set (CFNS) S on X is given by

$$S = \{(u, T_S(u), I_S(u), F_S(u)) : u \in X\}$$

with $0^- \le r_S^3(u) + t_S^3(u) + k_S^3(u) \le 2^+$ and $0^- \le \omega_S^3(u) + \theta_S^3(u) + \rho_S^3(u) \le 2\pi$. The triplet $\delta = (r \cdot e^{2\pi i\omega_S}, t \cdot e^{2\pi i\theta_S}, k \cdot e^{2\pi i\rho_S})$ is referred as a complex Fermatean Neutrosophic number (CFNN). A comparison of PNS, FNS, CNS and CFNS is made in Table 3.

Table 3. Comparison of PNS, FNS, CNS and CFNS

Extension	Condition	Example	Explanation
PNS	$0 \le \alpha^2 + \beta^2 \le 1$; $0 \le \gamma^2 \le 1$ *and* $0 \le \alpha^2 + \beta^2 + \le \gamma^2 \le 2$	$A = \{(a, 0.7, 0.6, 0.3), (b.0.6, 0.5, 0.6)\}$ $B = \{(a, 0.8, 0.8, 0.9), (b, 0.9, 0.7, 0.9)\}$	Condition for PNS satisfies for A but fails for B as $\alpha^2 + \beta^2 > 1$
FNS	$0 \le \alpha^3 + \beta^3 \le 1$; $0 \le \gamma^3 \le 1$ *and* $0 \le \alpha^3 + \beta^3 + \le \gamma^3 \le 2$	$B = \{(a, 0.8, 0.8, 0.9), (b, 0.9, 0.7, 0.9)\}$	Condition for FNS satisfies for A

Extension	Condition	Example	Explanation
CNS	$0 \le r+t+k \le 3;$ $0 \le \omega+\theta+ \le \rho \le 2\pi$	$A=\left\{(a,\left\langle \begin{matrix}0.1e^{i2\pi(0.3)},0.2e^{i2\pi(0.5)},\\ 0.7e^{i2\pi(0.2)}\end{matrix}\right\rangle),\right.$ $\left.(b,\left\langle \begin{matrix}0.3e^{i2\pi(0.4)},0.2e^{i2\pi(0.2)},\\ 0.4e^{i2\pi(0.2)}\end{matrix}\right\rangle)\right\}$ $B=\left\{(a,\left\langle \begin{matrix}0.8e^{i2\pi(0.7)},0.7e^{i2\pi(0.5)},\\ 0.7e^{i2\pi(0.2)}\end{matrix}\right\rangle),\right.$ $\left.(b,\left\langle \begin{matrix}0.6e^{i2\pi(0.8)},0.6e^{i2\pi(0.8)},\\ 0.4e^{i2\pi(0.5)}\end{matrix}\right\rangle)\right\}$	Condition for CNS satisfies for A but fails for B as $r+t>1$
Proposed CFNS	$0 \le r^3+t^3+k^3 \le 3;$ $0 \le \omega^3+\theta^3+ \le \rho^3 \le 2\pi$	$B=\left\{(a,\left\langle 0.8e^{i2\pi(0.7)},0.7e^{i2\pi(0.5)},0.7e^{i2\pi(0.2)}\right\rangle),\right.$ $\left.(b,\left\langle 0.6e^{i2\pi(0.8)},0.6e^{i2\pi(0.8)},0.4e^{i2\pi(0.5)}\right\rangle)\right\}$	Condition for CFNS satisfies for B

Definition 3.3: Let $\delta=\left(r\cdot e^{2\pi i\omega},t\cdot e^{2\pi i\theta},k\cdot e^{2\pi i\rho}\right),\delta_1=\left(r_1\cdot e^{2\pi i\omega_1},t_1\cdot e^{2\pi i\theta_1},k_1\cdot e^{2\pi i\rho_1}\right)$ and $\delta_2=\left(r_2\cdot e^{2\pi i\omega_2},t_2\cdot e^{2\pi i\theta_2},k_2\cdot e^{2\pi i\rho_2}\right)$ be three CFNNs, then

- $\delta_1\subseteq\delta_2$ iff $r_1\le r_2,t_1\le t_2,k_1\le k$ and $\omega_1\le\omega_2,\ \theta_1\le\theta_2,\rho_1\le\rho_2$
- $\delta_1=\delta_2$ iff $r_1=r_2,t_1=t_2,k_1=k_2$ and $\omega_1=\omega_2,\ \theta_1=\theta_2,\rho_1=\rho_2$
- $\overline{\delta}=\left(k\cdot e^{2\pi i\rho},t\cdot e^{2\pi i\theta},r\cdot e^{2\pi i\omega}\right)$

In the next definition we show how to perform operations of union and intersection on CFNNs.

Definition 3.4 Let $\delta_1=\left(r_1\cdot e^{2\pi i\omega_1},t_1\cdot e^{2\pi i\theta_1},k_1\cdot e^{2\pi i\rho_1}\right)$ and $\delta_2=\left(r_2\cdot e^{2\pi i\omega_2},t_2\cdot e^{2\pi i\theta_2},k_2\cdot e^{2\pi i\rho_2}\right)$ be two CFNNs, then

$$\delta_1\cup\delta_2=\left\{x,\left(\max\left(r_1,r_2\right)\cdot e^{2\pi i\max(\omega_1,\omega_2)},\max\left(t_1,t_2\right)\cdot e^{2\pi i\max(\theta_1,\theta_2)},\max\left(r_1,r_2\right)\cdot e^{2\pi i\max(\rho_1,\rho_2)}\right)\right\}$$

$$\delta_1\cap\delta_2=\left\{x,\left(\min\left(r_1,r_2\right)\cdot e^{2\pi i\min(\omega_1,\omega_2)},\min\left(t_1,t_2\right)\cdot e^{2\pi i\min(\theta_1,\theta_2)},\min\left(r_1,r_2\right)\cdot e^{2\pi i\min(\rho_1,\rho_2)}\right)\right\}$$

Definition 3.5 **:** A CFN relation in X is defined by a CFNS Q in $X\times X$ characterized by

$$Q=\left\{\left\langle(u,v),r_Q\left(u,v\right)\cdot e^{i\omega_Q(u,v)},t_Q\left(u,v\right)\cdot e^{i\theta_Q(u,v)},k_Q\left(u,v\right)\cdot e^{i\rho_Q(u,v)}\right\rangle\right\}$$

where $r_Q,t_Q,k_Q:X\times X\to[0,1]$ are defined such that $0\le r_Q^3\left(u,v\right)+t_Q^3\left(u,v\right)+k_Q^3\left(u,v\right)\le 2$ and $0\le\frac{\omega_Q^3\left(u,v\right)}{2\pi}+\frac{\theta_Q^3\left(u,v\right)}{2\pi}+\frac{\rho_Q^3\left(u,v\right)}{2\pi}\le 2$ where $(u,v)\in X\times X$ **.**

Now we shall connect our sets with some notions from graph theory.

Definition 3.6 **:** A complex fermatean neutrosophic graph (CFNG) defined on X is a pair $G=(P,\mathrm{Q})$ where P is an CFNS on vertex set V and Q is a CFNS on edge set $E\subseteq V\times V$ such that

$$\begin{cases} r_Q(u,v)\cdot e^{i\omega_Q(u,v)} \le \min\{t_P(\mathrm{u}),t_P(\mathrm{v})\}\cdot e^{i\min\{\omega_P(\mathrm{u}),\omega_P(\mathrm{v})\}} \\ t_Q(u,v)\cdot e^{i\theta_Q(u,v)} \le \min\{r_P(\mathrm{u}),r_P(\mathrm{v})\}\cdot e^{i\min\{\theta_P(\mathrm{u}),\theta_P(\mathrm{v})\}} \\ t_Q(u,v)\cdot e^{i\rho_Q(u,v)} \le \min\{k_P(\mathrm{u}),k_P(\mathrm{v})\}\cdot e^{i\min\{\rho_P(\mathrm{u}),\rho_P(\mathrm{v})\}} \end{cases}$$

Where $r_P(u),k_P(u),t_P(u)\in[0,1], 0\le r_Q^3(u,v),t_Q^3(u,v),k_Q^3(u,v)\le 2$ and $0\le \frac{\omega_Q^3(u,v)}{2\pi}+\frac{\theta_Q^3(u,v)}{2\pi}+\frac{\rho_Q^3(u,v)}{2\pi}\le 2$ for all $(u,v)\in E$.

Example 1: Consider a CFNG $G=(P,Q)$ defined on $G=(V,E)$ represented in Figure 1. The vertex set P is a CFNS defined on V given in Table 4 and Q is a CFNS on edge set $E\subseteq V\times V$ given in Table 5.

Table 4. Vertices of CFNG G

Node	CFNS
v_1	$\langle 0.5e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.5)}, 0.7e^{i2\pi(0.2)}\rangle$
v_2	$\langle 0.8e^{i2\pi(0.3)}, 0.4e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.9)}\rangle$
v_3	$\langle 0.3e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.8)}, 0.4e^{i2\pi(0.5)}\rangle$

Table 5. Edges of CFNG G

Edge	CFFS
e_{12}	$\langle 0.3e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.2)}\rangle$
e_{13}	$\langle 0.3e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.5)}\rangle$
e_{23}	$\langle 0.2e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.4)}\rangle$

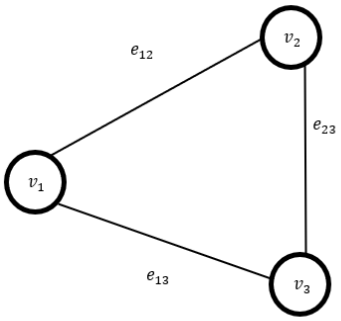

Figure 1. CFNG G

Definition 3.7*:* Consider a CFNG $G=(P,Q)$ whose vertex set is $P=\{\langle u, r_P(u)\cdot e^{i\omega_P(u)}, t_P(u)\cdot e^{i\theta_P(u)}, k_P(u)\cdot e^{i\rho_P(u)}\rangle \mid u\in V\}$ and edge set

$$Q=\{\langle u, r_Q(\mathrm{u},\mathrm{v})\cdot \mathrm{e}^{i\omega_Q(\mathrm{u},\mathrm{v})}, \mathrm{t}_Q(\mathrm{u},\mathrm{v})\cdot \mathrm{e}^{i\theta_Q(\mathrm{u},\mathrm{v})}, \mathrm{k}_Q(\mathrm{u},\mathrm{v})\cdot \mathrm{e}^{i\rho_Q(\mathrm{u},\mathrm{v})}\rangle \mid (u,v)\in E\}$$

then the order and size of a CFNG is defined by

$$O(G)=\left(\sum_{u\in V} r_P(u)\cdot e^{\sum_{u\in V}\omega_P(u)},\sum_{u\in V} t_P(u)\cdot e^{\sum_{u\in V}\theta_P(u)},\sum_{u\in V} k_P(u)\cdot e^{\sum_{u\in V}\rho_P(u)}\right)$$

$$S(G)=\left(\sum_{u\in V} r_Q(u,v)\cdot e^{\sum_{(u,v)\in E}\omega_Q(u,v)},\sum_{u\in V} t_Q(u,v)\cdot e^{\sum_{(u,v)\in E}\theta_Q(u,v)},\sum_{u\in V} k_Q(u,v)\cdot e^{\sum_{(u,v)\in E}\rho_Q(u,v)}\right)$$

Example 2: The order and the size of the CFNG G represented in Figure 1 is given by $O(G)=\langle 1.6e^{i2\pi(1.8)},1.4e^{i2\pi(1.9)},1.7e^{i2\pi(1.6)}\rangle$ and $S(G)=\langle 0.8e^{i2\pi(1)},0.6e^{i2\pi(1.6)},1.1e^{i2\pi(0.9)}\rangle$ respectively.

3.2. Basic operations of complex fermatean fuzzy graphs

This section introduces some of the fundamental CFNG operations and explores some of its associated properties.

Definition 3.8*:* The complement of a CFNG $G=(P,Q)$ on an underlying graph $G=(V,E)$ is a CFNG $\overline{G}=(\overline{P},\overline{Q})$ defined by

- $\overline{r_P(u)\cdot e^{l\omega_Q(u)}}=r_P(u)\cdot e^{l\omega_Q(u)}$, $\overline{t_P(u)\cdot e^{l\theta_Q(u)}}=t_P(u)\cdot e^{l\theta_Q(u)}$ and $\overline{k_P(u)\cdot e^{l\rho_Q(u)}}=k_P(u)\cdot e^{l\rho_Q(u)}$.
- $\overline{r_Q(u,v)\cdot e^{l\omega_Q(u,v)}}=\begin{cases}\min\left(r_P(u),r_P(v)\right)\cdot e^{i\min\left(\omega_P(u),\omega_P(v)\right)} & \text{if } r_Q(u,v)\cdot e^{i\omega_Q(u,v)}=0\\ \min\left(r_P(u),r_P(v)\right)\cdot e^{i\min\left(\omega_P(u),\omega_P(v)\right)}-r_Q(u,v)\cdot e^{i\omega_Q(u,v)} & \text{if } 0\le r_Q(u,v)\cdot e^{i\omega_Q(u,v)}\le 1\end{cases}$
- $\overline{t_Q(u,v)\cdot e^{l\theta_Q(u,v)}}=\begin{cases}\max\left(t_P(u),t_P(v)\right)\cdot e^{i\max\left(\theta_P(u),\theta_P(v)\right)} & \text{if } t_Q(u,v)\cdot e^{i\theta_Q(u,v)}=0\\ \max\left(t_P(u),t_P(v)\right)\cdot e^{i\max\left(\theta_P(u),\theta_P(v)\right)}-t_Q(u,v)\cdot e^{i\theta_Q(u,v)} & \text{if } 0\le t_Q(u,v)\cdot e^{i\theta_Q(u,v)}\le 1\end{cases}$
- $\overline{k_Q(u,v)\cdot e^{l\rho_Q(u,v)}}=\begin{cases}\max\left(k_P(u),k_P(v)\right)\cdot e^{i\max\left(\rho_P(u),\rho_P(v)\right)} & \text{if } k_Q(u,v)\cdot e^{i\rho_Q(u,v)}=0\\ \max\left(k_P(u),k_P(v)\right)\cdot e^{i\max\left(\rho_P(u),\rho_P(v)\right)}-k_Q(u,v)\cdot e^{i\rho_Q(u,v)} & \text{if } 0\le k_Q(u,v)\cdot e^{i\rho_Q(u,v)}\le 1\end{cases}$

Example 3: The complement of CFNG G is represented in Figure 2 whose vertex set $\overline{P}$ is shown in Table 6 and edge set $\overline{Q}$ is shown in Table 7 and is denoted by CFNG $\overline{G}$.

Table 6. Vertices of CFNG $\overline{G}$

Node	CFNS
v_1	$\langle 0.5e^{i2\pi(0.7)},0.2e^{i2\pi(0.5)},0.7e^{i2\pi(0.2)}\rangle$
v_2	$\langle 0.8e^{i2\pi(0.3)},0.4e^{i2\pi(0.6)},0.6e^{i2\pi(0.9)}\rangle$
v_3	$\langle 0.3e^{i2\pi(0.8)},0.2e^{i2\pi(0.8)},0.4e^{i2\pi(0.5)}\rangle$

Table 7. Edges of CFNG $\overline{G}$

Edge	CFNS
e'_{12}	$\left\langle 0.2e^{i2\pi(0.1)}, 0.5e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.7)} \right\rangle$
e'_{23}	$\left\langle 0.1e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.5)} \right\rangle$

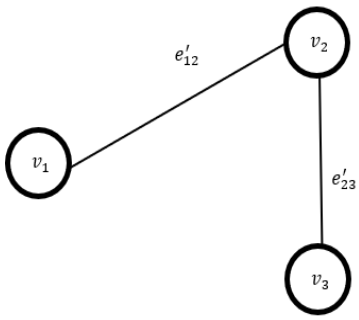

Figure 2. CFNG $\overline{G}$

Theorem 1: For a CFNG G, we have $\overline{\overline{G}} = G$.

Proof: Let G be a CFNG. Using definition 3.7 we have

$$\overline{\overline{r_P(u)\cdot e^{l\omega_P(u)}}} = \overline{r_P(u)\cdot e^{l\omega_P(u)}} = r_P(u)\cdot e^{i\omega_P(u)},$$

$$\overline{\overline{t_P(u)\cdot e^{l\theta_P(u)}}} = \overline{t_P(u)\cdot e^{l\theta_P(u)}} = t_P(u)\cdot e^{i\theta_P(u)}$$

and $\overline{\overline{k_P(u)\cdot e^{l\rho_P(u)}}} = \overline{k_P(u)\cdot e^{l\rho_P(u)}} = k_P(u)\cdot e^{i\rho_P(u)}$ for all $u \in V$

$$\overline{\overline{r_Q(u,v)\cdot e^{l\omega_Q(u,v)}}} = \overline{r_Q(u,v)\cdot e^{l\omega_Q(u,v)}}$$

$$= \begin{cases} \min\ r_P\ u\ , r_P\ v\ \cdot e^{i\min\ \omega_P\ u\ ,\omega_P\ v} & if\ r_Q(u,v)\cdot e^{i\omega_Q(u,v)} = 0 \\ \min\ r_P\ u\ , r_P\ v\ \cdot e^{i\min\ \omega_P\ u\ ,\omega_P\ v} - r_Q(u,v)\cdot e^{i\omega_Q(u,v)} & if\ 0 \le r_Q\ u,v\ \cdot e^{i\omega_{Q\ u,v}} \le 1 \end{cases}$$

$$\overline{\overline{t_Q(u,v)\cdot e^{l\theta_Q(u,v)}}} = \overline{t_Q(u,v)\cdot e^{l\theta_Q(u,v)}}$$

$$= \begin{cases} \max\ t_P\ u\ , t_P\ v\ \cdot e^{i\max\ \theta_P\ u\ ,\theta_P\ v} & if\ t_Q(u,v)\cdot e^{i\theta_Q(u,v)} = 0 \\ \max\ t_P\ u\ , t_P\ v\ \cdot e^{i\max\ \theta_P\ u\ ,\theta_P\ v} - t_Q(u,v)\cdot e^{i\theta_Q(u,v)} & if\ 0 \le t_Q\ u,v\ \cdot e^{i\theta_{Q\ u,v}} \le 1 \end{cases}$$

$$\overline{\overline{k_Q(u,v)\cdot e^{l\rho_Q(u,v)}}} = \overline{k_Q(u,v)\cdot e^{l\rho_Q(u,v)}}$$

$$= \begin{cases} \max\ k_P\ u\ , k_P\ v\ \cdot e^{i\max\ \rho_P\ u\ ,\rho_P\ v} & if\ k_Q(u,v)\cdot e^{i\rho_Q(u,v)} = 0 \\ \max\ k_P\ u\ , k_P\ v\ \cdot e^{i\max\ \rho_P\ u\ ,\rho_P\ v} - t_Q(u,v)\cdot e^{i\rho_Q(u,v)} & if\ 0 \le k_Q\ u,v\ \cdot e^{i\rho_{Q\ u,v}} \le 1 \end{cases}$$

$u,v \in E$

Definition 3.9*:* For $G_1 = P_1, Q_1$ and $G_2 = P_2, Q_2$, the union between two CFNGS is given by

- $$r_{P_1} \cup r_{P_2} \quad u \cdot e^{i\,\omega_{P_1} \cup \omega_{P_2} \quad u} = \begin{cases} r_{P_1} \quad u \cdot e^{i\,\omega_{P_1} \quad u}, \ if \ u \in P_1 - P_2 \\ r_{P_2} \quad u \cdot e^{i\,\omega_{P_2} \quad u}, \ if \ u \in P_2 - P_1 \\ \max \ r_{P_1} \quad u \ , r_{P_2} \quad u \quad \cdot e^{i \max \ \omega_{P_1} \quad u \ , \omega_{P_2} \quad u} \quad if \ u \in P_1 \cap P_2 \end{cases}$$

- $$t_{P_1} \cup t_{P_2} \quad u \cdot e^{i\,\theta_{P_1} \cup \theta_{P_2} \quad u} = \begin{cases} t_{P_1} \quad u \cdot e^{i\,\theta_{P_1} \quad u}, \ if \ u \in P_1 - P_2 \\ t_{P_2} \quad u \cdot e^{i\,\theta_{P_2} \quad u}, \ if \ u \in P_2 - P_1 \\ \min \ t_{P_1} \quad u \ , t_{P_2} \quad u \quad \cdot e^{i \min \ \theta_{P_1} \quad u \ , \theta_{P_2} \quad u} \quad if \ u \in P_1 \cap P_2 \end{cases}$$

- $$k_{P_1} \cup k_{P_2} \quad u \cdot e^{i\,\rho_{P_1} \cup \rho_{P_2} \quad u} = \begin{cases} k_{P_1} \quad u \cdot e^{i\,\rho_{P_1} \quad u}, \ if \ u \in P_1 - P_2 \\ k_{P_2} \quad u \cdot e^{i\,\rho_{P_2} \quad u}, \ if \ u \in P_2 - P_1 \\ \min \ k_{P_1} \quad u \ , k_{P_2} \quad u \quad \cdot e^{i \min \ \rho_{P_1} \quad u \ , \rho_{P_2} \quad u} \quad if \ u \in P_1 \cap P_2 \end{cases}$$

- $$\left(r_{Q_1} \cup r_{Q_2}\right)(u,v) \cdot e^{i(\omega_{Q_1} \cup \omega_{Q_2})(u,v)} = \begin{cases} \left(r_{Q_1}\right)(u,v) \cdot e^{i(\omega_{Q_1})(u,v)} & if \ (u,v) \in Q_1 - Q_2 \\ \left(r_{Q_2}\right)(u,v) \cdot e^{i(\omega_{Q_2})(u,v)} & if \ (u,v) \in Q_2 - Q_1 \\ \max\left(r_{Q_1}(u,v), r_{Q_2}(u,v)\right) \cdot e^{i \max\left(\omega_{Q_1}(u,v), \omega_{Q_2}(u,v)\right)} & if \ (u,v) \in Q_1 \cap Q_2 \end{cases}$$

- $$\left(t_{Q_1} \cup t_{Q_2}\right)(u,v) \cdot e^{i(\theta_{Q_1} \cup \theta_{Q_2})(u,v)} = \begin{cases} \left(t_{Q_1}\right)(u,v) \cdot e^{i(\theta_{Q_1})(u,v)} & if \ (u,v) \in Q_1 - Q_2 \\ \left(t_{Q_2}\right)(u,v) \cdot e^{i(\theta_{Q_2})(u,v)} & if \ (u,v) \in Q_2 - Q_1 \\ \min\left(t_{Q_1}(u,v), t_{Q_2}(u,v)\right) \cdot e^{i \min\left(\theta_{Q_1}(u,v), \theta_{Q_2}(u,v)\right)} & if \ (u,v) \in Q_1 \cap Q_2 \end{cases}$$

- $$\left(k_{Q_1} \cup k_{Q_2}\right)(u,v) \cdot e^{i(\rho_{Q_1} \cup \rho_{Q_2})(u,v)} = \begin{cases} \left(k_{Q_1}\right)(u,v) \cdot e^{i(\rho_{Q_1})(u,v)} & if \ (u,v) \in Q_1 - Q_2 \\ \left(k_{Q_2}\right)(u,v) \cdot e^{i(\rho_{Q_2})(u,v)} & if \ (u,v) \in Q_2 - Q_1 \\ \min\left(k_{Q_1}(u,v), k_{Q_2}(u,v)\right) \cdot e^{i \min\left(\rho_{Q_1}(u,v), \rho_{Q_2}(u,v)\right)} & if \ (u,v) \in Q_1 \cap Q_2 \end{cases}$$

Example 4: Let $G_1 = P_1, Q_1$ and $G_2 = P_2, Q_2$ be two CFNGs whose vertex sets and edge sets are provided in Table 8, Table 9, Table 10 and Table 11 and are represented in Figure 3 and Figure 4. respectively.

Table 8. Vertices of CFNG G_1

Node	CFNS
v_1	$\left\langle 0.5e^{i2\pi(0.3)}, 0.8e^{i2\pi(0.6)}, 0.4^{i2\pi(0.6)} \right\rangle$
v_2	$\left\langle 0.8e^{i2\pi(0.3)}, 0.6e^{i2\pi(0.6)}, 0.4^{i2\pi(0.8)} \right\rangle$
v_4	$\left\langle 0.3e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.5)}, 0.6^{i2\pi(0.4)} \right\rangle$

Table 9. Edges of CFNG G_1

Edge	CFNS
e_{12}	$\langle 0.6e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.7)}, 0.7^{i2\pi(0.9)} \rangle$
e_{14}	$\langle 0.4e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.8)}, 0.6^{i2\pi(0.8)} \rangle$

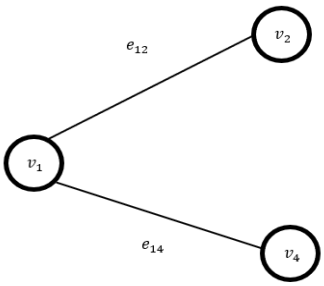

Figure 3. CFNG G_1

Table 10. Vertices of CFNG G_2

Node	CFNS
v_1	$\langle 0.7e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.3)}, 0.5^{i2\pi(0.6)} \rangle$
v_2	$\langle 0.8e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.8)}, 0.6^{i2\pi(0.4)} \rangle$
v_3	$\langle 0.6e^{i2\pi(0.8)}, 0.8e^{i2\pi(0.7)}, 0.3^{i2\pi(0.5)} \rangle$
v_4	$\langle 0.3e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.9)}, 0.4^{i2\pi(0.3)} \rangle$

Table 11. Edges of CFNG G_2

Edge	CFNS
e_{12}	$\langle 0.4e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.7)}, 0.6^{i2\pi(0.4)} \rangle$
e_{13}	$\langle 0.6e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.6)}, 0.5^{i2\pi(0.3)} \rangle$
e_{24}	$\langle 0.5e^{i2\pi(0.4)}, 0.8e^{i2\pi(0.7)}, 0.6^{i2\pi(0.8)} \rangle$
e_{34}	$\langle 0.7e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.5)}, 0.4^{i2\pi(0.3)} \rangle$

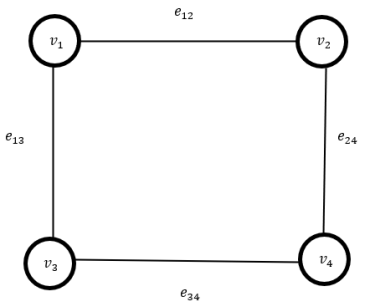

Figure 4. CFNG G_1

Then by definition 3.9, $v_3 \in P_2 \cap P_1$ and $v_1, v_2, v_4 \in P_1 \cap P_2$. The union of two CFNGs $G_1 \cup G_2$ is shown in Figure 5 whose vertex set and edge set given in Table 12 and Table 13 respectively.

Table 12. Vertices of $G_1 \cup G_2$

Node	CFNS
v_1	$\langle 0.7e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.3)}, 0.4^{i2\pi(0.6)} \rangle$
v_2	$\langle 0.8e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.6)}, 0.4^{i2\pi(0.4)} \rangle$
v_3	$\langle 0.6e^{i2\pi(0.8)}, 0.8e^{i2\pi(0.7)}, 0.3^{i2\pi(0.5)} \rangle$
v_4	$\langle 0.3e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.5)}, 0.4^{i2\pi(0.3)} \rangle$

Table 13. Edges of $G_1 \cup G_2$

Edge	CFFS
e_{12}	$\langle 0.6e^{i2\pi(0.8)}, 0.3e^{i2\pi(0.7)}, 0.6^{i2\pi(0.4)} \rangle$
e_{13}	$\langle 0.6e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.6)}, 0.5^{i2\pi(0.3)} \rangle$
e_{14}	$\langle 0.4e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.8)}, 0.6^{i2\pi(0.8)} \rangle$
e_{24}	$\langle 0.5e^{i2\pi(0.4)}, 0.8e^{i2\pi(0.7)}, 0.6^{i2\pi(0.8)} \rangle$
e_{34}	$\langle 0.7e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.5)}, 0.4^{i2\pi(0.3)} \rangle$

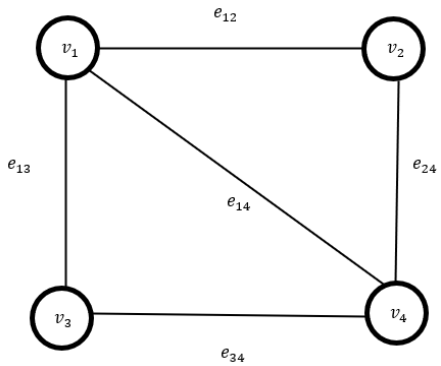

Figure 5. CFNG $G_1 \cup G_2$

Definition 3.10: Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two CFNSs, then the ring-sum $G_1 \oplus G_2 = (P_1 \oplus P_2, Q_1 \oplus Q_2)$ is defined as follows:

$$\begin{cases} (r_{P_1} \oplus r_{P_2})(u) \cdot e^{i(\omega_{P_1} \oplus \omega_{P_2})(u)} = (r_{P_1} \cup r_{P_2})(u) \cdot e^{i(\omega_{P_1} \cup \omega_{P_2})(u)} \\ (t_{P_1} \oplus t_{P_2})(u) \cdot e^{i(\theta_{P_1} \oplus \theta_{P_2})(u)} = (t_{P_1} \cup t_{P_2})(u) \cdot e^{i(\theta_{P_1} \cup \theta_{P_2})(u)} \\ (k_{P_1} \oplus k_{P_2})(u) \cdot e^{i(\rho_{P_1} \oplus \rho_{P_2})(u)} = (k_{P_1} \cup k_{P_2})(u) \cdot e^{i(\rho_{P_1} \cup \rho_{P_2})(u)} \ \text{if} \ u \in P_1 \cup P_2 \end{cases}$$

$$(r_{Q_1} \oplus r_{Q_2})(u,v) \cdot e^{i(\omega_{Q_1} \oplus \omega_{Q_2})(u,v)} = \begin{cases} r_{Q_1}(u,v) \cdot e^{i(\omega_{Q_1})(u,v)}, & f\ (u,v) \in Q_1 - Q_2 \\ r_{Q_2}(u,v) \cdot e^{i(\omega_{Q_2})(u,v)}, & f\ (u,v) \in Q_2 - Q_1 \\ 0, & f\ (u,v) \in Q_1 \cap Q \end{cases}$$

$$\left(t_{Q_1} \oplus t_{Q_2}\right)(u,v)\cdot e^{i(\theta_{Q_1}\oplus\theta_{Q_2})(u,v)} = \begin{cases} t_{Q_1}(u,v)\cdot e^{i(\theta_{Q_1})(u,v)}, & f(u,v)\in Q_1 - Q_2 \\ t_{Q_2}(u,v)\cdot e^{i(\theta_{Q_2})(u,v)}, & f(u,v)\in Q_2 - Q_1 \\ 0, & f(u,v)\in Q_1\cap Q \end{cases}$$

$$\left(k_{Q_1} \oplus k_{Q_2}\right)(u,v)\cdot e^{i(\rho_{Q_1}\oplus\rho_{Q_2})(u,v)} = \begin{cases} k_{Q_1}(u,v)\cdot e^{i(\rho_{Q_1})(u,v)}, & f(u,v)\in Q_1 - Q_2 \\ k_{Q_2}(u,v)\cdot e^{i(\rho_{Q_2})(u,v)}, & f(u,v)\in Q_2 - Q_1 \\ 0, & f(u,v)\in Q_1\cap Q \end{cases}$$

Remark: If $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ are the CFNGs, then $G_1 \oplus G_2$ is the CFNG,

Definition 3.11: Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two CFNSs, then the join $G_1 + G_2 = (P_1 + P_2, Q_1 + Q_2)$ is defined as:

$$\begin{cases} \left(r_{P_1} + r_{P_2}\right)(u)\cdot e^{i(\omega_{P_1}+\omega_{P_2})(u)} = \left(r_{P_1}\cup r_{P_2}\right)(u)\cdot e^{i(\omega_{P_1}\cup\omega_{P_2})(u)} \\ \left(t_{P_1} + t_{P_2}\right)(u)\cdot e^{i(\theta_{P_1}+\theta_{P_2})(u)} = \left(t_{P_1}\cup t_{P_2}\right)(u)\cdot e^{i(\theta_{P_1}\cup\theta_{P_2})(u)} \\ \left(k_{P_1} + k_{P_2}\right)(u)\cdot e^{i(\rho_{P_1}+\rho_{P_2})(u)} = \left(k_{P_1}\cup k_{P_2}\right)(u)\cdot e^{i(\rho_{P_1}\cup\rho_{P_2})(u)} \ \text{if}\ u\in P_1\cup P_2 \end{cases}$$

$$\begin{cases} \left(r_{Q_1} \oplus r_{Q_2}\right)(u)\cdot e^{i(\omega_{Q_1}\oplus\omega_{Q_2})(u)} = \left(r_{Q_1}\cup r_{Q_2}\right)(u)\cdot e^{i(\omega_{Q_1}\cup\omega_{Q_2})(u)} \\ \left(t_{P_1} \oplus t_{P_2}\right)(u)\cdot e^{i(\theta_{P_1}\oplus\theta_{P_2})(u)} = \left(t_{Q_1}\cup t_{Q_2}\right)(u)\cdot e^{i(\theta_{Q_1}\cup\theta_{PQ})(u)} \\ \left(k_{Q_1} \oplus k_{Q_2}\right)(u)\cdot e^{i(\rho_{Q_1}\oplus\rho_{Q_2})(u)} = \left(k_{Q_1}\cup k_{Q_2}\right)(u)\cdot e^{i(\rho_{Q_1}\cup\rho_{Q_2})(u)} \ \text{if}\ u\in Q_1\cup Q_2 \end{cases}$$

$$\begin{cases} \left(r_{Q_1} \oplus r_{Q_2}\right)(u)\cdot e^{i(\omega_{Q_1}\oplus\omega_{Q_2})(u)} = \min\left(r_{Q_1}\cup r_{Q_2}\right)(u)\cdot e^{i\min(\omega_{Q_1}\cup\omega_{Q_2})(u)} \\ \left(t_{Q_1} \oplus t_{Q_2}\right)(u)\cdot e^{i(\theta_{Q_1}\oplus\theta_{Q_2})(u)} = \min\left(t_{Q_1}\cup t_{Q_2}\right)(u)\cdot e^{i\min(\theta_{Q_1}\cup\theta_{PQ})(u)} \\ \left(k_{Q_1} \oplus k_{Q_2}\right)(u)\cdot e^{i(\rho_{Q_1}\oplus\rho_{Q_2})(u)} = \max\left(k_{Q_1}\cup k_{Q_2}\right)(u)\cdot e^{i\min(\rho_{Q_1}\cup\rho_{Q_2})(u)} \ \text{if}\ u\in Q' \end{cases}$$

where Q' is the set of all edges joining the vertices of P_1 and P_2, $P_1 \cap P_2 = \varnothing$.

Theorem 2: The join $G_1 + G_2$ of G_1 and G_2 is a CFNG of $G_1 + G_2$ iff G_1 and G_2 are CFNGs of G_1 and G_2, where $P_1 \cap P_2 = \varnothing$.

Definition 3.12: For CFNG G, the degree of a vertex $u \in V$ is denoted by $D(u)$ and is defined by

$$D(u) = \left(D_{re^{i\omega}}(u), D_{te^{i\theta}}(u), D_{ke^{i\rho}}(u)\right)$$

Where $D_{re^{i\omega}}(u) = \sum_{(u,v)\neq u\in V} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\omega_{Q(u,v)}}$; $D_{te^{i\theta}}(u) = \sum_{(u,v)\neq u\in V} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\theta_{Q(u,v)}}$

$D_{ke^{i\rho}}(u) = \sum_{(u,v)\neq u\in V} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\rho_{Q(u,v)}}$

Definition 3.13*:* For CFNG G, the total degree of a vertex $u \in V$ is denoted by $TD(u)$ and is defined by

$$TD(u) = \left(TD_{re^{i\omega}}(u), TD_{te^{i\theta}}(u), TD_{ke^{i\rho}}(u)\right)\left(TD_r e^{i\omega(u)}(u), TD_k(u)e^{i\rho(u)}\right),$$

Where $TD_{re^{i\omega}}(u) = \sum_{(u,v)\neq u\in V} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\omega_{Q(u,v)}} + r_P(u)\cdot e^{i\omega_P(u)}$

$TD_{te^{i\theta}}(u) = \sum_{(u,v)\neq u\in V} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\theta_{Q(u,v)}} + t_P(u)\cdot e^{i\theta_P(u)}$

$TD_{ke^{i\rho}}(u) = \sum_{(u,v)\neq u\in V} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\rho_{Q(u,v)}} + k_P(u)\cdot e^{i\rho_P(u)}$

Definition 3.14: Let G_1 and G_2 be two CFNGs. For any vertex $u \in P_1 \cup P_2$, there are three cases to consider.

Case 1: Either $u \in P_1 - P_2$ or $u \in P_2 - P_1$. Then, no edge incident at u lies in $Q_1 \cup Q_2$. Thus, for $u \in P_1 - P_2$

$$\left(D_{re^{i\omega}}\right)_{G_1 \cup G_2}(u) = \sum_{(u,v)\in Q_1} r_{Q_1}(u,v) \cdot e^{\sum_{(u,v)\in Q_1} \omega_{Q_1}(u,v)} = \left(D_{re^{i\omega}}\right)_{G_1}(u)$$

$$\left(D_{te^{i\theta}}\right)_{G_1 \cup G_2}(u) = \sum_{(u,v)\in Q_1} t_{Q_1}(u,v) \cdot e^{\sum_{(u,v)\in Q_1} \theta_{Q_1}(u,v)} = \left(D_{te^{i\theta}}\right)_{G_1}(u)$$

$$\left(D_{ke^{i\rho}}\right)_{G_1 \cup G_2}(u) = \sum_{(u,v)\in Q_1} k_{Q_1}(u,v) \cdot e^{\sum_{(u,v)\in Q_1} \rho_{Q_1}(u,v)} = \left(D_{te^{i\rho}}\right)_{G_1}(u)$$

$$\left(TD_{ke^{i\rho}}\right)_{G_1 \cup G_2}(u) = \left(TD_{re^{i\omega}}\right)_{G_1}(u); \left(TD_{ke^{i\rho}}\right)_{G_1 \cup G_2}(u) = \left(TD_{ke^{i\rho}}\right)_{G_1}(u)$$

Case 2: Either $u \in P_1 \cap P_2$ but no edge incident at $u \in Q_1 \cap Q_2$. Then, any edge incident at u is either in $Q_1 - Q_2$ or $Q_2 - Q_1$.

$$\left(D_{re^{i\omega}}\right)_{G_1 \cup G_2}(u) = \left(D_{re^{i\omega}}\right)_{G_1}(u) + \left(D_{re^{i\omega}}\right)_{G_2}(u)$$

$$\left(D_{te^{i\theta}}\right)_{G_1 \cup G_2}(u) = \left(D_{te^{i\theta}}\right)_{G_1}(u) + \left(D_{te^{i\theta}}\right)_{G_2}(u)$$

$$\left(D_{ke^{i\rho}}\right)_{G_1 \cup G_2}(u) = \left(D_{ke^{i\rho}}\right)_{G_1}(u) + \left(D_{ke^{i\rho}}\right)_{G_2}(u)$$

$$\left(TD_{re^{i\omega}}\right)_{G_1 \cup G_2}(u) = \left(TD_{re^{i\omega}}\right)_{G_1}(u) + \left(TD_{re^{i\omega}}\right)_{G_2}(u) - \min\left(r_{P_1}(u), r_{P_2}(u)\right) \cdot e^{i\min\left(\omega_{P_1}(u), \omega_{P_2}(u)\right)}$$

$$\left(TD_{te^{i\theta}}\right)_{G_1 \cup G_2}(u) = \left(TD_{te^{i\theta}}\right)_{G_1}(u) + \left(TD_{te^{i\theta}}\right)_{G_2}(u) - \min\left(t_{P_1}(u), t_{P_2}(u)\right) \cdot e^{i\min\left(\theta_{P_1}(u), \theta_{P_2}(u)\right)}$$

$$\left(TD_{ke^{i\rho}}\right)_{G_1 \cup G_2}(u) = \left(TD_{ke^{i\rho}}\right)_{G_1}(u) + \left(TD_{ke^{i\rho}}\right)_{G_2}(u) - \min\left(k_{P_1}(u), k_{P_2}(u)\right) \cdot e^{i\min\left(\rho_{P_1}(u), \rho_{P_2}(u)\right)}$$

Case 3: $u \in P_1 \cap P_2$ and some edges incident at u are in $Q_2 - Q_1$.

$$\left(D_{re^{i\omega}}\right)_{G_1 \cup G_2}(u) = \left(D_{re^{i\omega}}\right)_{G_1}(u) + \left(D_{re^{i\omega}}\right)_{G_2}(u) - \sum_{(u,v)\in Q_1 \cap Q_2} \min\left(r_{Q_1}(u,v), r_{Q_2}(u,v)\right) \cdot e^{i\sum_{(u,v)\in Q_1 \cap Q_2} \min\left(\omega_{Q_1}(u,v), \omega_{Q_2}(u,v)\right)}$$

$$\left(D_{te^{i\theta}}\right)_{G_1 \cup G_2}(u) = \left(D_{te^{i\theta}}\right)_{G_1}(u) + \left(D_{te^{i\theta}}\right)_{G_2}(u) - \sum_{(u,v)\in Q_1 \cap Q_2} \min\left(t_{Q_1}(u,v), t_{Q_2}(u,v)\right) \cdot e^{i\sum_{(u,v)\in Q_1 \cap Q_2} \min\left(\theta_{Q_1}(u,v), \theta_{Q_2}(u,v)\right)}$$

$$\left(D_{ke^{i\rho}}\right)_{G_1 \cup G_2}(u) = \left(D_{ke^{i\rho}}\right)_{G_1}(u) + \left(D_{ke^{i\rho}}\right)_{G_2}(u) - \sum_{(u,v)\in Q_1 \cap Q_2} \max\left(k_{Q_1}(u,v), k_{Q_2}(u,v)\right) \cdot e^{i\sum_{(u,v)\in Q_1 \cap Q_2} \max\left(\rho_{Q_1}(u,v), \rho_{Q_2}(u,v)\right)}$$

$$\left(TD_{re^{i\omega}}\right)_{G_1 \cup G_2}(u) = \left(TD_{re^{i\omega}}\right)_{G_1}(u) + \left(TD_{re^{i\omega}}\right)_{G_2}(u) - \sum_{(u,v)\in Q_1 \cap Q_2} \min\left(r_{Q_1}(u,v), r_{Q_2}(u,v)\right) \cdot e^{i\sum_{(u,v)\in Q_1 \cap Q_2} \min\left(\omega_{Q_1}(u,v), \omega_{Q_2}(u,v)\right)}$$

$$\left(TD_{te^{i\theta}}\right)_{G_1 \cup G_2}(u) = \left(TD_{te^{i\theta}}\right)_{G_1}(u) + \left(TD_{te^{i\theta}}\right)_{G_2}(u) - \sum_{(u,v)\in Q_1 \cap Q_2} \min\left(t_{Q_1}(u,v), t_{Q_2}(u,v)\right) \cdot e^{i\sum_{(u,v)\in Q_1 \cap Q_2} \min\left(\theta_{Q_1}(u,v), \theta_{Q_2}(u,v)\right)}$$

$$\left(TD_{ke^{i\rho}}\right)_{G_1 \cup G_2}(u) = \left(TD_{ke^{i\rho}}\right)_{G_1}(u) + \left(TD_{ke^{i\rho}}\right)_{G_2}(u) - \sum_{(u,v)\in Q_1 \cap Q_2} \max\left(k_{Q_1}(u,v), k_{Q_2}(u,v)\right) \cdot e^{i\sum_{(u,v)\in Q_1 \cap Q_2} \max\left(\rho_{Q_1}(u,v), \rho_{Q_2}(u,v)\right)}$$

Definition 3.15: For any vertex $u \in P_1 \oplus P_2$ of two CFNGs G_1 and G_2, then two cases are to be considered. If

Case 1: $u \in P_1 - P_2$ or $u \in P_2 - P_1$

Case 2: $u \in P_1 \cap P_2$, then any edge incident at u is either in $Q_1 - Q_2$ or $Q_2 - Q_1$. For these cases we have

$$\left(D_{re^{i\omega}}\right)_{G_1 \oplus G_2}(u) = \left(D_{re^{i\omega}}\right)_{G_1 \cup G_2}, \left(D_{ke^{i\omega}}\right)_{G_1 \oplus G_2}(u) = \left(D_{ke^{i\omega}}\right)_{G_1 \cup G_2}(u)$$

$$\left(TD_{re^{i\omega}}\right)_{G_1 \oplus G_2}(u) = \left(TD_{re^{i\omega}}\right)_{G_1 \cup G_2}, \left(TD_{ke^{i\omega}}\right)_{G_1 \oplus G_2}(u) = \left(TD_{ke^{i\omega}}\right)_{G_1 \cup G_2}(u)$$

Definition 3.16: $G_1 \times G_2$ is the cartesian product of two CFNGs and is defined as a pair $G_1 \times G_2 = (P_1 \times P_2, Q_1 \times Q_2)$, such that

- $r_{P_1 \times P_2}(u,v) \cdot e^{i\omega_{P_1 \times P_2}(u,v)} = \min\left\{r_{P_1}(u), r_{P_2}(v)\right\} \cdot e^{i \min\{\omega_{P_1}(u), \omega_{P_2}(v)\}}$

  $t_{P_1 \times P_2}(u,v) \cdot e^{i\theta_{P_1 \times P_2}(u,v)} = \max\left\{t_{P_1}(u), t_{P_2}(v)\right\} \cdot e^{i \max\{\theta_{P_1}(u), \theta_{P_2}(v)\}}$

  $k_{P_1 \times P_2}(u,v) \cdot e^{i\rho_{P_1 \times P_2}(u,v)} = \max\left\{k_{P_1}(u), k_{P_2}(v)\right\} \cdot e^{i \max\{\rho_{P_1}(u), \rho_{P_2}(v)\}}$ for all $u, v \in V$.
- $r_{Q_1 \times Q_2}\left\langle (u,u_2),(u,v_2)\right\rangle \cdot e^{i\omega_{Q_1 \times Q_2}\langle (u,u_2),(u,v_2)\rangle} = \min\left\{r_{P_1}(u), r_{Q_2}(u_2,v_2)\right\} \cdot e^{i \min\{\omega_{P_1}(u), \omega_{Q_2}(u_2,v_2)\}}$

  $t_{Q_1 \times Q_2}\left\langle (u,u_2),(u,v_2)\right\rangle \cdot e^{i\theta_{Q_1 \times Q_2}\langle (u,u_2),(u,v_2)\rangle} = \max\left\{t_{P_1}(u), t_{Q_2}(u_2,v_2)\right\} \cdot e^{i \max\{\theta_{P_1}(u), \theta_{Q_2}(u_2,v_2)\}}$

  $k_{Q_1 \times Q_2}\left\langle (u,u_2),(u,v_2)\right\rangle \cdot e^{i\rho_{Q_1 \times Q_2}\langle (u,u_2),(u,v_2)\rangle} = \max\left\{k_{P_1}(u), k_{Q_2}(u_2,v_2)\right\} \cdot e^{i \max\{\rho_{P_1}(u), \rho_{Q_2}(u_2,v_2)\}}$

  for all $u \in V_1, (u_2,v_2) \in E_2$
- $r_{Q_1 \times Q_2}\left\langle (u_1,w),(v_1,w)\right\rangle \cdot e^{i\omega_{Q_1 \times Q_2}\langle (u_1,w),(v_1,w)\rangle} = \min\left\{r_{P_1}(u_1,v_1), r_{Q_2}(w)\right\} \cdot e^{i \min\{\omega_{P_1}(u_1,v_1), \omega_{Q_2}(w)\}}$

  $t_{Q_1 \times Q_2}\left\langle (u_1,w),(v_1,w)\right\rangle \cdot e^{i\theta_{Q_1 \times Q_2}\langle (u_1,w),(v_1,w)\rangle} = \max\left\{t_{P_1}(u_1,v_1), t_{Q_2}(w)\right\} \cdot e^{i \max\{\theta_{P_1}(u_1,v_1), \theta_{Q_2}(w)\}}$

  $k_{Q_1 \times Q_2}\left\langle (u_1,w),(v_1,w)\right\rangle \cdot e^{i\rho_{Q_1 \times Q_2}\langle (u_1,w),(v_1,w)\rangle} = \max\left\{k_{P_1}(u_1,v_1), k_{Q_2}(w)\right\} \cdot e^{i \max\{\rho_{P_1}(u_1,v_1), \rho_{Q_2}(w)\}}$

  for all $w \in V_2, (u_1,v_1) \in E_1$

3.3. Regular and totally regular CFNGs

Regular and totally regular CFNGs are introduced in this section along with examples.

Definition 3.17: A CFNG $G = (P,Q)$ is with each vertex having same degree is said to be a regular CFNG. If the degree of each vertex $D_G(u_i) = \left(D_{re^{i\omega}}(u), D_{te^{i\theta}}(u), D_{ke^{i\rho}}(u)\right) = (u_1,u_2,u_3)$ for all $u_i \in V$, where

$$D_{re^{i\omega}}(u) = \sum_{(u,v) \neq u \in V} r_Q(u,v) \cdot e^{\sum_{(u,v) \neq u \in V} \omega_Q(u,v)} = u_1$$

$$D_{te^{i\theta}}(u) = \sum_{(u,v) \neq u \in V} t_Q(u,v) \cdot e^{\sum_{(u,v) \neq u \in V} \theta_Q(u,v)} = u_2$$

$$D_{ke^{i\rho}}(u) = \sum_{(u,v) \neq u \in V} k_Q(u,v) \cdot e^{\sum_{(u,v) \neq u \in V} \rho_Q(u,v)} = u_3$$

G is called regular of degree (u_1,u_2,u_3) or (u_1,u_2,u_3)-regular.

Example 5: Consider a CFNG $G = (P,Q)$ on $P = \{v_1,v_2,v_3,v_4\}$ as in Figure 6 whose vertex and edge set are given in Table 14 and Table 15 respectively.

Table 14. Vertices of regular CFNG G

Node	CFNS
v_1	$\left\langle 0.7e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.3)}, 0.5^{i2\pi(0.6)} \right\rangle$
v_2	$\left\langle 0.8e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.8)}, 0.6^{i2\pi(0.4)} \right\rangle$
v_3	$\left\langle 0.6e^{i2\pi(0.8)}, 0.8e^{i2\pi(0.7)}, 0.3^{i2\pi(0.5)} \right\rangle$
v_4	$\left\langle 0.3e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.9)}, 0.4^{i2\pi(0.3)} \right\rangle$

Table 15. Edges of regular CFNG G

Edge	CFNS
e_{12}	$\left\langle 0.3e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.4)}, 0.5^{i2\pi(0.4)} \right\rangle$
e_{13}	$\left\langle 0.2e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.3)}, 0.4^{i2\pi(0.8)} \right\rangle$
e_{24}	$\left\langle 0.2e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.3)}, 0.4^{i2\pi(0.8)} \right\rangle$
e_{34}	$\left\langle 0.3e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.4)}, 0.5^{i2\pi(0.4)} \right\rangle$

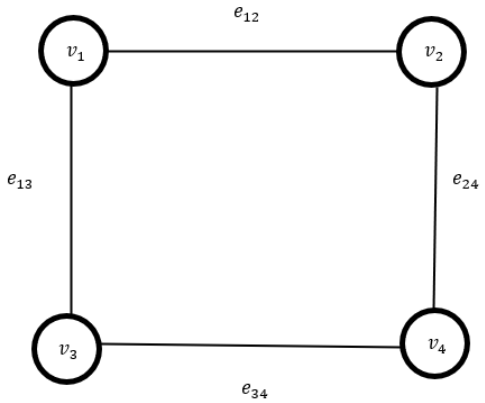

Figure 6. Regular CFNG G

Then the degrees of vertices v_1, v_2, v_3, v_4 are determined as:

$$D_G(v_1) = \left(r_Q(v_1, v_2) + r_Q(v_1, v_3)\right) \cdot e^{i\left(\omega_Q(v_1, v_2) + \omega_Q(v_1, v_2)\right)},$$
$$\left(t_Q(v_1, v_2) + t_Q(v_1, v_3)\right) \cdot e^{i\left(\theta_Q(v_1, v_2) + \theta_Q(v_1, v_2)\right)}, \left(k_Q(v_1, v_2) + k_Q(v_1, v_3)\right) \cdot e^{i\left(\rho_Q(v_1, v_2) + \rho_Q(v_1, v_2)\right)},$$
$$= \left\langle 0.5^{i2\pi(1.1)}, 0.6^{i2\pi(0.7)}, 0.9^{i2\pi(1.2)} \right\rangle$$

Similarly, $D_G(v_2) = D_G(v_3) = D_G(v_4) = \left\langle 0.5^{i2\pi(1.1)}, 0.6^{i2\pi(0.7)}, 0.9^{i2\pi(1.2)} \right\rangle$. Hence,

G is $\left\langle 0.5^{i2\pi(1.1)}, 0.6^{i2\pi(0.7)}, 0.9^{i2\pi(1.2)} \right\rangle$ - regular graph.

Definition 3.18: A CFNG $G = (P, Q)$ on G is called a totally regular CFNG if the total degree of its each vertex is same. If each vertex has total degree (m_1, m_2, m_3), that is, $TD_G(u_i) = \left(TD_{re^{i\omega}}(u), TD_{te^{i\theta}}(u), TD_{ke^{i\rho}}(u)\right) = (m_1, m_2, m_3)$ for all $u_i \in V$, where

$$TD_{re^{i\omega}}(u)=\sum_{(u,v)\neq u\in V} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\omega_Q(u,v)}+r_P(u)\cdot e^{i\omega_P(u)}=m_1$$
$$TD_{te^{i\theta}}(u)=\sum_{(u,v)\neq u\in V} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\theta_Q(u,v)}+t_P(u)\cdot e^{i\theta_P(u)}=m_2$$
$$TD_{ke^{i\rho}}(u)=\sum_{(u,v)\neq u\in V} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\rho_Q(u,v)}+k_P(u)\cdot e^{i\rho_P(u)}=m_3$$

G is called totally regular of degree (m_1,m_2,m_3) or (m_1,m_2,m_3) -totally regular.

Example 6: Consider a CFNG $G=(P,Q)$ on $P=\{v_1,v_2,v_3,v_4\}$ as in Figure 7 whose vertex and edge set are given in Table 16 and Table 17 respectively.

Table 16. Vertices of totally regular CFNG G

Node	CFNS
v_1	$\langle 0.7e^{i2\pi(0.5)},0.6e^{i2\pi(0.3)},0.5^{i2\pi(0.6)}\rangle$
v_2	$\langle 0.7e^{i2\pi(0.5)},0.6e^{i2\pi(0.3)},0.5^{i2\pi(0.6)}\rangle$
v_3	$\langle 0.7e^{i2\pi(0.5)},0.6e^{i2\pi(0.3)},0.5^{i2\pi(0.6)}\rangle$
v_4	$\langle 0.7e^{i2\pi(0.5)},0.6e^{i2\pi(0.3)},0.5^{i2\pi(0.6)}\rangle$

Table 17. Edges of totally regular CFNG G

Edge	CFNS
e_{12}	$\langle 0.3e^{i2\pi(0.6)},0.2e^{i2\pi(0.4)},0.5^{i2\pi(0.4)}\rangle$
e_{13}	$\langle 0.2e^{i2\pi(0.5)},0.4e^{i2\pi(0.3)},0.4^{i2\pi(0.8)}\rangle$
e_{24}	$\langle 0.2e^{i2\pi(0.5)},0.4e^{i2\pi(0.3)},0.4^{i2\pi(0.8)}\rangle$
e_{34}	$\langle 0.3e^{i2\pi(0.6)},0.2e^{i2\pi(0.4)},0.5^{i2\pi(0.4)}\rangle$

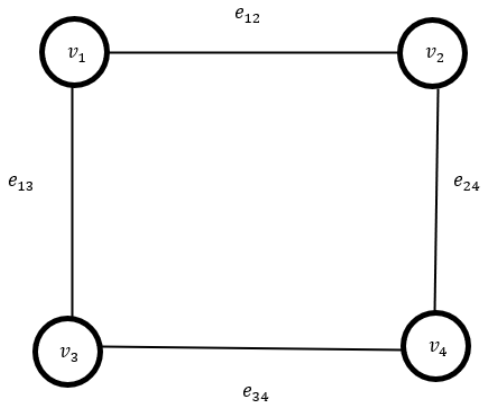

Figure 7. Totally regular CFNG G

$$TD_G(v_1)=\left(r_Q(v_1,v_2)+r_Q(v_1,v_3)+r_P(v_1)\right)\cdot e^{i\left(\omega_Q(v_1,v_2)+\omega_Q(v_1,v_2)+\omega_P(v_1)\right)},$$
$$\left(t_Q(v_1,v_2)+t_Q(v_1,v_3)+t_P(v_1)\right)\cdot e^{i\left(\theta_Q(v_1,v_2)+\theta_Q(v_1,v_2)+\theta_p(v_1),\right)},\quad,$$
$$\left(k_Q(v_1,v_2)+k_Q(v_1,v_3)+k_P(v_1)\right)\cdot e^{i\left(\rho_Q(v_1,v_2)+\rho_Q(v_1,v_2)+\rho_P(v_1)\right)}$$
$$=\langle 1.2^{i2\pi(1.6)},1.2^{i2\pi(1.0)},1.4^{i2\pi(1.8)}\rangle$$

Similarly, $TD_G(v_2) = TD_G(v_3) = TD_G(v_4) = \left\langle 1.2^{i2\pi(1.6)}, 1.2^{i2\pi(1.0)}, 1.4^{i2\pi(1.8)} \right\rangle$. Hence G is $\left\langle 1.2^{i2\pi(1.6)}, 1.2^{i2\pi(1.0)}, 1.4^{i2\pi(1.8)} \right\rangle$ - totally regular CFNG.

Theorem 3: Let $G = (P, Q)$ be a (u_1, u_2, u_3)-regular CFNG. Then $S(G) = \left(\frac{nu_1}{2}, \frac{nn_2}{2}, \frac{nn_3}{2} \right)$ where $|V| = n$.

Proof: Assume that G is a CFNG with size

$$S(G) = \left(\sum_{(u,v)\in E} r_Q(u,v) \cdot e^{\sum_{(u,v)\in E} \omega_Q(u,v)}, \sum_{(u,v)\in E} t_Q(u,v) \cdot e^{\sum_{(u,v)\in E} \theta_Q(u,v)}, \sum_{(u,v)\in E} k_Q(u,v) \cdot e^{\sum_{(u,v)\in E} \rho_Q(u,v)} \right)$$

Since $G(u_1, u_2, u_3)$-regular CFNG, that is

$D_G(u)_i = \left(D_{re^{i\omega}}(u), D_{te^{i\theta}}(u), D_{ke^{i\rho}}(u) \right) = (u_1, u_2, u_3)$ for all $u_i \in V$, where

$$D_{re^{i\omega}}(u) = \sum_{(u,v)\neq u\in V} r_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in V} \omega_Q(u,v)} = u_1$$
$$D_{te^{i\theta}}(u) = \sum_{(u,v)\neq u\in V} t_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in V} \theta_Q(u,v)} = u_2$$
$$D_{ke^{i\rho}}(u) = \sum_{(u,v)\neq u\in V} k_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in V} \rho_Q(u,v)} = u_3$$

Therefore,

$$D_G(u_i) = \left(\sum_{(u,v)\in E} r_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in E} \omega_Q(u,v)}, \sum_{(u,v)\in E} t_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in E} \theta_Q(u,v)}, \sum_{(u,v)\in E} k_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in E} \rho_Q(u,v)} \right)$$

$$\sum_{u_i\in V} D_G(u_i) = \left(\sum_{u_i\in V} \sum_{(u,v)\in E} r_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in E} \omega_Q(u,v)}, \sum_{u_i\in V} \sum_{(u,v)\in E} t_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in E} \theta_Q(u,v)}, \sum_{u_i\in V} \sum_{(u,v)\in E} k_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in E} \rho_Q(u,v)} \right)$$

Since each edge is considered twice, so

$$\left(\sum_{u_i\in V} D_{re^{i\omega}}(u_i), \sum_{u_i\in V} D_{te^{i\theta}}(u_i), \sum_{u_i\in V} D_{ke^{i\rho}}(u_i) \right) = 2\left(\sum_{(u,v)\in E} r_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in E} \omega_Q(u,v)}, \sum_{(u,v)\in E} t_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in E} \theta_Q(u,v)}, \sum_{(u,v)\in E} k_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in E} \rho_Q(u,v)} \right)$$

$$(nu_1, nu_2, nu_3) = 2S(G)$$
$$S(G) = \left(\frac{nu_1}{2}, \frac{nn_2}{2}, \frac{nn_3}{2} \right)$$

Theorem 4: Let $G = (P, Q)$ be a (m_1, m_2, m_3) totally regular graph CFNG. Then $2S(G) + O(G) = (nm_1, nm_2, nm_3)$ where $|V| - n$.

$$TD_{re^{i\omega}}(u) = \sum_{(u,v)\neq u\in V} r_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in V} \omega_Q(u,v)} + r_P(u) \cdot e^{i\omega_P(u)} = m_1$$
$$TD_{te^{i\theta}}(u) = \sum_{(u,v)\neq u\in V} t_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in V} \theta_Q(u,v)} + t_P(u) \cdot e^{i\theta_P(u)} = m_2$$
$$TD_{ke^{i\rho}}(u) = \sum_{(u,v)\neq u\in V} k_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in V} \rho_Q(u,v)} + k_P(u) \cdot e^{i\rho_P(u)} = m_3$$

Therefore,

$$TD_G\left(u_i\right)=\begin{pmatrix}\sum\limits_{(u,v)\neq u\in E} r_Q\left(u,v\right)\cdot e^{\sum_{(u,v)\neq u\in E}\omega_Q(u,v)}+r_P\left(u\right)\cdot e^{i\omega_P(u)},\\ \sum\limits_{(u,v)\neq u\in E} t_Q\left(u,v\right)\cdot e^{\sum_{(u,v)\neq u\in E}\theta_Q(u,v)}+t_P\left(u\right)\cdot e^{i\theta_P(u)},\\ \sum\limits_{(u,v)\neq u\in E} k_Q\left(u,v\right)\cdot e^{\sum_{(u,v)\neq u\in E}\rho_Q(u,v)}+k_P\left(u\right)\cdot e^{i\rho_P(u)}\end{pmatrix}$$

$$\sum_{u_i\in V} TD_G\left(u_i\right)=\begin{pmatrix}\sum\limits_{u_i\in V}\sum\limits_{(u,v)\neq u\in E} r_Q\left(u,v\right)\cdot e^{\sum_{(u,v)\neq u\in E}\omega_Q(u,v)}+r_P\left(u\right)\cdot e^{i\omega_P(u)},\\ \sum\limits_{u_i\in V}\sum\limits_{(u,v)\neq u\in E} t_Q\left(u,v\right)\cdot e^{\sum_{(u,v)\neq u\in E}\theta_Q(u,v)}+t_P\left(u\right)\cdot e^{i\theta_P(u)},\\ \sum\limits_{u_i\in V}\sum\limits_{(u,v)\neq u\in E} k_Q\left(u,v\right)\cdot e^{\sum_{(u,v)\neq u\in E}\rho_Q(u,v)}+k_P\left(u\right)\cdot e^{i\rho_P(u)}\end{pmatrix}$$

Since each edge is considered twice, so

$$\left(D_{re^{i\omega}}\left(u_i\right),D_{te^{i\theta}}\left(u_i\right),D_{ke^{i\rho}}\left(u_i\right)\right)=2\begin{pmatrix}\sum\limits_{(u,v)\neq u\in E} r_Q\left(u,v\right)\cdot e^{\sum_{(u,v)\neq u\in E}\omega_Q(u,v)},\sum\limits_{(u,v)\neq u\in E} t_Q\left(u,v\right)\cdot e^{\sum_{(u,v)\neq u\in E}\theta_Q(u,v)},\\ \sum\limits_{(u,v)\neq u\in E} k_Q\left(u,v\right)\cdot e^{\sum_{(u,v)\neq u\in E}\rho_Q(u,v)}\end{pmatrix}$$
$$+\left(r_P\left(u\right)\cdot e^{i\omega_P(u)},t_P\left(u\right)\cdot e^{i\theta_P(u)},k_P\left(u\right)\cdot e^{i\rho_P(u)}\right)$$

Corollary 1: Let $G=\left(P,Q\right)$ be a $\left(n_1,n_2,n_3\right)$- regular and $\left(m_1,m_2,m_3\right)$- totally regular CFNG. Then $O(G)=n\left\{\left(m_1,m_2,m_3\right)-\left(n_1,n_2,n_3\right)\right\}$.

Proof: Assume that G is a $\left(n_1,n_2,n_3\right)$- regular CFNG. Then the size of G is

$$S\left(G\right)=\left(\frac{nu_1}{2},\frac{nn_2}{2},\frac{nn_3}{2}\right)$$

As G is a $\left(m_1,m_2,m_3\right)$- totally regular CFNG. Using theorem 4, we have

$$2S\left(G\right)+O\left(G\right)=\left(ng_1,ng_2,ng_3\right)$$
$$O\left(G\right)=\left(nm_1,nm_2,nm_3\right)-2S\left(G\right)$$
$$=\left(nm_1,nm_2,nm_3\right)-2\left(\frac{nu_1}{2},\frac{nn_2}{2},\frac{nn_3}{2}\right)$$
$$=n\left\{\left(m_1,m_2,m_3\right)-\left(n_1,n_2,n_3\right)\right\}.$$

4. Decision making approach under CFN environment

In this section, based on the proposed CFNGs a decision-making approach is introduced. Further a real-life application on education system is discussed.

Definition 4.1: Let $F_j=\left\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\right\rangle\ \left(j=1,2,\cdots,m\right)$ be a collection of CFNNs and $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the weighted vector of $F_j\left(j=1,2,\cdots,m\right)$ with $w_j>0,\sum_{j=1}^{n}w_j=1.$ Then the complex fermatean neutrosophic weighted averaging (CFNWA) operator is a mapping $CFNWA:\Omega^n\to\Omega,$ such that

$$CFNWA\left(F_1,F_2,\cdots,F_m\right)=\overset{m}{\underset{j=1}{\oplus}}\left(w_jF_j\right)$$

Specifically, if we have $w_j=\frac{1}{m};\forall j,$ then CFNWA reduces to complex Fermatean neutrosophic averaging (CFNA) operator

$$CFNA\left(F_1,F_2,\cdots,F_m\right)=\frac{1}{m}\overset{m}{\underset{j=1}{\oplus}}\left(w_jF_j\right)$$

We get CFNWA operator using induction on n and is defined as

$$CFNWA\left(F_1,F_2,\cdots,F_m\right)=\left(\begin{array}{c}\sqrt[3]{1-\prod_{j=1}^{m}\left(1-r_j^3\right)^{wj}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\omega_j}{2\pi}}\right)^{wj}},\\ \prod_{j=1}^{m}t_j^{w_j}\cdot e^{i2\pi\left(\left(\frac{\theta_j}{2\pi}\right)^{w_j}\right)},\prod_{j=1}^{m}k_j^{w_j}\cdot e^{i2\pi\left(\left(\frac{\pi_j}{2\pi}\right)^{w_j}\right)}\end{array}\right)$$

Definition 4.2: Let $F_j=\left\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\right\rangle\ \left(j=1,2,\cdots,m\right)$ be a collection of CFNNs and $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the weighted vector of $F_j\left(j=1,2,\cdots,m\right)$ with $w_j>0,\sum_{j=1}^{n}w_j=1.$ Then the complex fermatean neutrosophic weighted geometric (CFNWG) operator is defined by $CFNWG:\Omega^n\to\Omega,$ such that

$$CFNWG\left(F_1,F_2,\cdots,F_m\right)=\overset{m}{\underset{j=1}{\otimes}}\left(w_jF_j\right)$$

Theorem 5: The aggregated value of CFNNs $F_j=\left\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\right\rangle\ \left(j=1,2,\cdots,m\right)$ using CFNWG operator is again a CFNN and is given by

$$CFNWG\left(F_1,F_2,\cdots,F_m\right)=\left(\begin{array}{c}\prod_{j=1}^{m}r_j^{w_j}\cdot e^{i2\pi\left(\left(\frac{\omega_j}{2\pi}\right)^{w_j}\right)},\sqrt[3]{1-\prod_{i=1}^{m}\left(1-t_j^3\right)^{w_j}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\theta_j}{2\pi}}\right)^{w_j}},\\ \sqrt[3]{1-\prod_{j=1}^{m}\left(1-k_j^3\right)^{w_j}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\rho_j}{2\pi}}\right)^{w_j}}\end{array}\right)$$

Definition 4.3: Let $F_j=\left\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\right\rangle\ \left(j=1,2,\cdots,m\right)$ be a collection of CFNNs and $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the weighted vector of $F_j\left(j=1,2,\cdots,m\right)$ with $w_j>0,\sum_{j=1}^{n}w_j=1.$ Then the complex fermatean neutrosophic ordered weighted averaging (CFNOWA) operator is a mapping $CFNOWA:\Omega^n\to\Omega,$ such that

$$CFNOWA\left(F_1,F_2,\cdots,F_m\right)=\overset{m}{\underset{j=1}{\oplus}}\left(w_jF_{\sigma(j)}\right)$$

Where $\left(\sigma(1),\sigma(2),\cdots,\sigma(m)\right)$ denotes the permutation of $(1,2,\cdots,m)$ and $F_{\sigma(j-1)}\geq F_{\sigma(j)},\forall j=1,2,\cdots,m$. Using induction on n, we get the following theorem.

Theorem 6: The aggregated value of CFNNs $F_j=\left\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\right\rangle\ \left(j=1,2,\cdots,m\right)$ using CFNOWA is again a CFNN and is given by

$$CFNOWA\left(F_1,F_2,\cdots,F_m\right)=\left(\begin{array}{c}\sqrt[3]{1-\prod_{j=1}^{m}\left(1-r_{\sigma(j)}^{3}\right)^{w_j}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\omega_{\sigma(j)}}{2\pi}}\right)^{w_j}},\\ \prod_{j=1}^{m}t_{\sigma(j)}^{w_j}\cdot e^{i2\pi\left(\left(\frac{\theta_{\sigma(j)}}{2\pi}\right)^{w_j}\right)},\prod_{j=1}^{m}k_{\sigma(j)}^{w_j}\cdot e^{i2\pi\left(\left(\frac{\pi_{\sigma(j)}}{2\pi}\right)^{w_j}\right)}\end{array}\right)$$

In which $\left(\sigma(1),\sigma(2),\cdots,\sigma(m)\right)$ denotes the permutation of $(1,2,\ldots,m)$ and $F_{\sigma(j-1)}\geq F_{\sigma(j)},\forall j=1,2,\cdots,m$ and $\sum_{j=1}^{n}w_j=1.$

Definition 4.4: Let $F_j=\left\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\right\rangle\ \left(j=1,2,\cdots,m\right)$ be a collection of CFNNs and $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the weighted vector of $F_j\left(j=1,2,\cdots,m\right)$ with $w_j>0,\sum_{j=1}^{n}w_j=1.$ Then the complex fermatean neutrosophic ordered weighted geometric (CFNOWG) operator is a mapping $CFNOWG:\Omega^n\to\Omega,$ such that

$$CFNOWG\left(F_1,F_2,\cdots,F_m\right)=\overset{m}{\underset{j=1}{\otimes}}\left(w_jF_{\sigma(j)}\right)$$

Where $\left(\sigma(1),\sigma(2),\cdots,\sigma(m)\right)$ denotes the permutation of $(1,2,\cdots,m)$ and $F_{\sigma(j-1)}\geq F_{\sigma(j)},\forall j=1,2,\cdots,m$. Using induction on n, we get the following theorem.

Theorem 7: The aggregated value of CFNNs $F_j=\left\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\right\rangle\ \left(j=1,2,\cdots,m\right)$ using CFNOWGO is again a CFNN and is given by

$$CFNOWG\left(F_1,F_2,\cdots,F_m\right)=\left(\begin{array}{c}\prod_{j=1}^{m}r_{\sigma(j)}^{w_j}\cdot e^{i2\pi\left(\left(\frac{\omega_{\sigma(j)}}{2\pi}\right)^{w_j}\right)},\sqrt[3]{1-\prod_{j=1}^{m}\left(1-t_{\sigma(j)}^{3}\right)^{w_j}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\theta_{\sigma(j)}}{2\pi}}\right)^{w_j}},\\ \sqrt[3]{1-\prod_{j=1}^{m}\left(1-k_{\sigma(j)}^{3}\right)^{w_j}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\rho_{\sigma(j)}}{2\pi}}\right)^{w_j}}\end{array}\right)$$

Now, we propose the idea of score function and accuracy function in order to compare two CFNN.

Definition 4.5: Let $A=\left\langle r\cdot e^{i2\pi\omega},t\cdot e^{i2\pi\theta},k\cdot e^{i2\pi\rho}\right\rangle$ be a CFNN, then the score function $Score\left(A\right)$ is defined by

$$Score\left(A\right)=\left(r^3-t^3-k^3\right)+\frac{1}{4\pi^2}\left(\omega^3-\theta^3-\rho^3\right)$$

where $Score\left(A\right)\in\left[-3,3\right].$

Definition 4.6: Let $A=\left\langle r\cdot e^{i2\pi\omega},t\cdot e^{i2\pi\theta},k\cdot e^{i2\pi\rho}\right\rangle$ be a CFNN, then the accuracy function $Accur\left(A\right)$ is defined as follows:

$$Accur\left(A\right)=\left(r^3+t^3+k^3\right)+\frac{1}{4\pi^2}\left(\omega^3+\theta^3+\rho^3\right)$$

where $Accur(A) \in [0,3]$.

Any two CFNNs are ranked using the following method:

Let $A = \langle r_1 \cdot e^{i2\pi\omega_1}, t_1 \cdot e^{i2\pi\theta_1}, k_1 \cdot e^{i2\pi\rho_1} \rangle$ and $B = \langle r_2 \cdot e^{i2\pi\omega_2}, t_2 \cdot e^{i2\pi\theta_2}, k_2 \cdot e^{i2\pi\rho_2} \rangle$ be two CFNNs, then

- If $Score(A) > Score(B)$ then $A > B$;
- If $Score(A) = Score(B)$ then
  - If $Accur(A) < Accur(B)$ then $A < B$;
  - If $Accur(A) > Accur(B)$ then $A > B$;
  - If $Accur(A) = Accur(B)$ then $A = B$;

4.1. Decision making approach

We developed a decision making approach under CFN environment using the proposed CFNWA, CFNOWA, CFNWG, CFNOWG operators. Let $A = \{A_1, A_2, \cdots, A_m\}$ and $C = \{C_1, C_2, \cdots, C_n\}$ be the number of alternatives and criteria respectively and $w = (w_1, w_2, \cdots, w_n)^T$ where $\sum_{i=1}^{n} w_i = 1$. So, the decision matrix $A = [X_{ij}]_{m\times n}$ $(i = 1,2,\cdots,m; j = 1,2,\cdots,n)$ is constructed where $X_{ij} = \langle r \cdot e^{i2\pi\omega}, t \cdot e^{i2\pi\theta}, k \cdot e^{i2\pi\rho} \rangle$.

Step 1: The decision matrix $A = [X_{ij}]_{m\times n}$ is transformed to a normalized matrix $B = [Y_{ij}]_{m\times n}$. For benefit and cost type of attributes the following normalization are used.

$$R_{ij} = \begin{cases} \langle r \cdot e^{i2\pi\omega}, t \cdot e^{i2\pi\theta}, k \cdot e^{i2\pi\rho} \rangle & for\ \ benefit \\ \langle k \cdot e^{i2\pi\rho}, t \cdot e^{i2\pi\theta}, r \cdot e^{i2\pi\omega} \rangle & for\ \ \cos t \end{cases}$$

Step 2: The alternatives are aggregated using the proposed operators CFNWA, CFNOWA, CFNWG, CFNOWG.

Step 3: The aggregated value's score function is calculated.

Step 4: The best alternative is selected based on ranking given in step 3.

4.2. Illustrative example

To provide evidence that the suggested arithmetic and geometric aggregation operator of CFNS, an illustrative example is discussed below. Teachers play a major role in imparting quality education to students in institutions. In this section, a decision-making approach to evaluate lecturer's research productivity at the end of year (Nguyen et al., 2020) is discussed. Assume that there are 3 lecturers to be evaluated. A committee of 4 decision makers are considered to make their assessment individually, according to their preferences of criteria and the ratings of alternative s. After a discussion with committee members five criteria are selected as in Table 18.

Table 18. Criteria for research productivity

	Criteria	Type
C_1	Number of research articles published	Benefit
C_2	Quality of journals published	Benefit
C_3	Number of books published	Benefit
C_4	Guiding postgraduate and Ph d students	Benefit
C_5	Project research grants secured	Benefit

The faculty members $A_m\left(m=1,2,3\right)$ are to be evaluated using CFNS by the expert under the set of 5 parameters $C_n\left(m=1,2,3,4,5\right)$ whose associating weighting vector is $w=\left(0.2,0.2,0.2,0.2,0.2\right)$. To select the most desirable teacher, we utilize the above step-wise procedure using CFNWA, CFNOWA, CFNWG, CFNOWG. Consider the CFNN decision matrix as stated in Table 19, Table 20, Table21 respectively.

Table 19. CFNN decision matrix for A_1

Criteria	Alternative A_1
C_1	$\left\langle 0.3e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.3)}, 0.5^{i2\pi(0.4)}\right\rangle$
C_2	$\left\langle 0.5e^{i2\pi(0.1)}, 0.6e^{i2\pi(0.4)}, 0.4^{i2\pi(0.3)}\right\rangle$
C_3	$\left\langle 0.4e^{i2\pi(0.3)}, 0.7e^{i2\pi(0.1)}, 0.5^{i2\pi(0.6)}\right\rangle$
C_4	$\left\langle 0.6e^{i2\pi(0.2)}, 0.5e^{i2\pi(0.2)}, 0.3^{i2\pi(0.5)}\right\rangle$
C_5	$\left\langle 0.7e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.3)}, 0.6^{i2\pi(0.3)}\right\rangle$

Table 20. CFNN decision matrix for A_2

Criteria	Alternative A_2
C_1	$\left\langle 0.6e^{i2\pi(0.1)}, 0.5e^{i2\pi(0.3)}, 0.4^{i2\pi(0.6)}\right\rangle$
C_2	$\left\langle 0.3e^{i2\pi(0.3)}, 0.8e^{i2\pi(0.2)}, 0.7^{i2\pi(0.4)}\right\rangle$
C_3	$\left\langle 0.7e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.6)}, 0.5^{i2\pi(0.7)}\right\rangle$
C_4	$\left\langle 0.5e^{i2\pi(0.1)}, 0.6e^{i2\pi(0.6)}, 0.3^{i2\pi(0.4)}\right\rangle$
C_5	$\left\langle 0.4e^{i2\pi(0.2)}, 0.5e^{i2\pi(0.1)}, 0.3^{i2\pi(0.3)}\right\rangle$

Table 21. CFNN decision matrix for A_3

Criteria	Alternative A_3
C_1	$\left\langle 0.7e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.2)}, 0.6^{i2\pi(0.4)}\right\rangle$
C_2	$\left\langle 0.4e^{i2\pi(0.3)}, 0.5e^{i2\pi(0.4)}, 0.7^{i2\pi(0.5)}\right\rangle$
C_3	$\left\langle 0.3e^{i2\pi(0.4)}, 0.6e^{i2\pi(0.5)}, 0.4^{i2\pi(0.3)}\right\rangle$
C_4	$\left\langle 0.2e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.1)}, 0.5^{i2\pi(0.6)}\right\rangle$
C_5	$\left\langle 0.1e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.3)}, 0.3^{i2\pi(0.7)}\right\rangle$

Now we aggregate the criteria using the proposed CFNWA, CFNOWA, CFNWG, CFNOWG and the values are stated in Table 22, Table 23, Table 24.

Table 22. Aggregated values of A_1

Aggregation operator	A_1
CFNWA	$\langle 0.05e^{i2\pi(0.01)}, 0.51e^{i2\pi(0.37)}, 0.45^{i2\pi(0.64)} \rangle$
CFNOWA	$\langle 0.05e^{i2\pi(0.01)}, 0.51e^{i2\pi(0.46)}, 0.45^{i2\pi(0.63)} \rangle$
CFNWG	$\langle 0.48e^{i2\pi(0.26)}, 0.06e^{i2\pi(0.03)}, 0.04^{i2\pi(0.15)} \rangle$
CFNOWG	$\langle 0.48e^{i2\pi(0.26)}, 0.06e^{i2\pi(0.03)}, 0.04^{i2\pi(0.65)} \rangle$

Table 23. Aggregated values of A_2

Aggregation operator	A_2
CFNWA	$\langle 0.05e^{i2\pi(0.01)}, 0.54e^{i2\pi(0.46)}, 0.42^{i2\pi(0.72)} \rangle$
CFNOWA	$\langle 0.05e^{i2\pi(0.01)}, 0.54e^{i2\pi(0.46)}, 0.42^{i2\pi(0.72)} \rangle$
CFNWG	$\langle 0.48e^{i2\pi(0.26)}, 0.08e^{i2\pi(0.18)}, 0.04^{i2\pi(0.05)} \rangle$
CFNOWG	$\langle 0.48e^{i2\pi(0.26)}, 0.08e^{i2\pi(0.18)}, 0.04^{i2\pi(0.05)} \rangle$

Table 24. Aggregated values of the CFNN information

Aggregation operator	A_3
CFNWA	$\langle 0.03e^{i2\pi(0.03)}, 0.43e^{i2\pi(0.41)}, 0.48^{i2\pi(0.75)} \rangle$
CFNOWA	$\langle 0.03e^{i2\pi(0.03)}, 0.43e^{i2\pi(0.41)}, 0.48^{i2\pi(0.75)} \rangle$
CFNWG	$\langle 0.28e^{i2\pi(0.39)}, 0.03e^{i2\pi(0.07)}, 0.05^{i2\pi(0.48)} \rangle$
CFNOWG	$\langle 0.28e^{i2\pi(0.39)}, 0.03e^{i2\pi(0.07)}, 0.05^{i2\pi(0.48)} \rangle$

Using the score function of CFNN, we arrive at the ranking order and stated in Table 25.

Table 25. Ranking order

Aggregation operator	Ranking
Proposed CFNWA	$A_2 > A_1 > A_3$
Proposed CFNOWA	$A_2 > A_1 > A_3$
Proposed CFNWG	$A_2 > A_3 > A_1$
Proposed CFNOWG	$A_2 > A_3 > A_1$

Clearly, we can see that the CFNWA and CFNOWA provides same ranking and CFNWG and CFNOWG provides same ranking.

5. Conclusion

To model uncertain decision-making problem, graph theory plays a very important role. The novel idea of CFNG is introduced and investigated different properties of it in this paper. We have defined the order, size, degree and total degree of CFNG with appropriate examples. Also, the primary operation such as complement, union, join and ring sum of CFNG are proposed with appropriate examples and some of their important theorems are also described. It is shown with

a comparison study how CFNG is more flexible than the CNG. Further, the regular graph under CFN environment is presented. Finally, we determined the use of CFNG in educational system involving decision-making and proposed the operators CFNWA, CFNOWA, CFNWG, and CFNOWG into the approach for selecting the best lecturer. In the future, we will introduce complex fermatean Dombi neutrosophic fuzzy graph and complex fermatean neutrosophic hypergraph with its applications.

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References

Ajay, D., & Chellamani, P. (2020). Pythagorean Neutrosophic Fuzzy Graphs. International Journal of Neutrosophic Science, 11 (2), 108-114.

Akram, M., & Naz, S. (2019). A novel decision-making approach under complex Pythagorean fuzzy environment. Mathematical and Computational Applications, 24(3),73

Ali, M., & Smarandache, F. (2017). Complex neutrosophic set. Neural computing and applications, 28(7), 1817-1834.

Alkouri, A., & Salleh, A. (2012). Complex intuitionistic fuzzy sets. AIP Conference Proceedings, 14, 464–470.

Antony, C. S. C., & Jansi, R. (2021). Fermatean Neutrosophic Sets. International Journal of Advanced Research in Computer and Communication Engineering, 10(6), 24-27.

Atanassov, K. (1986). Intuitionistic Fuzzy Sets. Fuzzy Sets and Systems, 20, 87-96.

Broumi, S., Sundareswaran, R., Shanmugapriya, M., Assia Bakali, Mohamed Talea. (2022). Theory and Applications of Fermatean Neutrosophic Graphs, Neutrosophic Sets and Systems, 50, 248-286.

Buckley, J.J. (1989). Fuzzy complex numbers. Fuzzy sets and systems, 33, 333-345.

Chinnadurai, V., Thayalan, S., &. Bobin, A. (2021). Multi-criteria decision-making in complex Fermatean fuzzy environment. Journal of Mathematical and Computational Science, 11(6), 7209-7227.

Ibrahim, H. Z., Tareq M. Al-shami, O. G. Elbarbary (2021). (3, 2)-Fuzzy Sets and Their Applications to Topology and Optimal Choices. Computational Intelligence and Neuroscience, volume. 2021.

Khan, V. A., Tuba, U., & Rahaman, A. S. (2022). Motivations and basic of fuzzy, intuitionistic fuzzy and neutrosophic sets and norms. *Yugoslav Journal of Operations Research*, 32(3), 299-323.

Nguyen, A. T., Truong, T. H., Luong, T. L., Truong, D. T., Nguyen, D. Q., Luu, H. V. & Luong, T. A. (2020). A new integrated MCDM approach for lecturers' research productivity evaluation. Decision Science Letters. 9, 355-364.

Ramot, D., Milo, R., Friedman, M., & Kandel, A. (2002). Complex fuzzy sets. IEEE Tranactions and. Fuzzy System, 10, 171-186.

Samina, A., Naz, S., Rashmanlou, H., & Malik, M.A. (2017). Regularity of graphs in single valued neutrosophic environment, Journal of Intelligent and Fuzzy Systems, 33, 529–542.

Saumya, Y. M., & Hegde, S. M. (2022). Construction and Analysis of Graph Models for Multiprocessor Interconnection Networks. *Yugoslav Journal of Operations Research*, *32*(1), 87-109.

Senapati, T., & Yager, R. R. (2019a). Some New Operations Over Fermatean Fuzzy Numbers and Application of Fermatean Fuzzy WPM in Multiple Criteria Decision Making. In Informatica, 30(2), 391–412.

Senapati. T., & Yager. R. R. (2019). Fermatean Fuzzy Sets. Journal of Ambient Intelligence and Humanized Computing, 11, 663-674.

Shanmathi R., & Nirmala I. F. (2022). (4,2)-Fuzzy Set and their Topological Space. International Journal of Creative Research Thoughts, 10(4), 441-446.

Shoaib, M., Mahmood, W., Albalawi, W., & Shami, F. A. (2022a). Notion of Complex Spherical Fuzzy Graph with Application. In H. A. Al-Olayan (Ed.), Journal of Function Spaces, 1–27.

Shoaib, M., Mahmood, W., Xin, Q., Tchier, F., & Tawfiq, F.M. (2022). Certain Operations on Complex Picture Fuzzy Graphs, IEEE Access, 10, 114284- 114296.

Smarandache, F. (1999). A unifying field in logics neutrosophy: neutrosophic probability, set and logic. American Research Press, Rehoboth.

Thirunavukarasu, P., Suresh, R., & Viswanathan, K. K. (2016). Energy of a complex fuzzy graph. Int. J. Math. Sci. Eng. Appl., 10, 243–248.

Ullah, K., Mahmood, T., Ali, Z., & Jan, N. (2019). On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition. Complex Intelligent System, 6, 15-27.

Yager, R.R. (2013). Pythagorean Fuzzy Subsets. In:Proc Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada, 57-61.

Yager, R.R. (2014). Pythagorean membership grades in multicriteria decision making. IEEE Transactions on Fuzzy Systems, 22, 958–965.

Yaqoob, N., & Akram, M. (2018). Complex neutrosophic graphs. Bull. Comput. Appl. Math., 6, 85–109.

Yaqoob, N., Gulistan, M., Kadry, S., & Wahab, H. (2019). Complex intuitionistic fuzzy graphs with application in cellular network provider companies. Mathematics, 7(1), 35.

Yazdanbakhsh, O., & Dick, S., (2018). A systematic review of complex fuzzy sets and logic. Fuzzy Sets and System. 338, 1–22.

Zadeh, L. A., (1965). Fuzzy Sets. Inform. and Control, 8, 338- 353.

Octagonal fuzzy neutrosophic number and its application to reusable container inventory model with container shrinkage

S. Rajeswari,[1] **C. Sugapriya**,[1] **D. Nagarajan**,[2] **S. Broumi**,[3] **F. Smarandache**[4]

[1] Department of Mathematics, Queen Mary's College, University of Madras, Chennai, Tamil Nadu, India

[2] Department of Mathematics, Hindustan Institute of Technology and Science, Chennai 603 103, India

[3] Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, Casablanca, Morocco

[4] Department of Mathematics, University of New Mexico, 705 Gurley Avenue, Gallup, NM 87301, USA

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Abstract

In supply chain management, the reusable containers are the main source of import and export trading. Import and export of the cargoes need some reusable transport items like containers for proper transportation process. Generally, the container management organiza-tion supplies the empty containers to multiple consigners for the purpose of transportation of cargoes. This paper scrutinizes the effect of inventory shrinkage on the reusable container inventory management system under neutrosophic arena. The octagonal fuzzy neutrosophic number (OFNN) is applied in this research and its de-neutrosophication is framed using 'removal area method'. The container inventory model is proposed and formulated under neutrosophic environment by treating the demand of each consigner and the return rate as octagonal fuzzy neutrosophic numbers and interpreted with numerical computation. The notion of neutrosophic arena leads the model to reach approximate solutions under uncertain parameters. The sensitivity analysis is conferred to show the impact of the proportion of returned units on optimal solution under both crisp and neutrosophic arena.

Keywords Inventory · Supplychain · Neutrosophic · Uncertainty

Mathematics Subject Classification 62A86

1 Introduction

In practicality, the necessary needs for everyone day-to-day life has been accomplished with the help of import and export trading. This depends upon the consumer's requirement which is uncertain and unpredictable. In accordance with this situation, many researchers utilized the concepts of fuzzy set, intuitionistic fuzzy set, neutrosophic set and so on. The notion of intuitionistic fuzzy set is flourished by Atanassov (1986) as the generalization of fuzzy set which introduced by Zadeh (1965). As the extension of crisp set, fuzzy set and intuitionistic fuzzy sets, Smarandache (1998) established the neutrosophic set. This research scrutinizes the octagonal fuzzy neutrosophic number in which the 'removal area method' is performed to compute the de-neutrosophication of OFNN and enforced this method to the reusable container inventory model under shrinkage of container.

Import and export of the cargoes like garments, electronics, fruits, vegetables, seafood, etc. needs some reusable transport items for safe and proper transportation process. Different types of containers are used for particular products, for example general purpose (GP) containers are used to carry general cargoes garments, rough blocks, electronics, etc. also for food items like fruits, vegetables, and seafood are transported by reefer containers and so on. In accordance with this, many shipping and logistic organizations allow the shipper to stuff the cargo in empty reusable containers and shipped the same to the particular discharge port. The exported laden containers are de-stuffed by the consignee and return the empty containers at particular empty yard and vice-versa. Some container inventory management issue says container inaccuracy or container shrinkage like receiving errors, clerical errors and misplacement of containers may occur. Because of container shrinkage, the return rate of the used container is reduced. As a part of the container shrinkage, the misplacement of containers plays a vital role in the Container Management Organization (CMO). This paper mainly focuses on the container misplacement that occurs due to clerical errors, invisibility of the container number (damage in the place of container number) and so on. Sometimes, the containers stored at some other yard because of the receiving errors at empty yard. Such containers and the delayed returns are restored to the CMO's yard as soon as possible but the misplaced containers will take more time to restore. This research is developed to compute the CMO's total cost under the situation of container misplacement. Since the misplaced units and the container demand are uncertain, the proposed research is designed in the neutrosophic environment to reach the maximum approximate outcome. Due to the misplacement of the container, the CMO unable to satisfy the entire demand and the revenue is reduced because of this situation. This reduced amount is computed as the penalty cost for misplaced units.

To the best of our knowledge, a few studies on the misplacement of inventories have been done. So far, no one has analyzed the misplacement of containers in the supply chain management problems. This study initiated the concept of misplacement of containers and developed the container inventory model for CMO with the knowledge of the misplacement of containers. When comparing with the product inventories, the containers are very costlier, and also without loss of generality, the proportion of misplaced containers is fewer than the proportion of misplaced product inventories. Since the quantity of misplaced containers is unpredictable, the present research utilizing the neutrosophic environment to obtain the most approximate result. The OFNN is considered and the new de-neutrosophication of the OFNN using the 'removal area method' is computed and implemented. In this paper, the single-vendor multi-consigner container inventory model is developed with the effect of container misplacement under crisp and neutrosophic environment. The storage charge is the main costs spent by the container management companies. The empty container storage

yard is fixing the storage price according to the size of the container, type of the containers, and the number of storage days, and also they offer some beneficial storage systems like free days, free pool, slab storage system, etc. The slab storage system utilized in this paper helps the CMO to minimize the container storage cost. The main contributions of the proposed study are:

- Computed the container inventory model under the situation of container misplacement.
- Container inventory model in the neutrosophic arena for unpredictable parameters.
- Slab storage system is utilized to reduce the container storage charge.

The major issue faced by any supply chain management system is the misplacement of inventory. In 2010, Camdereli and Swaminathan (2010) examined the effect of misplacement of inventory in supply chain management and rectified the issue of misplacement of inventory by RFID adoption. The benefits of adopting RFID in a coordinated supply chain have discussed. Zhang et al. (2014) studied the centralized and decentralized supply chain model under the misplacement of stock. They examined that while adopting the RFID technology, the lead time of the supply chain has reduced and also the problem of misplacement of stock rectified. In 2016, Wang et al. (2016) discussed about the stock inaccuracy and its effect by considering the demand as stock dependent. They formulated the inventory models under without knowledge of stock inaccuracy, with some rectification of known inaccuracy and partially adopting the RFID to reduce the inaccuracy of stock.

The following studies scrutinize the concept of container inventory management. In 2011, Yun et al. (2011) performed a container inventory model to repositioning of empty container with renting system under (s,S) strategy. A closed loop supply chain model with the implementation of RFID technology to containers is proposed by Kim and Glock (2014). They designed a container inventory model by presuming the proportion of returned unit as stochastic and discussed the benefits of RFID technology. Further, in 2016, Cobb (2016) formulated a closed supply chain container inventory model with the effect of loss ratio and restored the deficit units by purchasing the new containers. The remaining research conferred the usage of buffer containers which returned earlier before the cycle. Recently, in 2019, Hosseini and Sahlin (2019) presented a container management inventory model by discussing about ECR option with respect to time under unpredictable environment. In the same year, Göçen et al. (2019) explored an inventory model by utilizes some strategy to reduce the storage cost to minimize the cost of repositioning the empty containers. Recently, Rajeswari et al. (2021) analyzed (2016) and proposed a model by utilizing the empty container reposition option and the leasing of empty containers instead of purchasing new containers with price sensitive demand under fuzzy environment is presented by considering fuzzy return rate and fuzzy proportion of repaired units.

For extending (Zadeh 1965), many research works presented to determine various fuzzy numbers say triangular fuzzy number, trapezoidal fuzzy numbers, pentagonal fuzzy number and so on. Inspite of this, Malini and Felbin (2013) established the octagonal fuzzy number and its applications to assignment problem. In 2017, Menaga (2017) designed the octagonal intuitionistic fuzzy number and its ranking method and Saqlain (2020) formulated the octagonal neutrosophic number along with its α-cuts under linear and non-linear systems. For simple understanding, Wang et al. (2010) flourished the neutrosophic set into the singled valued neutrosophic set. For effective results, [2, 3 and 22] utilizes various measures on neutrosophic numbers to decision making models under various environments. Selvakumari and Lavanya (2018) framed the octagonal neutrosophic number and determined its ranking method using alpha cuts and also the effective outcome of game problem is attained under neutrosophic arena. In 2019, Chakraborty et al. (2019) introduced the new de-neutrosophication

formulation of pentagonal neutrosophic number by removal area method and utilized this method to minimal spanning tree. Further, (Mullai et al. 2020; Pal and Chakraborty 2020) designed the inventory models with various strategies under neutrosophic situation. Singh (2019) applied single-value neutrosophic attributes to multi-decision process on using the metric of multi-granulation and Galois connection. As a special categorization of a neutrosophic set, Freen et al. (2020) initiated the four-valued refined neutrosophic set and applied this notion to multi-objective optimization techniques.

In the below section, some basic definitions are provided and the de-neutrosophication of octagonal fuzzy neutrosophic number is derived by 'removal area method'. Followed by the notations, the container inventory model is developed and the proposed model under the octagonal neutrosophic arena is designed in Sect. 3 also the optimal cycle length is provided along with the convexity of total cost. The numerical computation and sensitivity analysis are provided in Sect. 4. The conclusions of the study are deliberated in the last section.

2 Preliminaries

Fuzzy set (Zadeh (1965)): A fuzzy set $\tilde{F}$ in the universal set $\mathcal{U}$ is defined as a set of ordered pairs and it is expressed as

$$\tilde{F} = \left(g, \mu_{\tilde{F}}(g)\right) : g \in \mathcal{U},$$

in which $\mu_{\tilde{F}}(g)$ is a membership function of g which assumes values in the range from 0 to 1 (ie.,) $\mu_{\tilde{F}}(g) \in [0, 1]$.

Fuzzy number (Zadeh (1965)): A fuzzy number $\tilde{\mathcal{F}}$ is a subset of real line R, which has the membership function $\mu_{\tilde{\mathcal{F}}}$ fullfilling the given features:

(1) $\mu_{\tilde{\mathcal{F}}}(g)$ is piecewise continuous in its domain.
(2) $\tilde{\mathcal{F}}$ is normal, i.e., there is a $g_0 \in \tilde{\mathcal{F}}$, such that $\mu_{\tilde{\mathcal{F}}}(g_0) = 1$.
(3) $\tilde{\mathcal{F}}$ is convex, i.e., $\mu_{\tilde{\mathcal{F}}}\left(\varepsilon g_1 + (1-\varepsilon) g_2\right) \geq \min\left(\mu_{\tilde{\mathcal{F}}}(g_1), \mu_{\tilde{\mathcal{F}}}(g_2)\right) \forall g_1, g_2$ in $\mathcal{U}$.

Octagonal fuzzy number (Malini and Kennedy Felbin (2013)): An octagonal fuzzy number denoted by $\tilde{\mathcal{F}}$ is defined to be the ordered quadruple $\tilde{\mathcal{F}} = \left(L_1(x), S_1(x'), S_2(x'), L_2(x)\right)$ for $x \in [0, \tau]$ and $x' \in [\tau, p]$ where

(i) $L_1(x)$ is a bounded left continuous non decreasing function over $[0, p_1]$, $0 \leq p_1 \leq \tau$.
(ii) $S_1(x')$ is a bounded left continuous non decreasing function over $[\tau, p_2]$, $\tau \leq p_2 \leq p$.
(iii) $S_2(x')$ is a bounded left continuous non increasing function over $[\tau, p_2]$, $\tau \leq p_2 \leq p$.
(iv) $L_2(x)$ is a bounded left continuous non increasing function over $[0, p_1]$, $0 \leq p_1 \leq \tau$.

If $p = 1$ then the above defined number is called a normal octagonal fuzzy number. That is, a fuzzy number $\tilde{\mathcal{F}} = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8)$ is a normal octagonal fuzzy number, where $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$ are real and its membership function $\mu_{\tilde{\mathcal{F}}}(g)$

is given by

$$\mu_{\tilde{\mathcal{F}}}(g)=\begin{cases}\tau\left(\frac{g-u_1}{u_2-u_1}\right), & u_1\le g\le u_2\\ \tau, & u_2\le g\le u_3\\ \tau+(1-\tau)\left(\frac{g-u_3}{u_4-u_3}\right), & u_3\le g\le u_4\\ 1, & u_4\le g\le u_5\\ \tau+(1-\tau)\left(\frac{u_6-g}{u_6-u_5}\right), & u_5\le g\le u_6\\ \tau, & u_6\le g\le u_7\\ \tau\left(\frac{u_8-g}{u_8-u_7}\right), & u_7\le g\le u_8\\ 0, & \text{otherwise}\end{cases}$$

where $\tau\in[0,1]$.

Remark: When $\tau=0$, the above octagonal fuzzy number reduces to trapezoidal fuzzy number (u_3,u_4,u_5,u_6) and if the value of τ is 1, it reduces to trapezoidal fuzzy number (u_1,u_4,u_5,u_8).

Intuitionistic fuzzy set (Atanassov (1986)): Let $\mathcal{U}$ be a finite universe of discourse then an intuitionistic fuzzy set $\tilde{\mathcal{A}}$ is defined as.

$\tilde{\mathcal{A}}=\{\langle g,\,'_{\tilde{\mathcal{A}}}(g),\psi_{\tilde{\mathcal{A}}}(g)\rangle|g\in\mathcal{U}\}$,

where $'_{\tilde{\mathcal{A}}}:\mathcal{U}\to[0,1]$ and $\psi_{\tilde{\mathcal{A}}}:\mathcal{U}\to[0,1]$ satisfying the condition $0\le\,'_{\tilde{\mathcal{A}}}(g)+\psi_{\tilde{\mathcal{A}}}(g)\le 1$. The number $'_{\tilde{\mathcal{A}}}(g)$ represents the degree of membership of $g\in\mathcal{U}$ in $\tilde{\mathcal{A}}$ and the number $\psi_{\tilde{\mathcal{A}}}(g)$ represents the degree of non-membership of $g\in\mathcal{U}$ in $\tilde{\mathcal{A}}$. In addition with this, the hesitance degree of $g\in\mathcal{U}$ in $\tilde{\mathcal{A}}$ is denoted as $\varepsilon_{\tilde{\mathcal{A}}}(g)$ and is given as $\varepsilon_{\tilde{\mathcal{A}}}(g)=1-\,'_{\tilde{\mathcal{A}}}(g)-\psi_{\tilde{\mathcal{A}}}(g)$. For convenience, the intuitionistic fuzzy number is considered as $\tilde{\mathcal{A}}=\left('_{\tilde{\mathcal{A}}}(g),\psi_{\tilde{\mathcal{A}}}(g)\right)$.

Octagonal intuitionistic fuzzy number (Menaga (2017)): A fuzzy number $\tilde{\mathcal{F}}=(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8),\left(u_1^1,u_2^1,u_3^1,u_4^1,u_5^1,u_6^1,u_7^1,u_8^1\right)$ is an octagonal intuitionistic fuzzy number where $u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8,u_1^1,u_2^1,u_3^1,u_4^1,u_5^1,u_6^1,u_7^1,u_8^1$ are real and its membership function $\mu_{\tilde{\mathcal{F}}}(g)$ and non-membership function $\vartheta_{\tilde{\mathcal{F}}}(g)$ are given as follows:

$$\mu_{\tilde{\mathcal{F}}}(g)=\begin{cases}\tau\left(\frac{g-u_1}{u_2-u_1}\right), & u_1\le g\le u_2\\ \tau, & u_2\le g\le u_3\\ \tau+(1-\tau)\left(\frac{g-u_3}{u_4-u_3}\right), & u_3\le g\le u_4\\ 1, & u_4\le g\le u_5\\ \tau+(1-\tau)\left(\frac{u_6-g}{u_6-u_5}\right), & u_5\le g\le u_6\\ \tau, & u_6\le g\le u_7\\ \tau\left(\frac{u_8-g}{u_8-u_7}\right), & u_7\le g\le u_8\\ 0, & \text{otherwise}\end{cases}$$

and

$$\vartheta_{\tilde{\mathcal{F}}}(g) = \begin{cases} \tau + (1-\tau)\left(\frac{u_2^{'}-g}{u_2^{'}-u_1^{'}}\right), & u_1^{'} \le g \le u_2^{'} \\ \tau, & u_2^{'} \le g \le u_3^{'} \\ \tau\left(\frac{u_4^{'}-g}{u_4^{'}-u_3^{'}}\right), & u_3^{'} \le g \le u_4^{'} \\ 0, & u_4^{'} \le g \le u_5^{'} \\ \tau\left(\frac{g-u_6^{'}}{u_6^{'}-u_5^{'}}\right), & u_5^{'} \le g \le u_6^{'} \\ \tau, & u_6^{'} \le g \le u_7^{'} \\ \tau + (1-\tau)\left(\frac{g-u_8^{'}}{u_8^{'}-u_7^{'}}\right), & u_7^{'} \le g \le u_8^{'} \\ 1, & \text{otherwise} \end{cases}$$

2.1 Some basic definitions of neutrosophic set

Neutrosophic set (Smarandache (1998)): Let $\mathcal{U}$ be a space of points (objects) with generic element$g \in \mathcal{U}$. Then a neutrosophic set $\tilde{\mathcal{A}}$ in $\mathcal{U}$ is characterized by a truth membership function$\mathrm{T}_{\tilde{\mathcal{A}}}$, an indeterminacy membership function $\mathrm{I}_{\tilde{\mathcal{A}}}$ and a falsity membership function$\mathrm{F}_{\tilde{\mathcal{A}}}$. The functions$\mathrm{T}_{\tilde{\mathcal{A}}}$, $\mathrm{I}_{\tilde{\mathcal{A}}}$ and $\mathrm{F}_{\tilde{\mathcal{A}}}$ are real standard or nonstandard subsets of $]^{-}0, 1^{+}[$ that is $\mathrm{T}_{\tilde{\mathcal{A}}} : \mathcal{U} \to]^{-}0, 1^{+}[$;$\mathrm{I}_{\tilde{\mathcal{A}}} : \mathcal{U} \to]^{-}0, 1^{+}[$;$\mathrm{F}_{\tilde{\mathcal{A}}} : \mathcal{U} \to]^{-}0, 1^{+}[$. $\mathrm{T}_{\tilde{\mathcal{A}}}(g)$, $\mathrm{I}_{\tilde{\mathcal{A}}}(g)$, and $\mathrm{F}_{\tilde{\mathcal{A}}}(g)$ satisfy the relation i.e., $0 \le \sup \mathrm{T}_{\tilde{\mathcal{A}}}(g) \le \sup \mathrm{I}_{\tilde{\mathcal{A}}}(g) \le \sup \mathrm{F}_{\tilde{\mathcal{A}}}(g) \le 3^{+}$

Fuzzy neutrosophic number (Biswas et al. (2016)): A neutrosophic set $\tilde{\mathcal{A}}_N$ in a universe of discourse $\mathcal{U}$ is defined as $\tilde{\mathcal{A}}_N = \left\{\langle \mathrm{T}_{\tilde{\mathcal{A}}_N}(g), \mathrm{I}_{\tilde{\mathcal{A}}_N}(g), \mathrm{F}_{\tilde{\mathcal{A}}_N}(g)\rangle | g \in \mathcal{U}\right\}$ where the truth membership function $\mathrm{T}_{\tilde{\mathcal{A}}}(g)$, an indeterminacy membership function $\mathrm{I}_{\tilde{\mathcal{A}}}(g)$ and a falsity membership function $\mathrm{F}_{\tilde{\mathcal{A}}}(g)$ are the real subsets of [0, 1]. Fuzzy neutrosophic number can be defined by extending a discrete set to a continuous set.

Let $\tilde{\mathcal{A}}_N$ be a fuzzy neutrosophic number in the set of real numbers R. Then its truth membership function, indeterminacy membership function and falsity membership function are, respectively, written as follows:

$$\mathrm{T}_{\tilde{\mathcal{A}}_N}(g) = \begin{Bmatrix} \mathrm{T}^{L}_{\tilde{\mathcal{A}}_N}(g) & l_1 \le g \le l_2 \\ 1 & l_2 \le g \le l_3 \\ \mathrm{T}^{U}_{\tilde{\mathcal{A}}_N}(g) & l_3 \le g \le l_4 \\ 0 & \text{otherwise} \end{Bmatrix},$$

$$\mathrm{I}_{\tilde{\mathcal{A}}_N}(\mathrm{y}) = \begin{Bmatrix} \mathrm{I}^{L}_{\tilde{\mathcal{A}}_N}(g) & m_1 \le g \le m_2 \\ 0 & m_2 \le g \le m_3 \\ \mathrm{I}^{U}_{\tilde{\mathcal{A}}_N}(g) & m_3 \le g \le m_4 \\ 1 & \text{otherwise} \end{Bmatrix},$$

$$\mathrm{F}_{\tilde{\mathcal{A}}_N}(\mathrm{y}) = \begin{Bmatrix} \mathrm{F}^{L}_{\tilde{\mathcal{A}}_N}(g) & n_1 \le g \le n_2 \\ 0 & n_2 \le g \le n_3 \\ \mathrm{F}^{U}_{\tilde{\mathcal{A}}_N}(g) & n_3 \le g \le n_4 \\ 1 & \text{otherwise} \end{Bmatrix},$$

where $0 \leq \sup T_{\tilde{\mathcal{A}}}(g) \leq \sup I_{\tilde{\mathcal{A}}}(g) \leq \sup F_{\tilde{\mathcal{A}}}(g) \leq 3$, $\forall g \in \mathcal{U}$ and for all$i = 1, 2, 3, 4; l_i, m_i, n_i \in R$, such that $l_1 \leq l_2 \leq l_3 \leq l_4; m_1 \leq m_2 \leq m_3 \leq m_4; n_1 \leq n_2 \leq n_3 \leq n_4$. Here $T^L_{\tilde{\mathcal{A}}_N}(g), I^U_{\tilde{\mathcal{A}}_N}(g), F^U_{\tilde{\mathcal{A}}_N}(g) \in [0, 1]$ are continuous monotonic increasing functions and $T^U_{\tilde{\mathcal{A}}_N}(g), I^L_{\tilde{\mathcal{A}}_N}(g), F^L_{\tilde{\mathcal{A}}_N}(g) \in [0, 1]$ are continuous monotonic increasing functions.

Single-valued neutrosophic set (Wang et al. (2010)): Let $\mathcal{U}$ be a space of points (objects) with generic element $g \in \mathcal{U}$. A single-valued neutrosophic set $\tilde{\mathcal{A}}$ in $\mathcal{U}$ is characterized by a truth membership function $T_{\tilde{\mathcal{A}}}(g)$, an indeterminacy membership function $I_{\tilde{\mathcal{A}}}(g)$ and a falsity membership function $F_{\tilde{\mathcal{A}}}(g)$ and is given by.

$$\tilde{\mathcal{A}} = \left\{ g, \langle T_{\tilde{\mathcal{A}}}(g), I_{\tilde{\mathcal{A}}}(g), F_{\tilde{\mathcal{A}}}(g) \rangle | g \in \mathcal{U} \right\}.$$

Here $T_{\tilde{\mathcal{A}}}(g), I_{\tilde{\mathcal{A}}}(g), F_{\tilde{\mathcal{A}}}(g) \in [0, 1]$ and the relation $0 \leq \sup T_{\tilde{\mathcal{A}}}(g) \leq \sup I_{\tilde{\mathcal{A}}}(g) \leq \sup F_{\tilde{\mathcal{A}}}(g) \leq 3$ holds for all $g \in \mathcal{U}$.

Octagonal neutrosophic number (Saqlain et al. (2020)) An octagonal neutrosophic number $\tilde{\mathcal{O}}_N$ in a universe of discourse Y is defined as follows:

$$\tilde{\mathcal{O}}_N = \left\langle \begin{array}{c} [(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8); \delta_1], \\ [(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8); \delta_2], \\ [(w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8); \delta_3] \end{array} \right\rangle,$$

where $\delta_1, \delta_2, \delta_3 \in [0, 1]$. Then its truth membership function$\Omega_{\tilde{\mathcal{O}}_N} : \mathbb{R} \to [0, \delta_1]$, indeterminacy membership function $\Psi_{\tilde{\mathcal{O}}_N} : \mathbb{R} \to [0, \delta_2]$ and falsity membership function $\Phi_{\tilde{\mathcal{O}}_N} : \mathbb{R} \to [0, \delta_3]$ are written as follows:

$$\Omega_{\tilde{\mathcal{O}}_N}(g) = \begin{cases} \Omega^{l1}_{\tilde{\mathcal{O}}_N}(g), & u_1 \leq g \leq u_2 \\ \Omega^{l2}_{\tilde{\mathcal{O}}_N}(g), & u_2 \leq g \leq u_3 \\ \Omega^{l3}_{\tilde{\mathcal{O}}_N}(g), & u_3 \leq g \leq u_4 \\ \delta_1, & u_4 \leq g \leq u_5 \\ \Omega^{r1}_{\tilde{\mathcal{O}}_N}(g), & u_5 \leq g \leq u_6 \\ \Omega^{r2}_{\tilde{\mathcal{O}}_N}(g), & u_6 \leq g \leq u_7 \\ \Omega^{r3}_{\tilde{\mathcal{O}}_N}(g), & u_7 \leq g \leq u_8 \\ 0, & \text{otherwise} \end{cases};$$

$$\Psi_{\tilde{\mathcal{O}}_N}(g) = \begin{cases} \Psi^{l1}_{\tilde{\mathcal{O}}_N}(g), & v_1 \leq g \leq v_2 \\ \Psi^{l2}_{\tilde{\mathcal{O}}_N}(g), & v_2 \leq g \leq v_3 \\ \Psi^{l3}_{\tilde{\mathcal{O}}_N}(g), & v_3 \leq g \leq v_4 \\ \delta_2, & v_4 \leq g \leq v_5 \\ \Psi^{r1}_{\tilde{\mathcal{O}}_N}(g), & v_5 \leq g \leq v_6 \\ \Psi^{r2}_{\tilde{\mathcal{O}}_N}(g), & v_6 \leq g \leq v_7 \\ \Psi^{r3}_{\tilde{\mathcal{O}}_N}(g), & v_7 \leq g \leq v_8 \\ 1, & \text{otherwise} \end{cases};$$

$$\Phi_{\tilde{\mathcal{O}}_N}(g) = \begin{cases} \Phi^{l1}_{\tilde{\mathcal{O}}_N}(g), & w_1 \le g \le w_2 \\ \Phi^{l2}_{\tilde{\mathcal{O}}_N}(g), & w_2 \le g \le w_3 \\ \Phi^{l3}_{\tilde{\mathcal{O}}_N}(g), & w_3 \le g \le w_4 \\ \delta_3, & w_4 \le g \le w_5 \\ \Phi^{r1}_{\tilde{\mathcal{O}}_N}(g), & w_5 \le g \le w_6 \\ \Phi^{r2}_{\tilde{\mathcal{O}}_N}(g), & w_6 \le g \le w_7 \\ \Phi^{r3}_{\tilde{\mathcal{O}}_N}(g), & w_7 \le g \le w_8 \\ 1, & \text{otherwise} \end{cases}.$$

3 De-neutrosophication of an octagonal fuzzy neutrosophic number

The crisp value of any neutrosophic number can be obtained by developing the de-neutrosophication strategy. In this paper, the de-neutrosophication of an octagonal fuzzy neutrosophic number is derived by 'removal area method', where the same method for a pentagonal neutrosophic number was developed by Chakraborty et al. (2019) (Fig. 1).

A linear octagonal fuzzy neutrosophic number is given as follows:

$$\tilde{\mathcal{O}}_{\mathcal{N}} = \left\langle \begin{matrix} (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8), \\ (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8), \\ (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8) \end{matrix} \right\rangle.$$

The above figure represents the octagonal fuzzy neutrosophic number with $\tau = 0.5$ and $p = 1$. The blue colored octagonal diagram shows the truth membership function, red colored octagonal diagram represents the indeterminacy membership function and the green lined

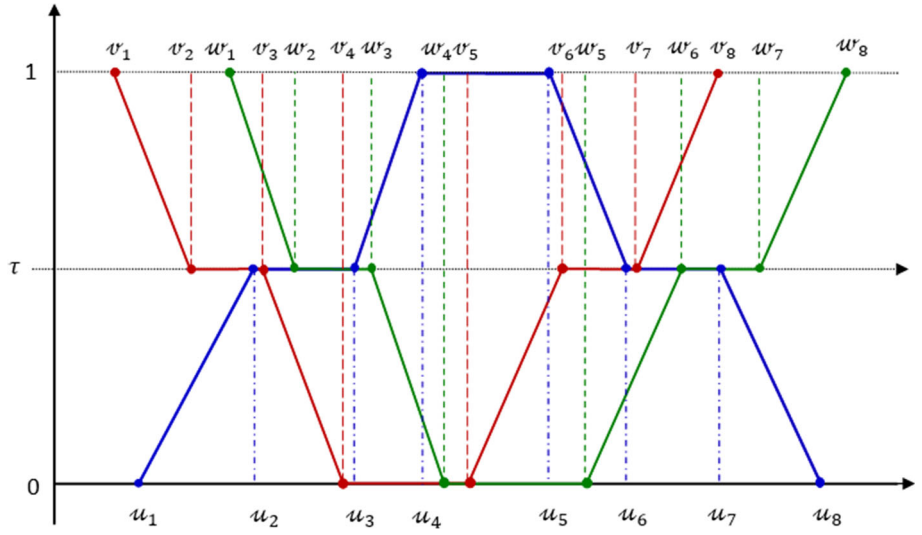


Fig. 1 Graphical representation of linear octagonal fuzzy neutrosophic number

octagonal diagram display the falsity membership function. It is noted that the value of τ lies between 0 and 1. When $\tau = 0 or 1$, this OFNN reduces to trapezoidal fuzzy neutrosophic number.

According to Chakraborty et al. (2019), the de-neutrosophication of an OFNN is given as,

$$\mathcal{O}_{\mathcal{N}}\left(\widetilde{\mathcal{D}e_{Oct}}, \ddagger\right) = \frac{\mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_1, \ddagger\right) + \mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_2, \ddagger\right) + \mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_3, \ddagger\right)}{3}$$

where $\mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_1, \ddagger\right) = \frac{\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_1, \ddagger\right) + \mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_1, \ddagger\right)}{2}$, $\mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_2, \ddagger\right) = \frac{\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_2, \ddagger\right) + \mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_2, \ddagger\right)}{2}$ and $\mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_3, \ddagger\right) = \frac{\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_3, \ddagger\right) + \mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_3, \ddagger\right)}{2}$.

Here, $\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_1, \ddagger\right)$ is the area of the left side of blue colored octagon and $\mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_1, \ddagger\right)$ is the area of the right side of the same octagon. Similarly, $\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_2, \ddagger\right)$ and $\mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_2, \ddagger\right)$ represent the left side area and the right side area of the red lined octagon. Also, $\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_3, \ddagger\right)$ and $\mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_3, \ddagger\right)$ denotes, respectively, the area of left side and the area of right side of the green lined octagon.

By setting $z = 0, \mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_1, 0\right) = \frac{\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_1, 0\right) + \mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_1, 0\right)}{2}$, $\mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_2, 0\right) = \frac{\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_2, 0\right) + \mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_2, 0\right)}{2}$ and $\mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_3, 0\right) = \frac{\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_3, 0\right) + \mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_3, 0\right)}{2}$.

Thus, $\mathcal{O}_{\mathcal{N}}\left(\tilde{\mathcal{D}}e_{Oct}, 0\right) = \frac{\mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_1, 0\right) + \mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_2, 0\right) + \mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_3, 0\right)}{3}$

Hence,

$\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_1, 0\right)$ = Shaded area of Fig. 2a $= \frac{(u_1+u_2)\tau}{2} + \frac{(u_3+u_4)(1-\tau)}{2}$,

$\mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_1, 0\right)$ = Shaded area of Fig. 2b $= \frac{(u_7+u_8)\tau}{2} + \frac{(u_5+u_6)(1-\tau)}{2}$,

$\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_2, 0\right)$ = Shaded area of Fig. 3a $= \frac{(v_1+v_2)(1-\tau)}{2} + \frac{(v_3+v_4)\tau}{2}$,

$\mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_2, 0\right)$ = Shaded area of Fig. 3b $= \frac{(v_7+v_8)(1-\tau)}{2} + \frac{(v_5+v_6)\tau}{2}$,

$\mathcal{O}^l_{\mathcal{N}}\left(\widetilde{\Delta}_3, 0\right)$ = Shaded area of Fig. 4a $= \frac{(w_1+w_2)(1-\tau)}{2} + \frac{(w_3+w_4)\tau,}{2}$

$\mathcal{O}^r_{\mathcal{N}}\left(\widetilde{\Delta}_3, 0\right)$ = Shaded area of Fig. 4b $= \frac{(w_7+w_8)(1-\tau)}{2} + \frac{(w_5+w_6)\tau}{2}$.

Therefore, $\mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_1, 0\right) = \frac{\frac{(u_1+u_2)\tau}{2} + \frac{(u_3+u_4)(1-\tau)}{2} + \frac{(u_7+u_8)\tau}{2} + \frac{(u_5+u_6)(1-\tau)}{2}}{2}$,

$\mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_2, 0\right) = \frac{\frac{(v_1+v_2)(1-\tau)}{2} + \frac{(v_3+v_4)\tau}{2} + \frac{(v_7+v_8)(1-\tau)}{2} + \frac{(v_5+v_6)\tau}{2}}{2}$ and $\mathcal{O}_{\mathcal{N}}\left(\widetilde{\Delta}_3, 0\right) = \frac{\frac{(w_1+w_2)(1-\tau)}{2} + \frac{(w_3+w_4)\tau}{2} + \frac{(w_7+w_8)(1-\tau)}{2} + \frac{(w_5+w_6)\tau}{2}}{2}$.

Thus, the de-neutrosophication of an octagonal fuzzy neutrosophic number is obtained as.

$$\mathcal{O}_{\mathcal{N}}\left(\widetilde{\mathcal{D}e_{Oct}}, 0\right) = \frac{(u_1+u_2+u_7+u_8+v_3+v_4+v_5+v_6+w_3+w_4+w_5+w_6)\tau + (u_3+u_4+u_5+u_6+v_1+v_2+v_7+v_8+w_1+w_2+w_7+w_8)(1-\tau)}{12}. \quad (1)$$

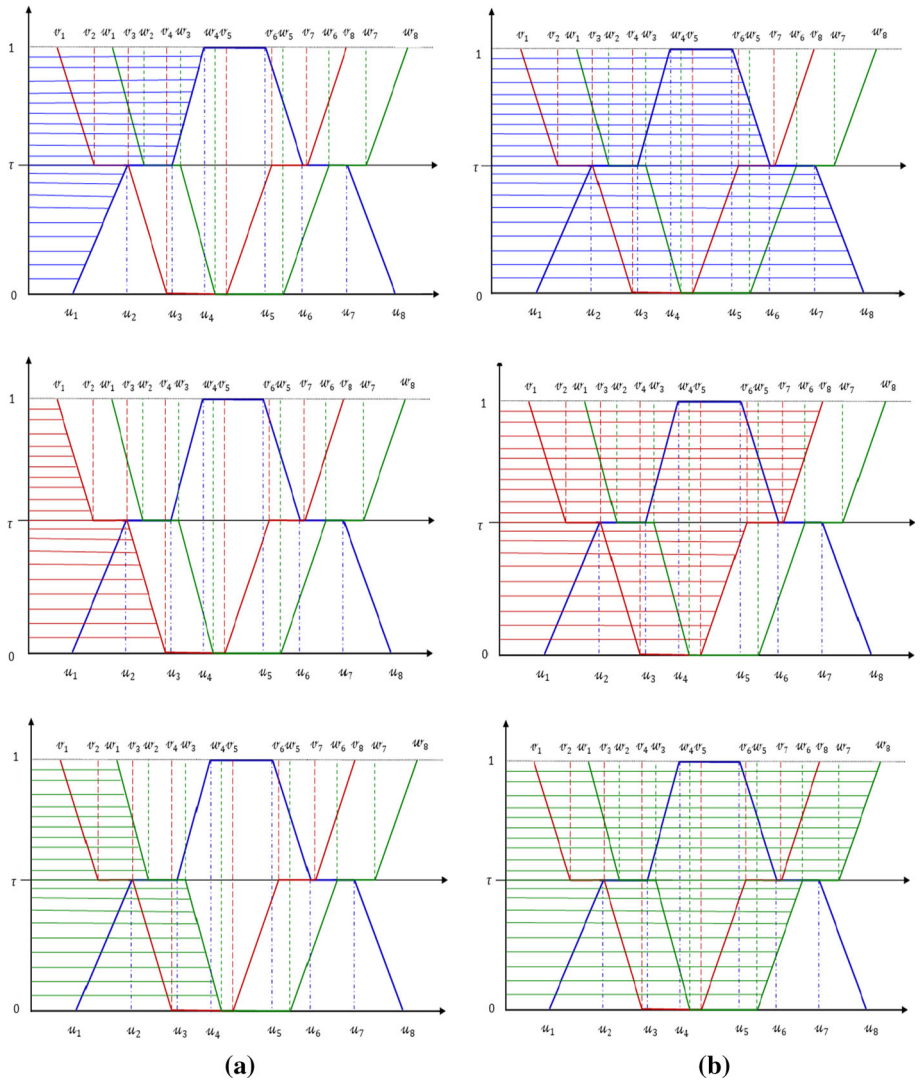


Fig. 2 Diagrammatic representation of removal area method of OFNN

4 Model development

Notation:

$\mathcal{D}$	Demand rate, where $\mathcal{D} = \sum_{k\in\mathcal{R}} \mathcal{D}_k$
$\mathcal{R}$	Number of consigner
$\mathcal{V}$	Set of containers, where $\mathcal{V} = \pi \sum_{k\in\mathcal{R}} \mathcal{D}_k$
π	Proportion of containers returned without misplacement, $0 \leq \pi \leq 1$
$\tilde{\pi}$	Neutrosophic proportion of containers returned without misplacement
$\mathcal{D}_k$	Demand rate for kth consigner
$\tilde{\mathcal{D}}_k$	Neutrosophic demand rate for kth consigner

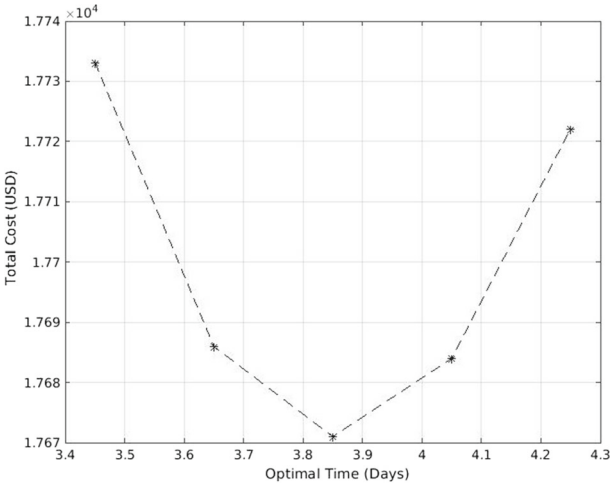
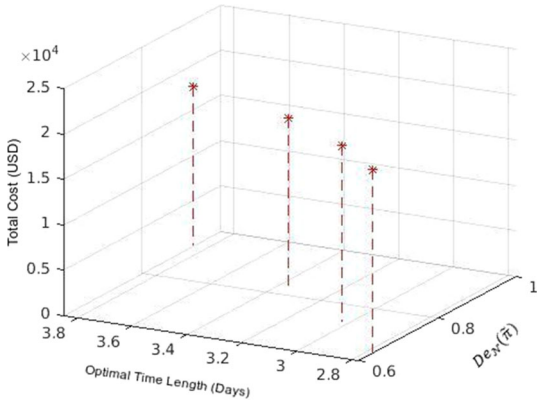


Fig. 3 Convexity of the inventory model

Fig. 4 Effect of neutrosophic proportion of returned units on T^* and TC^*



$\mathcal{M}$	Fixed maintenance and repair (M&R) cost
m	Variable M&R cost
d'	Depreciation cost per container/day
l	Lift-On Lift-Off (LOLO) charge per container
c'	Penalty charge for misplaced container
n_j	Number of storage days for container j, $j \in \mathcal{V}$
T	Cycle time
$\hbar_r$	Unit storage charge (where r varies from 1 to 4 based on the total number of storage days)
$\mathcal{S}_j, \mathcal{S}_j^{'}$	Decision variables depend upon the size of the container (where $\mathcal{S}_j = 1 \text{or} s_1 \text{and} \mathcal{S}_j^{'} = 1 \text{or} s_2$; for all $s_1, s_2 \in \mathbb{R}^+$)
$C_{j,k}$	Decision variable for the allocation of j th container to k th consigner
$\mathcal{S}_{j,\hbar_r}$	Storage charge for container j per day

$$\mathcal{S}_{j,\hbar_r} = \hbar_r = \begin{cases} \hbar_1, & n_j \leq 7 \\ \hbar_2, & 7 < n_j \leq 14 \\ \hbar_3, & 14 < n_j \leq 21 \\ \hbar_4, & n_j < 21 \end{cases} \tag{2}$$

Decision variables:

$$C_{j,k} = \begin{cases} 1, & \text{if container } j \text{ is allocated to consigner } k \\ 0, & \text{otherwise} \end{cases}$$

The type of containers discussed in this study is both 20 ft as well as 40 ft containers. Since, for particular cost such as handling cost, depreciation cost, storage cost etc. the charges assigned for 20 ft and 40 ft containers varies, the decision variables $\mathcal{S}_j$ and $\mathcal{S}_j^{'}$ are considered. The decision variables $\mathcal{S}_j$ and $\mathcal{S}_j^{'}$ represents the proportion assigned for the containers and its value depends upon the size of the containers, say, TEU (Twenty-foot Equivalent Unit).

$$\mathcal{S}_j = \begin{cases} 1, & \text{if TEU of container } j \text{ is 20} \\ s_1, & \text{if TEU of container } j \text{ is 40} \end{cases}$$

$$\mathcal{S}_j^{'} = \begin{cases} 1, & \text{if TEU of container } j \text{ is 20} \\ s_2, & \text{if TEU of container } j \text{ is 40} \end{cases}$$

4.1 Formulation of container inventory model

For import and export trading purpose, the CMO used to supply the empty container to multiple consigners ($\mathcal{R}$) at some price during the time (T) at the rate of demand $\mathcal{D} = \sum_{k\in\mathcal{R}} \mathcal{D}_k$, where $\mathcal{D}_k$ is the constant demand rate for kth consigner. In this study the shrinkage of container is considered, that is a proportion of container say, $1-\pi$ is misplaced because of some clerical and receiving errors. According to the hypothesis, $1-\pi$ fraction of container is misplaced and the CMO supplied only $\mathcal{V}$, ($\mathcal{V} = \pi\mathcal{D}$) units to their consigners per cycle. While managing the container inventory, the CMO should consume some expenses for management of empty containers. The costs involving in the management of container inventory in this model per cycle are maintenance and repair cost, depreciation cost, lift-on lift-off charge, penalty cost for misplaced units and storage cost which are discussed as follows.

Maintenance and repair (M & R) cost is the cost spent for the purpose of surveying, cleaning and repairing process of the containers and is given as

$$\text{MRC} = \frac{\mathcal{M}}{\text{T}} + \sum_{k\in\mathcal{R}} \pi\mathcal{D}_k m. \tag{3}$$

The value of an asset or the fixed product value is reduced over time due to the usage can be stated as depreciation cost. Each container is depreciated with respect to the time period and the cost of the depreciated value of reusable containers in this model per cycle is written as

$$\text{DC} = \sum_{k\in\mathcal{R}} \sum_{j\in\mathcal{V}} \pi\mathcal{S}_j\mathcal{D}_k d^{'}\text{T}. \tag{4}$$

Lift-on lift-off charge is the charge for loading and unloading the empty container of the trailer which is given as

$$\text{LOLO} = \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}} \pi \mathcal{S}_j^{'} \mathcal{D}_k l. \tag{5}$$

In this model, the revenue reduced due to the misplacement of containers is considered as the penalty cost. Thus, the penalty cost for misplaced containers during the cycle time T is derived as

$$\text{PC} = \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}} (1-\pi)\mathcal{S}_j \mathcal{D}_k c^{'} \text{T}. \tag{6}$$

The storage charge is the major expense for every CMO and the holding charge is based on the type and the size of the container. The storage cost is charged by the empty container storage yard according to the size of the container, type of the container and the number of storage days. To minimize the storage cost, the CMO should utilizes some beneficial storage systems like free days, free pool, slab storage system that are offered by the container storage yard. In this research, slab storage system is considered and the storage cost under slab system is derived as per the equation for $\mathcal{S}_{j,\hbar_r}$ given in (2). The derivation part of storage cost is provided in Appendix.

$$\begin{aligned}
\text{HC} = &\sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/n_j\le 7} n_j \mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_1} + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/7<n_j\le 14} 7\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_1} \\
&+ \left(\left(n_j - 7\right)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_2}\right) + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/14<n_j\le 21} 7\mathcal{S}_j C_{j,k}\left(\mathcal{S}_{j,\hbar_1} + \mathcal{S}_{j,\hbar_2}\right) \\
&+ \left(\left(n_j - 14\right)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_3}\right) + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/n_j>21} 7\mathcal{S}_j C_{j,k}\left(\mathcal{S}_{j,\hbar_1} + \mathcal{S}_{j,\hbar_2} + \mathcal{S}_{j,\hbar_3}\right) \\
&+ \left(\left(n_j - 21\right)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_4}\right).
\end{aligned} \tag{7}$$

The total cost is the sum of M & R cost, depreciation cost, LOLO cost, penalty cost and storage charge. The minimizing strategy of the total cost subject to the constraints under crisp model is obtained as

$$\begin{aligned}
\text{minTC} = &\frac{\mathcal{M}}{\text{T}} + \sum_{k\in\mathcal{R}} \pi \mathcal{D}_k m + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}} \pi \mathcal{S}_j \mathcal{D}_k d^{'} \text{T} + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}} \pi \mathcal{S}_j^{'} \mathcal{D}_k l \\
&+ \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}} (1-\pi)\mathcal{S}_j \mathcal{D}_k c^{'} \text{T} + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/n_j\le 7} n_j \mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_1} \\
&+ \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/7<n_j\le 14} 7\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_1} + \left(\left(n_j - 7\right)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_2}\right) \\
&+ \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/14<n_j\le 21} 7\mathcal{S}_j C_{j,k}\left(\mathcal{S}_{j,\hbar_1} + \mathcal{S}_{j,\hbar_2}\right) + \left(\left(n_j - 14\right)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_3}\right) \\
&+ \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/n_j>21} 7\mathcal{S}_j C_{j,k}\left(\mathcal{S}_{j,\hbar_1} + \mathcal{S}_{j,\hbar_2} + \mathcal{S}_{j,\hbar_3}\right) + \left(\left(n_j - 21\right)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_4}\right).
\end{aligned} \tag{8}$$

Subject to

$$\mathcal{V} < \sum_{k\in\mathcal{R}} \mathcal{D}_k, \tag{9}$$

$$n_j \leq \mathrm{T}, \quad \forall j \in \mathcal{V}, \tag{10}$$

$$\mathcal{S}_j, \mathcal{S}_j^{'} = 1, \quad \text{if TEU of container } j \text{ is } 20 \forall j \in \mathcal{V}, \tag{11}$$

$$\left.\begin{matrix} \mathcal{S}_j = s_1 = 2 \\ \mathcal{S}_j^{'} = s_2 = 1.5 \end{matrix}\right\}, \quad \text{if TEU of container } j \text{ is } 40, \forall j \in \mathcal{V}. \tag{12}$$

Note: Generally, most of the container management organization fixed the depreciation cost, storage cost etc. for 40 ft container is twice the cost fixed for 20 ft containers. Whereas, they fix the handling charges for 40 ft container is 1.5 times the cost fixed for 20 ft containers. Thus, the values of s_1 and s_2 in Eq. (12) are considered in such a way.

4.2 Container inventory model in octagonal neutrosophic arena

Since, the proportion of the returned units without misplacement and the container demand of each consigner are uncertain and unpredictable, the proposed model scrutinizes under neutrosophic environment. So that, the proposed container inventory model under neutrosophic arena is framed by presuming the proportion of returned containers without misplaced and the demand of each consigner as the octagonal fuzzy neutrosophic numbers. Thus, the octagonal neutrosophic proportion of containers returned without misplacement is denoted by $\tilde{\pi}$ and is given as $\tilde{\pi} = \left\langle \begin{matrix} (\pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}, \pi_{16}, \pi_{17}, \pi_{18}), \\ (\pi_{21}, \pi_{22}, \pi_{23}, \pi_{24}, \pi_{25}, \pi_{26}, \pi_{27}, \pi_{28}), \\ (\pi_{31}, \pi_{32}, \pi_{33}, \pi_{34}, \pi_{35}, \pi_{36}, \pi_{37}, \pi_{38}) \end{matrix} \right\rangle$

Also, the octagonal neutrosophic demand of kth consigner $(\tilde{\mathcal{D}_k})$ is written as $\tilde{\mathcal{D}_k} = \left\langle \begin{matrix} (\mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{13}, \mathcal{D}_{14}, \mathcal{D}_{15}, \mathcal{D}_{16}, \mathcal{D}_{17}, \mathcal{D}_{18}), \\ (\mathcal{D}_{21}, \mathcal{D}_{22}, \mathcal{D}_{23}, \mathcal{D}_{24}, \mathcal{D}_{25}, \mathcal{D}_{26}, \mathcal{D}_{27}, \mathcal{D}_{28}), \\ (\mathcal{D}_{31}, \mathcal{D}_{32}, \mathcal{D}_{33}, \mathcal{D}_{34}, \mathcal{D}_{35}, \mathcal{D}_{36}, \mathcal{D}_{37}, \mathcal{D}_{38}) \end{matrix} \right\rangle$.

The de-neutrosophication of neutrosophic proportion of returned containers and the neutrosophic demand of kth consigner is obtained using Eq. (1).

$$De_{\mathcal{N}}(\tilde{\pi}) = \frac{(\pi_{11}+\pi_{12}+\pi_{17}+\pi_{18}+\pi_{23}+\pi_{24}+\pi_{25}+\pi_{26}+\pi_{33}+\pi_{34}+\pi_{35}+\pi_{36})\tau+(\pi_{13}+\pi_{14}+\pi_{15}+\pi_{16}+\pi_{21}+\pi_{22}+\pi_{27}+\pi_{28}+\pi_{31}+\pi_{32}+\pi_{37}+\pi_{38})(1-\tau)}{12}$$

and

$$De_{\mathcal{N}}(\widetilde{\mathcal{D}_k}) = \frac{(\mathcal{D}_{11}+\mathcal{D}_{12}+\mathcal{D}_{17}+\mathcal{D}_{18}+\mathcal{D}_{23}+\mathcal{D}_{24}+\mathcal{D}_{25}+\mathcal{D}_{26}+\mathcal{D}_{33}+\mathcal{D}_{34}+\mathcal{D}_{35}+\mathcal{D}_{36})\tau+(\mathcal{D}_{13}+\mathcal{D}_{14}+\mathcal{D}_{15}+\mathcal{D}_{16}+\mathcal{D}_{21}+\mathcal{D}_{22}+\mathcal{D}_{27}+\mathcal{D}_{28}+\mathcal{D}_{31}+\mathcal{D}_{32}+\mathcal{D}_{37}+\mathcal{D}_{38})(1-\tau)}{12}$$

Thus, the minimizing strategy of total cost subject to the constraints under octagonal neutrosophic view is derived as follows:

$$\begin{aligned} \min T\tilde{C}_{\mathcal{N}}(\mathrm{T}) = {} & \frac{\mathcal{M}}{\mathrm{T}} + \sum_{k\in\mathcal{R}} \tilde{\pi}\tilde{\mathcal{D}}_k m + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}} \tilde{\pi}\mathcal{S}_j\tilde{\mathcal{D}}_k d^{'}\mathrm{T} + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}} \tilde{\pi}\mathcal{S}_j^{'}\tilde{\mathcal{D}}_k l \\ & + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}} (1-\tilde{\pi})\mathcal{S}_j\tilde{\mathcal{D}}_k c^{'}\mathrm{T} + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/n_j\leq 7} n_j\mathcal{S}_j C_{j,k}\mathcal{S}_{j,\hbar_1} \\ & + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/7<n_j\leq 14} 7\mathcal{S}_j C_{j,k}\mathcal{S}_{j,\hbar_1} + \left((n_j-7)\mathcal{S}_j C_{j,k}\mathcal{S}_{j,\hbar_2}\right) \\ & + \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/14<n_j\leq 21} 7\mathcal{S}_j C_{j,k}\left(\mathcal{S}_{j,\hbar_1}+\mathcal{S}_{j,\hbar_2}\right) + \left((n_j-14)\mathcal{S}_j C_{j,k}\mathcal{S}_{j,\hbar_3}\right) \end{aligned}$$

$$+\sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}/n_j>21} 7\mathcal{S}_j C_{j,k}\left(\mathcal{S}_{j,\hbar_1}+\mathcal{S}_{j,\hbar_2}+\mathcal{S}_{j,\hbar_3}\right)+\left((n_j-21)\mathcal{S}_j C_{j,k}\mathcal{S}_{j,\hbar_4}\right). \tag{13}$$

Subject to

$$\mathcal{V}<\sum_{k\in\mathcal{R}}\tilde{\mathcal{D}_k}, \tag{14}$$

$$n_j\leq \mathrm{T}, \forall j\in\mathcal{V}, \tag{15}$$

$$\mathcal{S}_j, \mathcal{S}_j^{'}=1, if\, TEU of\, container\, j\, is 20, \forall j\in\mathcal{V}, \tag{16}$$

$$\left.\begin{matrix}\mathcal{S}_j=s_1=2\\ \mathcal{S}_j^{'}=s_2=1.5\end{matrix}\right\}, if\, TEU of\, container\, j\, is 40, \forall j\in\mathcal{V}. \tag{17}$$

4.3 Convexity

On observing the above figure, it is clear that the total cost of the proposed model is strictly convex and the optimal cycle length is obtained by equating the first derivative of Eq. (8) with respect to T to zero. That is,

$$\frac{\partial TC}{\partial \mathrm{T}}=\frac{-\mathcal{M}}{\mathrm{T}^2}+\sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}}\pi\mathcal{S}_j\mathcal{D}_k d'+\sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}}(1-\pi)\mathcal{S}_j\mathcal{D}_k c'=0. \tag{18}$$

$$T^*=\sqrt{\frac{\mathcal{M}}{\sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}}\pi\mathcal{S}_j\mathcal{D}_k d'+\sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}}(1-\pi)\mathcal{S}_j\mathcal{D}_k c'}} \tag{19}$$

Similarly, by repeating the same procedure to Eq. (13), the optimal cycle length of neutrosophic inventory model is obtained as:

$$T^*=\sqrt{\frac{\mathcal{M}}{\sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}}\tilde{\pi}\mathcal{S}_j\tilde{\mathcal{D}}_k d'+\sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{V}}(1-\tilde{\pi})\mathcal{S}_j\tilde{\mathcal{D}}_k c'}} \tag{20}$$

4.4 Numerical computation

In reality, the storage charge is one of the major costs in the field of container inventory management and also the cost of holding a container is usually based on its type as well the size. The various types of the containers are general purpose container (GP), high cube, open top, flat rack, platform, reefer, and ISO tank with different size say 20 feet or 40 feet. The empty container storage yard is fixing the storage price according to the size of the container, type of the containers and the number of storage days. To minimize the storage cost, the container management companies should utilize some beneficial storage systems like free days, free pool, slab storage system, etc. offered by the container storage yard. In accordance with this, the storage option in the present model is treated as slab storage system which is adopted from Gocen et al. (2019) and is given as follows (Table 1).

For simple understanding, there are three consigners are considered in the proposed model (ie. the number of consigners $k=1, 2, 3$). The proportion of returned unit is adopted from (Cobb 2016) and some values of the parameters like LOLO, depreciation cost are analyzed and collected from the leading shipping companies like OOCL, COSCO, WAN HAI, etc. in Chennai branch.

Table 1 Variation of storage cost under slab system

Storage cost/day ($)	Storage period (days)
2	0–7
2.5	8–14
3	15–21
4	Above 21

The values of parameter for the proposed crisp inventory model are $\pi = 0.9287$, $\sum_{k\in\mathcal{R}} \mathcal{D}_k = 752; k = 1, 2, 3, d^{'} = \$1/\text{unit/day}, c^{'} = \$5/\text{unit/day}, l = \$2/\text{unit}, \hbar_1 = \$2/\text{day}, \hbar_2 = \$2.5/\text{day}, \hbar_3 = \$3/\text{day}, \hbar_4 = \$4/\text{day}$. The optimal solutions are $TC^* = \$17,685$ and $T^* = 3.8425$.

The values of parameter for OFNN container inventory model are same as in the crisp model except the proportion of returned units and the demand rate. The octagonal neutrosophic proportion of retuned container and the octagonal neutrosophic demand rate are given as $\tilde{\pi} = \left\langle \begin{matrix} (0.90, 0.925, 0.93, 0.945, 0.95, 0.965, 0.98, 0.99), \\ (0.86, 0.88, 0.90, 0.92, 0.94, 0.96, 0.98, 1.0), \\ (0.825, 0.85, 0.875, 0.90, 0.925, 0.95, 0.97, 1.0) \end{matrix} \right\rangle$,

$\sum_{k\in\mathcal{R}} \tilde{\mathcal{D}}_k = \tilde{\mathcal{D}}_1 + \tilde{\mathcal{D}}_2 + \tilde{\mathcal{D}}_3$, where $\tilde{\mathcal{D}}_1 = \left\langle \begin{matrix} (164, 172, 180, 190, 200, 212, 220, 230), \\ (125, 150, 175, 200, 225, 250, 275, 300), \\ (176, 180, 185, 190, 195, 200, 210, 220) \end{matrix} \right\rangle$;

$\tilde{\mathcal{D}}_2 = \left\langle \begin{matrix} (275, 280, 285, 290, 295, 300, 305, 310), \\ (290, 294, 297, 302, 305, 315, 321, 328), \\ (248, 260, 280, 295, 310, 325, 340, 350) \end{matrix} \right\rangle$ and

$\tilde{\mathcal{D}}_3 = \left\langle \begin{matrix} (230, 238, 242, 246, 255, 260, 270, 280), \\ (184, 200, 230, 245, 260, 275, 312, 322), \\ (214, 223, 231, 243, 250, 262, 271, 281) \end{matrix} \right\rangle$.

Thus, the optimal solutions of neutrosophic container inventory model are $TC^* = \$17,671$ and $T^* = 3.8503$. It is clearly observed that the total cost under neutrosophic arena is less than that under crisp inventory model.

4.5 Sensitivity analysis

The Table 2 shows that, when OFNN of return proportion reduces then the optimal total increases whereas the optimal time length decreases which is displayed in Fig. 4.

It is noted that from the Fig. 5a and b, the penalty charge for misplaced containers increases the optimal time duration reduces but the optimal total cost raises. Figure 5c indicates that the increase of LOLO charge leads to increase in total cost.

The variation of the number of 20 feet and 40 feet containers in total demand is discussed as follows. The consigners demand includes both 20 ft as well as the 40 ft containers, increase of the number 20 ft container leads to decrease of 40 ft container and vise-versa. Due to more demand of 20 ft container in local market, the return of 20 ft containers by the consignee takes more time when compare to the 40 ft container. And since, the cost for maintaining 20 ft container is less than that of the 40 ft container. Hence, without loss of generality, when the number of 20 ft containers increases the number of 40 ft containers decreases then the

Table 2 Impact of neutrosophic return rate on optimal solution

$\tilde{\pi}$	$De_{\mathcal{N}}(\tilde{\pi})$	T^*	TC^* (USD)
$\left\langle \begin{matrix}(0.90, 0.925, 0.93, 0.945, 0.95, 0.965, 0.98, 0.99),\\ (0.86, 0.88, 0.90, 0.92, 0.94, 0.96, 0.98, 1.0),\\ (0.825, 0.85, 0.875, 0.90, 0.925, 0.95, 0.97, 1.0)\end{matrix} \right\rangle$	0.93	3.8503	17,671
$\left\langle \begin{matrix}(0.8, 0.825, 0.8, 0.845, 0.85, 0.865, 0.88, 0.89),\\ (0.76, 0.78, 0.80, 0.82, 0.84, 0.86, 0.88, 0.90),\\ (0.725, 0.75, 0.775, 0.80, 0.825, 0.85, 0.87, 0.90)\end{matrix} \right\rangle$	0.83	3.3608	18,672
$\left\langle \begin{matrix}(0.7, 0.725, 0.73, 0.745, 0.75, 0.765, 0.78, 0.79),\\ (0.66, 0.68, 0.70, 0.72, 0.74, 0.76, 0.78, 0.80),\\ (0.625, 0.65, 0.675, 0.70, 0.725, 0.75, 0.77, 0.80)\end{matrix} \right\rangle$	0.73	3.0204	19,502
$\left\langle \begin{matrix}(0.6, 0.625, 0.63, 0.645, .065, 0.665, 0.68, 0.69),\\ (0.56, 0.58, 0.60, .062, 0.64, 0.66, 0.68, 0.70),\\ (0.525, 0.55, 0.575, 0.60, 0.625, 0.65, 0.67, 0.70)\end{matrix} \right\rangle$	0.6342	2.7754	20,180

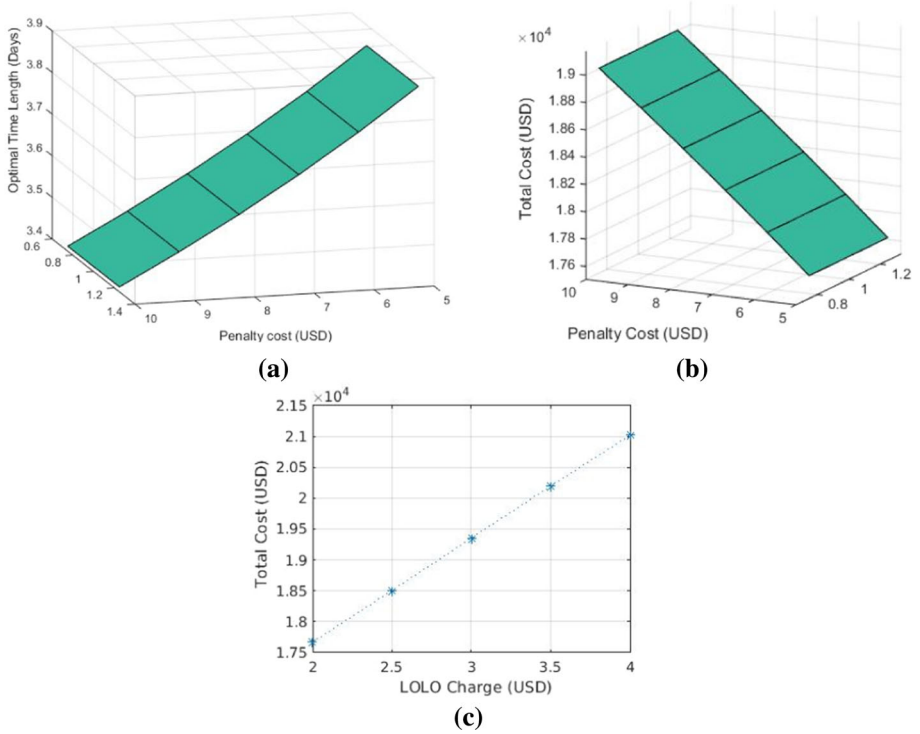


Fig. 5 Effect of variation of c', l, on optimal solution

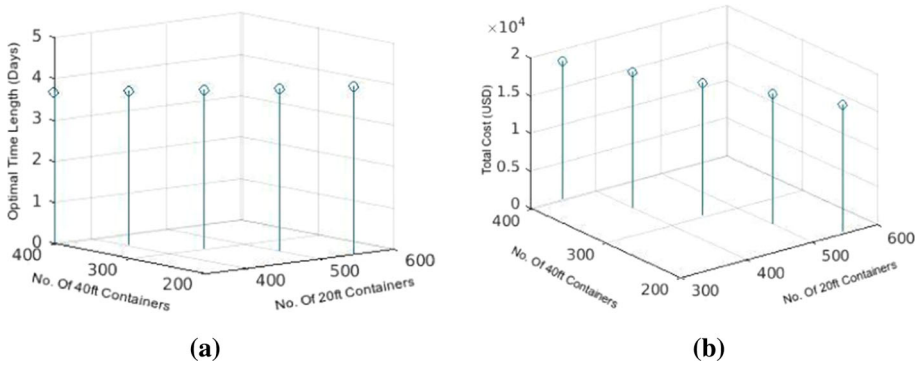


Fig. 6 Analysis of the number of 20 ft & 40 ft containers on TC^*

optimal time period increases slightly but the total cost reduces which is shown in Fig. 6a and b.

5 Conclusion

This research examined and enforced the notion of octagonal fuzzy neutrosophic number to the container inventory model. The current paper studied the container misplacement issue confronted by the container management company with multiple consigners under neutrosophic arena. The de-neutrosophication of OFNN is designed using the removal area method. Initially, the container inventory model under crisp view is developed then the proposed model is framed under neutrosophic situation by presuming the proportion of returned units and the demand of each consigner as OFNN. By knowing the situation of container misplacement, CMO is not necessarily waiting for the returns of the containers, which leads the CMO to decide to rectify the deficit issues as early as possible. The result obtained under the neutrosophic arena is more approximate for the uncertain return proportion and the unpredictable demand. Since the empty yard acquires a low charge for initial slab days under the slab storage option, the slab storage system utilized in this study helps the CMO to abate the container storage cost. This study also suggests the container management organization utilize the slab storage system offered by the empty yards under First In, First Out (FIFO) strategy to reduce the storage charge. The numerical computation demonstrates that the model under neutrosophic arena is comparatively better than the crisp model. The variation of the neutrosophic proportion of returned units, penalty cost for misplaced containers, LOLO charge and its effect are discussed in sensitivity analysis. Finally, the variation of number of 20ft containers and 40ft containers on consigners demand is analyzed and its impact to optimal solution is discussed.

Appendix

The storage cost under slab system is derived as per the equation for $\mathcal{S}_{j\hbar_r}$ is given in (2). Storage charges for container j per day under four categories are written as follows.

When the number of storage days $n_j \leq 7$, the storage cost is given as

$$\mathrm{HC}_1 = \sum_{k\in\mathcal{R}} \sum_{j\in\mathcal{V}/n_j\leq 7} n_j \mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_1}$$

When the number of storage days $7 < n_j \leq 14$, the storage cost is written as

$$\mathrm{HC}_2 = \sum_{k\in\mathcal{R}} \sum_{j\in\mathcal{V}/7<n_j\leq 14} 7\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_1} + \big((n_j - 7)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_2}\big)$$

When the number of storage days $14 < n_j \leq 21$, the storage cost is derived as

$$\begin{aligned}\mathrm{HC}_3 &= \sum_{k\in\mathcal{R}} \sum_{j\in\frac{\mathcal{V}}{14}<n_j\leq 21} \big(7\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_1}\big) + \big(7\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_2}\big) + \big((n_j - 14)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_3}\big) \\ &= \sum_{k\in\mathcal{R}} \sum_{j\in\mathcal{V}/14<n_j\leq 21} 7\mathcal{S}_j C_{j,k}\big(\mathcal{S}_{j,\hbar_1} + \mathcal{S}_{j,\hbar_2}\big) + \big((n_j - 14)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_3}\big).\end{aligned}$$

When the number of storage days $n_j > 21$, the storage cost is given as

$$\begin{aligned}\mathrm{HC}_4 &= \sum_{k\in\mathcal{R}} \sum_{j\in\frac{\mathcal{V}}{14}<n_j\leq 21} \big(7\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_1}\big) + \big(7\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_2}\big) \\ &\quad + \big(7\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_3}\big) + \big((n_j - 21)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_4}\big) \\ &= \sum_{k\in\mathcal{R}} \sum_{j\in\mathcal{V}/n_j>21} 7\mathcal{S}_j C_{j,k}\big(\mathcal{S}_{j,\hbar_1} + \mathcal{S}_{j,\hbar_2} + \mathcal{S}_{j,\hbar_3}\big) + \big((n_j - 21)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_4}\big).\end{aligned}$$

Thus, the storage cost for number of days $n_j, \forall j \in \mathcal{V}$ is obtained as

$$\begin{aligned}\mathrm{HC} = \mathrm{HC}_1 + \mathrm{HC}_2 + \mathrm{HC}_3 + \mathrm{HC}_4 &= \sum_{k\in\mathcal{R}} \sum_{j\in\mathcal{V}/n_j\leq 7} n_j \mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_1} \\ &+ \sum_{k\in\mathcal{R}} \sum_{j\in\mathcal{V}/7<n_j\leq 14} 7\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_1} + \big((n_j - 7)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_2}\big) \\ &+ \sum_{k\in\mathcal{R}} \sum_{j\in\mathcal{V}/14<n_j\leq 21} 7\mathcal{S}_j C_{j,k}\big(\mathcal{S}_{j,\hbar_1} + \mathcal{S}_{j,\hbar_2}\big) + \big((n_j - 14)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_3}\big) \\ &+ \sum_{k\in\mathcal{R}} \sum_{j\in\mathcal{V}/n_j>21} 7\mathcal{S}_j C_{j,k}\big(\mathcal{S}_{j,\hbar_1} + \mathcal{S}_{j,\hbar_2} + \mathcal{S}_{j,\hbar_3}\big) + \big((n_j - 21)\mathcal{S}_j C_{j,k} \mathcal{S}_{j,\hbar_4}\big).\end{aligned}$$

References

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20:87–96

Biswas P, Pramanik S, Giri BC (2016) Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making. Neutrosophic Sets Syst 12:20–40

Broumi S, Lathamaheswari M, Tan R, Nagarajan D, Mohamed T, Smarandache F, Bakali A (2020) A new distance measure for trapezoidal fuzzy neutrosophic numbers based on the centroids. Neutrosophic Sets Syst 35:478–502

Camdereli AZ, Swaminathan JM (2010) Misplaced inventory and radio-frequency identification (RFID) technology: information and coordination. Prod Operat Manage 19(1):1–18

Chakraborty A, Mondal S, Broumi S (2019) De-neutrosophication technique of pentagonal neutrosophic number and application in minimal spanning tree. Neutrosophic Sets Syst 29:1–18

Cobb BR (2016) Inventory control for returnable transport items in a closed-loop supply chain. Transp Res Part E 86:53–68
Freen G, Kousar S, Khalil S, Imran M (2020) Multi-objective non-linear four-valued refined neutrosophic optimization. Comput Appl Math 39:35. https://doi.org/10.1007/s40314-019-1012-4
Göçen MY, Çağlar Ö, Ercan E, Kizilay D (2019) Optimization of costs in empty container repositioning. Proc Int Sympos Prod Res 2019:732–747. https://doi.org/10.1007/978-3-030-31343-2_62
Hosseini A, Sahlin T (2019) An optimization model for management of empty containers in distribution network of a logistics company under uncertainity. J Ind Eng Int 15:585–602
Kim T, Glock CH (2014) On the use of RFID in the management of reusable containers in closed-loop supply chains under stochastic container return quantities. Transp Res Part E 64(1):12–27
MaliniKennedy Felbin SUC (2013) An approach for solving fuzzy assignment problem using Octagonal Fuzzy numbers, International J of math. Sci Eng Appl 7(3):441–449
Menagam T (2017) Ranking of octagonal intuitionistic fuzzy numbers. IOSR J Math 13(3):63–71
Mullai M, Sangeetha K, Surya R, Madhan Kumar G, Jeyabalan R, Broumi S (2020) A single valued neutrosophic inventory model with neutrosophic random variable. Int J Neutros Sci 1(2):52–63
Pal S, Chakraborty A (2020) Triangular neutrosophic based production reliability model of deteriorating item with ramp type demand under shortages and time discounting. Neutros Sets Syst 35(1):347–367
Rajeswari S, Sugapriya C, Nagarajan D (2021) Fuzzy inventory model for NVOCC's returnable containers under empty container repositioning with leasing option. Complex Intell Syst 7:753–764. https://doi.org/10.1007/s40747-020-00229-1
Saqlain M, Hamza A, Farooq S (2020) Linear and non-linear octagonal neutrosophic numbers: its representation, α-cut and applications. Int J Neutros Sci 3(1):29–43
Selvakumari K, Lavanya S (2018) Solving fuzzy game problem in octagonal neutrosophic numbers using heavy OWA operator. Int J Eng Technol 7(439):497–499
Singh PK (2019) Single-valued neutrosophic context analysis at distinct multi-granulation. Comp Appl Math 38:80. https://doi.org/10.1007/s40314-019-0842-4
Smarandache F (1998) A unifying field in logics neutrosophy: neutrosophic probability, set and logic. American Research Press Rehoboth
Wang H, Smarandache F, Zhang Y, Sunderraman R (2010) Single valued neutrosophic sets. Multispace Multistruct 4:410–413
Wang F, Fang X, Chen X, Li X (2016) Impact of inventory inaccuracies on products with inventory-dependent demand. Int J Prod Econ 177:118–130
Ye J (2015) Trapezoidal neutrosophic set and its application to multi attribute decision making. Neural Comput Appl 26:1157–1166
Yun WY, Lee YM, Choi YS (2011) Optimal inventory control of empty containers in inland transportation system. Int J Prod Econ 133(1):451–457
Zadeh L (1965) Fuzzy sets. Inf Control 8:338–353
Zhang LH, Fan TJ, Chiang WC, Tao F (2014) Misplaced inventory and lead-time in the supply chain: analysis of decision-making on RFID investment with service level. J Appl Math. https://doi.org/10.1155/2014/135284

On Single Valued Neutrosophic Soft Uniform Spaces

Yaser Saber,[1] **Hanan Alohali,**[2] **Tawfik Elmasry,**[1] **Florentin Smarandache**[3]

[1] Department of Business Administration, College of Science and Human Studies, Hotat Sudair, Majmaah University, Majmaah, 11952, Saudi Arabia

[2] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arab

[3] Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA

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Abstract: In this paper, we introduce the notion of single-valued neutrosophic soft uniform spaces as a view point of the entourage approach. We investigate the relationship among single-valued neutrosophic soft uniformities, single-valued neutrosophic soft topologies and single-valued neutrosophic soft interior operators. Also, we study several single-valued neutrosophic soft topologies induced by a single-valued neutrosophic soft uniform space.

Keywords: single-valued neutrosophic soft sets; single-valued neutrosophic soft uniformity; Stratification
Mathematics Subject Classification: 54A10, 54A40

1. Introduction

There are many theories that have been suggested for dealing with uncertainties in an efficient way such as the theory of fuzzy sets [1], the theory of intuitionistic fuzzy sets [2], the theory of rough sets [3], and the theory of neutrosophic sets [4]. However, the idea of fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets are not sufficient to cope with parametrization tools. In 1999, Molodtsov [5] proposed the idea of a soft set that has the ability to deal with this difficulty. The idea of fuzzy soft (FS) sets and neutrosophic soft sets was proposed by Maji et al. [6, 7], and some properties of FS sets were discussed by Ahmad and Kharal [8]. Wang et al. [9] proposed the idea of single-valued neutrosophic sets. Saber et al. [10–13] introduced several concepts including, r-single-valued neutrosophic compact modulo, and r-single-valued neutrosophic connected sets in single-valued neutrosophic topological spaces, single-valued neutrosophic ideal open local function, single-valued neutrosophic θ£-separated. Single-valued neutrosophic fuzzy set and multi-attribute

decision-making were introduced by Sasirekha et al. [14]. Masri et al. [15] introduced the idea of a single-valued trapezoidal neutrosophic number.

Šostak's single-valued neutrosophic soft topological spaces and single-valued neutrosophic soft sets were constructed by Saber et al. [16]. The concept of single-valued neutrosophic soft has been thoroughly explored and advanced by numerous researchers, such as (Shahzadi et al. [17], Cano et al. [18], Özkan et al. [19], Al-Hijjawi et al. [20], Jana et al. [21] and Kamal et al. [22]) There are three alternative approaches to uniformity in the fuzzy case: Lowen's [23] entourage approach based on power sets of the form $\zeta^{\mathcal{X}\times\mathcal{X}}$, Kotzé's [24] uniform covering approach, and Hutton's [25] uniform operator approach.

It is well known that the theory of neutrosophic sets has been regarded as a generalization of the theory of fuzzy sets, the theory of intuitionistic fuzzy sets and the theory of rough sets. Furthermore, this is an important mathematical tool to deal with uncertainty. One of the main contributions of this paper is to introduce the concepts of single-valued neutrosophic uniformity in the sense of entourage, which is a generalization of the concepts introduced in Lowen [23], Kotzé [24], Hutton [25] and Abbas et al. [26].

Motivated by the above discussion, the present work deals with the single-valued neutrosophic uniformity in the sense of entourage. We introduce the notions of single-valued neutrosophic soft uniform spaces and single-valued neutrosophic soft uniform bases. The notion of this single-valued neutrosophic soft uniformities to be stratified is ensured. We investigate the relationship among single-valued neutrosophic soft uniformities, single-valued neutrosophic soft topologies and single-valued neutrosophic soft interior operators. We study several single-valued neutrosophic soft topologies induced by a single-valued neutrosophic soft uniform structure. Finally, we introduce the product single-valued neutrosophic soft uniformity of a given family of single-valued neutrosophic soft uniform spaces.

2. Preliminaries

In this section, we give all the basic definitions and results that we need to go through our work. First, we give the definition of a single-valued neutrosophic set (svn-set) and a single-valued neutrosophic soft set (svns-set). For more details about svn-set theory and svns-set theory, we refer to [9, 16]. As usual, $(\widehat{\mathcal{X}, \mathrm{E}})$ denotes the family of all svns-sets on $\mathcal{X}$, and E is the set of all parameters. Additionally, $\mathcal{X}$ indicates an initial universe and $\zeta^{\mathcal{X}}$ are the sets of all svn-sets on $\mathcal{X}$ (where, $\zeta = [0, 1]$ and $\zeta_0 = (0, 1]$).

Definition 1. *[4]. Let $\mathcal{X}$ be a universe set. A neutrosophic set (n-set) Θ on $\mathcal{X}$ defined as*

$$\Theta = \{\langle y, \gamma_\Theta(y), \pi_\Theta(y), \varsigma_\Theta(y) \mid y \in \mathcal{X}, \gamma_\Theta(y), \pi_\Theta(y), \varsigma_\Theta(y) \in \rfloor^- 0, 1^+ \lfloor\},$$

where $\gamma_\Theta(y)$, $\pi_\Theta(y)$ and $\varsigma_\Theta(y)$ are the truth, the indeterminacy, and the falsity membership functions respectively.

Definition 2. *[9]. Let $\mathcal{X}$ be a non-null set. Then, svn-set Θ on $\mathcal{X}$ is defined as*

$$\Theta = \{\langle y, \gamma_\Theta(y), \pi_\Theta(y), \varsigma_\Theta(y) \mid y \in \mathcal{X}, \gamma_\Theta(y), \pi_\Theta(y), \varsigma_\Theta(y) \in \zeta\},$$

where $\gamma_\Theta, \pi_\Theta, \varsigma_\Theta : \mathcal{X} \to \zeta$ and $0 \leq \gamma_\Theta(y) + \pi_\Theta(y) + \varsigma_\Theta(y) \leq 3$.

Remark 1. *To clarify the relationship between intuitionistic fuzzy sets if-set, neutrosophic sets n-set, and single-valued neutrosophic sets svn-set, let us confirm that both neutrosophic sets and single-valued neutrosophic sets are a generalization of the concept of intuitionistic fuzzy sets, as follows:*

In IFS, paraconsistent, dialtheist and incomplete information cannot be characterized. This most important distinction between if-set and n-set is shown in the below neutrosophic cube A' B' C' D' E' F' G' H' introduced by J. Dezert [27].

Because only the classical interval [0,1]is used as a range for the neutrosophic parameters in technical applications (truth, indeterminacy and falsity), we call the cube ABCDEDGH the technical neutrosophic cube and its extension A' B' C' D' E' F' G' H' the neutrosophic cube or nonstandard neutrosophic cube, used in the fields where we need to differentiate between absolute and relative notions like philosophy.

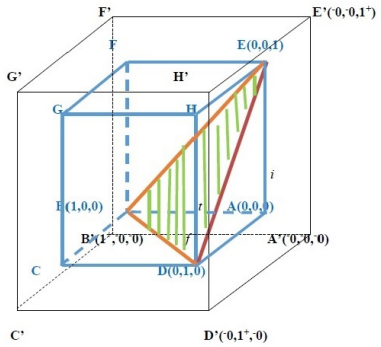

Figure 1. Neutrosophic cube.

Definition 3. *[16]. f_A is an svns-set on $\mathcal{X}$, where $f : E \to \zeta^{\mathcal{X}}$; i.e., $f_e \triangleq f(e)$ is an svn-set on $\mathcal{X}$, for all $e \in A$ and $f(e) = \langle 0, 1, 1\rangle$, if $e \notin A$.*

The svn-set $f(e)$ is termed as an element of the svns-set f_A. Thus, an svns-set f_E on $\mathcal{X}$ can be defined as:

$$\begin{aligned}(f, E) &= \left\{(e, f(e)) \mid e \in E, f(e) \in \zeta^{\mathcal{X}}\right\} \\ &= \left\{e, \langle \gamma_f(e), \pi_f(e), \varsigma_f(e)\rangle) \mid e \in E, f(e) \in \zeta^{\mathcal{X}}\right\},\end{aligned}$$

where $\gamma_f : E \to \zeta$ (γ_f is termed as a membership function), $\pi_f : E \to \zeta$ (π_f is termed as indeterminacy function) and $\varsigma_f : E \to \zeta$ (ς_f is termed as a nonmembership function) of svns-set.

An svns-set f_E on $\mathcal{X}$ is termed as a null svns-set (for short, $\widehat{\Phi}$), if $\gamma_f(e) = 0, \pi_f(e) = 1$ and $\varsigma_f(e) = 1$, for any $e \in E$.

An svns-set f_E on $\mathcal{X}$ is termed as an absolute svns-set (for short, $\widetilde{\widetilde{E}}$), if $\gamma_f(e) = 1, \pi_f(e) = 0$ and $\varsigma_f(e) = 0$, for any $e \in E$.

Definition 4. *[16]. Let $f_A, g_B \in (\widehat{\mathcal{X}, E})$ be an svns-sets on $\mathcal{X}$. Then,*

(1) Inclusion of two sets (for short, $f_A \preceq g_B$) defined as:

$$\gamma_f(e) \leq \gamma_g(e), \quad \pi_f(e) \geq \pi_g(e), \quad \varsigma_f(e) \geq \varsigma_g(e).$$

(2) The complemented of the set f_A denoted by (for short, f_A^c) defined as:

$$f_A^c = \left\{\left(e, \langle \varsigma_f(e), \tilde{\pi}_{f^c}(e), \gamma_f(e)\rangle\right) \mid e \in E\right\}.$$

Definition 5. *[16]. A mapping $\mathfrak{T}^\gamma, \mathfrak{T}^\pi, \mathfrak{T}^\varsigma : E \to \zeta^{\widehat{(X,E)}}$ is said to be a single-valued neutrosophic soft topology (svnst) on X if it meets the next criteria, for every $e \in E$:*

$(\mathfrak{T}_1)$ $\mathfrak{T}_e^\gamma(\widehat{\Phi}) = 1$, $\mathfrak{T}_e^\pi(\widehat{\Phi}) = 0$, $\mathfrak{T}_e^\varsigma(\widehat{\Phi}) = 0$ *and* $\mathfrak{T}_e^\gamma(\widehat{E}) = 1$ $\mathfrak{T}_e^\pi(\widehat{E}) = 0$, $\mathfrak{T}_e^\varsigma(\widehat{E}) = 0$,
$(\mathfrak{T}_2)$ $\mathfrak{T}_e^\gamma(f_A \sqcap g_B) \geq \mathfrak{T}_e^\gamma(f_A) \wedge \mathfrak{T}_e^\gamma(g_B)$, $\mathfrak{T}_e^\pi(f_A \sqcap g_B) \leq \mathfrak{T}_e^\pi(f_A) \vee \mathfrak{T}_e^\pi(g_B)$,
$\mathfrak{T}_e^\varsigma(f_A \sqcap g_B) \leq \mathfrak{T}_e^\varsigma(f_A) \vee \mathfrak{T}_e^\varsigma(g_B)$, $\forall\, f_A, g_B \in \widehat{(X,E)}$,
$(\mathfrak{T}_3)$ $\mathfrak{T}_e^\gamma(\bigsqcup_{j\in\Gamma}(f_A)_j) \geq \bigwedge_{j\in\Gamma} \mathfrak{T}_e^\gamma((f_A)_j)$, $\mathfrak{T}_e^\pi(\bigsqcup_{j\in\Gamma}(f_A)_j) \leq \bigvee_{j\in\Gamma} \mathfrak{T}_e^\pi((f_A)_j)$,
$\mathfrak{T}_e^\varsigma(\bigsqcup_{j\in\Gamma}(f_A)_j) \leq \bigvee_{j\in\Gamma} \mathfrak{T}_e^\varsigma((f_A)_j)$, $\forall\, (f_A)_j \in \widehat{(X,E)}$, $j \in \Gamma$.

(Note that $\sqcap$ and $\sqcup$ in the definition are clarified in Molodtsov [5]). The quadruple $(X, \mathfrak{T}^\gamma, \mathfrak{T}^\pi, \mathfrak{T}^\varsigma)$ is said to be a single-valued neutrosophic soft topological space (svnst-space), where $(\mathfrak{T}_e^\gamma(f_A))$ representing the degree of openness, $(\mathfrak{T}_e^\pi(f_A))$ the degree of indeterminacy and $(\mathfrak{T}_e^\varsigma(f_A))$ the degree of non-openness; of a svns-set with respect to that parameter $e \in E$. Sometimes, we will write $\mathfrak{T}^{\gamma\pi\varsigma}$ for $(\mathfrak{T}^\gamma, \mathfrak{T}^\pi, \mathfrak{T}^\varsigma)$.

Let $(X, \mathfrak{T}^{\gamma\pi\varsigma})$ and $(\mathcal{G}, \mathfrak{T}^{\star\gamma\pi\varsigma})$ be svnst-space. An svns-mapping $\psi_\varphi : \widehat{(X,E)} \to \widehat{(\mathcal{G},\mathcal{R})}$ is said to be a single-valued neutrosophic soft continuous mapping (svnsc-map) if

$$\mathfrak{T}_e^\gamma(\psi_\varphi^{-1}(g_B)) \geq \mathfrak{T}_{\varphi(e)}^{\star\gamma}(g_B), \qquad \mathfrak{T}_e^\pi(\psi_\varphi^{-1}(g_B)) \leq \mathfrak{T}_{\varphi(e)}^{\star\pi}(g_B),$$

$$\mathfrak{T}_e^\varsigma(\psi_\varphi^{-1}(g_B)) \leq \mathfrak{T}_{\varphi(e)}^{\star\varsigma}(g_B),$$

for all $g_B \in \widehat{(\mathcal{G},\mathcal{R})}$ and $e \in E$ [Saber et al. (2022) [16].

Definition 6. *A map $\mathtt{I} : E \times \widehat{(X,E)} \times \zeta_0 \to \widehat{(X,E)}$ is said to be single-valued neutrosophic soft interior operator (svnsi-operator) on X if it meets the next criteria, $\forall$, $e \in E$, $f_A, g_B \in \widehat{(X,E)}$ and $r, s \in \zeta$:*

$(\mathtt{I}_1)$ $\mathtt{I}(e, \widehat{E}, r) = \widehat{E}$,
$(\mathtt{I}_2)$ $\mathtt{I}(e, f_A, r) \leq f_A$,
$(\mathtt{I}_3)$ if $f_A \leq g_B$ and $r \leq s$ then $\mathtt{I}(e, f_A, r) \leq \mathtt{I}(e, g_B, s)$,
$(\mathtt{I}_4)$ $\mathtt{I}(e), f_A \sqcap g_B, r \wedge s) \geq \mathtt{I}(e, f_A, r) \sqcap \mathtt{I}(e, g_B, s)$,
$(\mathtt{I}_5)$ $\mathtt{I}(e, \mathtt{I}(e, f_A, r), r) = \mathtt{I}(e, f_A, r)$.

Definition 7. *[16]. A map $\mathtt{C} : E \times \widehat{(X,E)} \times \zeta_0 \to \widehat{(X,E)}$ is said to be single-valued neutrosophic soft closure operator (svnsc-operator) on X if it meets the next criteria, $\forall$, $e \in E$, $f_A, g_B \in \widehat{(X,E)}$ and $r, s \in \zeta$:*

$(\mathtt{C}_1)$ $\mathtt{C}(e, \widehat{\Phi}, r) = \widehat{\Phi}$,
$(\mathtt{C}_2)$ $\mathtt{C}(e, f_A, r) \geq f_A$,
$(\mathtt{C}_3)$ if $f_A \leq g_B$ and $r \leq s$ then $\mathtt{C}(e, f_A, r) \leq \mathtt{C}(e, g_B, s)$,
$(\mathtt{C}_4)$ $\mathtt{C}(e, f_A \sqcup g_B, r \wedge s) \leq \mathtt{C}(e, f_A, r) \sqcup \mathtt{C}(e, g_B, s)$,
$(\mathtt{C}_5)$ $\mathtt{C}(e, \mathtt{C}(e, f_A, r), r) \leq \mathtt{C}(e, f_A, r)$,
$(\mathtt{C}_6)$ $\mathtt{C}(e, f_A, r) = [\mathtt{I}(e, f_A^c, r)]^c$.

3. Single-valued neutrosophic soft soft uniform spaces

The main objective of this section is to define and discuss the concepts of single-valued neutrosophic soft uniformity (svns-uniformity), single-valued neutrosophic soft uniform base (svns-uniform base) and stratified single-valued neutrosophic soft uniform space (ssvns-uniform space). Several basic properties and theorems related to these concepts are explored.

In this section, we indicate that $(\mathcal{X}\widehat{\times}\mathcal{X}, \mathrm{E})$ is the family of all svns-sets on $\mathcal{X}\times\mathcal{X}$ and $\zeta^{\mathcal{X}\times\mathcal{X}}$ are the sets of all svn-sets on $\mathcal{X}\times\mathcal{X}$. Additionally, for $\varrho\in\zeta$, $\bar{\varrho}(x,y)=\varrho$ for any $(x,y)\in\mathcal{X}\times\mathcal{X}$.

Definition 8. *Let $\mathcal{X}$ be a set. A mappings $\pounds^{\gamma},\pounds^{\pi},\pounds^{\varsigma}: E\to\zeta^{(\mathcal{X}\widehat{\times}\mathcal{X},E)}$ is called an svns-uniformity on $\mathcal{X}$ if it meets the next criteria:*

($\pounds_1$) *for any $e\in E$, there exists $\upsilon_{_A}\in(\mathcal{X}\widehat{\times}\mathcal{X},E)$ such that $\pounds^{\gamma}_e(\upsilon_{_A})=1$, $\pounds^{\pi}_e(\upsilon_{_A})=0$, $\pounds^{\varsigma}_e(\upsilon_{_A})=0$,*
($\pounds_2$) *if $\upsilon_{_A}\preceq\mu_{_B}$, then $\pounds^{\gamma}_e(\upsilon_{_A})\le\pounds^{\gamma}_e(\mu_{_B})$, $\pounds^{\pi}_e(\upsilon_{_A})\ge\pounds^{\pi}_e(\mu_{_B})$, $\pounds^{\varsigma}_e(\upsilon_{_A})\ge\pounds^{\varsigma}_e(\mu_{_B})$,*
($\pounds_3$) *for every $\upsilon_{_A},\mu_{_B}\in(\mathcal{X}\widehat{\times}\mathcal{X},E)$, then*

$$\pounds^{\gamma}_e(\upsilon_{_A}\sqcap\mu_{_B})\ge\pounds^{\gamma}_e(\upsilon_{_A})\wedge\pounds^{\gamma}_e(\mu_{_B}),\quad \pounds^{\pi}_e(\upsilon_{_A}\sqcap\mu_{_B})\le\pounds^{\pi}_e(\upsilon_{_A})\vee\pounds^{\pi}_e(\mu_{_B})$$

$$\pounds^{\varsigma}_e(\upsilon_{_A}\sqcap\mu_{_B})\le\pounds^{\varsigma}_e(\upsilon_{_A})\vee\pounds^{\varsigma}_e(\mu_{_B}),$$

($\pounds_4$) $(\top)_C\not\preceq\upsilon_{_A}$ *implies that $\pounds^{\gamma}_e(\upsilon_{_A})=0$, $\pounds^{\pi}_e(\upsilon_{_A})=1$, $\pounds^{\varsigma}_e(\upsilon_{_A})=1$, where, $\forall\ e\in E$,*

$$(\top)_e(x,y)=\begin{cases}\langle 1,0,0\rangle, & \text{if } x=y,\\ \langle 0,1,1\rangle, & \text{otherwise},\end{cases}$$

($\pounds_5$) $\pounds^{\gamma}_e(\upsilon_{_A})\le\pounds^{\gamma}_e(\upsilon^{s}_{_A})$, $\pounds^{\pi}_e(\upsilon_{_A})\ge\pounds^{\pi}_e(\upsilon^{s}_{_A})$, $\pounds^{\varsigma}_e(\upsilon_{_A})\ge\pounds^{\varsigma}_e(\upsilon^{s}_{_A})$, *where $\upsilon^{s}_{e}(x,y)=\upsilon_{e}(y,x)$ for every $e\in E$,*
($\pounds_6$) *for each $\upsilon_{_A}\in(\mathcal{X}\widehat{\times}\mathcal{X},E)$, $e\in E$,*

$$\pounds^{\gamma}_e(\upsilon_{_A})\le\bigvee\{\pounds^{\gamma}_e(\mu_{_B}):\ (\mu_{_B}\circ\mu_{_B})\preceq\upsilon_{_A}\},\qquad \pounds^{\pi}_e(\upsilon_{_A})\ge\bigwedge\{\pounds^{\pi}_e(\mu_{_B}):\ (\mu_{_B}\circ\mu_{_B})\preceq\upsilon_{_A}\},$$

$$\pounds^{\varsigma}_e(\upsilon_{_A})\ge\bigwedge\{\pounds^{\varsigma}_e(\mu_{_B}):\ (\mu_{_B}\circ\mu_{_B})\preceq\upsilon_{_A}\},$$

where $(\upsilon_{_B}\circ\mu_{_B})=\bigvee\limits_{z\in\mathcal{X}}\{\upsilon_e(x,z)\wedge\mu_e(z,y)\}$ for each $x,y\in\mathcal{X}$.

A svns-uniformity $\pounds^{\gamma},\pounds^{\pi},\pounds^{\varsigma}: E\to\zeta^{(\mathcal{X}\widehat{\times}\mathcal{X},E)}$ is called stratified if

($\pounds_{st}$) $\pounds^{\gamma}_e(\widehat{E_{\varrho}})=1$, $\pounds^{\pi}_e(\widehat{E_{\varrho}})=0$, $\pounds^{\varsigma}_e(\widehat{E_{\varrho}})=0$, *where $\upsilon_{_E}=\widehat{E_{\varrho}}$ if $\upsilon_e=\bar{\varrho}\ \forall$, $e\in E$.*

After adding the last condition $(\mathcal{X},\pounds^{\gamma},\pounds^{\pi},\pounds^{\varsigma})$ is called ssvns-uniform space. Sometimes, we will write $\pounds^{\gamma\pi\varsigma}_{E}$ for $(\pounds^{\gamma},\pounds^{\pi},\pounds^{\varsigma})$.

Let $\pounds^{\gamma\pi\varsigma}_{E}$ and $\pounds^{\star\gamma\pi\varsigma}_{E}$ be two svns-uniformities on $\mathcal{X}$. $\pounds^{\gamma\pi\varsigma}_{E}$ is finer than $\pounds^{\star\gamma\pi\varsigma}_{E}$ ($\pounds^{\star\gamma\pi\varsigma}_{E}$ is coarser than $\pounds^{\gamma\pi\varsigma}_{E}$), indicated by $\pounds^{\star\gamma\pi\varsigma}_{E}\preceq\pounds^{\gamma\pi\varsigma}_{E}$ provided

$$\pounds^{\star\gamma}_{e}(\upsilon_{_A})\le\pounds^{\gamma}_{E}(\upsilon_{_A}),\qquad \pounds^{\star\pi}_{e}(\upsilon_{_A})\ge\pounds^{\pi}_{e}(\upsilon_{_A}),\qquad \pounds^{\star\varsigma}_{e}(\upsilon_{_A})\ge\pounds^{\varsigma}_{e}(\upsilon_{_A}),\quad \forall\ e\in E,\upsilon_{_A}\in(\mathcal{X}\widehat{\times}\mathcal{X},E).$$

Remark 2. *Suppose that $(\mathcal{X},\pounds^{\gamma\pi\varsigma}_{E})$ is an svns-uniform space. Then, by using the two conditions ($\pounds_1$) and ($\pounds_2$), we obtain, $\pounds^{\gamma}_e(\widehat{E})=1$, $\pounds^{\pi}_e(\widehat{E})=0$, $\pounds^{\varsigma}_e(\widehat{E})=0$ because $\upsilon_{_A}\preceq\widehat{E}$ for every $e\in E$, $\upsilon_{_A}\in(\mathcal{X}\widehat{\times}\mathcal{X},E)$.*

Theorem 1. *Let $(\mathcal{X}, \pounds_E^{\gamma\pi\varsigma})$ be an svns-uniform space. Define for any $e \in E$, $\upsilon_A \in (\mathcal{X}\widehat{\times}\mathcal{X}, E)$.*

$$(\pounds_{st}^{\gamma})_e(\upsilon_A) = \bigvee\{\pounds_e^{\gamma}((\mu_B)) : \ \mu_B \sqcap \widehat{E}_{\varrho} \preceq \upsilon_A, \ \varrho \in \zeta\},$$

$$(\pounds_{st}^{\pi})_e(\upsilon_A) = \bigwedge\{\pounds_e^{\pi}((\mu_B)) : \ \mu_B \sqcap \widehat{E}_{\varrho} \preceq \upsilon_A, \ \varrho \in \zeta\},$$

$$(\pounds_{st}^{\varsigma})_e(\upsilon_A) = \bigwedge\{\pounds_e^{\varsigma}((\mu_B)) : \ \mu_B \sqcap \widehat{E}_{\varrho} \preceq \upsilon_A, \ \varrho \in \zeta\}.$$

Then, $(\pounds_{st}^{\gamma\pi\varsigma})_E$ is the coarsest ssvns-uniformity which is finer than $\pounds_E^{\gamma\pi\varsigma}$.

Proof. ($\pounds_1$) There exists $\upsilon_A \in (\mathcal{X}\widehat{\times}\mathcal{X}, \mathrm{E})$ such that $\pounds_e^{\gamma}(\upsilon_A) = 1$, $\pounds_e^{\pi}(\upsilon_A) = 0$, $\pounds_e^{\varsigma}(\upsilon_A) = 0$ for every $e \in \mathrm{E}$. Since $\upsilon_A \sqcap \widehat{E}_{\varrho} \preceq \upsilon_A$, *then* $(\pounds_{st}^{\gamma})_e(\upsilon_A) = 1$, $(\pounds_{st}^{\pi})_e(\upsilon_A) = 0$, $(\pounds_{st}^{\varsigma})_e(\upsilon_A) = 0$.

($\pounds_2$) Direct from the definition.

($\pounds_3$) Let there exist $(\upsilon_1)_A, (\upsilon_2)_B \in (\mathcal{X}\widehat{\times}\mathcal{X}, \mathrm{E})$ such that for every $e \in \mathrm{E}$,

$$(\pounds_{st}^{\gamma})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \not\geq (\pounds_{st}^{\gamma})_e((\upsilon_1)_A) \wedge (\pounds_{st}^{\gamma})_e((\upsilon_2)_B),$$

$$(\pounds_{st}^{\pi})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \not\leq (\pounds_{st}^{\pi})_e((\upsilon_1)_A) \vee (\pounds_{st}^{\pi})_e((\upsilon_2)_B),$$

$$(\pounds_{st}^{\varsigma})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \not\leq (\pounds_{st}^{\varsigma})_e((\upsilon_1)_A) \vee (\pounds_{st}^{\varsigma})_e((\upsilon_2)_B).$$

By using the definition of $(\pounds_{st}^{\gamma\pi\varsigma})_{\mathrm{E}}$, then there exists $(\mu_1)_C, (\mu_2)_D \in (\mathcal{X}\widehat{\times}\mathcal{X}, \mathrm{E})$, $\varrho_1, \varrho_2 \in \zeta$ with $(\mu_1)_C \sqcap \widehat{E}_{\varrho_1} \preceq (\upsilon_1)_A$, $(\mu_2)_D \sqcap \widehat{E}_{\varrho_2} \preceq (\upsilon_2)_B$ such that

$$(\pounds_{st}^{\gamma})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \not\geq \pounds_e^{\gamma}((\mu_1)_C) \wedge \pounds_e^{\gamma}((\mu_2)_D),$$

$$(\pounds_{st}^{\pi})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \not\leq \pounds_e^{\pi}((\mu_1)_C) \vee \pounds_e^{\pi}((\mu_2)_D),$$

$$(\pounds_{st}^{\varsigma})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \not\leq \pounds_e^{\varsigma}((\mu_1)_C) \vee (\pounds_e^{\varsigma}((\mu_2)_D).$$

Otherwise, $(\mu_1)_C \sqcap (\mu_2)_D \sqcap \widehat{E}_{\varrho_1} \sqcap \widehat{E}_{\varrho_2} \preceq (\upsilon_1)_A \sqcap (\upsilon_2)_B$. Then, we have

$$(\pounds_{st}^{\gamma})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \geq \pounds_e^{\gamma}((\mu_1)_C \sqcap (\mu_2)_D) \geq \pounds_e^{\gamma}((\mu_1)_C) \wedge \pounds_e^{\gamma}((\mu_2)_D),$$

$$(\pounds_{st}^{\pi})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \leq \pounds_e^{\pi}((\mu_1)_C \sqcap (\mu_2)_D) \leq \pounds_e^{\pi}((\mu_1)_C) \vee \pounds_e^{\pi}((\mu_2)_D),$$

$$(\pounds_{st}^{\varsigma})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \leq \pounds_e^{\varsigma}((\mu_1)_C \sqcap (\mu_2)_D) \leq \pounds_e^{\varsigma}((\mu_1)_C) \vee (\pounds_e^{\varsigma}((\mu_2)_D).$$

This is a contradiction. Consequently, ($\pounds_3$) holds.

($\pounds_4$) Direct from the definition.

($\pounds_5$) Let

$$(\pounds_{st}^{\gamma})_e(\upsilon_A^s) \not\geq (\pounds_{st}^{\gamma})_e(\upsilon_A), \qquad (\pounds_{st}^{\pi})_e(\upsilon_A^s) \not\leq (\pounds_{st}^{\pi})_e(\upsilon_A), \qquad (\pounds_{st}^{\varsigma})_e(\upsilon_A^s) \not\leq (\pounds_{st}^{\varsigma})_e(\upsilon_A),$$

$\forall$, $e \in \mathrm{E}$, $\upsilon_A \in (\mathcal{X}\widehat{\times}\mathcal{X}, \mathrm{E})$. By using the definition of $(\pounds_{st}^{\gamma\pi\varsigma})_{\mathrm{E}}$, there exists $\mu_B \in (\mathcal{X}\widehat{\times}\mathcal{X}, \mathrm{E})$, $\varrho \in \zeta$ with $\mu_B \sqcap \widehat{E}_{\varrho} \preceq \upsilon_A$, such that

$$(\pounds_{st}^{\gamma})_e(\upsilon_A^s) \not\geq \pounds_e^{\gamma}(\mu_B), \qquad (\pounds_{st}^{\pi})_e(\upsilon_A^s) \not\leq \pounds_e^{\pi}(\mu_B), \qquad (\pounds_{st}^{\varsigma})_e(\upsilon_A^s) \not\leq \pounds_e^{\varsigma}(\mu_B).$$

Since $\pounds_{\mathrm{E}}^{\gamma\pi\varsigma}$ is svns-uniformity, then

$$\pounds_e^{\gamma}(\mu_B) \leq \pounds_e^{\gamma}(\mu_B^s), \qquad \pounds_e^{\pi}(\mu_B) \geq \pounds_e^{\pi}(\mu_B^s), \qquad \pounds_e^{\varsigma}(\mu_B) \geq \pounds_e^{\varsigma}(\mu_B^s),$$

It follows that

$$(\pounds^{\gamma}_{st})_e(\upsilon^s_A) \not\geq \pounds^{\gamma}_e(\mu^s_B), \qquad (\pounds^{\pi}_{st})_e(\upsilon^s_A) \not\leq \pounds^{\pi}_e(\mu^s_B), \qquad (\pounds^{\varsigma}_{st})_e(\upsilon^s_A) \not\leq \pounds^{\varsigma}_e(\mu^s_B).$$

On the other hand, $\mu^s_B \sqcap \widehat{E_\varrho} \preceq \upsilon^s_A$. Hence, for each $e \in \mathrm{E}$

$$(\pounds^{\gamma}_{st})_e(\upsilon^s_A) \geq \pounds^{\gamma}_e(\mu^s_B), \qquad (\pounds^{\pi}_{st})_e(\upsilon^s_A) \leq \pounds^{\pi}_e(\mu^s_B), \qquad (\pounds^{\varsigma}_{st})_e(\upsilon^s_A) \leq \pounds^{\varsigma}_e(\mu^s_B).$$

This is a contradiction. Therefore, $(\pounds_5)$ holds.

$(\pounds_6)$ Suppose that

$$(\pounds^{\gamma}_{st})_e(\upsilon_A) \not\leq \bigvee\{(\pounds^{\gamma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\},$$

$$(\pounds^{\pi}_{st})_e(\upsilon_A) \not\geq \bigwedge\{(\pounds^{\pi}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\},$$

$$(\pounds^{\varsigma}_{st})_e(\upsilon_A) \not\geq \bigwedge\{(\pounds^{\varsigma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\}.$$

for any $\upsilon_A \in (\mathcal{X}\widehat{\times\mathcal{X}}, \mathrm{E})$. From the definition of $(\pounds^{\gamma\pi\varsigma}_{st})_{\mathrm{E}}$, there exists $\mu_B \in (\mathcal{X}\widehat{\times\mathcal{X}}, \mathrm{E})$, $\varrho \in \zeta$ with $\mu_B \sqcap \widehat{E_\varrho} \preceq \upsilon_A$ such that

$$\pounds^{\gamma}_e(\mu_B) \not\leq \bigvee\{(\pounds^{\gamma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\},$$

$$\pounds^{\pi}_e(\mu_B) \not\geq \bigwedge\{(\pounds^{\pi}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\},$$

$$\pounds^{\varsigma}_e(\mu_B) \not\geq \bigwedge\{(\pounds^{\varsigma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\}.$$

Since $\pounds^{\gamma\pi\varsigma}_{\mathrm{E}}$ is svns-uniformity on $\mathcal{X}$, then

$$\pounds^{\gamma}_e(\mu_B) \leq \bigvee\{\pounds^{\gamma}_e(\sigma_D) : \ \sigma_D \circ \sigma_D \preceq \mu_B\},$$

$$\pounds^{\pi}_e(\mu_B) \geq \bigwedge\{\pounds^{\pi}_e(\sigma_D) : \ \sigma_D \circ \sigma_D \preceq \mu_B\},$$

$$\pounds^{\varsigma}_e(\mu_B) \geq \bigwedge\{\pounds^{\varsigma}_e(\sigma_D) : \ \sigma_D \circ \sigma_D \preceq \mu_B\}.$$

That means, there is $\sigma_D \in (\mathcal{X}\widehat{\times\mathcal{X}}, \mathrm{E})$ such that $\sigma_D \sqcap \sigma_D \preceq \mu_B$ and that

$$\pounds^{\gamma}_e(\sigma_D) \not\leq \bigvee\{(\pounds^{\gamma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\},$$

$$\pounds^{\pi}_e(\sigma_D) \not\geq \bigwedge\{(\pounds^{\pi}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\},$$

$$\pounds^{\varsigma}_e(\sigma_D) \not\geq \bigwedge\{(\pounds^{\varsigma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\}.$$

On the other hand,

$$(\sigma_D \sqcap \widehat{E_\varrho}) \circ (\sigma_D \sqcap \widehat{E_\varrho}) \preceq (\sigma_D \circ \sigma_D) \sqcap \widehat{E_\varrho} \preceq \mu_B \sqcap \widehat{E_\varrho} \preceq \upsilon_A,$$

which means that there is $(\upsilon_1)_C = \sigma_D \sqcap \widehat{E_\varrho}$ with $(\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A$,

$$\pounds^{\gamma}_e(\sigma_D) \leq (\pounds^{\gamma}_{st})_e((\upsilon_1)_C) \leq \bigvee\{(\pounds^{\gamma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\},$$

$$\pounds^{\pi}_e(\sigma_D) \geq (\pounds^{\pi}_{st})_e((\upsilon_1)_C) \geq \bigwedge\{(\pounds^{\pi}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\},$$

$$\pounds_e^{\varsigma}(\sigma_D) \geq (\pounds_{st}^{\varsigma})_e((\upsilon_1)_C) \geq \bigwedge\{(\pounds_{st}^{\varsigma})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\}.$$

It is a contradiction. Thus, $(\pounds_6)$ holds.

$(\pounds_{st})$ Since $\widehat{E_\varrho} \sqcap \widehat{E_1} = \widehat{E_\varrho}$ for each $\varrho \in \zeta$, then $(\pounds_{st}^{\gamma})_{\widehat{E}} = 1$, $(\pounds_{st}^{\pi})_{\widehat{E}} = 0$ and $(\pounds_{st}^{\varsigma})_{\widehat{E}} = 0$. Therefore, $(\pounds_{st}^{\gamma\pi\varsigma})_e$ is stratified.

For each $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$, $\upsilon_A \sqcap \widehat{E_1} = \upsilon_A$, we have for each $e \in \mathrm{E}$

$$(\pounds_{st}^{\gamma})_e(\upsilon_A) \geq \pounds_e^{\gamma}(\upsilon_A), \qquad (\pounds_{st}^{\pi})_e(\upsilon_A) \leq \pounds_e^{\pi}(\upsilon_A), \qquad (\pounds_{st}^{\varsigma})_e(\upsilon_A) \leq \pounds_e^{\varsigma}(\upsilon_A).$$

Hence, $(\pounds_{st}^{\gamma\pi\varsigma})_{\mathrm{E}}$ is finer than $\pounds_{\mathrm{E}}^{\gamma\pi\varsigma}$.

Finally, consider $\pounds_{\mathrm{E}}^{\star\gamma\pi\varsigma}$ is an ssvns-uniformity finer than $\pounds_{\mathrm{E}}^{\gamma\pi\varsigma}$. Let there exists $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$ such that

$$(\pounds_{st}^{\gamma})_e(\upsilon_A) \not\leq \pounds_e^{\star\gamma}(\upsilon_A), \qquad (\pounds_{st}^{\pi})_e(\upsilon_A) \not\geq \pounds_e^{\star\pi}(\upsilon_A), \qquad (\pounds_{st}^{\varsigma})_e(\upsilon_A) \not\geq \pounds_e^{\star\varsigma}(\upsilon_A).$$

From the definition of $\{(\pounds_{st}^{\gamma})_e(\upsilon_A), (\pounds_{st}^{\pi})_e(\upsilon_A), (\pounds_{st}^{\varsigma})_e(\upsilon_A)\}$, there exists $\mu_B \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$, $\varrho \in \zeta$ with $\mu_B \sqcap \widehat{E_\varrho} \preceq \upsilon_A$ and

$$\pounds_e^{\gamma}(\mu_B) \not\leq \pounds_e^{\star\gamma}(\upsilon_A), \qquad \pounds_e^{\pi}(\mu_B) \not\geq \pounds_e^{\star\pi}(\upsilon_A), \qquad \pounds_e^{\varsigma}(\mu_B) \not\geq \pounds_e^{\star\varsigma}(\upsilon_A).$$

Since $\pounds_{\mathrm{E}}^{\star\gamma\pi\varsigma}$ is stratified, then

$$\pounds_e^{\gamma}(\mu_B) \leq \pounds_e^{\star\gamma}(\mu_B) = \pounds_e^{\star\gamma}(\mu_B) \wedge \pounds_e^{\star\gamma}(\widehat{E_\varrho}) \leq \pounds_e^{\star\gamma}(\mu_B \sqcap \widehat{E_\varrho}) \leq \pounds_e^{\gamma}(\upsilon_A),$$

$$\pounds_e^{\pi}(\mu_B) \geq \pounds_e^{\star\pi}(\mu_B) = \pounds_e^{\star\pi}(\mu_B) \vee \pounds_e^{\star\pi}(\widehat{E_\varrho}) \geq \pounds_e^{\star\pi}(\mu_B \sqcup \widehat{E_\varrho}) \geq \pounds_e^{\pi}(\upsilon_A),$$

$$\pounds_e^{\varsigma}(\mu_B) \geq \pounds_e^{\star\varsigma}(\mu_B) = \pounds_e^{\star\varsigma}(\mu_B) \vee \pounds_e^{\star\varsigma}(\widehat{E_\varrho}) \geq \pounds_e^{\star\varsigma}(\mu_B \sqcup \widehat{E_\varrho}) \geq \pounds_e^{\varsigma}(\upsilon_A).$$

It is a contradiction. Hence,

$$(\pounds_{st}^{\gamma})_e(\upsilon_A) \leq \pounds_e^{\star\gamma}(\upsilon_A), \qquad (\pounds_{st}^{\pi})_e(\upsilon_A) \geq \pounds_e^{\star\pi}(\upsilon_A), \qquad (\pounds_{st}^{\varsigma})_e(\upsilon_A) \geq \pounds_e^{\star\varsigma}(\upsilon_A),$$

for each $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$, $e \in \mathrm{E}$. Hence, $(\pounds_{st}^{\gamma\pi\varsigma})_{\mathrm{E}}$ is the coarsest ssvns-uniformity which is finer than $\pounds_{\mathrm{E}}^{\gamma\pi\varsigma}$. □

Remark 3. *Let $\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma} : E \to \zeta^{(\widehat{\mathcal{X}\times\mathcal{X}}, E)}$ be a mapping and $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, E)$. Let us define $\langle \hbar_e^{\gamma} \rangle$, $\langle \hbar_e^{\pi} \rangle$ and $\langle \hbar_e^{\varsigma} \rangle$ as follows for each $e \in E$:*

$$\langle \hbar_e^{\gamma} \rangle(\upsilon_A) = \bigvee_{\upsilon_A \preceq \upsilon_B} \hbar_e^{\gamma}(\upsilon_B), \quad \langle \hbar_e^{\pi} \rangle(\upsilon_A) = \bigwedge_{\upsilon_A \preceq \upsilon_B} \hbar_e^{\pi}(\upsilon_B), \quad \langle \hbar_e^{\varsigma} \rangle(\upsilon_A) = \bigwedge_{\upsilon_A \preceq \upsilon_B} \hbar_e^{\varsigma}(\upsilon_B).$$

Definition 9. *A mappings $\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma} : E \to \zeta^{(\widehat{\mathcal{X}\times\mathcal{X}}, E)}$ is called a svns-uniform base on $\mathcal{X}$ if it meets the next criteria:*

$(\hbar_1)$ There exists $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, E)$ such that $\hbar_e^{\gamma}(\upsilon_A) = 1$, $\hbar_e^{\pi}(\upsilon_A) = 0$, $\hbar_e^{\varsigma}(\upsilon_A) = 0$, for all $e \in E$,
$(\hbar_2)$ for each $\upsilon_A, \mu_B \in (\widehat{\mathcal{X} \times \mathcal{X}}, E)$, $e \in E$, such that

$$\langle \hbar_e^{\gamma} \rangle(\upsilon_A \sqcap \mu_B) \geq \hbar_e^{\gamma}(\upsilon_A) \wedge \hbar_e^{\gamma}(\mu_B), \qquad \langle \hbar_e^{\pi} \rangle(\upsilon_A \sqcap \mu_B) \leq \hbar_e^{\pi}(\upsilon_A) \vee \hbar_e^{\pi}(\mu_B),$$

$$\langle \hbar_e^{\varsigma} \rangle(\upsilon_A \sqcap \mu_B) \leq \hbar_e^{\varsigma}(\upsilon_A) \vee \hbar_e^{\varsigma}(\mu_B),$$

($\hbar_3$) *If* $(\top)_A \not\leq \upsilon_B$, *then* $\hbar_e(\upsilon_B) = 0$, $\hbar_e(\upsilon_B) = 1$, $\hbar_e(\upsilon_B) = 1$.
($\hbar_4$) *For every* $\upsilon_A \in (\mathcal{X} \widehat{\times} \mathcal{X}, E)$, $\langle \hbar_e^{\gamma} \rangle(\upsilon_A^s) \geq \hbar_e^{\gamma}(\upsilon_A)$, $\langle \hbar_e^{\pi} \rangle(\upsilon_A^s) \leq \hbar_e^{\pi}(\upsilon_A)$ *and* $\langle \hbar_e^{\varsigma} \rangle(\upsilon_A^s) \leq \hbar_e^{\varsigma}(\upsilon_A)$,
($\hbar_5$) *For every* $\upsilon_A \in (\mathcal{X} \widehat{\times} \mathcal{X}, E)$,

$$\bigvee \{\hbar_e^{\gamma}(\mu_B) : \ (\mu_B \circ \mu_B) \leq \upsilon_A\} \geq \hbar_e^{\gamma}(\upsilon_A), ; \quad \bigwedge \{\hbar_e^{\pi}(\mu_B) : \ (\mu_B \circ \mu_B) \leq \upsilon_A\} \leq \hbar_e^{\pi}(\upsilon_A),$$

$$\bigwedge \{\hbar_e^{\varsigma}(\mu_B) : \ (\mu_B \circ \mu_B) \leq \upsilon_A\} \leq \hbar_e^{\varsigma}(\upsilon_A).$$

A svns-uniform base $(\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma})$ *is said to be stratified if and only if* $(\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma})$ *satisfies*
($\hbar_{st}$) $\hbar_e^{\gamma}(\widehat{E_{\varrho}}) = 1$, $\hbar_e^{\pi}(\widehat{E_{\varrho}}) = 0$, $\hbar_e^{\varsigma}(\widehat{E_{\varrho}}) = 0$, $\forall\ \varrho \in \zeta$, $e \in E$.

In this case $(\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma})$ *is stratified single-valued neutrosophic soft uniform base (for short, ssvns-uniform base). Sometimes, we will write* $\hbar_E^{\gamma\pi\varsigma}$ *for* $(\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma})$.

Let $\hbar_E^{\gamma\pi\varsigma}$ *and* $\hbar_E^{\star\gamma\pi\varsigma}$ *be two svns-uniform bases on* $\mathcal{X}$. *Then,* $\hbar_E^{\gamma\pi\varsigma}$ *is finer than* $\hbar_E^{\star\gamma\pi\varsigma}$ ($\hbar_E^{\star\gamma\pi\varsigma}$ *is coarser than* $\hbar_E^{\gamma\pi\varsigma}$), *denoted by* $\hbar_E^{\star\gamma\pi\varsigma} \preceq \hbar_E^{\gamma\pi\varsigma}$ *provided*

$$\langle \hbar_e^{\star\gamma} \rangle(\upsilon_A) \leq \langle \hbar_e^{\gamma} \rangle(\upsilon_A), \ \ \langle \hbar_e^{\star\pi} \rangle(\upsilon_A) \geq \langle \hbar_e^{\pi} \rangle(\upsilon_A), \quad \langle \hbar_e^{\star\varsigma} \rangle(\upsilon_A) \geq \langle \hbar_e^{\varsigma} \rangle(\upsilon_A),$$

for each $e \in E$, $\upsilon_A \in (\mathcal{X} \widehat{\times} \mathcal{X}, E)$. *Obviously, all svns-uniformity* $\pounds_E^{\gamma\pi\varsigma}$ *on* $\mathcal{X}$ *is a svns- uniform base with* $\langle \pounds_E^{\gamma\pi\varsigma} \rangle = \pounds_E^{\gamma\pi\varsigma}$.

Theorem 2. *Let* $\hbar_E^{\gamma\pi\varsigma}$ *be a svns-uniform base on* $\mathcal{X}$, *define the mappings* $\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma} : E \to \zeta^{(\mathcal{X} \widehat{\times} \mathcal{X}, E)}$, *for any* $\upsilon_A \in (\mathcal{X} \widehat{\times} \mathcal{X}, E)$, $e \in E$ *as follows:*

$$(\hbar_{st}^{\gamma})_e(\upsilon_A) = \bigvee \{\hbar_e^{\gamma}((\mu_B)) : \ \mu_B \sqcap \widehat{E_{\varrho}} \leq \upsilon_A, \ \varrho \in \zeta\},$$

$$(\hbar_{st}^{\pi})_e(\upsilon_A) = \bigwedge \{\hbar_e^{\pi}((\mu_B)) : \ \mu_B \sqcap \widehat{E_{\varrho}} \leq \upsilon_A, \ \varrho \in \zeta\},$$

$$(\hbar_{st}^{\varsigma})_e(\upsilon_A) = \bigwedge \{\hbar_e^{\varsigma}((\mu_B)) : \ \mu_B \sqcap \widehat{E_{\varrho}} \leq \upsilon_A, \ \varrho \in \zeta\}.$$

Then,
(1) $(\hbar_{st}^{\gamma\pi\varsigma})_E$ *is the coarsest ssvns-uniform base which is finer than* $\hbar_E^{\gamma\pi\varsigma}$,
(2) $\langle (\hbar_{st}^{\gamma\pi\varsigma})_E \rangle = \langle \hbar_E^{\gamma\pi\varsigma} \rangle_{st}$.

Proof. (1) Similar to Theorem 1.

(2) It becomes clear to us from (1), that

$$\langle \hbar_{\mathrm{E}}^{\gamma} \rangle_{st} \preceq \langle (\hbar_{st}^{\gamma})_{\mathrm{E}} \rangle, \qquad \langle \hbar_{\mathrm{E}}^{\pi} \rangle_{st} \succeq \langle (\hbar_{st}^{\pi})_{\mathrm{E}} \rangle, \qquad \langle \hbar_{\mathrm{E}}^{\gamma} \rangle_{st} \succeq \langle (\hbar_{st}^{\gamma})_{\mathrm{E}} \rangle.$$

Conversely, let

$$\langle (\hbar_{st}^{\gamma})_e \rangle(\upsilon_A) \not\leq \langle \hbar_e^{\gamma} \rangle_{st}(\upsilon_A), \qquad \langle (\hbar_{st}^{\pi})_e \rangle(\upsilon_A) \not\geq \langle \hbar_e^{\pi} \rangle_{st}(\upsilon_A), \qquad \langle (\hbar_{st}^{\varsigma})_e \rangle(\upsilon_A) \not\geq \langle \hbar_e^{\varsigma} \rangle_{st}(\upsilon_A),$$

for some $\upsilon_A \in (\mathcal{X} \widehat{\times} \mathcal{X}, \mathrm{E})$. By the concept of $\langle (\hbar_{st}^{\gamma\pi\varsigma})_{\mathrm{E}} \rangle$, there exists $\mu_B \in (\mathcal{X} \widehat{\times} \mathcal{X}, \mathrm{E})$ with $\mu_B \preceq \upsilon_A$ such that

$$(\hbar_{st}^{\gamma})_e(\mu_B) \not\leq \langle \hbar_e^{\gamma} \rangle_{st}(\upsilon_A), \qquad (\hbar_{st}^{\pi})_e(\mu_B) \not\geq \langle \hbar_e^{\pi} \rangle_{st}(\upsilon_A), \qquad (\hbar_{st}^{\varsigma})_e(\mu_B) \not\geq \langle \hbar_e^{\varsigma} \rangle_{st}(\upsilon_A).$$

By the concept of $\langle \hbar_{\mathrm{E}}^{\gamma\pi\varsigma} \rangle_{st}$, there exists $\sigma_C \in (\mathcal{X} \widehat{\times} \mathcal{X}, \mathrm{E})$, $\varrho \in \zeta$ with $\sigma_C \sqcap \widehat{E_{\varrho}} \preceq \mu_B$ such that

$$\hbar_e^{\gamma}(\sigma_C) \not\leq \langle \hbar_e^{\gamma} \rangle_{st}(\upsilon_A), \qquad \hbar_e^{\pi}(\sigma_C) \not\geq \langle \hbar_e^{\pi} \rangle_{st}(\upsilon_A), \qquad \hbar_e^{\varsigma}(\sigma_C) \not\geq \langle \hbar_e^{\varsigma} \rangle_{st}(\upsilon_A).$$

On the other hand, $\sigma_C \sqcap \widehat{E_\varrho} \preceq \upsilon_A$ implies that

$$\langle \hbar_e^\gamma \rangle_{st}(\upsilon_A) \geq \langle \hbar_e^\gamma \rangle(\sigma_C) \geq \hbar_e^\gamma(\sigma_C), \qquad \langle \hbar_e^\pi \rangle_{st}(\upsilon_A) \leq \langle \hbar_e^\pi \rangle(\sigma_C) \leq \hbar_e^\gamma(\sigma_C),$$

$$\langle \hbar_e^\varsigma \rangle_{st}(\upsilon_A) \leq \langle \hbar_e^\varsigma \rangle(\sigma_C) \leq \hbar_e^\gamma(\sigma_C).$$

It is a contradiction. Hence, $\langle \hbar_e^\gamma \rangle_{st}(\upsilon_A) \geq \langle (\hbar_{st}^\gamma)_e \rangle(\upsilon_A)$, $\langle \hbar_e^\pi \rangle_{st}(\upsilon_A) \leq \langle (\hbar_{st}^\pi)_e \rangle(\upsilon_A)$, $\langle \hbar_e^\varsigma \rangle_{st}(\upsilon_A) \leq \langle (\hbar_{st}^\varsigma)_e \rangle(\upsilon_A)$, and $\langle (\hbar_{st}^{\gamma\pi\varsigma})_{\mathrm{E}} \rangle = \langle \hbar_{\mathrm{E}}^{\gamma\pi\varsigma} \rangle_{st}$. □

Theorem 3. *Let $(\mathcal{X}, \pounds_E^{\gamma\pi\varsigma})$ be an svns-uniform space. For all $f_B \in (\widehat{\mathcal{X}, E})$ and $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}, E})$, the image $\upsilon_A[f_B]$ of f_B with respect to υ_A is the svns of $\mathcal{X}$ defined by*

$$(\upsilon_e[f_e])(x) = \bigvee_{y \in \mathcal{X}} [f_e(y) \wedge \upsilon_e(y, x)], \forall,\ e \in A \cap B \text{ and } x \in \mathcal{X}.$$

For $f_C, (f_D)_j \in (\widehat{\mathcal{X}, E})$, $\upsilon_A, \mu_B \in (\widehat{\mathcal{X} \times \mathcal{X}, E})$, we have:
(1) $f_C \preceq \upsilon_A[f_C]$ *whenever* $\pounds_e^\gamma(\upsilon_A) > 0$, $\pounds_e^\pi(\upsilon_A) < 1$, $\pounds_e^\varsigma(\upsilon_A) < 1$,
(2) $\upsilon_A \preceq \upsilon_A \circ \upsilon_A$ *whenever* $\pounds_e^\gamma(\upsilon_A) > 0$, $\pounds_e^\pi(\upsilon_A) < 1$, $\pounds_e^\varsigma(\upsilon_A) < 1$,
(3) $(\mu_B \circ \upsilon_A)[f_C] = \mu_B[\upsilon_A[f_C]]$,
(4) $\upsilon_A[\bigsqcup_j (f_D)_j] = \bigsqcup_j \upsilon_A[(f_D)_j]$,
(5) $(\upsilon_A \sqcap \mu_B)[(f_D)_1 \sqcap (f_D)_2] \preceq \upsilon_A[(f_D)_1] \sqcap \mu_B[(f_D)_2]$,
(6) $(\upsilon_A \sqcup \mu_B)[(f_D)_1 \sqcup (f_D)_2] \preceq \upsilon_A[(f_D)_1] \sqcup \mu_B[(f_D)_2]$,
(7) $\upsilon_A[(\upsilon_A^s[f_C])^c] \preceq f_C^c$.

Proof. Obvious. □

Theorem 4. *Let $\hbar_E^{\gamma\pi\varsigma}$ be a svns-uniform base on $\mathcal{X}$. define the operator $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}} : E \times (\widehat{\mathcal{X}, E}) \times \zeta_0 \to (\widehat{\mathcal{X}, E})$ as next for every $e \in E$, $r \in \zeta$, $f_B \in (\widehat{\mathcal{X}, E})$,*

$$\mathcal{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_B, r) = \bigsqcup \{\mathcal{R}_C : \upsilon_A[\mathcal{R}_C] \preceq f_B, \hbar_e^\gamma(\upsilon_A) \geq r,\ \hbar_e^\pi(\upsilon_A) \leq 1 - r,\ \hbar_e^\varsigma(\upsilon_A) \leq 1 - r\}.$$

Then, $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}$ is an svnsi-operator on $\mathcal{X}$.

Proof. ($\mathtt{I}_1$) Since $\widehat{E} = \upsilon_E[\widehat{E}]$, for all $\hbar_e^\gamma(\upsilon_E) \geq r$, $\hbar_e^\pi(\upsilon_E) \leq 1 - r$, $\hbar_e^\varsigma(\upsilon_E) \leq 1 - r$, then $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \widehat{E}, r) = \widehat{E}$.

($\mathtt{I}_2$) Whenever $\mathcal{R}_C \preceq \upsilon_A[\mathcal{R}_C] \preceq f_B$, $\forall\ \hbar_e^\gamma(\upsilon_A) \geq r$, $\hbar_e^\pi(\upsilon_A) \leq 1 - r$, $\hbar_e^\varsigma(\upsilon_A) \leq 1 - r$, we get that $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_B, r) \preceq f_B$ for all $f_B \in (\widehat{\mathcal{X}, \mathrm{E}})$.

($\mathtt{I}_3$) Clearly, $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_B, r) \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \mathcal{R}_D, s)$ for every $f_B \preceq \mathcal{R}_D$, $f_B, \mathcal{R}_D \in (\widehat{\mathcal{X}, \mathrm{E}})$ and $r \leq s$.

($\mathtt{I}_4$) Assume that

$$\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, (f_C)_1, r) \sqcap \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, (f_C)_2, s) \not\preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, (f_C)_1 \sqcap (f_C)_2, r \wedge s).$$

Then, there exists $(\mathcal{R}_D)_1, (\mathcal{R}_D)_2 \in (\widehat{\mathcal{X}, \mathrm{E}})$ with $\upsilon_A[(\mathcal{R}_D)_1] \preceq (f_C)_1$ $\mu_B[(\mathcal{R}_D)_2] \preceq (f_C)_2$ and

$$\hbar_e^\gamma(\upsilon_A) \geq r, \qquad \hbar_e^\pi(\upsilon_A) \leq 1 - r, \qquad \hbar_e^\varsigma(\upsilon_A) \leq 1 - r,$$

$$\hbar_e^\gamma(\mu_B) \geq s, \qquad \hbar_e^\pi(\mu_B) \leq 1 - s, \qquad \hbar_e^\varsigma(\mu_B) \leq 1 - s,$$

such that

$$(\mathcal{R}_D)_1 \sqcap (\mathcal{R}_D)_2 \not\preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(\widehat{e}, (f_C)_1 \sqcap (f_C)_2, r \wedge s).$$

Since

$$\hbar^{\gamma}_{e}(\upsilon_{A} \sqcap \mu_{B}) \geq \hbar^{\gamma}_{e}(\upsilon_{A}) \wedge \hbar^{\gamma}_{e}(\mu_{B}), \qquad \hbar^{\pi}_{e}(\upsilon_{A} \sqcap \mu_{B}) \leq \hbar^{\pi}_{e}(\upsilon_{A}) \vee \hbar^{\pi}_{e}(\mu_{B}),$$
$$\hbar^{\varsigma}_{e}(\upsilon_{A} \sqcap \mu_{B}) \leq \hbar^{\varsigma}_{e}(\upsilon_{A}) \vee \hbar^{\varsigma}_{e}(\mu_{B}),$$

we get than

$$(\upsilon_{A} \sqcap \mu_{B})[(\mathcal{R}_{D})_{1} \sqcap (\mathcal{R}_{D})_{2}] \leq \upsilon_{A}[(\mathcal{R}_{D})_{1}] \sqcap \mu_{B}[(\mathcal{R}_{D})_{2}] \leq (f_{C})_{1} \sqcap (f_{C})_{2}.$$

Then,

$$(\mathcal{R}_{D})_{1} \sqcap (\mathcal{R}_{D})_{2} \leq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(\widehat{e}, (f_{C})_{1} \sqcap (f_{C})_{2}, r \wedge s).$$

This is a contradiction. Consequently, ($\mathtt{I}_4$) holds.

($\mathtt{I}_5$) Assume that $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_{C}, r) \not\leq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_{C}, r), r)$. By using the definition of $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_{C}, r)$, there exists $\upsilon_{A} \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$ and $\mathcal{R}_{D} \in (\widehat{\mathcal{X}, \mathrm{E}})$, such that

$$\hbar^{\gamma}_{e}(\upsilon_{A}) \geq r,\ \hbar^{\pi}_{e}(\upsilon_{A}) \leq 1 - r,\ \hbar^{\varsigma}_{e}(\upsilon_{A}) \leq 1 - r,\ \upsilon_{A}[\mathcal{R}_{D}] \leq f_{C},$$

and $\mathcal{R}_{D} \not\leq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_{C}, r), r)$. Otherwise, since

$$\bigvee\{\hbar^{\gamma}_{e}(\mu_{B}) :\ \mu_{B} \circ \mu_{B} \leq \upsilon_{A}\} \geq \hbar^{\gamma}_{e}(\upsilon_{A}) \geq r,$$

$$\bigwedge\{\hbar^{\pi}_{e}(\mu_{B}) :\ \mu_{B} \circ \mu_{B} \leq \upsilon_{A}\} \leq \hbar^{\pi}_{e}(\upsilon_{A}) \leq 1 - r,$$

$$\bigwedge\{\hbar^{\varsigma}_{e}(\mu_{B}) :\ \mu_{B} \circ \mu_{B} \leq \upsilon_{A}\} \leq \hbar^{\varsigma}_{e}(\upsilon_{A}) \leq 1 - r,$$

there exists $\mu_{B} \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$ with $\mu_{B} \circ \mu_{B} \leq \upsilon_{A}$ such that

$$\hbar^{\gamma}_{e}(\mu_{B}) \geq r, \quad \hbar^{\pi}_{e}(\mu_{B}) \leq 1 - r, \quad \hbar^{\varsigma}_{e}(\mu_{B}) \leq 1 - r, \qquad \mu_{B}[\mu_{B}[\mathcal{R}_{D}]] \leq \upsilon_{A}[\mathcal{R}_{D}] \leq f_{C}.$$

By using the definition of $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_{C}, r)$, we obtain $\mu_{B}[\mathcal{R}_{D}] \leq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_{C}, r)$. By the concept of $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_{C}, r), r)$, it follows that

$$\mathcal{R}_{D} \leq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_{C}, r), r).$$

This is a contradiction. Consequently, ($\mathtt{I}_5$) holds. □

Theorem 5. *Let $\hbar^{\gamma\pi\varsigma}_{E}$ be a svns-uniform base on $\mathcal{X}$. Define the operator $\mathtt{C}_{\hbar^{\gamma\pi\varsigma}} : E \times (\widehat{\mathcal{X}, E}) \times \zeta_{0} \to (\widehat{\mathcal{X}, E})$ as next for every $e \in E$, $f_{B} \in (\widehat{\mathcal{X}, E})$, $r \in \zeta$,*

$$\mathtt{C}_{\hbar^{\gamma\pi\varsigma}}(e, f_{B}, r) = \bigsqcap\{\upsilon^{s}_{A}[f_{B}] : \hbar^{\gamma}_{e}(\upsilon_{A}) \geq r,\ \hbar^{\pi}_{e}(\upsilon_{A}) \leq 1 - r,\ \hbar^{\varsigma}_{e}(\upsilon_{A}) \leq 1 - r\},$$

Then, $\mathtt{C}_{\hbar^{\gamma\pi\varsigma}}$ is a svnsc-operator on $\mathcal{X}$.

Proof. ($\mathtt{C}_1$) Since $\widehat{\Phi} = \upsilon_{A}[\widehat{\Phi}]$, for all $\hbar^{\gamma}_{e}(\upsilon_{A}) \geq r,\ \hbar^{\pi}_{e}(\upsilon_{A}) \leq 1 - r,\ \hbar^{\varsigma}_{e}(\upsilon_{A}) \leq 1 - r$, then $\mathtt{C}_{\hbar^{\gamma\pi\varsigma}}(e, \widehat{\Phi}, r) = \widehat{\Phi}$.

($\mathtt{C}_2$) Whenever $\mathcal{R}_{C} \leq \upsilon_{A}[\mathcal{R}_{C}] \leq f_{B}$, for all $\hbar^{\gamma}_{e}(\upsilon_{A}) \geq r,\ \hbar^{\pi}_{e}(\upsilon_{A}) \leq 1 - r,\ \hbar^{\varsigma}_{e}(\upsilon_{A}) \leq 1 - r$, we get that $\mathtt{C}_{\hbar^{\gamma\pi\varsigma}}(e, f_{B}, r) \geq f_{B}$ for each $f_{B} \in (\widehat{\mathcal{X}, \mathrm{E}})$.

($\mathtt{C}_3$) It is established that $\mathtt{C}_{\hbar^{\gamma\pi\varsigma}}(e, f_{B}, r) \leq \mathtt{C}_{\hbar^{\gamma\pi\varsigma}}(e, \mathcal{R}_{D}, s)$ for every $f_{B} \leq \mathcal{R}_{D}$, $f_{B}, \mathcal{R}_{D} \in (\widehat{\mathcal{X}, \mathrm{E}})$ and $r \leq s$.

($\mathtt{C}_4$) Assume that

$$\mathtt{C}_{\hbar^{\gamma\pi\varsigma}}(e, f_{C}, r) \sqcup \mathtt{C}_{\hbar^{\gamma\pi\varsigma}}(e, \mathcal{R}_{D}, s) \not\geq \mathtt{C}_{\hbar^{\gamma\pi\varsigma}}(e, f_{C} \sqcup \mathcal{R}_{D}, r \wedge s).$$

Then, there exists $r, s \in \zeta_0$, $\upsilon_A, \mu_B \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$ with

$$\hbar_e^{\gamma}(\upsilon_A) \geq r \wedge s, \qquad \hbar_e^{\pi}(\upsilon_A) \leq 1 - (r \wedge s), \qquad \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - (r \wedge s),$$

$$\hbar_e^{\gamma}(\mu_B) \geq s \wedge s, \qquad \hbar_e^{\pi}(\mu_B) \leq 1 - (s \wedge s), \qquad \hbar_e^{\varsigma}(\mu_B) \leq 1 - (r \wedge s),$$

such that

$$\upsilon_A^s[f_C] \sqcup \mu_B^s[\mathcal{R}_D] \not\geq \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C \sqcup \mathcal{R}_D, r \wedge s).$$

Since $\hbar_e^{\gamma}(\upsilon_A \sqcup \mu_B) \geq \hbar_e^{\gamma}(\upsilon_A) \sqcap \hbar_e^{\gamma}(\mu_B) \geq r \wedge s$, $\hbar_e^{\pi}(\upsilon_A \sqcup \mu_B) \leq \hbar_e^{\pi}(\upsilon_A) \sqcup \hbar_e^{\pi}(\mu_B) \leq 1 - (r \vee s)$, $\hbar_e^{\varsigma}(\upsilon_A \sqcup \mu_B) \leq \hbar_e^{\varsigma}(\upsilon_A) \sqcup \hbar_e^{\varsigma}(\mu_B) \leq 1 - (r \vee s)$ and $(\upsilon_A \sqcup \mu_B)^s[f_C \sqcup \mathcal{R}_D] \leq \upsilon_A^s[f_C] \sqcup \mu_B^s[\mathcal{R}_D]$, then $\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C \sqcup \mathcal{R}_D, r \wedge s) \leq \upsilon_A^s[f_C] \sqcup \mu_B^s[\mathcal{R}_D]$. It is a contradiction. Thus, (C_4) holds.

(C_5) Assume that there exists $r \in \zeta_0$, $e \in \mathrm{E}$, $f_C \in (\widehat{\mathcal{X}, \mathrm{E}})$, such that

$$\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r) \not\geq \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r), r).$$

Using the concept of $\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r)$, there exist $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$ with

$$\hbar_e^{\gamma}(\upsilon_A) \geq r, \qquad \hbar_e^{\pi}(\upsilon_A) \leq 1 - r, \qquad \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r,$$

such that $\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r), r) \not\leq \upsilon_A^s[f_C]$. Otherwise, from $(\hbar_5)$, we have

$$\bigvee\{\hbar_e^{\gamma}(\mu_B) : (\mu_B \circ \mu_B) \leq \upsilon_A\} \geq \hbar_e^{\gamma}(\upsilon_A) \geq r, \qquad \bigwedge\{\hbar_e^{\pi}(\mu_B) : (\mu_B \circ \mu_B) \leq \upsilon_A\} \leq \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,$$

$$\bigwedge\{\hbar_e^{\varsigma}(\mu_B) : (\mu_B \circ \mu_B) \leq \upsilon_A\} \leq \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r,$$

which leads to the existence of $\mu_B \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$ with $\mu_B \circ \mu_B \preceq \upsilon_A$ and

$$\hbar_e^{\gamma}(\mu_B) \geq r, \qquad \hbar_e^{\pi}(\mu_B) \leq 1 - r, \qquad \hbar_e^{\varsigma}(\mu_B) \leq 1 - r.$$

It follows that

$$\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r), r) \preceq \mu_B^s[\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r)] \preceq \mu_B^s[\mu_B^s[f_C]] \preceq \upsilon_A^s[f_C].$$

It is a contradiction. Thus, (C_5) holds.

(C_6) We want, for each $e \in \mathrm{E}$, $f_C \in (\widehat{\mathcal{X}, \mathrm{E}})$, $r \in \zeta_0$, to verify that $\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r) = (\mathtt{I}_{\hbar\gamma\pi\varsigma}(e, f_C^c, r))^c$. This means that we need to prove it:

$$\begin{aligned}
&\bigsqcap\{\upsilon_A^s[f_C] : \hbar_e^{\gamma}(\upsilon_A) \geq r, \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r\} \\
&= \bigsqcap\{\mathcal{R}_D^c : \upsilon_A[\mathcal{R}_D] \leq f_C^c, \hbar_e^{\gamma}(\upsilon_A) \geq r,\ \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r\}.
\end{aligned}$$

Since $\upsilon_A[(\upsilon_A^s[f_C])^c] \leq f_C^c$, from (7) in Theorem 3, we obtain

$$\begin{aligned}
&\bigsqcap\{\upsilon_A^s[f_C] : \hbar_e^{\gamma}(\upsilon_A) \geq r, \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r\} \\
&= \bigsqcap\{\mathcal{R}_D^c : \upsilon_A[\mathcal{R}_D] \leq f_C^c, \hbar_e^{\gamma}(\upsilon_A) \geq r,\ \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r\}.
\end{aligned}$$

Since $(\upsilon_A[\mathcal{R}_D])^c \geq f_C$, we obtain $\upsilon_A^s[f_C] \leq \upsilon_A^s[(\upsilon_A[\mathcal{R}_D])^c]$. Then,

$$\begin{aligned}
&\bigsqcap\{\upsilon_A^s[f_C] : \hbar_e^{\gamma}(\upsilon_A) \geq r, \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r\} \\
&= \bigsqcap\{\mathcal{R}_D^c : \upsilon_A[\mathcal{R}_D] \leq f_C^c, \hbar_e^{\gamma}(\upsilon_A) \geq r,\ \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r\}.
\end{aligned}$$

Thus, (C_6) holds. □

4. Single-valued neutrosophic soft topologies induced by a single-valued neutrosophic soft uniformity

In this section, we study several single-valued neutrosophic soft topologies induced by a single-valued neutrosophic soft uniform structure. We have proved that single-valued neutrosophic soft uniform base and single-valued neutrosophic soft uniform space are single-valued neutrosophic soft topological spaces.

Theorem 6. *Let $\hbar_E^{\gamma\pi\varsigma}$ be an svns-uniform base on $\mathcal{X}$, define the mappings $\mathfrak{T}_\hbar^\gamma : E \to \zeta^{(\widehat{\mathcal{X},E})}$, $\mathfrak{T}_\hbar^\pi : E \to \zeta^{(\widehat{\mathcal{X},E})}$, $\mathfrak{T}_\hbar^\varsigma : E \to \zeta^{(\widehat{\mathcal{X},E})}$ as follows for each $e \in E$, $r \in \zeta_0$, $f_A \in (\widehat{\mathcal{X},E})$,*

$$(\mathfrak{T}_\hbar^\gamma)_e(f_A) = \bigvee\{r : \quad f_A \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\},$$

$$(\mathfrak{T}_\hbar^\pi)_e(f_A) = \bigwedge\{1 - r : \quad f_A \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\},$$

$$(\mathfrak{T}_\hbar^\varsigma)_e(f_A) = \bigwedge\{1 - r : \quad f_A \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\}.$$

Then, $\mathfrak{T}_\hbar^{\gamma\pi\varsigma}$ is an svnst on $\mathcal{X}$.

Proof. $(\mathfrak{T}_1)$ Since $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \widehat{E}, r) = \widehat{E}$ and $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \widehat{\Phi}, r) = \widehat{\Phi}$ for each $r \in \zeta_0$, $e \in$ E, then

$$\mathfrak{T}_e^\gamma(\widehat{\Phi}) = 1, \quad \mathfrak{T}_e^\pi(\widehat{\Phi}) = 0, \quad = \mathfrak{T}_e^\varsigma(\widehat{\Phi}) = 0,$$

$$\mathfrak{T}_e^\gamma(\widehat{E}) = 1, \quad \mathfrak{T}_e^\pi(\widehat{E}) = 0, \quad = \mathfrak{T}_e^\varsigma(\widehat{E}) = 0.$$

$(\mathfrak{T}_2)$ To prove the second condition, we follow as follows:

$$\begin{aligned}
(\mathfrak{T}_\hbar^\gamma)_e(f_A) \quad \wedge (\mathfrak{T}_\hbar^\gamma)_e(g_B) &= \bigvee\{r \mid f_A \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\} \wedge \bigvee\{s \mid g_B \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\} \\
&\leq \bigvee\{r \wedge s \mid f_A \sqcap g_B \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r) \sqcap \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\} \\
&\leq \bigvee\{r \wedge s \mid f_A \sqcap g_B \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A \sqcap g_B, r \wedge s)\} \\
&\leq (\mathfrak{T}_\hbar^\gamma)_e(f_A \sqcap g_B),
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{T}_\hbar^\pi)_e(f_A) \vee (\mathfrak{T}_\hbar^\pi)_e(g_B) &= \bigwedge\{1 - r \mid f_A \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\} \vee \bigwedge\{1 - s \mid g_B \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\} \\
&\geq \bigwedge\{1 - r \vee 1 - s) \mid f_A \sqcup g_B \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r) \sqcup \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\} \\
&\geq \bigwedge\{1 - (r \wedge s) \mid f_A \sqcup g_B \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A \sqcup g_B, r \wedge s)\} \\
&\geq \bigwedge\{1 - (r \wedge s) \mid f_A \sqcap g_B \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A \sqcap g_B, r \wedge s)\} \\
&\geq (\mathfrak{T}_\hbar^\pi)_e(f_A \sqcap g_B),
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{T}_\hbar^\varsigma)_e(f_A) \vee (\mathfrak{T}_\hbar^\varsigma)_e(g_B) &= \bigwedge\{1 - r \mid f_A \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\} \vee \bigwedge\{1 - s \mid g_B \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\} \\
&\geq \bigwedge\{1 - r \vee 1 - s) \mid f_A \sqcup g_B \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r) \sqcup \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\} \\
&\geq \bigwedge\{1 - (r \wedge s) \mid f_A \sqcup g_B \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A \sqcup g_B, r \wedge s)\}
\end{aligned}$$

$$\geq \quad \{1-(r\wedge s) \mid f_A \sqcap g_B \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A \sqcap g_B, r\wedge s)\}$$
$$\geq (\mathfrak{T}^{\varsigma}_{\hbar})_e(f_A \sqcap g_B).$$

($\mathfrak{T}_3$) Assume that there exists a collection $\{(f_A)_j : j\in\Gamma\}$ such that

$$\mathfrak{T}^{\gamma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \not\geq \bigwedge_{j\in\Gamma}\mathfrak{T}^{\gamma}_e((f_A)_j), \qquad \mathfrak{T}^{\pi}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \not\leq \bigvee_{j\in\Gamma}\mathfrak{T}^{\pi}_e((f_A)_j),$$

$$\mathfrak{T}^{\varsigma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \not\leq \bigvee_{j\in\Gamma}\mathfrak{T}^{\varsigma}_e((f_A)_j).$$

For every $j\in\Gamma$, there exist $r_j\in\zeta_0$ such that $(f_A)_j \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)$ and that

$$\mathfrak{T}^{\gamma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \not\geq \bigwedge_{j\in\Gamma} r_j, \qquad \mathfrak{T}^{\pi}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \not\leq \bigvee_{j\in\Gamma}(1-r)_j, \qquad \mathfrak{T}^{\varsigma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \not\leq \bigvee_{j\in\Gamma}(1-r)_j.$$

Putting $r = \bigwedge\limits_{j\in\Gamma} r_j$ and $1-r = \bigvee\limits_{j\in\Gamma}(1-r)_j$, from Theorem 4, we get that

$$\bigsqcup_{j\in\Gamma}(f_A)_j \preceq \bigsqcup_{j\in\Gamma}\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e,(f_A)_j, r_j) \preceq \bigsqcup_{j\in\Gamma}\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e,(f_A)_j, r) \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \bigsqcup_{j\in\Gamma}(f_A)_j, r).$$

It follows that

$$\mathfrak{T}^{\gamma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \geq \bigwedge_{j\in\Gamma} r_j = r, \qquad \mathfrak{T}^{\pi}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \leq \bigvee_{j\in\Gamma}(1-r)_j = 1-r,$$

$$\mathfrak{T}^{\varsigma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \leq \bigvee_{j\in\Gamma}(1-r)_j = 1-r.$$

It is a contradiction. Thus, $\mathfrak{T}_3$ holds. □

Definition 10. *Let $f_B\in\widehat{(\mathcal{X},E)}$ and $\upsilon_A\in\widehat{(\mathcal{X}\times\mathcal{X},E)}$. Define $\upsilon_A^{f_B}\in\widehat{(\mathcal{X}\times\mathcal{X},E)}$, for each $e\in A\cap B$ related with f_B by*

$$(\upsilon_A^{f_B})_e(x,y) = \begin{cases} \langle 1,0,0\rangle, & \text{if } x=y,\\ \gamma_{f_e(x)\wedge f_e(y)}, \pi_{f_e(x)\vee f_e(y)}, \varsigma_{f_e(x)\vee f_e(y)}, & \text{otherwise.}\end{cases}$$

Theorem 7. *Let $(\mathcal{X}, \pounds_E^{\gamma\pi\varsigma})$ be an svns-uniform space, define the mappings $\mathfrak{T}^{*\gamma}_{\pounds}, \mathfrak{T}^{*\pi}_{\pounds}, \mathfrak{T}^{*\varsigma}_{\pounds} : E\to\zeta^{\widehat{(\mathcal{X},E)}}$ as follows:*

$$(\mathfrak{T}^{*\gamma}_{\pounds})_e(f_B) = \begin{cases} 1, & \text{if } f_B=\widehat{\Phi},\\ \pounds^{\gamma}_e(\upsilon_A^{f_B}), & \text{if } f_B\in\widehat{(\mathcal{X},E)}-\widehat{\Phi},\end{cases}$$

$$(\mathfrak{T}^{*\pi}_{\pounds})_e(f_B) = \begin{cases} 0, & \text{if } f_B=\widehat{\Phi},\\ \pounds^{\pi}_e(\upsilon_A^{f_B}), & \text{if } f_B\in\widehat{(\mathcal{X},E)}-\widehat{\Phi},\end{cases}$$

$$(\mathfrak{T}^{*\varsigma}_{\pounds})_e(f_B) = \begin{cases} 0, & \text{if } f_B=\widehat{\Phi},\\ \pounds^{\varsigma}_e(\upsilon_A^{f_B}), & \text{if } f_B\in\widehat{(\mathcal{X},E)}-\widehat{\Phi}.\end{cases}$$

Then, $\mathfrak{T}^{*\gamma\pi\varsigma}_{\pounds}$ is an svnst on $\mathcal{X}$.

Proof. $(\mathfrak{T}_1)$ $(\mathfrak{T}^{*\gamma}_{\pounds})_e(\widehat{\Phi}) = 1$, $(\mathfrak{T}^{*\pi}_{\pounds})_e(\widehat{\Phi}) = 0$, $(\mathfrak{T}^{*\varsigma}_{\pounds})_e(\widehat{\Phi}) = 0$ and $(\mathfrak{T}^{*\gamma}_{\pounds})_e(\widehat{E}) = \pounds^{\gamma}_{e}(\upsilon^{\widehat{E}}_{e}) = 1$, $(\mathfrak{T}^{*\pi}_{\pounds})_e(\widehat{E}) = \pounds^{\pi}_{e}(\upsilon^{\widehat{E}}_{e}) = 0$, $(\mathfrak{T}^{*\varsigma}_{\pounds})_e(\widehat{E}) = \pounds^{\varsigma}_{e}(\upsilon^{\widehat{E}}_{e}) = 0$.

$(\mathfrak{T}_2)$ Since $\upsilon^{f_B}_A \sqcap \upsilon^{\mathcal{R}_C}_A = \upsilon^{f_B \sqcap \mathcal{R}_C}_A$ for every $f_B, \mathcal{R}_C \in (\widehat{\mathcal{X}, \mathrm{E}})$, by $(\pounds_3)$, we have

$$\pounds^{\gamma}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_A) = \pounds^{\gamma}_{e}(\upsilon^{f_B}_A \sqcap \upsilon^{\mathcal{R}_C}_A) \geq \pounds^{\gamma}_{e}(\upsilon^{f_B}_A) \wedge \pounds^{\gamma}_{e}(\upsilon^{\mathcal{R}_C}_A),$$

$$\pounds^{\pi}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_A) = \pounds^{\pi}_{e}(\upsilon^{f_B}_A \sqcap \upsilon^{\mathcal{R}_C}_A) \leq \pounds^{\pi}_{e}(\upsilon^{f_B}_A) \vee \pounds^{\pi}_{e}(\upsilon^{\mathcal{R}_C}_A),$$

$$\pounds^{\varsigma}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_A) = \pounds^{\varsigma}_{e}(\upsilon^{f_B}_A \sqcap \upsilon^{\mathcal{R}_C}_A) \leq \pounds^{\varsigma}_{e}(\upsilon^{f_B}_A) \vee \pounds^{\varsigma}_{e}(\upsilon^{\mathcal{R}_C}_A).$$

Thus,

$$(\mathfrak{T}^{*\gamma}_{\pounds})_e(f_B \sqcap \mathcal{R}_C) = \pounds^{\gamma}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_A) \geq \pounds^{\gamma}_{e}(\upsilon^{f_B}_A) \wedge \pounds^{\gamma}_{e}(\upsilon^{\mathcal{R}_C}_A) = (\mathfrak{T}^{*\gamma}_{\pounds})_e(f_B) \wedge (\mathfrak{T}^{*\gamma}_{\pounds})_e(\mathcal{R}_C),$$

$$(\mathfrak{T}^{*\pi}_{\pounds})_e(f_B \sqcap \mathcal{R}_C) = \pounds^{\pi}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_A) \leq \pounds^{\pi}_{e}(\upsilon^{f_B}_A) \vee \pounds^{\pi}_{e}(\upsilon^{\mathcal{R}_C}_A) = (\mathfrak{T}^{*\pi}_{\pounds})_e(f_B) \vee (\mathfrak{T}^{*\pi}_{\pounds})_e(\mathcal{R}_C),$$

$$(\mathfrak{T}^{*\varsigma}_{\pounds})_e(f_B \sqcap \mathcal{R}_C) = \pounds^{\varsigma}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_A) \leq \pounds^{\varsigma}_{e}(\upsilon^{f_B}_A) \vee \pounds^{\varsigma}_{e}(\upsilon^{\mathcal{R}_C}_A) = (\mathfrak{T}^{*\varsigma}_{\pounds})_e(f_B) \vee (\mathfrak{T}^{*\varsigma}_{\pounds})_e(\mathcal{R}_C).$$

$(\mathfrak{T}_3)$ Similar to the proof in $(\mathfrak{T}_3)$ from Theorem 6. □

Theorem 8. *Let $(\mathcal{X}, \pounds^{\gamma\pi\varsigma}_{E})$ be a svns-uniform space, define the mappings $\mathfrak{T}^{\star\star\gamma}_{\pounds}, \mathfrak{T}^{\star\star\pi}_{\pounds}, \mathfrak{T}^{\star\star\varsigma}_{\pounds} : E \to \zeta^{(\widehat{\mathcal{X},E})}$ as follows:*

$$(\mathfrak{T}^{\star\star\gamma}_{\pounds})_e(f_B) = \bigwedge_{x\in\mathcal{X}} \left[(f_e)^c(x) \vee \bigvee_{\upsilon_A[x]\preceq f_A} \pounds^{\gamma}_{e}(\upsilon_A) \right],$$

$$(\mathfrak{T}^{\star\star\pi}_{\pounds})_e(f_B) = \bigvee_{x\in\mathcal{X}} \left[(f_e)^c(x) \wedge \bigvee_{\upsilon_A[x]\preceq f_A} \pounds^{\pi}_{e}(\upsilon_A) \right],$$

$$(\mathfrak{T}^{\star\star\varsigma}_{\pounds})_e(f_B) = \bigvee_{x\in\mathcal{X}} \left[(f_e)^c(x) \wedge \bigvee_{\upsilon_A[x]\preceq f_A} \pounds^{\varsigma}_{e}(\upsilon_A) \right].$$

Then, $\mathfrak{T}^{\star\star\gamma\pi\varsigma}_{\pounds}$ is an svnst on $\mathcal{X}$, where $(\upsilon_A[x])(y) \preceq \upsilon_A(y, x)$ for all $e \in A$.

Proof. $(\mathfrak{T}_1)$ Obvious.

$(\mathfrak{T}_2)$ Assume that

$$\bigvee_{\upsilon_A[x]\preceq(f_D)_1} \pounds^{\gamma}_{e}(\upsilon_A) \wedge \bigvee_{\mu_B[x]\preceq(f_D)_2} \pounds^{\gamma}_{e}(\mu_B) \not\leq \bigvee_{\kappa_C[x]\preceq(f_D)_1\sqcap(f_D)_2} \pounds^{\gamma}_{e}(\kappa_C),$$

$$\bigwedge_{\upsilon_A[x]\preceq(f_D)_1} \pounds^{\pi}_{e}(\upsilon_A) \vee \bigwedge_{\mu_B[x]\preceq(f_D)_2} \pounds^{\pi}_{e}(\mu_B) \not\geq \bigwedge_{\kappa_C[x]\preceq(f_D)_1\sqcap(f_D)_2} \pounds^{\pi}_{e}(\kappa_C),$$

$$\bigwedge_{\upsilon_A[x]\preceq(f_D)_1} \pounds^{\varsigma}_{e}(\upsilon_A) \vee \bigwedge_{\mu_B[x]\preceq(f_D)_2} \pounds^{\varsigma}_{e}(\mu_B) \not\geq \bigwedge_{\kappa_C[x]\preceq(f_D)_1\sqcap(f_D)_2} \pounds^{\varsigma}_{e}(\kappa_C).$$

Then, there exists υ_A, μ_B with $\upsilon_A[x] \preceq (f_D)_1$, $\mu_B[x] \preceq (f_D)_2$ such that

$$\pounds^{\gamma}_{e}(\upsilon_A) \wedge \pounds^{\gamma}_{e}(\mu_B) \not\leq \bigvee_{\kappa_C[x]\preceq(f_D)_1\sqcap(f_D)_2} \pounds^{\pi}_{e}(\kappa_C), \qquad \pounds^{\pi}_{e}(\upsilon_A) \vee \pounds^{\pi}_{e}(\mu_B) \not\geq \bigwedge_{\kappa_C[x]\preceq(f_D)_1\sqcap(f_D)_2} \pounds^{\pi}_{e}(\kappa_A),$$

$$\pounds_e^{\varsigma}(\upsilon_A) \vee \pounds_e^{\varsigma}(\mu_B) \ngeq \bigwedge_{\kappa_C[x] \preceq (f_D)_1 \sqcap (f_D)_2} \pounds_e^{\varsigma}(\kappa_C).$$

This results in $(\upsilon_A \sqcap \mu_B)[x] \preceq (f_D)_1 \sqcap (f_D)_2$ such that

$$\bigvee_{\kappa_C[x] \preceq (f_D)_1 \sqcap (f_D)_2} \pounds_e^{\gamma}(\kappa_C) \geq \pounds_e^{\gamma}(\upsilon_A \sqcap \mu_B) \geq \pounds_e^{\gamma}(\upsilon_A) \wedge \pounds_e^{\gamma}(\mu_B),$$

$$\bigwedge_{\kappa_C[x] \preceq (f_D)_1 \sqcap (f_D)_2} \pounds_e^{\pi}(\kappa_C) \leq \pounds_e^{\pi}(\upsilon_A \sqcap \mu_B) \leq \pounds_e^{\pi}(\upsilon_A) \vee \pounds_e^{\pi}(\mu_B),$$

$$\bigwedge_{\kappa_C[x] \preceq (f_D)_1 \sqcap (f_D)_2} \pounds_e^{\varsigma}(\kappa_C) \leq \pounds_e^{\varsigma}(\upsilon_A \sqcap \mu_B) \leq \pounds_e^{\varsigma}(\upsilon_A) \vee \pounds_e^{\varsigma}(\mu_B).$$

It is a contradiction. Thus,

$$\begin{aligned}
&(\mathfrak{T}_{\pounds}^{\star\star\gamma})_e((f_D)_1) \wedge (\mathfrak{T}_{\pounds}^{\star\star\gamma})_e((f_D)_2) \\
&= \left(\bigwedge_{x \in \mathcal{X}} \left[(f_e^c)_1(x) \vee \bigvee_{\upsilon_A[x] \preceq (f_D)_1} \pounds_e^{\gamma}(\upsilon_A) \right] \right) \wedge \left(\bigwedge_{x \in \mathcal{X}} \left[(f_e^c)_2(x) \vee \bigvee_{\mu_B[x] \preceq (f_D)_2} \pounds_e^{\gamma}(\mu_B) \right] \right) \\
&\leq \left(\bigwedge_{x \in \mathcal{X}} \left[(f_e^c)_1(x) \vee \bigvee_{\upsilon_A[x] \preceq (f_D)_1} \pounds_e^{\gamma}(\upsilon_A) \right] \wedge \left[(f_e^c)_2(x) \vee \bigvee_{\mu_B[x] \preceq (f_D)_2} \pounds_e^{\gamma}(\mu_B) \right] \right) \\
&\leq \bigwedge_{x \in \mathcal{X}} \left[((f_e^c)_1 \sqcup (f_e^c)_2)(x) \vee \bigvee_{\upsilon_A[x] \preceq (f_D)_1} \pounds_e^{\gamma}(\upsilon_A) \wedge \bigvee_{\mu_B[x] \preceq (f_D)_2} \pounds_e^{\gamma}(\mu_B) \right] \\
&\leq \bigwedge_{x \in \mathcal{X}} \left[((f_e^c)_1 \sqcup (f_e^c)_2)(x) \vee \bigvee_{(\upsilon_A \sqcap \mu_B)[x] \preceq (f_D)_1 \sqcap (f_D)_2} \pounds_e^{\gamma}(\upsilon_A \sqcap \mu_B) \right] \\
&\leq (\mathfrak{T}_{\pounds}^{\star\star\gamma})_e((f_D)_1 \sqcap (f_D)_2),
\end{aligned}$$

$$\begin{aligned}
&(\mathfrak{T}_{\pounds}^{\star\star\pi})_e((f_D)_1) \vee (\mathfrak{T}_{\pounds}^{\star\pi})_e((f_D)_2) \\
&= \left(\bigvee_{x \in \mathcal{X}} \left[(f_e^c)_1(x) \wedge \bigvee_{\upsilon_A[x] \preceq (f_D)_1} \pounds_e^{\pi}(\upsilon_A) \right] \right) \vee \left(\bigvee_{x \in \mathcal{X}} \left[(f_e^c)_2(x) \wedge \bigvee_{\mu_B[x] \preceq (f_D)_2} \pounds_e^{\pi}(\mu_B) \right] \right) \\
&\geq \bigvee_{x \in \mathcal{X}} \left(\left[(f_e^c)_1(x) \wedge \bigvee_{\upsilon_A[x] \preceq (f_D)_1} \pounds_e^{\pi}(\upsilon_A) \right] \vee \left[(f_e^c)_2(x) \wedge \bigvee_{\mu_B[x] \preceq (f_D)_2} \pounds_e^{\pi}(\mu_B) \right] \right) \\
&\geq \bigvee_{x \in \mathcal{X}} \left[((f_e^c)_1 \sqcap (f_e^c)_2)(x) \wedge \bigvee_{\upsilon_A[x] \preceq (f_D)_1} \pounds_e^{\pi}(\upsilon_A) \vee \bigvee_{\mu_B[x] \preceq (f_D)_2} \pounds_e^{\pi}(\mu_B) \right] \\
&\geq \bigvee_{x \in \mathcal{X}} \left[((f_e^c)_1 \sqcap (f_e^c)_2)(x) \wedge \bigvee_{(\upsilon_A \sqcap \mu_B)[x] \preceq (f_D)_1 \sqcap (f_D)_2} \pounds_e^{\pi}(\upsilon_A \sqcap \mu_B) \right] \\
&\geq (\mathfrak{T}_{\pounds}^{\star\star\pi})_e((f_D)_1 \sqcap (f_D)_2).
\end{aligned}$$

Likewise, we can establish through a similar line of reasoning that

$$(\mathfrak{T}_{\pounds}^{\star\star\varsigma})_e((f_D)_1) \vee (\mathfrak{T}_{\pounds}^{\star\star\varsigma})_e((f_D)_2) \geq (\mathfrak{T}_{\pounds}^{\star\star\varsigma})_e((f_D)_1 \sqcap (f_D)_2).$$

($\mathfrak{T}_3$) For $e \in \mathrm{E}$

$$\begin{aligned}
(\mathfrak{T}_{\pounds}^{\star\star\gamma})_e\left(\bigvee_{j\in\Gamma}(f_B)_j\right) &= \bigwedge_{x\in\mathcal{X}}\left[\left(\bigvee_{j\in\Gamma}(f_e)_j\right)^c(x)\right] \vee \left[\bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} \pounds_e^{\gamma}(\upsilon_A)\right] \\
&= \bigwedge_{x\in\mathcal{X}}\left[\bigwedge_{j\in\Gamma}(f_e^c)_j(x) \vee \bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} \pounds_e^{\gamma}(\upsilon_A)\right] \\
&= \bigwedge_{j\in\Gamma}\left[\bigwedge_{x\in\mathcal{X}}(f_e^c)_j(x) \vee \bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} \pounds_e^{\gamma}(\upsilon_A)\right] \\
&\geq \bigwedge_{j\in\Gamma}\left[\bigwedge_{x\in\mathcal{X}}(f_e^c)_j(x) \vee \bigvee_{\upsilon_A[x]\leq(f_B)_j} \pounds_e^{\gamma}(\upsilon_A)\right] \\
&= \bigwedge_{j\in\Gamma}(\mathfrak{T}_{\pounds}^{\star\star\gamma})_e((f_B)_j),
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{T}_{\pounds}^{\star\star\pi})_e\left(\bigvee_{j\in\Gamma}(f_B)_j\right) &= \bigvee_{x\in\mathcal{X}}\left[\left(\bigvee_{j\in\Gamma}(f_e)_j\right)^c(x)\right] \wedge \left[\bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} \pounds_e^{\pi}(\upsilon_A)\right] \\
&= \bigvee_{x\in\mathcal{X}}\left[\bigwedge_{j\in\Gamma}(f_e^c)_j(x) \wedge \bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} \pounds_e^{\pi}(\upsilon_A)\right] \\
&\leq \bigvee_{x\in\mathcal{X}}\left[\bigvee_{j\in\Gamma}(f_e^c)_j(x) \wedge \bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} \pounds_e^{\pi}(\upsilon_A)\right] \\
&= \bigvee_{j\in\Gamma}\left[\bigvee_{x\in\mathcal{X}}(f_e^c)_j(x) \wedge \bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} \pounds_e^{\pi}(\upsilon_A)\right] \\
&\leq \bigvee_{j\in\Gamma}\left[\bigvee_{x\in\mathcal{X}}(f_e^c)_j(x) \wedge \bigvee_{\upsilon_A[x]\leq(f_B)_j} \pounds_e^{\pi}(\upsilon_A)\right] \\
&= \bigvee_{j\in\Gamma}(\mathfrak{T}_{\pounds}^{\star\star\pi})_e((f_B)_j).
\end{aligned}$$

In a similar vein, we can demonstrate through a parallel line of reasoning that

$$(\mathfrak{T}_{\mathfrak{£}}^{\star\star\varsigma})_e\left(\bigvee_{j\in\Gamma}(f_B)_j\right) \leq \bigvee_{j\in\Gamma}(\mathfrak{T}_{\mathfrak{£}}^{\star\star\varsigma})_e((f_B)_j).$$

Therefore, $\mathfrak{T}_{\mathfrak{£}}^{\star\star\gamma\pi\varsigma}$ is an svnst on $\mathcal{X}$. □

5. Single-valued neutrosophic soft uniformly continuous mappings

In this section, we obtain crucial results in introducing and characterizing single-valued neutrosophic soft uniformly continuous, on single-valued neutrosophic soft uniformly topological spaces. Moreover, the relationship between single-valued neutrosophic soft uniformly continuous and single-valued neutrosophic soft continuous is studied.

Definition 11. *Let $(\mathcal{X}, \mathfrak{£}_E^{\gamma\pi\varsigma})$ and $(\mathcal{G}, \mathfrak{£}_R^{\star\gamma\pi\varsigma})$ be two svns-uniform spaces and $\psi : \mathcal{X} \to \mathcal{G}$ and $\vartheta : E \to \mathcal{R}$ be two mappings. Then, an svns-map $\psi_\vartheta : (\widehat{\mathcal{X}\times\mathcal{X}}, E) \to (\widehat{\mathcal{G}\times\mathcal{G}}, \mathcal{R})$ is called single-valued neutrosophic soft uniformly continuous (svns-uniformly continuous) if*

$$\mathfrak{£}_e^{\gamma}((\psi\times\psi)_\vartheta^{-1}(\mu_B)) \geq \mathfrak{£}_{\vartheta(e)}^{\star\gamma}(\mu_B), \quad \mathfrak{£}_e^{\pi}((\psi\times\psi)_\vartheta^{-1}(\mu_B)) \leq \mathfrak{£}_{\vartheta(e)}^{\star\pi}(\mu_B),$$

$$\mathfrak{£}_e^{\varsigma}((\psi\times\psi)_\vartheta^{-1}(\mu_B)) \leq \mathfrak{£}_{\vartheta(e)}^{\star\varsigma}(\mu_B),$$

for each $\mu_B \in (\widehat{\mathcal{G}\times\mathcal{G}}, \mathcal{R})$, $e \in E$.

Proposition 1. *Let $(\mathcal{X}, \mathfrak{£}_E^{\gamma\pi\varsigma})$ and $(\mathcal{G}, \mathcal{F}_R^{\gamma\pi\varsigma})$ be svns-uniform spaces. If $\psi_\vartheta : (\mathcal{X}, \mathfrak{£}^{\gamma\pi\varsigma}) \to (\mathcal{G}, \mathcal{F}^{\gamma\pi\varsigma})$ is svns-uniformly continuous, then $\psi_\vartheta : (\mathcal{X}, (\mathfrak{£}_{st}^{\gamma\pi\varsigma}) \to (\mathcal{G}, \mathcal{F}_{st}^{\gamma\pi\varsigma})$ is svns-uniformly continuous.*

Proof. To prove this theorem, we need to prove that

$$(\mathfrak{£}_{st}^{\gamma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \geq (\mathcal{F}_{st}^{\gamma})_{\vartheta(e)}(\upsilon_A), \qquad (\mathfrak{£}_{st}^{\pi})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \leq (\mathcal{F}_{st}^{\pi})_{\vartheta(e)}(\upsilon_A),$$

$$(\mathfrak{£}_{st}^{\varsigma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \leq (\mathcal{F}_{st}^{\varsigma})_{\vartheta(e)}(\upsilon_A),$$

for each $\upsilon_A \in (\widehat{\mathcal{G}\times\mathcal{G}}, \mathcal{R})$, $e \in$ E.

Assume that

$$(\mathfrak{£}_{st}^{\gamma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \ngeq (\mathcal{F}_{st}^{\gamma})_{\vartheta(e)}(\upsilon_A), \qquad (\mathfrak{£}_{st}^{\pi})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A) \nleq (\mathcal{F}_{st}^{\pi})_{\vartheta(e)}(\upsilon_A),$$

$$(\mathfrak{£}_{st}^{\varsigma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \nleq (\mathcal{F}_{st}^{\varsigma})_{\vartheta(e)}(\upsilon_A).$$

From the concept of $(\mathcal{F}_{st}^{\star\gamma\pi\varsigma})_{\vartheta(e)}(\upsilon_A)$, there exists $\mu_B \in (\widehat{\mathcal{G}\times\mathcal{G}}, \mathcal{R})$, $e \in$ E, $\varrho \in \zeta$ with $\mu_B \sqcap \widehat{E}_\varrho \leq \upsilon_A$ such that

$$(\mathfrak{£}_{st}^{\gamma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \ngeq \mathcal{F}_{\vartheta(e)}^{\gamma}(\mu_B), \qquad (\mathfrak{£}_{st}^{\pi})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \nleq \mathcal{F}_{\vartheta(e)}^{\pi}(\mu_B),$$

$$(\mathfrak{£}_{st}^{\varsigma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \nleq \mathcal{F}_{\vartheta(e)}^{\varsigma}(\mu_B).$$

Since $\psi_\vartheta : (\mathcal{X}, (\mathfrak{£}_{st}^{\gamma\pi\varsigma}) \to (\mathcal{G}, \mathcal{F}_{st}^{\gamma\pi\varsigma})$ is svns-uniformly continuous,

$$\mathfrak{£}_e^{\gamma}((\psi\times\psi)_\vartheta^{-1}(\mu_B)) \geq \mathcal{F}_{\vartheta(e)}^{\gamma}(\mu_B), \qquad \mathfrak{£}_e^{\pi}((\psi\times\psi)_\vartheta^{-1}(\mu_B)) \leq \mathcal{F}_{\vartheta(e)}^{\pi}(\mu_B),$$

$$\pounds_e^{\varsigma}((\psi\times\psi)_{\vartheta}^{-1}(\mu_B)) \leq \mathcal{F}_{\vartheta(e)}^{\varsigma}(\mu_B).$$

From the concept of $\pounds_e^{\gamma\pi\varsigma}((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A))$, we get

$$(\pounds_{st}^{\gamma})_e((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A)) \geq \pounds_e^{\gamma}((\psi\times\psi)_{\vartheta}^{-1}(\mu_B)) \geq \mathcal{F}_{\vartheta(e)}^{\gamma}(\mu_B),$$

$$(\pounds_{st}^{\pi})_e((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A)) \leq \pounds_e^{\pi}((\psi\times\psi)_{\vartheta}^{-1}(\mu_B)) \leq \mathcal{F}_{\vartheta(e)}^{\pi}(\mu_B),$$

$$(\pounds_{st}^{\varsigma})_e((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A)) \leq \pounds_e^{\pi}((\psi\times\psi)_{\vartheta}^{-1}(\mu_B)) \leq \mathcal{F}_{\vartheta(e)}^{\varsigma}(\mu_B).$$

This is a conflict with the hypothesis. □

Proposition 2. *Let $\psi : \mathcal{X} \to \mathcal{G}$, and $\vartheta : E \to \mathcal{R}$ be two mappings, and let $f_D \in (\widehat{\mathcal{X}, E})$, $\upsilon_A, \mu_B, \kappa_C \in (\mathcal{G}\times\widehat{\mathcal{G}}, \mathcal{R})$. Then, the following results hold in general:*
(1) $\psi_{\vartheta}^{-1}(\upsilon_A[\psi_{\vartheta}(f_D)]) = ((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A))[f_D]$,
(2) $((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A^s))[f_D] = ((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A))^s[f_D]$,
(3) $(\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A \sqcap \mu_B) = (\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A) \sqcap (\psi\times\psi)_{\vartheta}^{-1}(\mu_B)$,
(4) $(\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A) \circ (\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A) \preceq (\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A \circ \upsilon_A)$.

Proof. (1) For $\omega \in \psi(\mathrm{E})$, we get that

$$\begin{aligned}
\psi_{\vartheta}^{-1}(\upsilon_\omega[\psi_{\vartheta}(f_{\vartheta^{-1}(\omega)})])(x) &= \psi_{\vartheta}^{-1}(\upsilon_\omega[(\psi(f))_\omega])(x) = (\upsilon_\omega[(\psi(f))_\omega])(\psi(x)) \\
&= \bigvee_{y\in\mathcal{G}}[(\psi(f))_\omega(y) \wedge \upsilon_\omega(y,\psi(x))] \\
&= \bigvee_{z\in\mathcal{X}}[(\psi(f))_\omega(\psi(z)) \wedge \upsilon_\omega(\psi(z),\psi(x))] \\
&= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z) \wedge (\psi\times\psi)^{-1}(\upsilon_\omega(z,x))] \\
&= (\psi\times\psi)_{\vartheta}^{-1}(\upsilon_\omega[(f_{\vartheta^{-1}(\omega)}])(x).
\end{aligned}$$

(2) For $\omega \in \psi(\mathrm{E})$, we have

$$\begin{aligned}
((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_\omega^s))[f_{\vartheta^{-1}(\omega)}](x) &= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z) \wedge ((\psi\times\psi^{-1})_{\vartheta}(\upsilon_\omega^s))(z,x)] \\
&= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z) \wedge \upsilon_\omega^s(\psi(z),\psi(x)] \\
&= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z) \wedge \upsilon_\omega(\psi(x),\psi(z)] \\
&= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z) \wedge ((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_\omega))(x,z)] \\
&= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z) \wedge ((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_\omega))^s(z,x)] \\
&= ((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_\omega))^s[(f_{\vartheta^{-1}(\omega)}](x).
\end{aligned}$$

(3) Direct.

(4) For $\omega \in \psi(\mathrm{E})$, we have

$$\begin{aligned}((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_{\omega})\circ(\psi\times\psi)_{\vartheta}^{-1}(\upsilon_{\omega}))(x_1,x_2) &= \bigvee_{z\in\mathcal{X}}[(\psi\times\psi)_{\vartheta}^{-1}(\upsilon_{\omega})(x_1,z)\wedge(\psi\times\psi)_{\vartheta}^{-1}(\upsilon_{\omega})(z,x_2)]\\ &= \bigvee_{z\in\mathcal{X}}[\upsilon_{\omega}(\psi(x_1),\psi(z))\wedge\upsilon_{\omega}(\psi(z),\psi(x_2))]\\ &\leq \bigvee_{z\in\mathcal{X}}[\upsilon_{\omega}(\psi(x_1),y)\wedge\upsilon_{\omega}(y,\psi(x_2))]\\ &= (\upsilon_{\omega}\circ\upsilon_{\omega})(\psi(x_1),\psi(x_2)) = (\psi\times\psi)_{\vartheta}^{-1}(\upsilon_{\omega}\circ\upsilon_{\omega})(x_1,x_2).\end{aligned}$$

□

Theorem 9. *Let $(\mathcal{X},\pounds^{\gamma\pi\varsigma})$ and $(\mathcal{G},\mathcal{F}^{\gamma\pi\varsigma})$ be svns-uniform spaces, $\psi_{\vartheta}:(\widehat{\mathcal{X}\times\mathcal{X}},E)\to(\widehat{\mathcal{G}\times\mathcal{G}},\mathcal{R})$ be svns-uniformly continuous. Then, the following results hold in general.*
(1) $\psi_{\vartheta}^{-1}(\mathcal{I}_{\mathcal{F}^{\gamma\pi\varsigma}}(\omega,f_C,r))\leq\mathcal{I}_{\pounds^{\gamma\pi\varsigma}}(\vartheta^{-1}(\omega),\psi_{\vartheta}^{-1}(f_C),r))$, *for each* $f_C\in(\widehat{\mathcal{G},\mathcal{R}})$, $r\in\xi$, $\omega\in\mathcal{R}$,
(2) $\mathcal{C}_{\pounds^{\gamma\pi\varsigma}}(\vartheta^{-1}(\omega),\psi_{\vartheta}^{-1}(f_C),r))\leq\psi_{\vartheta}^{-1}(\mathcal{C}_{\mathcal{F}^{\gamma\pi\varsigma}}(\omega,f_C,r))$, *for each* $f_C\in(\widehat{\mathcal{G},\mathcal{R}})$, $r\in\xi$, $\omega\in\mathcal{R}$,
(3) $\psi_{\vartheta}(\mathcal{C}_{\pounds^{\gamma\pi\varsigma}}(e,g_D,r))\leq\mathcal{C}_{\mathcal{F}^{\gamma\pi\varsigma}}(\vartheta(\omega),\psi_{\vartheta}(g_D),r))$, *for each* $g_D\in(\widehat{\mathcal{X},E})$, $r\in\xi$, $e\in E$.

Proof. (1) For each $\upsilon_A,\in(\widehat{\mathcal{G}\times\mathcal{G}},\mathcal{R})$ and $f_C,g_D\in(\widehat{\mathcal{G},\mathcal{R}})$, from Proposition 2, $\upsilon_A[f_C]\leq g_D$ implies that

$$((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A))[\psi_{\vartheta}^{-1}(f_C)]=\psi_{\vartheta}^{-1}(\upsilon_A[\psi_{\vartheta}(\psi_{\vartheta}^{-1}(f_C))])\leq\psi_{\vartheta}^{-1}(\upsilon_A[f_C])\leq\psi_{\vartheta}^{-1}(g_D).$$

Since

$$\pounds^{\gamma}_{\vartheta^{-1}(\omega)}(\mu_B)\geq\mathcal{F}^{\gamma}_{\omega}(\upsilon_A),\quad \pounds^{\pi}_{\vartheta^{-1}(\omega)}(\mu_B)\leq\mathcal{F}^{\pi}_{\omega}(\upsilon_A),\quad \pounds^{\varsigma}_{\vartheta^{-1}(\omega)}(\mu_B)\leq\mathcal{F}^{\varsigma}_{\omega}(\upsilon_A),$$

for every $\mu_B\in(\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A)$, we obtain

$$\begin{aligned}&\psi_{\vartheta}^{-1}(\mathtt{I}_{\mathcal{F}^{\gamma\pi\varsigma}}(\omega,f_C,r))\\ &=\psi_{\vartheta}^{-1}(\bigsqcup\{g_D\in(\widehat{\mathcal{G},\mathcal{R}}):\ \upsilon_A[g_D]\leq f_C,\mathcal{F}^{\gamma}(\upsilon_A)\geq r,\mathcal{F}^{\pi}_{\omega}(\upsilon_A)\leq 1-r,\mathcal{F}^{\varsigma}_{\omega}(\upsilon_A)\leq 1-r\})\\ &=\bigsqcup\{\psi_{\vartheta}^{-1}(g_D)\in(\widehat{\mathcal{X},\mathrm{E}}):\ \upsilon_A[g_D]\\ &\leq f_C,\mathcal{F}^{\gamma}(\upsilon_A)\geq r,\mathcal{F}^{\pi}_{\omega}(\upsilon_A)\leq 1-r,\mathcal{F}^{\varsigma}_{\omega}(\upsilon_A)\leq 1-r\}\\ &\leq\bigsqcup\{\psi_{\vartheta}^{-1}(f_C)\in(\widehat{\mathcal{X},\mathrm{E}}):\ \mu_B[\psi_{\vartheta}^{-1}(g_D)]\\ &\leq\psi_{\vartheta}^{-1}(f_C),\pounds^{\gamma}_{\vartheta^{-1}(\omega)}(\mu_B)\geq r,\pounds^{\pi}_{\vartheta^{-1}(\omega)}(\mu_B)\leq 1-r,\pounds^{\varsigma}_{\vartheta^{-1}(\omega)}(\mu_B)\leq 1-r\}\\ &\leq\mathtt{I}_{\pounds^{\gamma\pi\varsigma}}(\vartheta^{-1}(\omega),\psi_{\vartheta}^{-1}(f_C),r)).\end{aligned}$$

In a similar vein, we can demonstrate (2) and (3) through a parallel line of reasoning. □

Theorem 10. *Let $(\mathcal{X},\pounds_E^{\gamma\pi\varsigma})$ and $(\mathcal{G},\mathcal{F}_{\mathcal{R}}^{\gamma\pi\varsigma})$ be svns-uniform spaces, and $\psi_{\vartheta}:(\widehat{\mathcal{X},E})\to(\widehat{\mathcal{G},\mathcal{R}})$ an injective svns-uniformly continuous. Then, $\psi_{\vartheta}:(\mathcal{X},\mathfrak{T}_{\pounds}^{\star\gamma\pi\varsigma})\to(\mathcal{G},\mathfrak{T}_{\mathcal{F}}^{\star\gamma\pi\varsigma})$ is svns-continuous.*

Proof. Since ψ_{ϑ} injective and by applying Theorem 4, we get that:
For each $\upsilon_A,\in(\widehat{\mathcal{G}\times\mathcal{G}},\mathcal{R})$ and $f_B\in(\widehat{\mathcal{G},\mathcal{R}})$, $\omega\in A\cap B$. Then,

$$((\psi\times\psi)_{\vartheta}^{-1}((\upsilon_A^{f_B})_{\omega}))(x_1,x_2) \quad=(\upsilon_A^{f_B})_{\omega}(\psi(x_1),\psi(x_2))$$

$$= \begin{cases} 1, & \text{if } \psi(x_1) = \psi(x_2), \\ f_\omega(\psi(x_1)) \wedge f_\omega(\psi(x_2)), & \text{if } \psi(x_1) \neq \psi(x_2), \end{cases}$$
$$= \begin{cases} 1, & \text{if } \psi(x_1) = \psi(x_2), \\ \psi_\vartheta^{-1}(f_\omega)(x_1) \wedge \psi_\vartheta^{-1}(f_\omega)(x_2), & \text{if } \psi(x_1) \neq \psi(x_2), \end{cases}$$
$$= \left(\upsilon_{\vartheta^{-1}(A)}^{\psi_\vartheta^{-1}(f_B)}\right)_{\vartheta^{-1}(\omega)} (x_1, x_2).$$

Therefore, $\forall\ e \in \mathrm{E}$

$$(\mathfrak{T}_{\pounds}^{\star\gamma})_e(\psi_\vartheta^{-1}(f_B)) = \pounds_e^{\gamma}(\upsilon_{\vartheta^{-1}(A)}^{\psi_\vartheta^{-1}(f_B)}) = \pounds_e^{\gamma}((\psi \times \psi)_\vartheta^{-1}(\upsilon_A^{f_B})) \geq \mathcal{F}_{\vartheta(e)}^{\gamma}(\upsilon_A^{f_B}) = (\mathfrak{T}_{\mathcal{F}}^{\star\gamma})_{\vartheta(e)}(f_B)$$

$$(\mathfrak{T}_{\pounds}^{\star\pi})_e(\psi_\vartheta^{-1}(f_B)) = \pounds_e^{\pi}(\upsilon_{\vartheta^{-1}(A)}^{\psi_\vartheta^{-1}(f_B)}) = \pounds_e^{\pi}((\psi \times \psi)_\vartheta^{-1}(\upsilon_A^{f_B})) \leq \mathcal{F}_{\vartheta(e)}^{\pi}(\upsilon_A^{f_B}) = (\mathfrak{T}_{\mathcal{F}}^{\star\pi})_{\vartheta(e)}(f_B)$$

$$(\mathfrak{T}_{\pounds}^{\star\varsigma})_e(\psi_\vartheta^{-1}(f_B)) = \pounds_e^{\varsigma}(\upsilon_{\vartheta^{-1}(A)}^{\psi_\vartheta^{-1}(f_B)}) = \pounds_e^{\varsigma}((\psi \times \psi)_\vartheta^{-1}(\upsilon_A^{f_B})) \leq \mathcal{F}_{\vartheta(e)}^{\varsigma}(\upsilon_A^{f_B}) = (\mathfrak{T}_{\mathcal{F}}^{\star\varsigma})_{\vartheta(e)}(f_B).$$

□

Theorem 11. *Let $(\mathcal{X}, \pounds_E^{\gamma\pi\varsigma})$ and $(\mathcal{G}, \mathcal{F}_{\mathcal{R}}^{\gamma\pi\varsigma})$ be two svns-uniform spaces and $\psi_\vartheta : (\widehat{\mathcal{X}, E}) \to (\widehat{\mathcal{G}, \mathcal{R}})$ be an svns-uniformly continuous mapping. Then, $\psi_\vartheta : (\mathcal{X}, \mathfrak{T}_{\pounds}^{\star\star\gamma\pi\varsigma}) \to (\mathcal{G}, \mathfrak{T}_{\mathcal{F}}^{\star\star\gamma\pi\varsigma})$ is svns-continuous.*

Proof. Initially, it is clear that $\psi_\vartheta^{-1}(\upsilon_A[\psi(x)]) = (\psi \times \psi)_\vartheta^{-1}(\upsilon_A[x])$ from that:

$$[\psi_\vartheta^{-1}(\upsilon_A[\psi(x)])](z) = (\upsilon_A[\psi(x)])(\psi(z)) = \upsilon_A(\psi(z), \psi(x)) = ((\psi \times \psi)_\vartheta^{-1}(\upsilon_A))(z, x)$$
$$= [((\psi \times \psi)_\vartheta^{-1}(\upsilon_A))[x]](z).$$

Thus, $\upsilon_A[\psi(x)] \leq f_B$ implies that $\psi_\vartheta^{-1}(\upsilon_A[\psi(x)]) = ((\psi \times \psi)_\vartheta^{-1}(\upsilon_A))[x] \leq \psi_\vartheta^{-1}(f_B)$. By applying Theorem 8, we obtain

$$(\mathfrak{T}_{\mathcal{F}}^{\star\star\gamma})_\omega(f_B) = \bigwedge_y \left[(f_B^c)(y) \vee \bigvee_{\upsilon_A[y] \leq f_B} \mathcal{F}_\omega^{\gamma}(\upsilon_A)\right] \leq \bigwedge_x \left[f_B^c(\psi(x)) \vee \bigvee_{\upsilon_A[\psi(x)] \leq f_B} \mathcal{F}_\omega^{\gamma}(\upsilon_A)\right]$$
$$\leq \bigwedge_x \left[(\psi_\vartheta^{-1}(f_B))^c(x) \vee \bigvee_{((\psi\times\psi)_\vartheta^{-1}(\upsilon_A))[x] \leq \psi_\vartheta^{-1}(f_B)} \pounds_{\vartheta^{-1}(\omega)}^{\gamma}((\psi \times \psi)_\vartheta^{-1}(\upsilon_A))\right]$$
$$\leq (\mathfrak{T}_{\pounds}^{\star\star\gamma})_{\vartheta^{-1}(\omega)}(\psi_\vartheta^{-1}(f_B)),$$

$$(\mathfrak{T}_{\pounds}^{\star\star\pi})_\omega(f_B) = \bigvee_y \left[(f_B^c)(y) \wedge \bigvee_{\upsilon_A[y] \leq f_B} \mathcal{F}_\omega^{\pi}(\upsilon_A)\right] \geq \bigvee_x \left[f_B^c(\psi(x)) \wedge \bigvee_{\upsilon_A[\psi(x)] \leq f_B} \mathcal{F}_\omega^{\pi}(\upsilon_A)\right]$$
$$\geq \bigvee_x \left[(\psi_\vartheta^{-1}(f_B))^c(x) \wedge \bigvee_{((\psi\times\psi)_\vartheta^{-1}(\upsilon_A))[x] \leq \psi_\vartheta^{-1}(f_B)} \pounds_{\vartheta^{-1}(\omega)}^{\pi}((\psi \times \psi)_\vartheta^{-1}(\upsilon_A))\right]$$
$$\geq (\mathfrak{T}_{\pounds}^{\star\star\pi})_{\vartheta^{-1}(\omega)}(\psi_\vartheta^{-1}(f_B)),$$

Likewise, we can establish through a similar line of reasoning that $(\mathfrak{T}_{\pounds}^{\star\star\varsigma})_\omega(f_B) \geq (\mathfrak{T}_{\pounds}^{\star\star\varsigma})_{\vartheta^{-1}(\omega)}(\psi_\vartheta^{-1}(f_B))$. □

Theorem 12. *Let* $\{(\mathcal{X}_j, (\pounds_j^{\gamma\pi\varsigma})_{E_j}) : j \in \Gamma\}$ *be a family of svns- uniform spaces and, for all* $j \in \Gamma$, $\psi_j : \mathcal{X} \to \mathcal{X}_j$, *and* $\vartheta_j : E \to E_j$ *are mappings. Define* $\pounds^{\gamma} : E \to \zeta^{(\mathcal{X}\widehat{\times}\mathcal{X},E)}$, $\pounds^{\pi} : E \to \zeta^{(\mathcal{X}\widehat{\times}\mathcal{X},E)}$ *and* $\pounds^{\varsigma} : E \to \zeta^{(\mathcal{X}\widehat{\times}\mathcal{X},E)}$ *on* $\mathcal{X}$ *by:*

$$\pounds_e^{\gamma}(\upsilon_A) = \bigvee \left[\bigwedge_{j=1}^{n} ((\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) \mid \upsilon_A \geq \sqcap_{j=1}^{n} (\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}} ((\mu_B)_{\omega_j}) \right],$$

$$\pounds_e^{\pi}(\upsilon_A) = \bigwedge \left[\bigvee_{j=1}^{n} ((\pounds_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) \mid \upsilon_A \geq \sqcap_{j=1}^{n} (\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}} ((\mu_B)_{\omega_j}) \right],$$

$$\pounds_e^{\varsigma}(\upsilon_A) = \bigwedge \left[\bigvee_{j=1}^{n} ((\pounds_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) \mid \upsilon_A \geq \sqcap_{j=1}^{n} (\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}} ((\mu_B)_{\omega_j}) \right].$$

where $\bigvee$ *is taken over all finite subsets* $\Omega = \{\omega_1, \omega_2, ..., \omega_n\} \subseteq \Gamma$. *Then,*
(1) $\pounds^{\gamma\pi\varsigma}$ *is the coarsest svns-uniformity on* $\mathcal{X}$ *for which all* $\{(\psi_\vartheta)_j : j \in \Gamma\}$ *are svns-uniformly continuous.*
(2) *A map* $\psi_\vartheta : (\mathcal{X}^\star, \pounds_{\mathcal{R}}^{\gamma\pi\varsigma}) \to (\mathcal{X}, \pounds_E^{\gamma\pi\varsigma})$ *is svns-uniformly continuous if for all* $j \in \Gamma$, $(\psi_\vartheta)_j \circ \psi_\vartheta : (\mathcal{X}^\star, \pounds_{\mathcal{R}}^{\gamma\pi\varsigma}) \to (\mathcal{X}_j, (\pounds_j^{\gamma\pi\varsigma})_{E_j})$ *is svns-uniformly continuous.*

Proof. (1) Initially, we indication that $\pounds^{\gamma\pi\varsigma}$ is an svns-uniformity on $\mathcal{X}$ for which all $\{(\psi_\vartheta)_j : j \in \Gamma\}$ are svns-uniformly continuous.

($\pounds_1$) For every $\omega_j \in \Omega$, there exists $(\upsilon_A)_{\omega_j} \in (\mathcal{X}_{\omega_j} \times \widehat{\mathcal{X}}_{\omega_j}, \mathrm{E}_{\omega_j})$ such that, for $e \in \mathrm{E}$, we obtain that

$$(\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) = 1, \quad (\pounds_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) = 0, \quad (\pounds_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) = 0.$$

Put $(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}} ((\mu_B)_{\omega_j}) = \upsilon_A$. Then, $\pounds_e^{\gamma}(\upsilon_A) = 1$, $\pounds_e^{\pi}(\upsilon_A) = 0$ and $\pounds_e^{\varsigma}(\upsilon_A) = 0$.

($\pounds_2$) It is obvious from the definition of $\pounds^{\gamma\pi\varsigma}$.

($\pounds_3$) For all limited subsets $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$, $T = \{t_1, t_2, ..., t_m\}$ of Γ such that

$$\sqcap_{j=1}^{n} (\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}} ((\upsilon_A)_{\omega_j}) \preceq \upsilon_A, \quad \sqcap_{j=1}^{m} (\psi_{t_j} \times \psi_{t_j})^{-1}_{\vartheta_{t_j}} ((\mu_B)_{t_j}) \preceq \mu_B$$

we have

$$\sqcap_{j=1}^{m} (\psi_{t_j} \times \psi_{t_j})^{-1}_{\vartheta_{t_j}} ((\mu_B)_{t_j}) \sqcap \sqcap_{j=1}^{n} (\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}} ((\upsilon_A)_{\omega_j}) \preceq \mu_B \sqcap \upsilon_A.$$

Moreover, for all $\omega \in \Omega \cap T$ we have

$$(\psi_\omega \times \psi_\omega)^{-1}_{\vartheta_\omega}((\mu_B)_\omega) \sqcap (\psi_\omega \times \psi_\omega)^{-1}_{\vartheta_\omega}((\upsilon_A)_\omega) = (\psi_\omega \times \psi_\omega)^{-1}_{\vartheta_\omega}((\mu_B)_\omega \sqcap (\upsilon_A)_\omega).$$

Put $(\psi_{m_j} \times \psi_{m_j})^{-1}_{\vartheta_{m_j}} ((\mathcal{W}_C)_{m_j}) \preceq \mu_B \sqcap \upsilon_A$, where

$$\gamma_{(\mathcal{W}_C)_{m_j}}(x) = \begin{cases} \gamma_{(\upsilon_A)_{m_j}}(x), & \text{if } m_j \in \Omega - (\Omega \cap T), \\ \gamma_{(\mu_B)_{m_j}}(x), & \text{if } m_j \in \Omega - (\Omega \cap T), \\ \gamma_{(\upsilon_A)_{m_j}}(x) \cap \gamma_{(\mu_B)_{m_j}}(x), & \text{if } m_j \in \Omega \cap T, \end{cases}$$

$$\pi_{(\mathcal{W}_C)_{m_j}}(x)=\begin{cases}\pi_{(\upsilon_A)_{m_j}}(x), & \text{if } m_j\in\Omega-(\Omega\cap T),\\ \pi_{(\mu_B)_{m_j}}(x), & \text{if } m_j\in\Omega-(\Omega\cap T),\\ \pi_{(\upsilon_A)_{m_j}}(x)\cup\pi_{(\mu_B)_{m_j}}(x), & \text{if } m_j\in\Omega\cap T,\end{cases}$$

$$\varsigma_{(\mathcal{W}_C)_{m_j}}(x)=\begin{cases}\varsigma_{(\upsilon_A)_{m_j}}(x), & \text{if } m_j\in\Omega-(\Omega\cap T),\\ \varsigma_{(\mu_B)_{m_j}}(x), & \text{if } m_j\in\Omega-(\Omega\cap T),\\ \varsigma_{(\upsilon_A)_{m_j}}(x)\cup\varsigma_{(\mu_B)_{m_j}}(x), & \text{if } m_j\in\Omega\cap T,\end{cases}$$

Therefore, we obtain

$$\pounds_e^{\gamma}(\upsilon_A\sqcap\mu_B)\geq\bigwedge_{j\in\Omega\cup T}(\pounds_j^{\gamma})_{\vartheta_j(e)}((\mathcal{W}_C)_j)\geq\left[\bigwedge_{j=1}^{n}(\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\upsilon_A)_{\omega_j})\right]\wedge\left[\bigwedge_{j=1}^{m}(\pounds_{t_j}^{\gamma})_{\vartheta_{t_j}(e)}((\mu_B)_{t_j})\right]$$

$$\pounds_e^{\pi}(\upsilon_A\sqcap\mu_B)\leq\bigvee_{j\in\Omega\cup T}(\pounds_j^{\pi})_{\vartheta_j(e)}((\mathcal{W}_C)_j)\leq\left[\bigvee_{j=1}^{n}(\pounds_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}((\upsilon_A)_{\omega_j})\right]\vee\left[\bigvee_{j=1}^{m}(\pounds_{t_j}^{\pi})_{\vartheta_{t_j}(e)}((\mu_B)_{t_j})\right],$$

$$\pounds_e^{\varsigma}(\upsilon_A\sqcap\mu_B)\leq\bigvee_{j\in\Omega\cup T}(\pounds_j^{\varsigma})_{\vartheta_j(e)}((\mathcal{W}_C)_j)\leq\left[\bigvee_{j=1}^{n}(\pounds_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)}((\upsilon_A)_{\omega_j})\right]\vee\left[\bigvee_{j=1}^{m}(\pounds_{t_j}^{\varsigma})_{\vartheta_{t_j}(e)}((\mu_B)_{t_j})\right].$$

Taking the supremum on the families $\sqcap_{j=1}^{n}(\psi_{\omega_j}\times\psi_{\omega_j})_{\vartheta_{\omega_j}}^{-1}((\upsilon_A)_{\omega_j})\leq\upsilon_A$ and $\sqcap_{j=1}^{m}(\psi_{t_j}\times\psi_{t_j})_{\vartheta_{t_j}}^{-1}((\mu_B)_{t_j})\leq\mu_B$ we obtain

$$\pounds_e^{\gamma}(\upsilon_A\sqcap\mu_B)\geq\pounds_e^{\gamma}(\upsilon_A)\wedge\pounds_e^{\gamma}(\mu_B),\qquad\pounds_e^{\pi}(\upsilon_A\sqcap\mu_B)\leq\pounds_e^{\pi}(\upsilon_A)\vee\pounds_e^{\pi}(\mu_B),$$
$$\pounds_e^{\varsigma}(\upsilon_A\sqcap\mu_B)\leq\pounds_e^{\varsigma}(\upsilon_A)\vee\pounds_e^{\varsigma}(\mu_B),\quad\forall\, e\in\mathrm{E}.$$

($\pounds_4$) If $\pounds_e^{\gamma}(\upsilon_A)\neq 0$, $\pounds_e^{\pi}(\upsilon_A)\neq 1$ and $\pounds_e^{\varsigma}(\upsilon_A)\neq 1$, then there exists $\Omega=\{\omega_1,\omega_2,...,\omega_p\}$ of Γ with $\sqcap_{j=1}^{p}(\psi_{\omega_j}\times\psi_{\omega_j})_{\vartheta_{\omega_j}}^{-1}((\mu_B)_{\omega_j})\leq\upsilon_A$ such that

$$\pounds_e^{\gamma}(\upsilon_A)\geq\bigwedge_{j=1}^{p}(\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 0,\qquad\pounds_e^{\pi}(\upsilon_A)\leq\bigvee_{j=1}^{p}(\pounds_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 1,$$

$$\pounds_e^{\pi}(\upsilon_A)\leq\bigvee_{j=1}^{p}(\pounds_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 1.$$

Since, $(\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 0$, $(\pounds_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 1$, $(\pounds_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 0\ \forall\ \omega_j\in\Omega$, then $(\top)_C\not\leq(\upsilon_B)_{\omega_j}$. Thus,

$$(\top)_C\leq(\psi_{\omega_j}\times\psi_{\omega_j})_{\vartheta_{\omega_j}}^{-1}((\top)_C)\leq\sqcap_{j=1}^{p}(\psi_{\omega_j}\times\psi_{\omega_j})_{\vartheta_{\omega_j}}^{-1}((\mu_B)_{\omega_j})\leq\upsilon_A.$$

($\pounds_5$) Assume that $\pounds_e^{\gamma}(\upsilon_A^s)\not\geq\pounds_e^{\gamma}(\upsilon_A)$, $\pounds_e^{\pi}(\upsilon_A^s)\not\leq\pounds_e^{\pi}(\upsilon_A)$ and $\pounds_e^{\varsigma}(\upsilon_A^s)\not\leq\pounds_e^{\varsigma}(\upsilon_A)$. From the concept of $\pounds^{\gamma\pi\varsigma}$, there exists $\Omega=\{\omega_1,\omega_2,...,\omega_p\}$ of Γ with $\sqcap_{j=1}^{p}(\psi_{\omega_j}\times\psi_{\omega_j})_{\vartheta_{\omega_j}}^{-1}((\mu_B)_{\omega_j})\leq\upsilon_A$ such that

$$\pounds_e^{\gamma}(\upsilon_A^s)\not\geq\bigwedge_{j=1}^{p}(\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}),\qquad\pounds_e^{\pi}(\upsilon_A^s)\not\leq\bigvee_{j=1}^{p}(\pounds_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}),$$

$$\mathfrak{L}_e^{\varsigma}(\upsilon_A^s) \not\leq \bigvee_{j=1}^{p} (\mathfrak{L}_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}).$$

Since $\mathfrak{L}_{\omega_j}^{\gamma\pi\varsigma}$ is an svns-uniformity on $\mathcal{X}$ for each ω_j

$$(\mathfrak{L}_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}) \geq (\mathfrak{L}_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}), \quad (\mathfrak{L}_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}) \leq (\mathfrak{L}_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}),$$

$$(\mathfrak{L}_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}) \leq (\mathfrak{L}_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}).$$

It follows that

$$\mathfrak{L}_e^{\gamma}(\upsilon_A^s) \not\geq \bigwedge_{j=1}^{p} (\mathfrak{L}_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}), \quad \mathfrak{L}_e^{\pi}(\upsilon_A^s) \not\leq \bigvee_{j=1}^{p} (\mathfrak{L}_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}),$$

$$\mathfrak{L}_e^{\varsigma}(\upsilon_A^s) \not\leq \bigvee_{j=1}^{p} (\mathfrak{L}_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}).$$

On the other hand,

$$\sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B^s)_{\omega_j}) = \sqcap_{j=1}^{n}((\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}))^s \leq \upsilon_A^s.$$

Hence,

$$\mathfrak{L}_e^{\gamma}(\upsilon_A^s) \geq \bigwedge_{j=1}^{p} \mathfrak{L}^{\gamma}_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}), \quad \mathfrak{L}_e^{\pi}(\upsilon_A^s) \leq \bigvee_{j=1}^{p} \mathfrak{L}^{\pi}_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}), \quad \mathfrak{L}_e^{\varsigma}(\upsilon_A^s) \leq \bigvee_{j=1}^{p} \mathfrak{L}^{\varsigma}_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}).$$

It is a contradiction. Hence, $(\mathfrak{L}_5)$ holds.

$(\mathfrak{L}_6)$ Suppose that for each $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$

$$\mathfrak{L}_e^{\gamma}(\upsilon_A) \not\leq \bigvee\{\mathfrak{L}_e^{\gamma}((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \preceq \upsilon_A\}, \quad \mathfrak{L}_e^{\pi}(\upsilon_A) \not\geq \bigwedge\{\mathfrak{L}_e^{\pi}((\upsilon_A)_1)) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \preceq \upsilon_A\},$$

$$\mathfrak{L}_e^{\varsigma}(\upsilon_A) \not\geq \bigwedge\{\mathfrak{L}_e^{\varsigma}((\upsilon_A)_1)) \mid (\upsilon_A)_1) \circ (\upsilon_A)_1) \preceq \upsilon_A\}.$$

By the concept of $\mathfrak{L}^{\gamma\pi\varsigma}$, there exists $\Omega = \{\omega_1, \omega_2, ..., \omega_p\}$ of Γ with $\sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \preceq \upsilon_A$ such that

$$\bigwedge_{j=1}^{p} (\mathfrak{L}_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}(\mu_{\omega_j}) \not\leq \bigvee\{\mathfrak{L}_e^{\gamma}((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \preceq \upsilon_A\},$$

$$\bigvee_{j=1}^{p} (\mathfrak{L}_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}(\mu_{\omega_j}) \not\geq \bigwedge\{\mathfrak{L}_e^{\pi}((\upsilon_A)_1)) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \preceq \upsilon_A\},$$

$$\bigvee_{j=1}^{p} (\mathfrak{L}_{\varsigma_j}^{\pi})_{\vartheta_{\omega_j}(e)}(\mu_{\omega_j}) \not\geq \bigwedge\{\mathfrak{L}_e^{\varsigma}((\upsilon_A)_1)) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \preceq \upsilon_A\}.$$

Since $\mathfrak{L}_{\omega_j}^{\gamma\pi\varsigma}$ is *svns- uniformity* on $\mathcal{X}_{\omega_j}$ for each $\omega_j \in \Omega$

$$(\mathfrak{L}_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\mu_B)_j) \not\leq \bigvee\{(\mathfrak{L}_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}(\mathcal{W}_C) \mid \mathcal{W}_C \circ \mathcal{W}_C \preceq \mu_B\},$$

$$(\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\mu_B)_j) \not\geq \bigwedge\{(\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j}(e)}(\mathcal{W}_C) \mid \mathcal{W}_C \circ \mathcal{W}_C \preceq \mu_B\},$$

$$(\pounds^{\varsigma}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\mu_B)_j) \not\geq \bigwedge\{(\pounds^{\varsigma}_{\omega_j})_{\vartheta_{\omega_j}(e)}(\mathcal{W}_C) \mid \mathcal{W}_C \circ \mathcal{W}_C \preceq \mu_B\}.$$

Thus, there exists $\mathcal{W}_C \in (\mathcal{X}_{\omega_j} \times \widehat{\mathcal{X}_{\omega_j}}, \mathrm{E}_{\omega_j})$, $\mathcal{W}_C \circ \mathcal{W}_C \preceq \upsilon_A$ such that

$$\bigwedge_{j=1}^{p}(\pounds^{\gamma}_{\omega_j})_{\vartheta_{\omega_j}(e)}(\mathcal{W}_C) \not\leq \bigvee\{\pounds^{\gamma}_e((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \preceq \upsilon_A\},$$

$$\bigvee_{j=1}^{p}(\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j}(e)}(\mathcal{W}_C) \not\geq \bigwedge\{\pounds^{\pi}_e((\upsilon_A)_1)) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \preceq \upsilon_A\},$$

$$\bigvee_{j=1}^{p}(\pounds^{\pi}_{\varsigma_j})_{\vartheta_{\omega_j}(e)}(\mathcal{W}_C) \not\geq \bigwedge\{\pounds^{\varsigma}_e((\upsilon_A)_1)) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \preceq \upsilon_A\}.$$

On the other hand,

$$\begin{aligned}\sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C) \circ \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C) &\preceq \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C \circ \mathcal{W}_C)\\ &\preceq \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \preceq \upsilon_A.\end{aligned}$$

Therefore, we have

$$\bigvee\{\pounds^{\gamma}_e((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \preceq \upsilon_A\} \geq \pounds^{\gamma}_e((\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C)) \geq (\pounds^{\gamma}_{\omega_j})_{\vartheta_{\omega_j}(e)}(\mathcal{W}_C),$$

$$\bigwedge\{\pounds^{\pi}_e((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \preceq \upsilon_A\} \leq \pounds^{\pi}_e((\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C)) \leq (\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j}(e)}(\mathcal{W}_C),$$

$$\bigwedge\{\pounds^{\varsigma}_e((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \preceq \upsilon_A\} \leq \pounds^{\varsigma}_e((\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C)) \leq (\pounds^{\varsigma}_{\omega_j})_{\vartheta_{\omega_j}(e)}(\mathcal{W}_C).$$

It is a contradiction. Hence, $\pounds_6$ holds.

Next by the concept of $\pounds^{\gamma\pi\varsigma}$ it is easily proved that, for each $j \in \Gamma$

$$\pounds^{\gamma}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}(\mu_B)) \geq (\pounds^{\gamma}_j)_{\vartheta_{j(e)}}(\mu_B), \qquad \pounds^{\pi}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}(\mu_B)) \leq (\pounds^{\pi}_j)_{\vartheta_{j(e)}}(\mu_B),$$

$$\pounds^{\varsigma}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}(\mu_B)) \leq (\pounds^{\varsigma}_j)_{\vartheta_{j(e)}}(\mu_B),\ \forall\ \mu_B \in (\mathcal{X}_j \widehat{\times \mathcal{X}_j}, \mathrm{E}_j).$$

Thus, $(\psi_j)_{\vartheta_j} : \mathcal{X} \to \mathcal{X}_j$ is svns-uniformly continuous.

Lastly, let us say that $\pounds^{\star\gamma\pi\varsigma}$ is an svns-uniformity on $\mathcal{X}$ and $(\psi_j)_{\vartheta_j} : (\mathcal{X}, \pounds^{\star\gamma\pi\varsigma}) \to (\mathcal{X}_j, \pounds^{\gamma\pi\varsigma}_j)$ is svns-uniformly continuous, that is, for every $j \in \Gamma$ and $(\mu_B)_j \in (\mathcal{X}_j \widehat{\times \mathcal{X}_j}, \mathrm{E}_j)$,

$$\pounds^{\star\gamma}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}((\mu_B)_j)) \geq (\pounds^{\gamma}_j)_{\vartheta_{j(e)}}((\mu_B)_j), \qquad \pounds^{\star\pi}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}((\mu_B)_j)) \leq (\pounds^{\pi}_j)_{\vartheta_{j(e)}}((\mu_B)_j),$$

$$\pounds^{\star\varsigma}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}((\mu_B)_j)) \leq (\pounds^{\varsigma}_j)_{\vartheta_{j(e)}}((\mu_B)_j).$$

For every finite subset $\Omega = \{\omega_1, \omega_2, ..., \omega_p\}$ of Γ with $\sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \preceq \upsilon_A$, we have

$$\begin{aligned}
\pounds_e^{\gamma}(\upsilon_A) &= \bigvee \left[\bigwedge_{j=1}^{p} (\pounds_j^{\gamma})_{\vartheta_{\omega_j(e)}}((\mu_B)_{\omega_j}) \mid \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \preceq \upsilon_A \right] \\
&\leq \bigvee \left[\bigwedge_{j=1}^{p} (\pounds_e^{\star\gamma}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \mid \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \preceq \upsilon_A \right] \\
&\leq \bigvee \left[\bigwedge_{j=1}^{p} (\pounds_e^{\star\gamma}(\sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j})) \mid \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \preceq \upsilon_A \right] \\
&\leq \pounds_e^{\star\gamma}(\upsilon_A).
\end{aligned}$$

In a similar vein, we can demonstrate through a parallel line of reasoning that $\pounds_e^{\pi}(\upsilon_A) \geq \pounds_e^{\star\pi}(\upsilon_A)$ and $\pounds_e^{\varsigma}(\upsilon_A) \geq \pounds_e^{\star\varsigma}(\upsilon_A)$.

(2) It can be easily proved. □

6. Conclusions

Many scientists have studied the soft set theory and easily applied it to many problems in social life. In the present work, we defined t he s ingle-valued n eutrosophic s oft u niform s paces and single-valued neutrosophic soft uniform bases. The relationships between them were also investigated. Next, the relationship among single-valued neutrosophic soft uniformities, single-valued neutrosophic soft topologies, and single-valued neutrosophic soft interior operators were introduced and studied. Finally, we proved crucial results in introducing and characterizing single-valued neutrosophic soft uniformly continuous, on single-valued neutrosophic soft uniformly topological spaces. Moreover, the relationship between single-valued neutrosophic soft uniformly continuous and single-valued neutrosophic soft continuous was studied. This paper can form the theoretical basis for further applications of single-valued neutrosophic soft topology, potentially leading to the development of other scientific areas.

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References

1. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
2. K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, **20** (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3
3. Z. Pawlak, Rough sets, *Int. J. Comput. Inform. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956
4. F. Smarandache, *A unifying field in logics: Neutrosophic logic*, Rehoboth: American Research Press, 1999.
5. D. Molodtsov, Soft set theory-first result, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
6. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, *J. Fuzzy Math.*, **9** (2001), 589–602.
7. P. K. Maji, Neutrosophic soft set, *Ann. Fuzzy Math. Inf.*, **5** (2013), 157–168.
8. B. Ahmad, A. Kharal, On fuzzy soft sets, *Adv. Fuzzy Syst.*, **2009** (2009), 586507. https://doi.org/10.1155/2009/586507
9. H. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, *Multispace Multistruct*, **4** (2010), 410–413.
10. Y. M. Saber, F. Alsharari, F. Smarandache, On Single-valued neutrosophic ideals in Šostak sense, *Symmetry*, **12** (2020), 193. https://doi.org/10.3390/sym12020193
11. Y. M. Saber, F. Alsharari, F. Smarandache, A. Abdel-Sattar, Connectedness and stratification of single-valued neutrosophic topological spaces, *Symmetry*, **12** (2020), 1464. https://doi.org/10.3390/sym12091464
12. F. Alsharari, F. Smarandache, Y. M. Saber, Compactness on single-valued neutrosophic ideal topological spaces, *Neutrosophic Sets Syst.*, **41** (2021), 127–145.
13. Y. M. Saber, F. Alsharari, F. Smarandache, A. Abdel-Sattar, On single valued neutrosophic regularity spaces, *Comput. Model. Eng. Sci.*, **130** (2022), 1625–1648. https://doi.org/10.32604/cmes.2022.017782
14. D. Sasirekha, P. Senthilkumar, Determining the Best Plastic Recycling Technology Using the MABAC Method in a Single-Valued Neutrosophic Fuzzy Approach, *Neutrosophic Sets Syst.*, **58** (2023), 194–210
15. F. Masri, M. Zeina, O. Zeitouny, Some Single Valued Neutrosophic Queueing Systems with Maple Code, *Neutrosophic Sets Syst.*, **53** (2023), 251–273. https://doi.org/10.5281/zenodo.7536023
16. Y. M. Saber, F. Alsharari, F. Smarandache, An Introduction to Single-Valued Neutrosophic Soft Topological Structure, *Soft Comput.*, **26** (2022), 7107–7122. https://doi.org/10.1007/s00500-022-07150-4
17. S. Shahzadi , A. Rasool, G. Santos-Garcìa, Methods to find strength of job competition among candidates under single-valued neutrosophic soft model, *Math. Bio. Eng.*, **20** (2023), 4609–4642. https://doi.org/10.3934/mbe.2023214

18. A. Cano, G. Petalcorin, Single-valued Neutrosophic Soft sets in Hyper UP-Algebra, *Eur. J. Pure. Appl. Math.*, **16** (2023), 548–576. https://doi.org/10.29020/nybg.ejpam.v16i1.4637
19. A. Özkan, Ş. Yazgan, S. Kaur, Neutrosophic Soft Generalized b-Closed Sets in Neutrosophic Soft Topological Spaces, *Neutrosophic Sets Syst.*, **56** (2023), 48–69.
20. S. Al-Hijjawi, A. Ahmad, S. Alkhazaleh, A generalized effective neurosophic soft set and its applications, *AIMS Mathematics*, **18** (2023), 29628–29666. https://doi.org/10.3934/math.20231517
21. C. Jana, M. Pal, A Robust Single-Valued Neutrosophic Soft Aggregation Operators in Multi-Criteria Decision Making, *Symmetry*, **11** (2019), 110. https://doi.org/10.3390/sym11010110
22. N. L. A. Mohd Kamal, L. Abdullah, Multi-Valued Neutrosophic Soft Set, *Malays. J. Math. Sci.*, **13** (2019), 153–168.
23. R. Lowen, Fuzzy uniform spaces, *J. Math. Anal. Appl.*, **82** (1981), 370–385. https://doi.org/10.1016/0022-247X(81)90202-X
24. W. Kotzé, Uniform spaces, In: *Mathematics of Fuzzy Sets*, Boston: Springer, 553–580.
25. B. Hutton, Uniformities in fuzzy topological spaces, *J. Math. Anal. Appl.*, **58** (1977), 559–571. https://doi.org/10.1016/0022-247X(77)90192-5
26. S. E. Abbas, I. Ibedou, Fuzzy soft uniform spaces, *Soft Comput.*, **21** (2017), 6073–6083. https://doi.org/10.1007/s00500-016-2327-3
27. J. Dezert, Open Questions to Neutrosophic Inferences, *Multi. Val. Logic.*, **8** (2001), 439–472.

A New Type of Neutrosophic Set in Pythagorean Fuzzy Environment and Applications to Multi-Criteria Decision Making

Mahmut Can Bozyıgıt,[1] Murat Olgun,[2] Florentin Smarandache,[3] Mehmet Unver[2]

[1]Ankara Yıldırım Beyazıt University, Faculty of Engineering and Natural Sciences, Department of Mathematics, 06010 Ankara Turkey
[2]Ankara University, Faculty of Science, Department of Mathematics, 06100 Ankara Turkey
[3]Mathematics Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA

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Abstract

In this paper, we introduce the concepts of Pythagorean fuzzy valued neutrosophic set (PFVNS) and Pythagorean fuzzy valued neutrosophic (PFVNV) constructed by considering Pythagorean fuzzy values (PFVs) instead of numbers for the degrees of the truth, the indeterminacy and the falsity, which is a new extension of intuitionistic fuzzy valued neutrosophic set (IFVNS). By means of PFVNSs, the degrees of the truth, the indeterminacy and the falsity can be given in Pythagorean fuzzy environment and more sensitive evaluations are made by a decision maker in decision making problems compared to IFVNSs. In other words, such sets enable a decision maker to evaluate the degrees of the truth, the indeterminacy and the falsity as PFVs to model the uncertainty in the evaluations. First of all, we propose the concepts of Pythagorean fuzzy t-norm and t-conorm and show that some Pythagorean fuzzy t-norms and t-conorms are expressed via ordinary continuous Archimedean t-norms and t-conorms. Then we define the concepts of PFVNS and PFVNV and provide a tool to construct a PFVNV from an ordinary neutrosophic fuzzy value. We also define some set theoretic operations between PFVNSs and some algebraic operations between PFVNVs via t-norms and t-conorms. With the help of these algebraic operations we propose some weighted aggregation operators. To measure discrimination information of PFVNVs, we define a simplified neutrosophic valued modified fuzzy cross-entropy measure. Moreover, we introduce a multi-criteria decision making method in Pythagorean fuzzy valued neutrosophic environment and practice the proposed theory to a real life multi-criteria decision making problem. Finally, we study the comparison analysis and the time complexity of the proposed method.

Keywords: Pythagorean fuzzy valued neutrosophic set; aggregation operators; multi-criteria decision making

1 Introduction

Zadeh[42] introduced the concept of fuzzy set (FS) defined by a membership function μ_A from a universal set X to closed unit interval $[0, 1]$ in order to handle the uncertainty in various real-world problems. Then using a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\nu_A : X \to [0, 1]$ under the condition $\mu_A(x) + \nu_A(x) \leq 1$ for $x \in X$, Atanassov[2] proposed the concept of intuitionistic fuzzy set (IFS) which is an extension of the concept of FS. The pair $\langle \mu_A(x), \nu_A(x) \rangle$ is called an intuitionistic fuzzy value (IFV) for a fixed $x \in X$. The theory of IFS has been used to solve problems in many applications as multi-criteria decision making (MCDM), classification, pattern recognition and clustering and it has been studied in many areas by researchers. The concepts of entropy and cross-entropy are important measurement methods used in the information theory and these concepts were proposed by Shannon.[23,24] Then the concept of cross-entropy was improved by Kullback and Leibler[13,14] and it was modified by Lin.[15] In order to measure

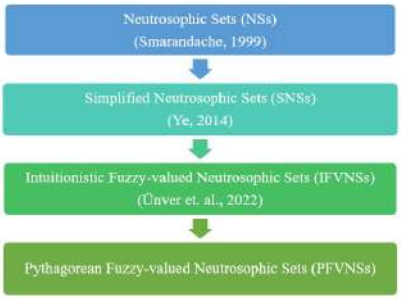

Figure 1: The development of theory of neutrosophic set

discrimination information between IFSs, Wei and Ye[31] introduced an improved intuitionistic fuzzy cross-entropy to overcome the drawback of intuitionistic fuzzy cross-entropy given by Vlachos and Sergiadis.[30] The concept of aggregation operator that transforms several input values into a single output value is an important tool in the decision making theory. A series of aggregation operators have been proposed by many researchers. Particularly, various generalizations of aggregation operators for IFSs (see e.g.[3,7]) were defined via several types of t-norms and t-conorms on $[0,1]$. Further studies on MCDM with FSs and aggregation operators can be found in.[1,9,17,27,29,39–41]

As a generalization of IFS, the concept of Pythagorean fuzzy set (PFS) was developed by Yager,[33,34] which is defined by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$ under the condition $\mu_A^2(x) + \nu_A^2(x) \leq 1$ for $x \in X$. A PFS is more capable than an IFS to express uncertainty. In other words, PFSs enhance flexibility and practicability of IFSs and have wider area than IFSs while describing the uncertainty. Therefore theory of PFS has been studied by many researchers to model the uncertainty. Later Yager[34,35] proposed a range of aggregation operators for PFSs. After that, Peng et. al.[20] presented the axiomatic definitions of distance measure, similarity measure, entropy and inclusion measure for PFSs.

Smarandache[26] introduced the concept of neutrosophic set (NS) where each element has the degrees of truth, indeterminacy and falsify in the non-standard unit interval. There is no restriction on the membership functions for NSs. Due to the difficulty of applying neutrosophic sets to practical problems, the concept of simplified neutrosophic sets (SNSs) was proposed by Ye.[37] A SNS is constructed by a truth, an indeterminacy and a falsify membership function defined from X to $[0,1]$. It has been used in various numerical applications. Moreover, the concept of single valued neutrosophic multi-set (SVNMS) that is characterized by sequences of truth, indeterminacy and falsify membership functions has been proposed by Ye and Ye.[36]

Schweizer and Sklar[21] introduced the concept of triangular norm (t-norm) and triangular conorm (t-conorm), following the concept of probabilistic metric spaces proposed by Menger,[16] which is a generalization of metric spaces. These notions are useful tools to define algebraic operations and aggregation operators for FSs. Deschrijver et. al.[6] proposed the concepts of intuitionistic fuzzy t-norm and intuitionistic fuzzy t-conorm, which are extensions of the concepts of t-norm and t-conorm, respectively, by turning into interval $[0,1]$ to $\{(x_1, x_2) : x_1, x_2 \in [0,1] \text{ and } x_1 + x_2 \leq 1\}$. In this paper, we introduce the concepts of Pythagorean fuzzy t-norm and Pythagorean fuzzy t-conorm which are extensions of notions of both ordinary and intuitionistic fuzzy t-norm and t-conorm, respectively. Ünver et. al.[28] proposed the concept of intuitionistic fuzzy valued neutrosophic multi-set (IFVNMS) by combining NS theory and IFS theory with the help of intuitionistic fuzzy values instead of numbers in membership sequences. Motivating from this idea, we propose the concepts of Pythagorean fuzzy valued neutrosophic set (PFVNS) and Pythagorean fuzzy valued neutrosophic value (PFVNV) by combining NS theory and PFS theory. A PFVNV consists of a triple of Pythagorean fuzzy values (PFVs) and a PFVNS consists of PFVNVs. Then we develop a method to transform fuzzy values and give some set theoretic and algebraic operations with the help of Pythagorean fuzzy t-norms and t-conorms. By using these algebraic operations we define some weighted arithmetic and geometric aggregation operators.

Entropy is a useful tool for measuring uncertain information. The concept of fuzzy entropy was introduced by Zadeh.[43] Later, a fuzzy cross-entropy measure and a symmetric discrimination information measure between FSs were proposed by Shang and Jiang.[22] Wei and Ye[31] defined a modified intuitionistic fuzzy cross-entropy measure between IFSs. With similar motivation, we introduce a simplified neutrosophic valued modified fuzzy

cross-entropy for PFVNVs. Finally, using these concepts we provide a MCDM method in the Pythagorean fuzzy valued neutrosophic environment.

Some main contributions of the present study can be listed as follows:

- With the help of Pythagorean fuzzy t-norms and t-conorms we study in the fuzzy environment while conducting the aggregation process in decision making problems rather than defuzzified environment that provides us with more sensitive solutions.
- By means of PFVNSs, the degrees of the truth, the indeterminacy and the falsity can be given in the Pythagorean fuzzy environment and more sensitive evaluations are made by decision makers in real life problems. In other words, such sets enable a decision maker evaluate the degrees of the truth, the indeterminacy and the falsity as PFVs in decision making process. Thus, the uncertainty in the decision maker's evaluations is represented with the help of more capable FS notion.
- The proposed cross-entropy measure measures the discrimination of the information between PFVNVs in a fuzzy environment. Thus more sensitive evaluations can be made in decision making problems before defuzzifying the environment.

The rest of the paper is organized as follows. In Section 2, we introduce the concepts of Pythagorean fuzzy t-norm and t-conorm and show that some Pythagorean fuzzy t-norms and t-conorms are expressed via ordinary continuous Archimedean t-norms and t-conorms. In Section 3, we introduce the concept of PFVNS, which is an extension of the notion of intuitionistic fuzzy valued neutrosophic set (IFVNS) where an IFVNS is an IFVNMS with sequence length 1. Then we develop a method to transform simplified neutrosophic values (SNVs) to PFVNVs. We also give some set theoretical and algebraic operations via Pythagorean fuzzy t-norms and t-conorms for PFVNVs. In Section 4 we define some weighted aggregation operators using these algebraic operations. In Section 5, motivating by the modified intuitionistic fuzzy cross-entropy measure defined by Wei and Ye,[31] we give a simplified neutrosophic valued modified fuzzy cross-entropy. In Section 6, we propose a MCDM method and apply the proposed theory to a MCDM problem from the literature. We also compare the results of the proposed method with the existing results. Finally, we calculate the time complexity of the MCDM method. In Section 7, we conclude the paper.

2 Pythagorean Fuzzy t-norms and t-conorms

In this section, we recall some fundamental definitions about t-norms and t-conorms and define the concepts of Pythagorean fuzzy t-norm and t-conorm.

Definition 2.1. [10,21] A t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions:

(T1) $T(x,1) = x$ for all $x \in [0,1]$,

(T2) $T(x,y) = T(y,x)$ for all $x, y \in [0,1]$,

(T3) $T(x,T(y,z)) = T(T(x,y),z)$ for all $x, y, z \in [0,1]$,

(T4) $T(x,y) \leq T(x',y')$ whenever $x \leq x'$ and $y \leq y'$.

Definition 2.2. [10,21] A t-conorm is a function $S : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions:

(S1) $S(x,0) = x$ for all $x \in [0,1]$,

(S2) $S(x,y) = S(y,x)$ for all $x, y \in [0,1]$,

(S3) $S(x,S(y,z)) = S(S(x,y),z)$ for all $x, y, z \in [0,1]$,

(S4) $S(x,y) \leq S(x',y')$ whenever $x \leq x'$ and $y \leq y'$.

Definition 2.3. [33,34] A fuzzy complement is a function $N : [0,1] \to [0,1]$ satisfying the following conditions:

(N1) $N(0) = 1$ and $N(1) = 0$,

(N2) $N(a) \geq N(b)$ whenever $a \leq b$ for all $a, b \in [0,1]$,

(N3) Continuity,

(N4) $N(N(a)) = a$ for all $a \in [0,1]$.

The function $N : [0,1] \to [0,1]$ defined by $N(a) = (1 - a^p)^{1/p}$ where $p \in (0, \infty)$ is a fuzzy complement introduced by Yager.[33,34] When $p = 2$, $N(a) = \sqrt{1 - a^2}$ is called the Pythagorean fuzzy complement.

Definition 2.4. [12] Let T be a t-norm and let S be a t-conorm on $[0,1]$. If $T(x,y) = N(S(N(x), N(y)))$ and $S(x,y) = N(T(N(x), N(y)))$ for a fuzzy negator N, then T and S are said to be dual with respect to N.

Remark 2.5. Let T be a t-norm on $[0,1]$ and let $N(a) = \sqrt{1 - a^2}$. Then the dual t-conorm S with respect to N is

$$S(x,y) = \sqrt{1 - T^2\left(\sqrt{1 - x^2}, \sqrt{1 - y^2}\right)}.$$

A strictly decreasing function $g : [0,1] \to [0, \infty]$ with $g(1) = 0$ is called the additive generator of a t-norm T if we have $T(x,y) = g^{-1}(g(x) + g(y))$ for all $(x,y) \in [0,1] \times [0,1]$.

Proposition 2.6. [12] *Let $g : [0,1] \to [0, \infty]$ be the additive generator of a t-norm T, let S be the dual t-conorm of T and let N be a fuzzy complement. The strictly increasing function $h : [0,1] \to [0, \infty]$ defined by $h(t) = g(N(t))$ is the additive generator of S and so $S(x,y) = h^{-1}(h(x) + h(y))$.*

Note that T is an Archimedean t-norm if and only if $T(x,x) < x$ for all $x \in (0,1)$ and S is an Archimedean t-conorm if and only if $S(x,x) > x$ for all $x \in (0,1)$.[10] Klement and Mesiar[11] proved that continuous Archimedean t-norms have useful representations via their additive generators as follows.

Theorem 2.7. [11] *Let T be a t-norm on $[0,1]$. The following are equivalent:*
(i) T is a continuous Archimedean t-norm.
(ii) T has a continuous additive generator.

Before introducing the concepts of Pythagorean fuzzy t-norm and t-conorm we recall the concept of PFS. Throughout this paper we assume $X = \{x_1, ..., x_n\}$ is a finite set.

Definition 2.8. [33,34] A PFS A on X is defined by

$$A = \{\langle x_j, \mu_A(x_j), \nu_A(x_j)\rangle : j = 1, ..., n\}$$

where $\mu_A, \nu_A : X \to [0,1]$ are the membership and non-membership functions respectively with condition that $\mu_A^2(x_j) + \nu_A^2(x_j) \leq 1$ for any $j = 1, ..., n$. For a fixed $j = 1, ..., n$ a PFV is denoted by

$$\alpha = \langle \mu_\alpha, \nu_\alpha \rangle = \langle \mu_A(x_j), \nu_A(x_j)\rangle.$$

Motivating by[6] we now introduce the notions of Pythagorean fuzzy t-norm and Pythagorean fuzzy t-conorm. Consider the set P^* defined by

$$P^* = \{x = (x_1, x_2) : x_1, x_2 \in [0,1] \text{ and } x_1^2 + x_2^2 \leq 1\}.$$

We use the partial order $\preceq$ on P^* that is defined by

$$x \preceq y \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2$$

for $x = (x_1, x_2), y = (y_1, y_2) \in P^*$.

Definition 2.9. A Pythagorean fuzzy t-norm is a function $\mathcal{T} : P^* \times P^* \to P^*$ that satisfies the following conditions for any $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in P^*$.

(PT1) $\mathcal{T}(x, (1, 0)) = x$ for all $x \in P^*$,

(PT2) $\mathcal{T}(x, y) = \mathcal{T}(y, x)$ for all $x, y \in P^*$,

(PT3) $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(y, \mathcal{T}(x, z))$ for all $x, y, z \in P^*$,

(PT4) $\mathcal{T}(x, y) \preceq \mathcal{T}(x', y')$ whenever $x \preceq x'$ and $y \preceq y'$.

Definition 2.10. A Pythagorean fuzzy t-conorm is a function $\mathcal{S} : P^* \times P^* \to P^*$ that satisfies the following conditions for any $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in P^*$.

(PS1) $\mathcal{S}(x, (0, 1)) = x$ for all $x \in P^*$,

(PS2) $\mathcal{S}(x, y) = \mathcal{S}(y, x)$ for all $x, y \in P^*$,

(PS3) $\mathcal{S}(x, \mathcal{S}(y, z)) = \mathcal{S}(y, \mathcal{S}(x, z))$ for all $x, y, z \in P^*$,

(PS4) $\mathcal{S}(x, y) \preceq \mathcal{S}(x', y')$ whenever $x \preceq x'$ and $y \preceq y'$.

We define the concept of Pythagorean fuzzy negator as an extension of the notion of fuzzy negator.

Definition 2.11. A Pythagorean fuzzy negator is a function $\mathcal{N} : P^* \to P^*$ satisfying the following conditions:

(PN1) $\mathcal{N}((1, 0)) = (0, 1)$ and $\mathcal{N}((0, 1)) = (1, 0)$,

(PN2) $\mathcal{N}(x) \succeq \mathcal{N}(y)$ whenever $x \preceq y$.

If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in P^*$, then we call $\mathcal{N}$ involutive. The mapping $\mathcal{N}_s : P^* \to P^*$ given by $\mathcal{N}_s((x_1, x_2)) = (x_2, x_1)$ is an involutive Pythagorean fuzzy negator which is called the standard negator.

Following theorem shows that a Pythagorean fuzzy t-conorm can be obtained from a Pythagorean fuzzy t-norm via Pythagorean fuzzy negators.

Theorem 2.12. *Let $\mathcal{T}$ be a Pythagorean fuzzy t-norm and let $\mathcal{N}$ be an involutive Pythagorean fuzzy negator. The function $\mathcal{S}$ defined by $\mathcal{S}(x, y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)))$ for any $x, y \in P^*$ is a t-conorm.*

Proof. (PS1) For all $x = (x_1, x_2) \in P^*$ we get

$$\begin{aligned} \mathcal{S}\big(x, (0, 1)\big) &= \mathcal{N}(\mathcal{T}(\mathcal{N}(x_1, x_2), \mathcal{N}(0, 1))) \\ &= \mathcal{N}(\mathcal{T}(\mathcal{N}(x_1, x_2), (1, 0))) \\ &= \mathcal{N}(\mathcal{N}(x_1, x_2)) \\ &= x. \end{aligned}$$

(PS2) For all $x, y \in P^*$ we obtain

$$\begin{aligned} \mathcal{S}(x, y) &= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y))\big) \\ &= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(y), \mathcal{N}(x))\big) \\ &= \mathcal{S}(y, x). \end{aligned}$$

(PS3) For all $x, y, z \in P^*$ we have

$$\begin{aligned}
\mathcal{S}(x, \mathcal{S}(y,z)) &= \mathcal{S}\big(x, \mathcal{N}(\mathcal{T}(\mathcal{N}(y), \mathcal{N}(z))\big) \\
&= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(x), \mathcal{N}\big(\mathcal{N}(\mathcal{T}(\mathcal{N}(y), \mathcal{N}(z))\big)\big) \\
&= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(x), (\mathcal{T}(\mathcal{N}(y), \mathcal{N}(z))\big) \\
&= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(y), (\mathcal{T}(\mathcal{N}(x), \mathcal{N}(z))\big) \\
&= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(y), \mathcal{N}\big(\mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(z))\big)\big) \\
&= \mathcal{S}\big(y, \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(z))\big) \\
&= \mathcal{S}(y, \mathcal{S}(x,z)).
\end{aligned}$$

(PS4) For all $x, x', y, y' \in P^*$ we have

$$\begin{aligned}
x \preceq x' \text{ and } y \preceq y' &\Rightarrow \mathcal{N}(x) \succeq \mathcal{N}(x') \text{ and } \mathcal{N}(y) \succeq \mathcal{N}(y') \\
&\Rightarrow \mathcal{T}\big(\mathcal{N}(x), \mathcal{N}(y)\big) \succeq \mathcal{T}\big(\mathcal{N}(x'), \mathcal{N}(y')\big) \\
&\Rightarrow \mathcal{N}\big(\mathcal{T}\big(\mathcal{N}(x), \mathcal{N}(y)\big) \preceq \mathcal{N}\big(\mathcal{T}\big(\mathcal{N}(x'), \mathcal{N}(y')\big)\big) \\
&\Rightarrow \mathcal{S}(x,y) \preceq \mathcal{S}(x', y').
\end{aligned}$$

Therefore, $\mathcal{S}$ is a Pythagorean fuzzy t-conorm. □

The t-conorm $\mathcal{S}$ in Theorem 2.12 is called the dual Pythagorean fuzzy t-conorm of $\mathcal{T}$ with respect to negator $\mathcal{N}$. Following theorem states that Pythagorean fuzzy t-norms and t-conorms can be produced by ordinary t-norms and t-conorms.

Theorem 2.13. *Let T be a t-norm and let S be a t-conorm on $[0,1]$. If*

$$T(a,b) \leq \sqrt{1 - S^2\big(\sqrt{1-a^2}, \sqrt{1-b^2}\big)} \text{ for all } a, b \in [0,1], \tag{1}$$

then the function $\mathcal{T} : P^* \times P^* \to P^*$ defined by

$$\mathcal{T}(x,y) = \big(T(x_1, y_1), S(x_2, y_2)\big)$$

is a Pythagorean fuzzy t-norm and the function $\mathcal{S} : P^* \times P^* \to P^*$ defined by

$$\mathcal{S}(x,y) = \big(S(x_1, y_1), T(x_2, y_2)\big)$$

is the dual Pythagorean fuzzy t-conorm of $\mathcal{T}$ with respect to $\mathcal{N}_s$.

Proof. As S is increasing, from (1) we have for any $x, y \in P^*$ that

$$\begin{aligned}
T^2(x_1, x_2) + S^2(x_2, y_2) &\leq 1 - S^2\Big(\sqrt{1-x_1^2}, \sqrt{1-x_2^2}\Big) + S^2(x_2, y_2) \\
&\leq 1 - S^2\Big(\sqrt{1-x_1^2}, \sqrt{1-x_2^2}\Big) + S^2\Big(\sqrt{1-x_1^2}, \sqrt{1-x_2^2}\Big) \\
&= 1
\end{aligned}$$

which yields that $\mathcal{T}(x,y)$ is in P^*.
(PT1) For any $x = (x_1, x_2) \in P^*$ we have

$$\begin{aligned}
\mathcal{T}(x, (1,0)) &= \left(T\left(x_1, 1\right), S\left(x_2, 0\right)\right) \\
&= (x_1, x_2) \\
&= x.
\end{aligned}$$

(PT2) For any $x, y \in P^*$ we get

$$\begin{aligned}
\mathcal{T}(x,y) &= \left(T\left(x_1, y_1\right), S\left(x_2, y_2\right)\right) \\
&= \left(T\left(y_1, x_1\right), S\left(y_2, x_2\right)\right) \\
&= \mathcal{T}(y,x).
\end{aligned}$$

(PT3) For all $x, y, z \in P^*$ we obtain

$$\begin{aligned}\mathcal{T}(x, \mathcal{T}(y,z)) &= \mathcal{T}\big((x_1,x_2), \big(T(y_1,z_1), S(y_2,z_2)\big)\big)\\ &= \Big(T\big(x_1, T(y_1,z_1)\big), S\big(x_2, S(y_2,z_2)\big)\Big)\\ &= \Big(T\big(y_1, T(x_1,z_1)\big), S\big(y_2, S(x_2,z_2)\big)\Big)\\ &= \mathcal{T}\big((y_1,y_2), \big(T(x_1,z_1), S(x_2,z_2)\big)\big)\\ &= \mathcal{T}(y, \mathcal{T}(x,z)).\end{aligned}$$

(PT4) For all $x, x', y, y' \in P^*$ we have

$$\begin{aligned}x \preceq x' \text{ and } y \preceq y' &\Rightarrow x_1 \le x_1' \text{ , } x_2 \ge x_2' \text{ and } y_1 \le y_1' \text{ , } y_2 \ge y_2'\\ &\Rightarrow T(x_1,y_1) \le T(x_1', y_1') \text{ and } S(x_2,y_2) \ge S(x_2', y_2')\\ &\Rightarrow \mathcal{T}(x,y) \preceq \mathcal{T}(x', y').\end{aligned}$$

Therefore $\mathcal{T}$ is a Pythagorean fuzzy t-norm. Similarly, it can be shown that $\mathcal{S}$ is a Pythagorean fuzzy t-conorm. □

Continuous Archimedean t-norms and t-conorms on $[0,1]$ can be generated by their additive generators. So we can construct a Pythagorean fuzzy t-norm and a Pythagorean fuzzy t-conorm using these additive generators. Note that if a t-norm T and a t-conorm S are dual with respect to the Pythagorean fuzzy complement, then (1) is satisfied.

Corollary 2.14. *Let $g : [0,1] \to [0,\infty]$ be the additive generator of a continuous Archimedean t-norm T and let $h : [0,1] \to [0,\infty]$ be the additive generator of the dual continuous Archimedean t-conorm S where $h(t) = g(\sqrt{1-t^2})$. Then the function $\mathcal{T} : P^* \times P^* \to P^*$ defined by*

$$\mathcal{T}(x,y) = \big(g^{-1}(g(x_1) + g(y_1)), h^{-1}(h(x_2) + h(y_2))\big)$$

is a Pythagorean fuzzy t-norm and the function $\mathcal{S} : P^* \times P^* \to P^*$ defined by

$$\mathcal{S}(x,y) = \big(h^{-1}(h(x_1) + h(y_1)), g^{-1}(g(x_2) + g(y_2))\big)$$

is the dual Pythagorean fuzzy t-conorm of $\mathcal{S}$ with respect to $\mathcal{N}_{\mathcal{S}}$. In this case $\mathcal{T}$ and $\mathcal{S}$ are called Pythagorean fuzzy t-norm and Pythagorean fuzzy t-conorm generated by g and h, respectively.

Proof. It is trivial from Theorem 2.13 and Remark 2.5. □

Example 2.15. Consider the functions $g : [0,1] \to [0,\infty]$ defined by $g(t) = -\log t^2$ and $h : [0,1] \to [0,\infty]$ defined by $h(t) = -\log(1-t^2)$. Then the Algebraic Pythagorean fuzzy t-norm is

$$\mathcal{T}(x,y) = \Big(x_1 y_1, \sqrt{x_2^2 + y_2^2 - x_2^2 y_2^2}\Big)$$

and the Algebraic dual Pythagorean fuzzy t-conorm with respect to $\mathcal{N}_s$ is

$$\mathcal{S}(x,y) = \Big(\sqrt{x_1^2 + y_1^2 - x_1^2 y_1^2}, x_2 y_2\Big).$$

3 Pythagorean Fuzzy Valued Neutrosophic Sets

We start this section recalling the concept of IFVNS obtained by taking the sequence length equal to 1 in IFVNMSs. Then we define the concepts of PFVNS and PFVNV.

Definition 3.1. [28] An IFVNS A on X is defined by

$$A = \left\{ \langle x_j, \mathbb{T}_A^j, \mathbb{I}_A^j, \mathbb{F}_A^j \rangle : j = 1, \ldots, n \right\}$$

where $\mathbb{T}_A^j$, $\mathbb{I}_A^j$ and $\mathbb{F}_A^j$ are the truth, the indeterminacy and the falsity membership pairs of IFVs, respectively, i.e., for $j = 1, \ldots, n$

$$\mathbb{T}_A^j = \big(\mu_{A,t}(x_j), \nu_{A,t}(x_j)\big) \text{ with } \mu_{A,t}(x_j), \nu_{A,t}(x_j) \in [0,1] \text{ such that } \mu_{A,t}(x_j) + \nu_{A,t}(x_j) \leq 1$$
$$\mathbb{I}_A^j = \big(\mu_{A,i}(x_i), \nu_{A,i}(x_j)\big) \text{ with } \mu_{A,i}(x_j), \nu_{A,i}(x_j) \in [0,1] \text{ such that } \mu_{A,i}(x_j) + \nu_{A,i}(x_j) \leq 1$$
$$\mathbb{F}_A^j = \big(\mu_{A,f}(x_j), \nu_{A,f}(x_j)\big) \text{ with } \mu_{A,f}(x_j), \nu_{A,f}(x_j) \in [0,1] \text{ such that } \mu_{A,f}(x_j) + \nu_{A,f}(x_j) \leq 1.$$

An IFVNV is denoted by

$$\alpha = \langle \mathbb{T}_\alpha, \mathbb{I}_\alpha, \mathbb{F}_\alpha \rangle := \langle \mathbb{T}_A^j, \mathbb{I}_A^j, \mathbb{F}_A^j \rangle$$

for a fixed $j = 1, ..., n$.

Example 3.2. Let $X = \{x_1, x_2\}$.

$$A = \{\langle x_1, (0.7, 0.3), (0.3, 0.5), (0.1, 0.8)\rangle, \langle x_2, (0.4, 0.3), (0.1, 0.6), (0.2, 0.75)\rangle\}$$

is an IFVNS.

Remark 3.3. The notion of IFVNS is a new extension of the notion of SNS, since each SNS has the form

$$\begin{aligned} A &= \left\{ \left\langle x_j, T_A(x_j), I_A(x_j), F_A(x_j) \right\rangle : j = 1, \ldots, n \right\} \\ &= \left\{ \left\langle x_j, \big(\mu_{A,t}(x_j), 0\big), \big(\mu_{A,i}(x_j), 0\big), \big(\mu_{A,f}(x_j), 0\big) \right\rangle : j = 1, \ldots, n \right\}. \end{aligned}$$

However, an IFVNS cannot be represented as a standard SNS (see, e.g., Example 3.2).

In some decision making problems, the sum of the membership and non-membership degrees that are determined by the decision makers can be larger than 1. Therefore, PFVs are more capable than IFVs for modelling vagueness in the degrees of the truth, the indeterminacy and the falsity in the practical problems as shown in Figure 2. Using PFVs for the degree of the truth, the indeterminacy and the falsity, we define the concept of PFVNS and propose some set theoretical and algebraic operations between PFVNSs. By utilizing PFVNSs, the uncertainty in the decision maker's evaluations can be modeled in a more capable environment.

Definition 3.4. A PFVNS A on X is defined by

$$A = \left\{ \langle x_j, \mathbb{T}_A^j, \mathbb{I}_A^j, \mathbb{F}_A^j \rangle : j = 1, \ldots, n \right\}$$

where $\mathbb{T}_A^j$, $\mathbb{I}_A^j$ and $\mathbb{F}_A^j$ are the truth, the indeterminacy and the falsity membership pairs of PFVs, respectively, i.e., for $j = 1, \ldots, n$

$$\mathbb{T}_A^j = \big(\mu_{A,t}(x_j), \nu_{A,t}(x_j)\big) \text{ with } \mu_{A,t}(x_j), \nu_{A,t}(x_j) \in [0,1] \text{ such that } \mu_{A,t}^2(x_j) + \nu_{A,t}^2(x_j) \leq 1$$
$$\mathbb{I}_A^j = \big(\mu_{A,i}(x_i), \nu_{A,i}(x_j)\big) \text{ with } \mu_{A,i}(x_j), \nu_{A,i}(x_j) \in [0,1] \text{ such that } \mu_{A,i}^2(x_j) + \nu_{A,i}^2(x_j) \leq 1$$
$$\mathbb{F}_A^j = \big(\mu_{A,f}(x_j), \nu_{A,f}(x_j)\big) \text{ with } \mu_{A,f}(x_j), \nu_{A,f}(x_j) \in [0,1] \text{ such that } \mu_{A,f}^2(x_j) + \nu_{A,f}^2(x_j) \leq 1.$$

A PFVNV is denoted by

$$\alpha = \langle \mathbb{T}_\alpha, \mathbb{I}_\alpha, \mathbb{F}_\alpha \rangle := \langle \mathbb{T}_A^j, \mathbb{I}_A^j, \mathbb{F}_A^j \rangle$$

for a fixed $j = 1, ..., n$.

Example 3.5. Let $X = \{x_1, x_2\}$.

$$A = \{\langle x_1, (0.4, 0.3), (0.5, 0.5), (0.1, 0.9)\rangle, \langle x_2, (0.3, 0.7), (0.7, 0.6), (0.2, 0.2)\rangle\}$$

is a PFVNS.

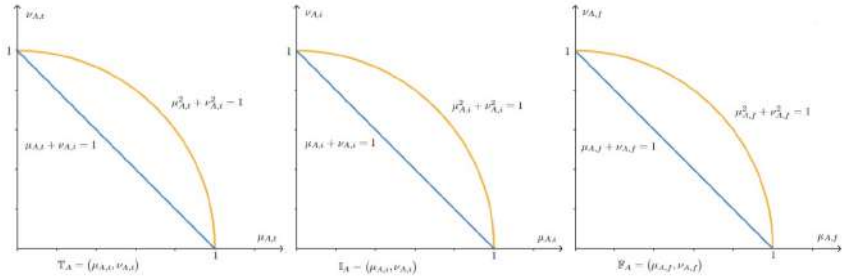

Figure 2: Comparison of concepts of PFV and IFV for the truth, the indeterminacy and the falsity membership degrees

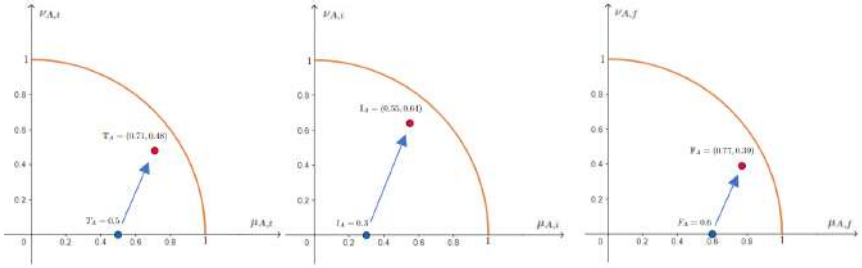

Figure 3: The SNV and PFVNV in Example 3.8

Remark 3.6. Each IFVNS is a PFVNS but the converse of this statement is not true in general. For example, A is a PFVNS, but A is not a $IFVNS$ in Example 3.5 since $0.7 + 0.6 = 1.3 > 1$ for $\mathbb{I}_A^2 = (0.7, 0.6)$.

SNVs can be converted to PFVNVs with the following method. A similar method was proposed for IFSs in.[25]

Proposition 3.7. *Let $\rho : [0,1] \to [0,1]$ be a function such that $\rho(t) \leq \sqrt{t}$ for any $t \in [0,1]$. Consider a number $\mu_\alpha \in [0,1]$. Then*

$$\alpha = \langle \sqrt{\mu_\alpha}, \rho(1-\mu_\alpha) \rangle$$

is a PFV.

Proof. Let $\mu_\alpha \in [0,1]$

$$\begin{aligned} \sqrt{\mu_\alpha} + \rho^2(1-\mu_\alpha) &\leq \mu_\alpha + \left(\sqrt{1-\mu_\alpha}\right)^2 \\ &= \mu_\alpha + 1 - \mu_\alpha \\ &= 1. \end{aligned}$$

So α is a PFV. □

We can transform a SNV into a $PFVNV$ by using Proposition 3.7 as in the following example.

Example 3.8. Consider the SNV $A = \langle 0.5, 0.3, 0.6 \rangle$ and the function $\rho : [0,1] \to [0,1]$ defined by $\rho(t) = sint$. From Proposition 3.7, we obtain a PFVNV from A as follows:

$$A = \langle (0.71, 0.48), (0.55, 0.64), (0.77, 0.39) \rangle.$$

We visualize this transformation in Figure 3.

Now we define the set operations between PFVNSs. Throughout this paper the set operations for PFSs are denoted by $\subset_{(pyt)}, \cup_{(pyt)}, \cap_{(pyt)}, (.)^{c_{(pyt)}}$" (see,[34]).

Definition 3.9. Let

$$A = \left\{ \langle x_j, \mathbb{T}_A^j, \mathbb{I}_A^j, \mathbb{F}_A^j \rangle : j = 1, \ldots, n \right\}$$

and

$$B = \left\{ \langle x_j, \mathbb{T}_B^j, \mathbb{I}_B^j, \mathbb{F}_B^j \rangle : j = 1, \ldots, n \right\}$$

be two PFVNSs. Set operations among A and B are defined as follows.
a) $A \subset B$ if and only if for all j=1,. . . ,n

$$\begin{aligned} \mathbb{T}_A^j \subset_{(pyt)} \mathbb{T}_B^j \text{ i.e. } \mu_{A,t}(x_j) \leq \mu_{B,t}(x_j) \text{ and } \nu_{A,t}(x_j) \geq \nu_{B,t}(x_j) \\ \mathbb{I}_A^j \supset_{(pyt)} \mathbb{I}_B^j \text{ i.e. } \mu_{A,i}(x_j) \geq \mu_{B,i}(x_j) \text{ and } \nu_{A,i}(x_j) \leq \nu_{B,i}(x_j) \\ \mathbb{F}_A^j \supset_{(pyt)} \mathbb{F}_B^j \text{ i.e. } \mu_{A,f}(x_j) \geq \mu_{B,f}(x_j) \text{ and } \nu_{A,f}(x_j) \leq \nu_{B,f}(x_j). \end{aligned}$$

b) $A = B$ if and only if $A \subset B$ and $B \subset A$.
c) $A^c = \left\{ \langle x_j, \mathbb{F}_A^j, \left(\mathbb{I}_A^j\right)^{c_{(pyt)}}, \mathbb{T}_A^j \rangle : j = 1, \ldots, n \right\}$ where for all $j = 1, \ldots, n$

$$\left(\mathbb{I}_A^j\right)^{c_{(pyt)}} = \left(\nu_{A,i}(x_j), \mu_{A,i}(x_j)\right).$$

d) $A \cup B = \left\{ \langle x_j, \mathbb{T}_A^j \cup_{(pyt)} \mathbb{T}_B^j, \mathbb{I}_A^j \cap_{(pyt)} \mathbb{I}_B^j, \mathbb{F}_A^j \cap_{(pyt)} \mathbb{F}_B^j \rangle : j = 1, \ldots, n \right\}$ where

$$\begin{aligned} \mathbb{T}_A^j \cup_{(pyt)} \mathbb{T}_B^j &= \left(\max\left(\mu_{A,t}(x_j), \mu_{B,t}(x_j)\right), \min\left(\nu_{A,t}(x_j), \nu_{B,t}(x_j)\right) \right) \\ \mathbb{I}_A^j \cap_{(pyt)} \mathbb{I}_B^j &= \left(\min\left(\mu_{A,i}(x_j), \mu_{B,i}(x_j)\right), \max\left(\nu_{A,i}(x_j), \nu_{B,i}(x_j)\right) \right) \\ \mathbb{F}_A^j \cap_{(pyt)} \mathbb{F}_B^j &= \left(\min\left(\mu_{A,f}(x_j), \mu_{B,f}(x_j)\right), \max\left(\nu_{A,f}(x_j), \nu_{B,f}(x_j)\right) \right). \end{aligned}$$

e) $A \cap B = \left\{ \langle x_j, \mathbb{T}_A^j \cap_{(pyt)} \mathbb{T}_B^j, \mathbb{I}_A^j \cup_{(pyt)} \mathbb{I}_B^j, \mathbb{F}_A^j \cup_{(pyt)} \mathbb{F}_B^j \rangle : j = 1, \ldots, n \right\}$ where

$$\begin{aligned} \mathbb{T}_A^j \cap_{(pyt)} \mathbb{T}_B^j &= \left(\min\left(\mu_{A,t}(x_j), \mu_{B,t}(x_j)\right), \max\left(\nu_{A,t}(x_j), \nu_{B,t}(x_j)\right) \right) \\ \mathbb{I}_A^j \cup_{(pyt)} \mathbb{I}_B^j &= \left(\max\left(\mu_{A,i}(x_j), \mu_{B,i}(x_j)\right), \min\left(\nu_{A,i}(x_j), \nu_{B,i}(x_j)\right) \right) \\ \mathbb{F}_A^j \cup_{(pyt)} \mathbb{F}_B^j &= \left(\max\left(\mu_{A,f}(x_j), \mu_{B,f}(x_j)\right), \min\left(\nu_{A,f}(x_j), \nu_{B,f}(x_j)\right) \right). \end{aligned}$$

Example 3.10. Let $X = \{x_1, x_2\}$ and consider the PFVNSs

$$A = \left\{ \langle x_1, (0.2, 0.3), (0.7, 0.4), (0.4, 0.7) \rangle, \langle x_2, (0.4, 0.8), (0.7, 0.3), (0.7, 0.1) \rangle \right\}$$

and

$$B = \left\{ \langle x_1, (0.4, 0.3), (0.4, 0.5), (0.1, 0.9) \rangle, \langle x_2, (0.6, 0.5), (0.2, 0.6), (0.2, 0.2) \rangle \right\}.$$

Then $A \subset B$. On the other hand, it is easy to obtain that

$$A^c = \left\{ \langle x_1, (0.4, 0.7), (0.4, 0.7), (0.2, 0.3) \rangle, \langle x_2, (0.7, 0.1), (0.3, 0.7), (0.4, 0.8) \rangle \right\}$$

$$A \cup B = \left\{ \langle x_1, (0.4, 0.3), (0.4, 0.5), (0.1, 0.9) \rangle, \langle x_2, (0.6, 0.5), (0.2, 0.6), (0.2, 0.2) \rangle \right\}$$

and

$$A \cap B = \left\{ \langle x_1, (0.2, 0.3), (0.7, 0.4), (0.4, 0.7) \rangle, \langle x_2, (0.4, 0.8), (0.7, 0.3), (0.7, 0.1) \rangle \right\}.$$

Next we show that set operations defined in Definition 3.9 satisy De Morgan's rules.

Theorem 3.11. *Let*

$$A = \left\{ \langle x_j, \mathbb{T}_A^j, \mathbb{I}_A^j, \mathbb{F}_A^j \rangle : j = 1, \ldots, n \right\}$$

and

$$B = \left\{ \langle x_j, \mathbb{T}_B^j, \mathbb{I}_B^j, \mathbb{F}_B^j \rangle : j = 1, \ldots, n \right\}$$

be two PFVNSs. The following are valid.
a) $(A \cup B)^c = A^c \cap B^c$
b) $(A \cap B)^c = A^c \cup B^c$.

Proof. a) We get

$$\begin{aligned}(A\cup B)^c &= \left(\left\{\left\langle x_j,\mathbb{T}_A^j\cup_{(pyt)}\mathbb{T}_B^j,\mathbb{I}_A^j\cap_{(pyt)}\mathbb{I}_B^j,\mathbb{F}_A^j\cap_{(pyt)}\mathbb{F}_B^j\right\rangle : j=1,\dots,n\right\}\right)^c\\ &= \left\{\left\langle x_j,\mathbb{F}_A^j\cap_{(pyt)}\mathbb{F}_B^j,\left(\mathbb{I}_A^j\cap_{(pyt)}\mathbb{I}_B^j\right)^{c_{(pyt)}},\mathbb{T}_A^j\cup_{(pyt)}\mathbb{T}_B^j\right\rangle : j=1,\dots,n\right\}\\ &= \left\{\left\langle x_j,\mathbb{F}_A^j\cap_{(pyt)}\mathbb{F}_B^j,\left(\mathbb{I}_A^j\right)^{c_{(pyt)}}\cup_{(pyt)}\left(\mathbb{I}_B^j\right)^{c_{(pyt)}},\mathbb{T}_A^j\cup_{(pyt)}\mathbb{T}_B^j\right\rangle : j=1,\dots,n\right\}\\ &= A^c\cap B^c.\end{aligned}$$

b) We get

$$\begin{aligned}(A\cap B)^c &= \left(\left\{\left\langle x_j,\mathbb{T}_A^j\cap_{(pyt)}\mathbb{T}_B^j,\mathbb{I}_A^j\cup_{(pyt)}\mathbb{I}_B^j,\mathbb{F}_A^j\cup_{(pyt)}\mathbb{F}_B^j\right\rangle : j=1,\dots,n\right\}\right)^c\\ &= \left\{\left\langle x_j,\mathbb{F}_A^j\cup_{(pyt)}\mathbb{F}_B^j,\left(\mathbb{I}_A^j\cup_{(pyt)}\mathbb{I}_B^j\right)^{c_{(pyt)}},\mathbb{T}_A^j\cap_{(pyt)}\mathbb{T}_B^j\right\rangle : j=1,\dots,n\right\}\\ &= \left\{\left\langle x_j,\mathbb{F}_A^j\cup_{(pyt)}\mathbb{F}_B^j,\left(\mathbb{I}_A^j\right)^{c_{(pyt)}}\cap_{(pyt)}\left(\mathbb{I}_B^j\right)^{c_{(pyt)}},\mathbb{T}_A^j\cap_{(pyt)}\mathbb{T}_B^j\right\rangle : j=1,\dots,n\right\}\\ &= A^c\cup B^c.\end{aligned}$$

□

Now we introduce some algebraic operations for PFVNVs via Pythagorean fuzzy t-norms and t-conorms.

Definition 3.12. Let $\alpha=\langle\mathbb{T}_\alpha,\mathbb{I}_\alpha,\mathbb{F}_\alpha\rangle$ and $\beta=\langle\mathbb{T}_\beta,\mathbb{I}_\beta,\mathbb{F}_\beta\rangle$ be two PFVNVs, let $\mathcal{T}$ be a Pythagorean fuzzy t-norm and let $\mathcal{S}$ be the dual Pythagorean fuzzy t-conorm of $\mathcal{T}$ with respect to a Pythagorean fuzzy negator $\mathcal{N}$. Then

$$\alpha\oplus\beta:=\left\langle\left(\mathcal{S}\big(\mathbb{T}_\alpha,\mathbb{T}_\beta\big),\mathcal{T}\big(\mathbb{I}_\alpha,\mathbb{I}_\beta\big),\mathcal{T}\big(\mathbb{F}_\alpha,\mathbb{F}_\beta\big)\right)\right\rangle$$

and

$$\alpha\otimes\beta:=\left\langle\left(\mathcal{T}\big(\mathbb{T}_\alpha,\mathbb{T}_\beta\big),\mathcal{S}\big(\mathbb{I}_\alpha,\mathbb{I}_\beta\big),\mathcal{S}\big(\mathbb{F}_\alpha,\mathbb{F}_\beta\big)\right)\right\rangle$$

Remark 3.13. If g is the additive generator of a continuous Archimedean t-norm and $h(t)=g(\sqrt{1-t^2})$, then from Corollary 2.14 we get

$$\begin{aligned}\alpha\oplus\beta &= \left\langle\left(h^{-1}\big(h(\mu_{\alpha,t})+h(\mu_{\beta,t})\big),g^{-1}\big(g(\nu_{\alpha,t})+g(\nu_{\beta,t})\big)\right),\right.\\ &\quad\left(g^{-1}\big(g(\mu_{\alpha,i})+g(\mu_{\beta,i})\big),h^{-1}\big(h(\nu_{\alpha,i})+h(\nu_{\beta,i})\big)\right),\\ &\quad\left.\left(g^{-1}\big(g(\mu_{\alpha,f})+g(\mu_{\beta,f})\big),h^{-1}\big(h(\nu_{\alpha,f})+h(\nu_{\beta,f})\big)\right)\right\rangle\end{aligned}$$

and

$$\begin{aligned}\alpha\otimes\beta &= \left\langle\left(g^{-1}\big(g(\mu_{\alpha,t})+g(\mu_{\beta,t})\big),h^{-1}\big(h(\nu_{\alpha,t})+h(\nu_{\beta,t})\big)\right),\right.\\ &\quad\left(h^{-1}\big(h(\mu_{\alpha,i})+h(\mu_{\beta,i})\big),g^{-1}\big(g(\nu_{\alpha,i})+g(\nu_{\beta,i})\big)\right),\\ &\quad\left.\left(h^{-1}\big(h(\mu_{\alpha,f})+h(\mu_{\beta,f})\big),g^{-1}\big(g(\nu_{\alpha,f})+g(\nu_{\beta,f})\big)\right)\right\rangle.\end{aligned}$$

Following proposition is the validation of that the sum and the product of two PFVNVs are also PFVNVs.

Proposition 3.14. *Let α and β be two PFVNVs, let $\mathcal{T}$ be A Pythagorean fuzzy t-norm and let $\mathcal{S}$ be the dual Pythagorean fuzzy t-conorm with respect to a Pythagorean fuzzy negator $\mathcal{N}$. Then, $\alpha\oplus\beta$ and $\alpha\otimes\beta$ are also PFVNVs.*

Proof. Since $\mathcal{T}$ and $\mathcal{S}$ have range P^*, $\alpha\oplus\beta$ and $\alpha\otimes\beta$ are PFVNVs. □

Next we define multiplication by non-negative constant and non-negative power of PFVNVs using additive generators of continuous Archimedean t-norm and t-conorm on $[0,1]$.

Definition 3.15. Let $\alpha = \langle \mathbb{T}_\alpha, \mathbb{I}_\alpha, \mathbb{F}_\alpha \rangle$ be a PFVNV, let $\lambda \geq 0$, let $g : [0,1] \to [0,\infty]$ be the additive generator of a continuous Archimedean t-norm and let $h(t) = g(\sqrt{1-t^2})$. Then

$$\begin{aligned} \lambda\alpha \quad = \Big\langle & \Big(h^{-1}\big(\lambda h\big(\mu_{\alpha,t}\big)\big), g^{-1}\big(\lambda g\big(\nu_{\alpha,t}\big)\big)\Big), \\ & \Big(g^{-1}\big(\lambda g\big(\mu_{\alpha,i}\big)\big), h^{-1}\big(\lambda h\big(\nu_{\alpha,i}\big)\big)\Big), \\ & \Big(g^{-1}\big(\lambda g\big(\mu_{\alpha,f}\big)\big), h^{-1}\big(\lambda h\big(\nu_{\alpha,f}\big)\big)\Big)\Big\rangle \end{aligned}$$

and

$$\begin{aligned} \alpha^\lambda \quad = \Big\langle & \Big(g^{-1}\big(\lambda g\big(\mu_{\alpha,t}\big)\big), h^{-1}\big(\lambda h\big(\nu_{\alpha,t}\big)\big)\Big), \\ & \Big(h^{-1}\big(\lambda h\big(\mu_{\alpha,i}\big)\big), g^{-1}\big(\lambda g\big(\nu_{\alpha,i}\big)\big)\Big), \\ & \Big(h^{-1}\big(\lambda h\big(\mu_{\alpha,f}\big)\big), g^{-1}\big(\lambda g\big(\nu_{\alpha,f}\big)\big)\Big)\Big\rangle. \end{aligned}$$

The following proposition verifies that multiplication by constant and power of PFVNVs are also PFVNVs.

Proposition 3.16. *Let α be a PFVNV, let $\lambda \geq 0$, let $g : [0,1] \to [0,\infty]$ be the additive generator of a continuous Archimedean t-norm and let $h(t) = g(\sqrt{1-t^2})$. Then $\lambda\alpha$ and α^λ are PFVNVs.*

Proof. We know that $h^{-1}(t) = \sqrt{1-[g^{-1}(t)]^2}$ and $g(t) = h(\sqrt{1-t^2})$. Since $\mu_{\alpha,t} \leq \sqrt{1-\nu_{\alpha,t}^2}$ and h, h^{-1} are increasing, we obtain

$$\begin{aligned} 0 \quad &\leq \quad \big[h^{-1}(\lambda h(\mu_{\alpha,t}))\big]^2 + \big[g^{-1}(\lambda g(\nu_{\alpha,t}))\big]^2 \\ &\leq \quad \Big[h^{-1}\Big(\lambda h\Big(\sqrt{1-\nu_{\alpha,t}^2}\Big)\Big)\Big]^2 + \big[g^{-1}(\lambda g(\nu_{\alpha,t}))\big]^2 \\ &= \quad 1 - \Big[g^{-1}\Big(\lambda h\Big(\sqrt{1-\nu_{\alpha,t}^2}\Big)\Big]^2 + \Big[g^{-1}(\lambda g(\nu_{\alpha,t}))\Big]^2 \\ &= \quad 1 - \big[g^{-1}(\lambda g(\nu_{\alpha,t}))\big]^2 + \big[g^{-1}(\lambda g(\nu_{\alpha,t}))\big]^2 \\ &= \quad 1 \end{aligned}$$

which yields that $\mathbb{T}_{\lambda\alpha}$ is a PFV. Similarly, it can be easily shown that $\mathbb{I}_{\lambda\alpha}, \mathbb{F}_{\lambda\alpha}.\mathbb{T}_{\alpha^\lambda}, \mathbb{I}_{\alpha^\lambda}$ and $\mathbb{F}_{\alpha^\lambda}$ are PFVs. Therefore $\lambda\alpha$ is a PFVNV. □

Following theorem gives some basic properties of algebraic operations.

Theorem 3.17. *Let $\alpha = \langle \mathbb{T}_\alpha, \mathbb{I}_\alpha, \mathbb{F}_\alpha \rangle$, $\beta = \langle \mathbb{T}_\beta, \mathbb{I}_\beta, \mathbb{F}_\beta \rangle$ and $\theta = \langle \mathbb{T}_\theta, \mathbb{I}_\theta, \mathbb{F}_\theta \rangle$ be PFVNVs, let $\lambda, \gamma \geq 0$, let g be the additive generator of a continuous Archimedean t-norm and let $h(t) = g(\sqrt{1-t^2})$. Then,*
i) $\alpha \oplus \beta = \beta \oplus \alpha$
ii) $\alpha \otimes \beta = \beta \otimes \alpha$
iii) $(\alpha \oplus \beta) \oplus \theta = \alpha \oplus (\beta \oplus \theta)$
iv) $(\alpha \otimes \beta) \otimes \theta = \alpha \otimes (\beta \otimes \theta)$
v) $\lambda(\alpha \oplus \beta) = \lambda\alpha \oplus \lambda\beta$
vi) $(\lambda + \gamma)\alpha = \alpha\lambda \oplus \gamma\alpha$
vii) $(\alpha \otimes \beta)^\lambda = \alpha^\lambda \otimes \beta^\lambda$
viii) $\alpha^\lambda \otimes \alpha^\gamma = \alpha^{\lambda+\gamma}$.

Proof. (i) and (ii) are trivial.
iii) It is clear from Definition 3.12 that

$$\mathbb{T}_{\alpha\oplus\beta} = \Big(h^{-1}\big(h(\mu_{\alpha,t}) + h(\mu_{\beta,t})\big), g^{-1}\big(g(\nu_{\alpha,t}) + g(\nu_{\beta,t})\big)\Big).$$

Therefore, we obtain

$$\begin{aligned}
\mathbb{T}_{(\alpha\oplus\beta)\oplus\theta} &= \mathcal{S}(\mathbb{T}_{\alpha\oplus\beta}, \mathbb{T}_{\theta}) \\
&= \Big(h^{-1}\Big(h\big(h^{-1}\big(h(\mu_{\alpha,t})+h(\mu_{\beta,t})\big)+h\big(\mu_{\theta,t}\big)\Big)\Big), \\
&\quad g^{-1}\Big(g\big(g^{-1}\big(g(\nu_{\alpha,t})+g(\nu_{\beta,t})\big)+g\big(\nu_{\theta,t}\big)\Big)\Big) \\
&= \Big(h^{-1}\Big(h(\mu_{\alpha,t})+h(\mu_{\beta,t})+h(\mu_{\theta,t})\Big), \\
&\quad g^{-1}\Big(g(\nu_{\alpha,t})+g(\nu_{\beta,t})+g\big(\nu_{\theta,t}\big)\Big)\Big) \\
&= \Big(h^{-1}\Big(h(\mu_{\alpha,t})+h\Big(h^{-1}\big(h(\mu_{\beta,t})+h(\mu_{\theta,t})\big)\Big)\Big), \\
&\quad g^{-1}\Big(g(\nu_{\alpha,t})+g\Big(g^{-1}\big(g(\nu_{\beta,t})+g\big(\nu_{\theta,t}\big)\big)\Big)\Big) \\
&= \mathcal{S}(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta\oplus\theta}) \\
&= \mathbb{T}_{\alpha\oplus(\beta\oplus\theta)}.
\end{aligned}$$

Similarly it can be easily obtained that $\mathbb{I}_{(\alpha\oplus\beta)\oplus\theta} = \mathbb{I}_{\alpha\oplus(\beta\oplus\theta)}$ and $\mathbb{F}_{(\alpha\oplus\beta)\oplus\theta} = \mathbb{F}_{\alpha\oplus(\beta\oplus\theta)}$.
iv) It is clear from Definition 3.12 that

$$\mathbb{T}_{\alpha\otimes\beta} = \Big(g^{-1}\big(g(\mu_{\alpha,t})+g(\mu_{\beta,t})\big), h^{-1}\big(h(\nu_{\alpha,t})+h(\nu_{\beta,t})\big)\Big).$$

Therefore, we have

$$\begin{aligned}
\mathbb{T}_{(\alpha\otimes\beta)\otimes\theta} &= \mathcal{T}(\mathbb{T}_{\alpha\otimes\beta}, \mathbb{T}_{\theta}) \\
&= \Big(g^{-1}\Big(g\big(g^{-1}\big(g(\mu_{\alpha,t})+g(\mu_{\beta,t})\big)+g\big(\mu_{\theta,t}\big)\Big)\Big), \\
&\quad h^{-1}\Big(h\big(h^{-1}\big(h(\nu_{\alpha,t})+h(\nu_{\beta,t})\big)+h\big(\nu_{\theta,t}\big)\Big)\Big) \\
&= \Big(g^{-1}\Big(g(\mu_{\alpha,t})+g(\mu_{\beta,t})+g(\mu_{\theta,t})\Big), \\
&\quad h^{-1}\Big(h(\nu_{\alpha,t})+h(\nu_{\beta,t})+h\big(\nu_{\theta,t}\big)\Big)\Big) \\
&= \Big(g^{-1}\Big(g(\mu_{\alpha,t})+g\Big(g^{-1}\big(g(\mu_{\beta,t})+g(\mu_{\theta,t})\big)\Big)\Big), \\
&\quad h^{-1}\Big(h(\nu_{\alpha,t})+h\Big(h^{-1}\big(h(\nu_{\beta,t})+h\big(\nu_{\theta,t}\big)\big)\Big)\Big) \\
&= \mathcal{T}(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta\otimes\theta}) \\
&= \mathbb{T}_{\alpha\otimes(\beta\otimes\theta)}.
\end{aligned}$$

Similarly it can be seen that $\mathbb{I}_{(\alpha\otimes\beta)\otimes\theta} = \mathbb{I}_{\alpha\otimes(\beta\otimes\theta)}$ and $\mathbb{F}_{(\alpha\otimes\beta)\otimes\theta} = \mathbb{F}_{\alpha\otimes(\beta\otimes\theta)}$.
v) It is clear from Definition 3.12 and Definition 3.15 that

$$\mathbb{T}_{\alpha\oplus\beta} = (\mu_{\alpha\oplus\beta,t}, \nu_{\alpha\oplus\beta,t}) = \Big(h^{-1}\big(h(\mu_{\alpha,t})+h(\mu_{\beta,t})\big), g^{-1}\big(g(\nu_{\alpha,t})+g(\nu_{\beta,t})\big)\Big)$$

and

$$\mathbb{T}_{\lambda\alpha} = (\mu_{\lambda\alpha,t}, \nu_{\lambda\alpha,t}) = \Big(h^{-1}\big(\lambda h\big(\mu_{\alpha,t}\big)\big), g^{-1}\big(\lambda g\big(\nu_{\alpha,t}\big)\big)\Big).$$

Therefore, we get

$$\begin{aligned}
\mathbb{T}_{\lambda(\alpha\oplus\beta)} &= \Big(h^{-1}\big(\lambda h(\mu_{\alpha\oplus\beta,t})\big), g^{-1}\big(\lambda g(\nu_{\alpha\oplus\beta,t})\big)\Big) \\
&= \Big(h^{-1}\big(\lambda h(h^{-1}\big(h(\mu_{\alpha,t})+h(\mu_{\beta,t})\big))\big), g^{-1}\big(\lambda g(g^{-1}\big(g(\nu_{\alpha,t})+g(\nu_{\beta,t})\big))\big)\Big) \\
&= \Big(h^{-1}\big(\lambda h(\mu_{\alpha,t})+\lambda h(\mu_{\beta,t})\big)\big), g^{-1}\big(\lambda g(\nu_{\alpha,t})+\lambda g(\nu_{\beta,t})\big)\big)\Big) \\
&= \Big(h^{-1}\big(h\big(h^{-1}\big(\lambda h(\mu_{\alpha,t})\big)+h\big(h^{-1}\big(\lambda h(\mu_{\beta,t})\big)\big)\big), \\
&\quad g^{-1}\big(g\big(g^{-1}\big(\lambda g(\nu_{\alpha,t})\big)+g\big(g^{-1}\big(\lambda g(\nu_{\beta,t})\big)\big)\big)\Big) \\
&= \Big(h^{-1}\big(h\big(\mu_{\lambda\alpha,t}\big)+h\big(\mu_{\lambda\beta,t}\big)\big), g^{-1}\big(g\big(\nu_{\lambda\alpha,t}\big)+g\big(\nu_{\lambda\beta,t}\big)\big)\Big) \\
&= \mathbb{T}_{\lambda\alpha\oplus\lambda\beta}.
\end{aligned}$$

Similarly it can be shown that $\mathbb{I}_{\lambda(\alpha\oplus\beta)} = \mathbb{I}_{\lambda\alpha\oplus\lambda\beta}$ and $\mathbb{F}_{\lambda(\alpha\oplus\beta)} = \mathbb{F}_{\lambda\alpha\oplus\lambda\beta}$.
vi) It is clear that

$$\begin{aligned}
\mathbb{T}_{(\lambda+\gamma)\alpha} &= \Big(h^{-1}\big((\lambda+\gamma)h\big(\mu_{\alpha,t}\big)\big), g^{-1}\big((\lambda+\gamma)g\big(\nu_{\alpha,t}\big)\big)\Big) \\
&= \Big(h^{-1}\big(\lambda h\big(\mu_{\alpha,t}\big) + \gamma h\big(\mu_{\alpha,t}\big)\big), g^{-1}\big(\lambda g\big(\nu_{\alpha,t}\big) + \gamma g\big(\nu_{\alpha,t}\big)\big)\Big) \\
&= \Big(h^{-1}\big(h\big(h^{-1}\big(\lambda h\big(\mu_{\alpha,t}\big)\big)\big) + h\big(h^{-1}\big(\gamma h\big(\mu_{\alpha,t}\big)\big)\big)\big), \\
&\quad g^{-1}\big(g\big(g^{-1}\big(\lambda g\big(\nu_{\alpha,t}\big)\big)\big) + g\big(g^{-1}\big(\gamma g\big(\nu_{\alpha,t}\big)\big)\big)\big)\Big) \\
&= \Big(h^{-1}\big(h\big(\mu_{\lambda\alpha,t}\big) + h\big(\mu_{\gamma\alpha,t}\big)\big), g^{-1}\big(g\big(\nu_{\lambda\alpha,t}\big) + g\big(\nu_{\gamma\alpha,t}\big)\big)\Big) \\
&= \mathbb{T}_{\lambda\alpha\oplus\gamma\alpha}.
\end{aligned}$$

Similarly it can be obtained that $\mathbb{I}_{(\lambda+\gamma)\alpha} = \mathbb{I}_{\lambda\alpha\oplus\gamma\alpha}$ and $\mathbb{F}_{(\lambda+\gamma)\alpha} = \mathbb{F}_{\lambda\alpha\oplus\gamma\alpha}$.
vii) We know that

$$\mathbb{T}_{\alpha\otimes\beta} = (\mu_{\alpha\otimes\beta,t}, \nu_{\alpha\otimes\beta,t}) = \Big(g^{-1}\big(g(\mu_{\alpha,t}) + g(\mu_{\beta,t})\big), h^{-1}\big(h(\nu_{\alpha,t}) + h(\nu_{\beta,t})\big)\Big)$$

and

$$\mathbb{T}_{\alpha^\lambda} = (\mu_{\alpha^\lambda,t}, \nu_{\alpha^\lambda,t}) = \Big(g^{-1}\big(\lambda g\big(\mu_{\alpha,t}\big)\big), h^{-1}\big(\lambda h\big(\nu_{\alpha,t}\big)\big)\Big).$$

Therefore, we obtain

$$\begin{aligned}
\mathbb{T}_{(\alpha\otimes\beta)^\lambda} &= \Big(g^{-1}\big(\lambda g(\mu_{\alpha\otimes\beta,t})\big), h^{-1}\big(\lambda h(\nu_{\alpha\otimes\beta,t})\big)\Big) \\
&= \Big(g^{-1}\big(\lambda g(g^{-1}\big(g(\mu_{\alpha,t}) + g(\mu_{\beta,t})\big)\big), h^{-1}\big(\lambda h(h^{-1}\big(h(\nu_{\alpha,t}) + h(\nu_{\beta,t})\big)\big)\Big) \\
&= \Big(g^{-1}\big(\lambda g(\mu_{\alpha,t}) + \lambda g(\mu_{\beta,t})\big)\big), h^{-1}\big(\lambda h(\nu_{\alpha,t}) + \lambda h(\nu_{\beta,t})\big)\big)\Big) \\
&= \Big(g^{-1}\big(g\big(g^{-1}\big(\lambda g(\mu_{\alpha,t})\big)\big) + g\big(g^{-1}\big(\lambda g(\mu_{\beta,t})\big)\big)\big), \\
&\quad h^{-1}\big(h\big(h^{-1}\big(\lambda h(\nu_{\alpha,t})\big)\big) + h\big(h^{-1}\big(\lambda h(\nu_{\beta,t})\big)\big)\big)\Big) \\
&= \Big(g^{-1}\big(g\big(\mu_{\alpha^\lambda,t}\big) + g\big(\mu_{\beta^\lambda,t}\big)\big), h^{-1}\big(h\big(\nu_{\alpha^\lambda,t}\big) + h\big(\nu_{\beta^\lambda,t}\big)\big)\Big) \\
&= \mathbb{T}_{\alpha^\lambda\oplus\beta^\lambda}.
\end{aligned}$$

Similarly it can be seen that $\mathbb{I}_{(\alpha\otimes\beta)^\lambda} = \mathbb{I}_{\alpha^\lambda\oplus\beta^\lambda}$ and $\mathbb{F}_{(\alpha\otimes\beta)^\lambda} = \mathbb{F}_{\alpha^\lambda\oplus\beta^\lambda}$.
viii) We have

$$\begin{aligned}
\mathbb{T}_{\alpha^{\lambda+\gamma}} &= \Big(g^{-1}\big((\lambda+\gamma)g\big(\mu_{\alpha,t}\big)\big), h^{-1}\big((\lambda+\gamma)h\big(\nu_{\alpha,t}\big)\big)\Big) \\
&= \Big(g^{-1}\big(\lambda g\big(\mu_{\alpha,t}\big) + \gamma g\big(\mu_{\alpha,t}\big)\big), h^{-1}\big(\lambda h\big(\nu_{\alpha,t}\big) + \gamma h\big(\nu_{\alpha,t}\big)\big)\Big) \\
&= \Big(g^{-1}\big(g\big(g^{-1}\big(\lambda g\big(\mu_{\alpha,t}\big)\big)\big) + g\big(h^{-1}\big(\gamma g\big(\mu_{\alpha,t}\big)\big)\big)\big), \\
&\quad h^{-1}\big(h\big(h^{-1}\big(\lambda h\big(\nu_{\alpha,t}\big)\big)\big) + h\big(h^{-1}\big(\gamma h\big(\nu_{\alpha,t}\big)\big)\big)\big)\Big) \\
&= \Big(g^{-1}\big(g\big(\mu_{\alpha^\lambda,t}\big) + g\big(\mu_{\alpha^\gamma,t}\big)\big), h^{-1}\big(h\big(\nu_{\alpha^\lambda,t}\big) + h\big(\nu_{\alpha^\gamma,t}\big)\big)\Big) \\
&= \mathbb{T}_{\alpha^\lambda\otimes\alpha^\gamma}.
\end{aligned}$$

Similarly it can be obtained that $\mathbb{I}_{\alpha^{\lambda+\gamma}} = \mathbb{I}_{\alpha^\lambda\otimes\alpha^\gamma}$ and $\mathbb{F}_{\alpha^{\lambda+\gamma}} = \mathbb{F}_{\alpha^\lambda\otimes\alpha^\gamma}$. □

4 Weighted Aggregation Operators for PFVNVs

Aggregation operators have a vital role while transforming input values represented by fuzzy values to a single output value. In this section, we introduce a weighted arithmetic aggregation operator and a weighted geometric aggregation operator for collections of PFVNVs by using algebraic operations given in Section 3.

4.1 A Weighted Arithmetic Aggregation Operator

Definition 4.1. Let $\{\alpha_j = \langle \mathbb{T}_{\alpha_j}, \mathbb{I}_{\alpha_j}, \mathbb{F}_{\alpha_j} \rangle : j = 1, \ldots, n\}$ be a collection of PFVNVs. A weighted arithmetic aggregation operator $WA - PFVNV$ is defined by

$$WA - PFVNV(\alpha_1, \ldots, \alpha_n) := \bigoplus_{j=1}^{n} w_j \alpha_j$$

where $w = (w_1, \ldots, w_n)^T$ is a weight vector such that $0 \leq w_j \leq 1$ for any $j = 1, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$.

As seen in the next theorem, weighted arithmetic aggregation operators can be expressed via additive generators whenever t-norm and t-conorm are continuous Archimedean.

Theorem 4.2. *Let $\{\alpha_j = \langle \mathbb{T}_{\alpha_j}, \mathbb{I}_{\alpha_j}, \mathbb{F}_{\alpha_j} \rangle : j = 1, \ldots, n\}$ be a collection of PFVNVs, let $w = (w_1, \ldots, w_n)^T$ be a weight vector such that $0 \leq w_j \leq 1$ for any $j = 1, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$ and let g be the additive generator of a continuous Archimedean t-norm and let $h(t) = g(\sqrt{1-t^2})$. Then $WA - PFVVNV(\alpha_1, \ldots, \alpha_n)$ is a $PFVNV$ and we have*

$$\begin{aligned} WA - PFVNV(\alpha_1, \ldots, \alpha_n) \quad = \quad & \Big\langle \Big(h^{-1}\Big(\sum_{j=1}^{n} w_j h(\mu_{\alpha_j,t}) \Big), g^{-1}\Big(\sum_{j=1}^{n} w_j g(\nu_{\alpha_j,t}) \Big) \Big), \\ & \Big(g^{-1}\Big(\sum_{j=1}^{n} w_j g(\mu_{\alpha_j,i}) \Big), h^{-1}\Big(\sum_{j=1}^{n} w_j h(\nu_{\alpha_j,i}) \Big) \Big), \\ & \Big(g^{-1}\Big(\sum_{j=1}^{n} w_j g(\mu_{\alpha_j,f}) \Big), h^{-1}\Big(\sum_{j=1}^{n} w_j h(\nu_{\alpha_j,f}) \Big) \Big) \Big\rangle. \end{aligned}$$

Proof. It is clear from Propositions 3.14 and 3.16 that $WA - PFVVNV(\alpha_1, \ldots, \alpha_n)$ is a PFVNV. Using mathematical induction we show that the second part is valid. If $n = 2$, we obtain

$$\begin{aligned} \mathbb{T}_{w_1\alpha_1 \oplus w_2\alpha_2} \quad &= \quad \mathcal{S}(\mathbb{T}_{w_1\alpha_1}, \mathbb{T}_{w_2\alpha_2}) \\ &= \quad \Big(h^{-1}\big(h(\mu_{w_1\alpha_1}) + h(\mu_{w_2\alpha_2}) \big), g^{-1}\big(g(\nu_{w_1\alpha_1}) + g(\nu_{w_2\alpha_2}) \Big) \\ &= \quad \Big(h^{-1}\big(h(h^{-1}(w_1 h(\mu_{\alpha_1}))) + h(h^{-1}(w_2 h(\mu_{\alpha_2}))) \Big), \\ &\quad\ g^{-1}\big(g(g^{-1}(w_1 g(\nu_{\alpha_1})) + g(g^{-1}(w_2 g(\nu_{\alpha_2})))) \big) \\ &= \quad \Big(h^{-1}\big(w_1 h(\mu_{\alpha_1}) + w_2 h(\mu_{\alpha_2}) \big), g^{-1}\big(w_1 g(\nu_{\alpha_1}) + w_2 g(\nu_{\alpha_2}) \big) \Big) \\ &= \quad \Big(h^{-1}\Big(\sum_{j=1}^{2} w_j h(\mu_{\alpha_j}) \Big), g^{-1}\Big(\sum_{j=1}^{2} w_j g(\nu_{\alpha_j}) \Big) \Big). \end{aligned}$$

Now assume that

$$\mathbb{T}_{A_{n-1}} = \left(h^{-1}\left(\sum_{j=1}^{n-1} w_j h(\mu_{\alpha_j}) \right), g^{-1}\left(\sum_{j=1}^{n-1} w_j g(\nu_{\alpha_j}) \right) \right),$$

where $A_n := \bigoplus_{j=1}^{n} w_j\alpha_j$. Then we obtain

$$\begin{aligned}
\mathbb{T}_{A_n} &= \mathbb{T}_{A_{n-1}} \oplus w_n\alpha_n \\
&= \mathcal{S}\big(\mathbb{T}_{A_{n-1}}, \mathbb{T}_{w_n\alpha_n}\big) \\
&= \mathcal{S}\Big(\Big(h^{-1}\Big(\sum_{j=1}^{n-1} w_j h(\mu_{\alpha_j})\Big), g^{-1}\Big(\sum_{j=1}^{n-1} w_j g(\nu_{\alpha_j})\Big)\Big), \\
&\Big(h^{-1}(w_n h(\mu_{\alpha_n,t})), g^{-1}(w_n g(\mu_{\alpha_n,t}))\Big)\Big) \\
&= \Big(h^{-1}\Big(h\Big(h^{-1}\Big(\sum_{j=1}^{n-1} w_j h(\mu_{\alpha_j})\Big)\Big) + h\Big(h^{-1}(w_n h(\mu_{\alpha_n,t}))\Big)\Big), \\
&g^{-1}\Big(g\Big(g^{-1}\Big(\sum_{j=1}^{n-1} w_j g(\nu_{\alpha_j})\Big)\Big) + g\Big(g^{-1}(w_n g(\mu_{\alpha_n,t}))\Big)\Big) \\
&= \Big(h^{-1}\Big(\sum_{j=1}^{n-1} w_j h(\mu_{\alpha_j}) + w_n h(\mu_{\alpha_n,t})\Big), g^{-1}\Big(\sum_{j=1}^{n-1} w_j g(\nu_{\alpha_j}) + w_n g(\mu_{\alpha_n,t})\Big)\Big) \\
&= \Big(h^{-1}\Big(\sum_{j=1}^{n} w_j h(\mu_{\alpha_j})\Big), g^{-1}\Big(\sum_{j=1}^{n} w_j g(\nu_{\alpha_j})\Big)\Big).
\end{aligned}$$

Similar proof is also valid for indeterminacy and falsity. □

Remark 4.3. Using particular additive generators we obtain some particular cases of $WA - PFVNV$ as follows:

a) Let g be the additive generator defined by $g(t) = -\log t^2$. Then we get Algebraic weighted arithmetic aggregation operator given by

$$\begin{aligned}
WA_A - PFVNV(\alpha_1, \ldots, \alpha_n) \quad = \quad & \Big\langle \Big(\sqrt{1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j,t}^2)^{w_j}}, \prod_{j=1}^{n} \nu_{\alpha_j,t}^{w_j} \Big), \\
& \Big(\prod_{j=1}^{n} \mu_{\alpha_j,i}^{w_j}, \sqrt{1 - \prod_{j=1}^{n} (1 - \nu_{\alpha_j,i}^2)^{w_j}} \Big), \\
& \Big(\prod_{j=1}^{n} \mu_{\alpha_j,f}^{w_j}, \sqrt{1 - \prod_{j=1}^{n} (1 - \nu_{\alpha_j,f}^2)^{w_j}} \Big) \Big\rangle.
\end{aligned}$$

b) Let g be the additive generator defined by $g(t) = \log(\frac{2-t^2}{t^2})$. Then we get Einstein weighted arithmetic aggregation operator given by

$$\begin{aligned}
WA_E - PFVNV(\alpha_1, \ldots, \alpha_n) \quad = \quad & \Big\langle \Big(\sqrt{\frac{\prod_{j=1}^{n} (1 + \mu_{\alpha_j,t}^2)^{w_j} - \prod_{j=1}^{n} (1 - \mu_{\alpha_j,t}^2)^{w_j}}{\prod_{j=1}^{n} (1 + \mu_{\alpha_j,t}^2)^{w_j} + \prod_{j=1}^{n} (1 - \mu_{\alpha_j,t}^2)^{w_j}}}, \frac{\sqrt{2} \prod_{j=1}^{n} \nu_{\alpha_j,t}^{w_i}}{\sqrt{\prod_{j=1}^{n} (2 - \nu_{\alpha_j,t}^2)^{w_j} + \prod_{j=1}^{n} (\nu_{\alpha_i,t}^2)^{w_j}}} \Big), \\
& \Big(\frac{\sqrt{2} \prod_{j=1}^{n} \mu_{\alpha_j,i}^{w_j}}{\sqrt{\prod_{j=1}^{n} (2 - \mu_{\alpha_j,i}^2)^{w_j} + \prod_{j=1}^{n} (\mu_{\alpha_j,i}^2)^{w_j}}}, \sqrt{\frac{\prod_{j=1}^{n} (1 + \nu_{\alpha_j,i}^2)^{w_j} - \prod_{j=1}^{n} (1 - \nu_{\alpha_j,i}^2)^{w_j}}{\prod_{j=1}^{n} (1 + \nu_{\alpha_j,i}^2)^{w_j} + \prod_{j=1}^{n} (1 - \nu_{\alpha_j,i}^2)^{w_j}}} \Big), \\
& \Big(\frac{\sqrt{2} \prod_{j=1}^{n} \mu_{\alpha_j,f}^{w_j}}{\sqrt{\prod_{j=1}^{n} (2 - \mu_{\alpha_j,f}^2)^{w_j} + \prod_{j=1}^{n} (\mu_{\alpha_j,f}^2)^{w_j}}}, \sqrt{\frac{\prod_{j=1}^{n} (1 + \nu_{\alpha_j,f}^2)^{w_j} - \prod_{j=1}^{n} (1 - \nu_{\alpha_j,f}^2)^{w_j}}{\prod_{j=1}^{n} (1 + \nu_{\alpha_j,f}^2)^{w_j} + \prod_{j=1}^{n} (1 - \nu_{\alpha_j,f}^2)^{w_j}}} \Big) \Big\rangle.
\end{aligned}$$

4.2 Weighted Geometric Aggregation Operator

Definition 4.4. Let $\{\alpha_j = \langle \mathbb{T}_{\alpha_j}, \mathbb{I}_{\alpha_j}, \mathbb{F}_{\alpha_j} \rangle : j = 1, \ldots, n\}$ be a collection of PFVNVs. A weighted geometric aggregation operator $WG - PFVVNV$ is defined by

$$WG - PFVVNV(\alpha_1, \ldots, \alpha_n) := \bigotimes^{n} \alpha_j^{w_j}$$

where $w = (w_1, \ldots, w_n)^T$ is a weight vector such that $0 \leq w_j \leq 1$ for any $j = 1, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$.

As seen in the next theorem, weighted geometric aggregation operators can be expressed via additive generators whenever t-norm and t-conorm are continuous Archimedean.

Theorem 4.5. *Let $\{\alpha_j = \langle \mathbb{T}_{\alpha_j}, \mathbb{I}_{\alpha_j}, \mathbb{F}_{\alpha_j} \rangle : j = 1, \ldots, n\}$ be a collection of PFVNVs, let $w = (w_1, \ldots, w_n)^T$ be a weight vector such that $0 \leq w_j \leq 1$ for any $j = 1, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$, let g be the additive generator of a continuous Archimedean t-norm and let $h(t) = g(\sqrt{1-t^2})$. Then $WG - PFVVNV(\alpha_1, \ldots, \alpha_n)$ is a PFVNV and*

$$\begin{aligned} WG - PFVVNV(\alpha_1, \ldots, \alpha_n) &= \Big\langle \Big(g^{-1}\Big(\sum_{j=1}^{n} w_j g(\mu_{\alpha_j,t}) \Big), h^{-1}\Big(\sum_{j=1}^{n} w_j h(\nu_{\alpha_j,t}) \Big) \Big), \\ &\quad \Big(h^{-1}\Big(\sum_{j=1}^{n} w_j h(\mu_{\alpha_j,i}) \Big), g^{-1}\Big(\sum_{j=1}^{n} w_j g(\nu_{\alpha_j,i}) \Big) \Big), \\ &\quad \Big(h^{-1}\Big(\sum_{j=1}^{n} w_j h(\mu_{\alpha_j,f}) \Big), g^{-1}\Big(\sum_{j=1}^{n} w_j g(\nu_{\alpha_j,f}) \Big) \Big) \Big\rangle. \end{aligned}$$

Proof. It can be proved similar to Theorem 4.2. □

Remark 4.6. By using particular additive generators we also obtain some particular cases of $WG-PFVNV$ as follows.

a) Let g be the additive generator defined by $g(t) = -\log t^2$. Then we get Algebraic weighted geometric aggregation operator given by

$$\begin{aligned} WG_A - PFVVNV(\alpha_1, \ldots, \alpha_n) &= \Big\langle \Big(\prod_{j=1}^{n} \mu_{\alpha_j,t}^{w_j}, \sqrt{1 - \prod_{j=1}^{n} (1 - \nu_{\alpha_j,t}^2)^{w_j}} \Big), \\ &\quad \Big(\sqrt{1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j,i}^2)^{w_j}}, \prod_{j=1}^{n} \nu_{\alpha_j,i}^{w_j} \Big), \\ &\quad \Big(\sqrt{1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j,f}^2)^{w_j}}, \prod_{j=1}^{n} \nu_{\alpha_j,f}^{w_j} \Big) \Big\rangle. \end{aligned}$$

b) Let g be the additive generator defined by $g(t) = \log(\frac{2-t^2}{t^2})$. Then we get Einstein weighted geometric aggregation operator given by

$$\begin{aligned} WG_E - PFVVNV(\alpha_1, \ldots, \alpha_n) &= \Big\langle \Big(\frac{\sqrt{2} \prod_{j=1}^{n} \mu_{\alpha_j,t}^{w_j}}{\sqrt{\prod_{j=1}^{n} (2 - \mu_{\alpha_j,t}^2)^{w_j} + \prod_{j=1}^{n} (\mu_{\alpha_j,t}^2)^{w_j}}}, \sqrt{\frac{\prod_{j=1}^{n} (1 + \nu_{\alpha_j,t}^2)^{w_j} - \prod_{j=1}^{n} (1 - \nu_{\alpha_j,t}^2)^{w_j}}{\prod_{j=1}^{n} (1 + \nu_{\alpha_j,t}^2)^{w_j} + \prod_{j=1}^{n} (1 - \nu_{\alpha_j,t}^2)^{w_j}}} \Big), \\ &\quad \Big(\sqrt{\frac{\prod_{j=1}^{n} (1 + \mu_{\alpha_j,i}^2)^{w_j} - \prod_{j=1}^{n} (1 - \mu_{\alpha_j,i}^2)^{w_j}}{\prod_{j=1}^{n} (1 + \mu_{\alpha_j,i}^2)^{w_j} + \prod_{j=1}^{n} (1 - \mu_{\alpha_j,i}^2)^{w_j}}}, \frac{\sqrt{2} \prod_{j=1}^{n} \nu_{\alpha_j,i}^{w_i}}{\sqrt{\prod_{j=1}^{n} (2 - \nu_{\alpha_j,i}^2)^{w_j} + \prod_{j=1}^{n} (\nu_{\alpha_i,i}^2)^{w_j}}} \Big), \\ &\quad \Big(\sqrt{\frac{\prod_{j=1}^{n} (1 + \mu_{\alpha_j,f}^2)^{w_j} - \prod_{j=1}^{n} (1 - \mu_{\alpha_j,f}^2)^{w_j}}{\prod_{j=1}^{n} (1 + \mu_{\alpha_j,f}^2)^{w_j} + \prod_{j=1}^{n} (1 - \mu_{\alpha_j,f}^2)^{w_j}}}, \frac{\sqrt{2} \prod_{j=1}^{n} \nu_{\alpha_j,f}^{w_i}}{\sqrt{\prod_{j=1}^{n} (2 - \nu_{\alpha_j,f}^2)^{w_j} + \prod_{j=1}^{n} (\nu_{\alpha_i,f}^2)^{w_j}}} \Big) \Big\rangle. \end{aligned}$$

5 A Simplified Neutrosophic Valued Modified Fuzzy Cross-Entropy Measure for PFVNVs

The notion of SNS was proposed by Ye[37] to relieve the difficulty of applying NSs to practical problems.

Definition 5.1. [37] A SNS A on X is given by

$$A = \{\langle x_j, T_A(x_j), I_A(x_j), F_A(x_j)\rangle : j = 1, ..., n\}$$

where $T_A, I_A, F_A : X \to [0, 1]$ are the truth, indeterminacy and falsify membership functions. For convenience, a SNV is denoted by

$$\alpha = \langle T_\alpha, I_\alpha, F_\alpha\rangle = \langle T_A(x_j), I_A(x_j), F_A(x_j)\rangle$$

for a fixed $x_j \in X$.

Some algebraic operations of SNV is recalled in the next definition.

Definition 5.2. [18] Let $A = \langle T_A, I_A, F_A\rangle$ and $B = \langle T_B, I_B, F_B\rangle$ be SNVs. Assume that $g : [0, 1] \to [0, \infty]$ is the additive generator of a continuous Archimedean t-norm and $h : [0, 1] \to [0, \infty]$ is the additive generator of a continuous Archimedean t-conorm. Some algebraic operations between SNVs are defined as follow:
i) $A \oplus_{SNV} B = \big\langle h^{-1}(h(T_A) + h(T_B)), g^{-1}(g(I_A) + g(I_B)), g^{-1}(g(F_A) + g(F_B))\big\rangle$
ii) $\lambda A = \big\langle h^{-1}(\lambda h(T_A)), g^{-1}(\lambda g(I_A)), g^{-1}(\lambda g(F_A))\big\rangle$.

Definition 5.3. Let $\alpha = \langle \mathbb{T}_\alpha, \mathbb{I}_\alpha, \mathbb{F}_\alpha\rangle$ and $\beta = \langle \mathbb{T}_\beta, \mathbb{I}_\beta, \mathbb{F}_\beta\rangle$ be two PFVNVs. A simplified neutrosophic valued modified fuzzy cross-entropy measure between α and β is defined by

$$\begin{aligned} E(\alpha, \beta) \quad := \quad & \left\langle \frac{1}{\ln 2}\left[\mu^2_{\alpha,t} \ln \frac{2\mu^2_{\alpha,t}}{\mu^2_{\alpha,t} + \mu^2_{\beta,t}} + \nu^2_{\alpha,t} \ln \frac{2\nu^2_{\alpha,t}}{\nu^2_{\alpha,t} + \nu^2_{\beta,t}} + \pi^2_{\alpha,t} \ln \frac{2\pi^2_{\alpha,t}}{\pi^2_{\alpha,t} + \pi^2_{\beta,t}}\right], \right. \\ & 1 - \frac{1}{\ln 2}\left[\mu^2_{\alpha,i} \ln \frac{2\mu^2_{\alpha,i}}{\mu^2_{\alpha,i} + \mu^2_{\beta,i}} + \nu^2_{\alpha,i} \ln \frac{2\nu^2_{\alpha,i}}{\nu^2_{\alpha,i} + \nu^2_{\beta,i}} + \pi^2_{\alpha,i} \ln \frac{2\pi^2_{\alpha,i}}{\pi^2_{\alpha,i} + \pi^2_{\beta,i}}\right], \\ & \left. 1 - \frac{1}{\ln 2}\left[\mu^2_{\alpha,f} \ln \frac{2\mu^2_{\alpha,f}}{\mu^2_{\alpha,f} + \mu^2_{\beta,f}} + \nu^2_{\alpha,f} \ln \frac{2\nu^2_{\alpha,f}}{\nu^2_{\alpha,f} + \nu^2_{\beta,f}} + \pi^2_{\alpha,f} \ln \frac{2\pi^2_{\alpha,f}}{\pi^2_{\alpha,f} + \pi^2_{\beta,f}}\right] \right\rangle \end{aligned}$$

where $\pi^2_{\alpha,.} = 1 - \mu^2_{\alpha,.} - \nu^2_{\alpha,.}$, and $\pi^2_{\beta,.} = 1 - \mu^2_{\beta,.} - \nu^2_{\beta,.}$. As in,[5] we use the convention (based on continuity arguments) that $0 \ln \frac{0}{p} = 0$ for $p \geq 0$.

The following proposition verifies that a simplified neutrosophic valued modified fuzzy cross-entropy measure between α and β is a SNV.

Proposition 5.4. *$E(\alpha, \beta)$ is a SNV.*

Proof. According to Shannon's inequality,[15] we know that each component of $E(\alpha, \beta)$ is equal or greater than 0. On the other hand, since the relation

$$\begin{aligned} \frac{1}{\ln 2}\left[\mu^2_{\alpha,.} \ln \frac{2\mu^2_{\alpha,.}}{\mu^2_{\alpha,.} + \mu^2_{\beta,.}} + \nu^2_{\alpha,.} \ln \frac{2\nu^2_{\alpha,.}}{\nu^2_{\alpha,.} + \nu^2_{\beta,.}} + \pi^2_{\alpha,.} \ln \frac{2\pi^2_{\alpha,.}}{\pi^2_{\alpha,.} + \pi^2_{\beta,.}}\right] \quad &\leq \quad \frac{1}{\ln 2}\Big(\mu^2_{\alpha,.} \ln 2 + \nu^2_{\alpha,.} \ln 2 + \pi^2_{\alpha,.} \ln 2\Big) \\ &= \quad 1 \end{aligned}$$

is true, each component of $E(\alpha, \beta)$ is equal or less than 1. Therefore $E(\alpha, \beta)$ is a SNV. □

It is clear that $E(\alpha, \beta) \neq E(\beta, \alpha)$ in general. To make the concept symmetric we can define the following.

Definition 5.5. Let α and β be two PFVNVs. A simplified neutrosophic valued symmetric discrimination information measure based on E is given by

$$E^*(\alpha, \beta) = \frac{1}{2}E(\alpha, \beta) \oplus_{SNV} \frac{1}{2}E(\beta, \alpha)$$

where multiplication by scalar $\frac{1}{2}$ is in the sense of (iii) of Definition 5.2.

E^* can be expressed via additive generators of continuous Archimedean t-norms and t-conorms.

Theorem 5.6. *Let α and β be two PFVNVs, let g be the additive generator of a continuous Archimedean t-norm and $h(t) = g(1-t)$. Then*

$$\begin{aligned} E^*(\alpha,\beta) &= \Bigg\langle h^{-1}\left(\frac{1}{2}\left[h\Big(T_{E(\alpha,\beta)}\Big)+h\Big(T_{E(\beta,\alpha)}\Big)\right]\right), g^{-1}\left(\frac{1}{2}\left[g\Big(I_{E(\alpha,\beta)}\Big)+g\Big(I_{E(\beta,\alpha)}\Big)\right]\right), \\ &\quad g^{-1}\left(\frac{1}{2}\left[g\Big(F_{E(\alpha,\beta)}\Big)+g\Big(F_{E(\beta,\alpha)}\Big)\right]\right)\Bigg\rangle. \end{aligned}$$

Proof. We have

$$\begin{aligned} E^*(\alpha,\beta) &= \frac{1}{2}E(\alpha,\beta)\oplus_{SNV}\frac{1}{2}E(\beta,\alpha) \\ &= \Bigg\langle h^{-1}\left(h\Big(T_{\frac{1}{2}E(\alpha,\beta)}\Big)+h\Big(T_{\frac{1}{2}E(\beta,\alpha)}\Big)\right), g^{-1}\left(g\Big(I_{\frac{1}{2}E(\alpha,\beta)}\Big)+g\Big(I_{\frac{1}{2}E(\beta,\alpha)}\Big)\right), \\ &\quad g^{-1}\left(g\Big(F_{\frac{1}{2}E(\alpha,\beta)}\Big)+g\Big(F_{\frac{1}{2}E(\beta,\alpha)}\Big)\right)\Bigg\rangle \\ &= \Bigg\langle h^{-1}\left(h\Big(h^{-1}\Big\langle\frac{1}{2}h(T_{E(\alpha,\beta)})\Big)+h\Big(h^{-1}\Big(\frac{1}{2}h\big(T_{E(\beta,\alpha)}\big)\Big)\right)\right), \\ &\quad g^{-1}\left(g\Big(g^{-1}\Big(\frac{1}{2}g(I_{E(\alpha,\beta)})\Big)\Big)+g\Big(g^{-1}\Big(\frac{1}{2}g\big(I_{E(\beta,\alpha)}\big)\Big)\right)\right), \\ &\quad g^{-1}\left(g\Big(g^{-1}\Big(\frac{1}{2}g(I_{E(\alpha,\beta)})\Big)\Big)+g\Big(g^{-1}\Big(\frac{1}{2}g\big(I_{E(\beta,\alpha)}\big)\Big)\right)\right)\Bigg\rangle \end{aligned}$$

which yields that

$$\begin{aligned} E^*(\alpha,\beta) &= \Bigg\langle h^{-1}\left(\frac{1}{2}\left[h\Big(T_{E(\alpha,\beta)}\Big)+h\Big(T_{E(\beta,\alpha)}\Big)\right]\right), g^{-1}\left(\frac{1}{2}\left[g\Big(I_{E(\alpha,\beta)}\Big)+g\Big(I_{E(\beta,\alpha)}\Big)\right]\right), \\ &\quad g^{-1}\left(\frac{1}{2}\left[g\Big(F_{E(\alpha,\beta)}\Big)+g\Big(F_{E(\beta,\alpha)}\Big)\right]\right)\Bigg\rangle. \end{aligned}$$

□

By using particular additive generators we can obtain some particular cases of E^*.

Remark 5.7. Let g be the additive generator defined by $g(t) = -\log t$. Then we get Algebraic simplified neutrosophic valued symmetric discrimination information measure based on E given by

$$E^*_A(\alpha,\beta) = \Big\langle 1-\sqrt{(1-T_{E(\alpha,\beta)})(1-T_{E(\beta,\alpha)})}, \sqrt{I_{E(\alpha,\beta)}I_{E(\beta,\alpha)}}, \sqrt{F_{E(\alpha,\beta)}F_{E(\beta,\alpha)}}\Big\rangle.$$

Let g be the additive generator defined by $g(t) = \log(\frac{2-t}{t})$. Then we get Einstein simplified neutrosophic valued symmetric discrimination information measure based on E given by

$$\begin{aligned} E^*_E(\alpha,\beta) &= \Bigg\langle \frac{\sqrt{(1+T_{E(\alpha,\beta)})(1+T_{E(\beta,\alpha)})}-\sqrt{(1-T_{E(\alpha,\beta)})(1-T_{E(\beta,\alpha)})}}{\sqrt{(1+T_{E(\alpha,\beta)})(1+T_{E(\beta,\alpha)})}+\sqrt{(1-T_{E(\alpha,\beta)})(1-T_{E(\beta,\alpha)})}}, \\ &\quad \frac{2\sqrt{I_{E(\alpha,\beta)}I_{E(\beta,\alpha)}}}{\sqrt{I_{E(\alpha,\beta)}I_{E(\beta,\alpha)}}+\sqrt{(2-I_{E(\alpha,\beta)})(2-I_{E(\beta,\alpha)})}}, \\ &\quad \frac{2\sqrt{F_{E(\alpha,\beta)}F_{E(\beta,\alpha)}}}{\sqrt{F_{E(\alpha,\beta)}F_{E(\beta,\alpha)}}+\sqrt{(2-F_{E(\alpha,\beta)})(2-F_{E(\beta,\alpha)})}}\Bigg\rangle. \end{aligned}$$

Since E^* is a SNV, a score function is needed to rank the values of E^*. In this paper, we use the following score function.

Definition 5.8. [32] Let $A = (T_A, I_A, F_A)$ be a SNV. A score function is defined by

$$N(A) = \frac{T_A + 1 - I_A + 1 - F_A}{3}.$$

Theorem 5.9. *Let α and β be two PFVNVs. The modified cross-entropy measure E^* satisfies the following properties.*

i) $E^*(\alpha, \alpha) = \langle 0, 1, 1\rangle$,

ii) $0 \leq N(E^*(\alpha, \beta)) \leq 1$,

iii) $E^*(\alpha, \beta) = E^*(\beta, \alpha)$,

iv) $N(E^*(\alpha, \alpha)) = 0$.

Proof. **i)** We know that

$$\begin{aligned} E(\alpha,\alpha) &= \left\langle \frac{1}{\ln 2}\left[\mu_{\alpha,t}^2 \ln \frac{2\mu_{\alpha,t}^2}{\mu_{\alpha,t}^2+\mu_{\alpha,t}^2} + \nu_{\alpha,t}^2 \ln \frac{2\nu_{\alpha,t}^2}{\nu_{\alpha,t}^2+\nu_{\alpha,t}^2} + \pi_{\alpha,t}^2 \ln \frac{2\pi_{\alpha,t}^2}{\pi_{\alpha,t}^2+\pi_{\alpha,t}^2}\right], \right. \\ &\quad 1 - \frac{1}{\ln 2}\left[\mu_{\alpha,i}^2 \ln \frac{2\mu_{\alpha,i}^2}{\mu_{\alpha,i}^2+\mu_{\alpha,i}^2} + \nu_{\alpha,i}^2 \ln \frac{2\nu_{\alpha,i}^2}{\nu_{\alpha,i}^2+\nu_{\alpha,i}^2} + \pi_{\alpha,i}^2 \ln \frac{2\pi_{\alpha,i}^2}{\pi_{\alpha,i}^2+\pi_{\alpha,i}^2}\right], \\ &\quad \left. 1 - \frac{1}{\ln 2}\left[\mu_{\alpha,f}^2 \ln \frac{2\mu_{\alpha,f}^2}{\mu_{\alpha,f}^2+\mu_{\alpha,f}^2} + \nu_{\alpha,f}^2 \ln \frac{2\nu_{\alpha,f}^2}{\nu_{\alpha,f}^2+\nu_{\alpha,f}^2} + \pi_{\alpha,f}^2 \ln \frac{2\pi_{\alpha,f}^2}{\pi_{\alpha,f}^2+\pi_{\alpha,f}^2}\right] \right\rangle \\ &= \langle 0, 1, 1\rangle. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} E^*(\alpha,\alpha) &= \left\langle h^{-1}\left(\frac{1}{2}\left[h\Big(T_{E(\alpha,\alpha)}\Big) + h\Big(T_{E(\alpha,\alpha)}\Big)\right]\right), g^{-1}\left(\frac{1}{2}\left[g\Big(I_{E(\alpha,\alpha)}\Big) + g\Big(I_{E(\alpha,\alpha)}\Big)\right]\right), \right. \\ &\quad \left. g^{-1}\left(\frac{1}{2}\left[g\Big(F_{E(\alpha,\alpha)}\Big) + g\Big(F_{E(\alpha,\alpha)}\Big)\right]\right)\right\rangle \\ &= \left\langle T_{E(\alpha,\alpha)}, I_{E(\alpha,\alpha)}, F_{E(\alpha,\alpha)}\right\rangle \\ &= \langle 0, 1, 1\rangle. \end{aligned}$$

ii) Since $E^*(\alpha, \beta)$ is a SNV we obtain that $0 \leq N(E^*(\alpha, \beta)) \leq 1$.

iii) It is trivial from definition of E^*.

iv) It is trivial from (i) and definition of N.

□

6 An Application of $PFVNVs$ To A MCDM Problem

In this section, we provide a MCDM method in Pythagorean fuzzy valued neutrosophic environment. Then we solve a MCDM problem from the literature by using the proposed method.

	C_1	C_2	C_3
A_1	$\langle 0.4, 0.2, 0.3\rangle$	$\langle 0.4, 0.2, 0.3\rangle$	$\langle 0.2, 0.2, 0.5\rangle$
A_2	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.5, 0.2, 0.2\rangle$
A_3	$\langle 0.3, 0.2, 0.3\rangle$	$\langle 0.5, 0.2, 0.3\rangle$	$\langle 0.5, 0.3, 0.2\rangle$
A_4	$\langle 0.7, 0.0, 0.1\rangle$	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.4, 0.3, 0.2\rangle$

Table 1: SNVs of alternatives according to criteria

6.1 A MCDM method

In this sub-section, we give a MCDM method in the Pythagorean fuzzy valued neutrosophic environment. The proposed method is applied to a MCDM problem adapted from the literature[37,38] to see the effectiveness of the proposed method. The steps can be summarized as follows.

Step 1: Form a MCDM problem with alternatives $A = \{A_1, \ldots, A_n\}$ and criteria $C = \{C_1, \ldots, C_k\}$.

Step 2: The weights of criteria are determined.

Step 3: The evaluation results of the alternatives are expressed by the expert as $SNVs$ for each criterion.

Step 4: Using Proposition 3.7, evaluation results of alternatives in Step 3 are converted from SNVs to PFVNVs.

Step 5: Evaluation values expressed as PFVNVs for each alternative according to criteria are transformed to a value expressed as a PFVNV by utilizing proposed weighted aggregation operators.

Step 6: Positive ideal alternative for each sample is defined by

$$A^+ = \langle (\mu_t^+, \nu_t^+), (\mu_i^+, \nu_i^+), (\mu_f^+, \nu_f^+)\rangle$$

with alternatives $A = \{A_1, \ldots, A_n\} = \{\langle (\mu_t^1, \nu_t^1), (\mu_i^1, \nu_i^1), (\mu_f^1, \nu_f^1)\rangle, \ldots, \langle (m_t^n, \nu_t^n), (\mu_i^n, \nu_i^n), (\mu_f^n, \nu_f^n)\rangle\}$ and

$$\mu_t^+ = \max_{j\in\{1,\ldots,n\}} \mu_t^j, \nu_t^+ = \min_{j\in\{1,\ldots,n\}} \nu_t^j, \mu_i^+ = \min_{j\in\{1,\ldots,n\}} \mu_i^j, \nu_i^+ = \max_{j\in\{1,\ldots,n\}} \nu_i^j, \mu_f^+ = \min_{i\in\{1,\ldots,n\}} \mu_f^j, \nu_f^+ = \max_{i\in\{1,\ldots,n\}} \nu_f^j.$$

Here, all of the criteria are assumed to be benefit criteria. If there exists a cost criterion, then we can take compliment.

Step 7: By using E^* defined in Definition 5.5 the symmetric discrimination information measure between aggregated value of each alternative and positive ideal alternative are calculated.

Step 8: The scores of these $SNVs$ are obtained by using a score function.

Step 9: Alternatives are ranked so that the minimum score value is the best alternative.

6.2 An application

As an application, we consider an investment company which wants to invest a sum of money in the best opinion. This problem is adapted from.[37,38] There are four alternatives for investing the money: (1) A_1 (a car company); (2) A_2 (a food company); (3) A_3 (a computer company); and (4) A_4 (an arms company). The investment company wants to make a decision using the following three criteria: (1) C_1 (the risk); (2) C_2 (the growth); and (3) C_3 (the the environmental impact). The weight vector of the criteria is considered by $w = (0.35, 0.25, 0.4)$ and the decision matrix given in Table 1 is used as in.[37,38]

Let $f_\delta(t) = sin\delta t$ with $\delta \in [0, 1]$. It is clear that $f_\delta(t) = sin\delta t \leq \sqrt{t}$ for $t \in [0, 1]$ (see also Figure 4). Now we convert $SNVs$ of alternatives in Table 1 to $PFVNVs$ by using Proposition 3.7. For $\delta = 1$, we

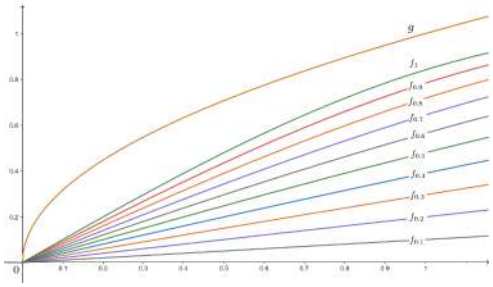

Figure 4: The graphs of $y = \sqrt{t}$ and f_δ with $\delta \in [0, 1]$

	C_1	C_2
A_1	$\langle((0.63, 0.56), (0.45, 0.72), (0.55, 0.64)\rangle$	$\langle((0.63, 0.56), (0.45, 0.72), (0.55, 0.64)\rangle$
A_2	$\langle((0.77, 0.39), (0.32, 0.78), (0.45, 0.72)\rangle$	$\langle((0.77, 0.39), (0.32, 0.78), (0.45, 0.72)\rangle$
A_3	$\langle((0.55, 0.64), (0.45, 0.72), (0.55, 0.64)\rangle$	$\langle((0.71, 0.48), (0.45, 0.72), (0.55, 0.64)\rangle$
A_4	$\langle((0.84, 0.3), (0.0, 0.84), (0.32, 0.78)\rangle$	$\langle((0.77, 0.39), (0.32, 0.78), (0.45, 0.72)\rangle$
	C_3	
A_1	$\langle((0.45, 0.72), (0.45, 0.72), (0.71, 0.48)\rangle$	
A_2	$\langle((0.71, 0.48), (0.45, 0.72), (0.45, 0.72)\rangle$	
A_3	$\langle((0.71, 0.48), (0.55, 0.64), (0.45, 0.72)\rangle$	
A_4	$\langle((0.63, 0, 56), (0.55, 0.64), (0.45, 0.72)\rangle$	

Table 2: PFVNVs of alternatives obtained for $\delta = 1$

	PFVNV
A_1	$\langle((0.57, 0.62), (0.43, 0.76), (0.6, 0.62)\rangle$
A_2	$\langle((0.76, 0.41), (0.37, 0.76), (0.43, 0.74)\rangle$
A_3	$\langle((0.6, 0.59), (0.55, 0.74), (0.45, 0.68)\rangle$
A_4	$\langle((0.74, 0.44), (0.0, 0.68), (0.48, 0.69)\rangle$
A^+	$\langle((0.76, 0.41), (0.0, 0.76), (0.43, 0.74)\rangle$

Table 3: Aggregated PFVNV according to $WA_A - PFVNV$ and positive ideal alternative A^+

Algebraic symmetric discrimination information measures	SNV
$E^*_A(A^+, A_1)$	$\langle 0.06, 0.97, 0.97\rangle$
$E^*_A(A^+, A_2)$	$\langle 0.0, 0.99, 1\rangle$
$E^*_A(A^+, A_3)$	$\langle 0.04, 0.93, 0.99\rangle$
$E^*_A(A^+, A_4)$	$\langle 0.003, 0.99, 0.99\rangle$

Table 4: Algebraic symmetric discrimination information measures between positive ideal alternative and alternatives

	Score
$N(E_A^*(A^+, A_1))$	0.039
$N(E_A^*(A^+, A_2))$	**0.0048**
$N(E_A^*(A^+, A_3))$	0.041
$N(E_A^*(A^+, A_4))$	0.0051

Table 5: Scores of Algebraic symmetric discrimination information measures between A^+ and $A_j (j = 1, 2, 3, 4)$

Method	Ranking order	Best Alternative
Ye[37]	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
Ye[38]	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
Peng et al.[18]	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
Proposed method	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2

Table 6: The comparison of some previous methods with proposed method

obtain Table 2. When $PFVNVs$ of each alternative with respect to criteria are aggregated by utilizing $WA_A - PFVNV$, Table 3 containing the positive ideal alternative is obtained.

Algebraic symmetric discrimination information measure E_A^* between positive ideal alternative A^+ and alternatives are calculated. The results are shown in Table 4.

Score function N recalled in Definition 5.8 is used to rank the value of E_A^*. The smaller the value of $N(E_A^*(A_j, A^+))$ is, the better the alternative A_j is. In the other words, the alternative A_j is closer to positive ideal alternative A^+ as $N(E_A^*(A_j, A^+))$ gets smaller. The results of score function are shown in Table 5. As a result, A_2 is the best alternative.

6.3 Comparative analysis

6.3.1 Comparison with the literature

We compare the results of some existing methods with the results of the proposed method in Sub-section 6.1. So as to solve the MCDM problem given in Sub-section 6.2, some methods were proposed under different fuzzy environments. Ye[38] gave a weighted correlation coefficient for SVNSs, utilized it to solve the same investment problem and obtained the ranking $A_1 \prec A_3 \prec A_4 \prec A_2$. Using different aggregation operators and a similarity measure for SNSs, Ye[37] solved the same problem and obtained the ranking $A_1 \prec A_3 \prec A_2 \prec A_4$. Similarly, Peng[18] proposed some aggregation operators via t-norms and t-conorms, applied them to investigate same problem and got the ranking $A_1 \prec A_3 \prec A_2 \prec A_4$. The best alternative obtained with the present method remains same with the one of[37] and[38] for some δ. The comparison results of proposed method with the existing ones are shown in Table 6 and illustrated in Figure 5.

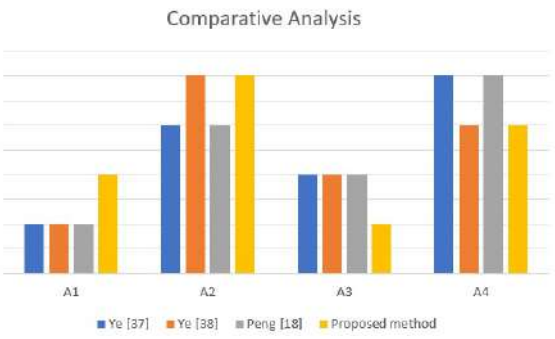

Figure 5: The column chart comparison of the other methods and proposed method

δ	Ranking order	Best alternative
$\delta = 1$	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2
$\delta = 0.9$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.8$	$A_3 \prec A_1 \prec A_2 \prec A_4$	A_4
$\delta = 0.7$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.6$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.5$	$A_3 \prec A_1 \prec A_2 \prec A_4$	A_4
$\delta = 0.4$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.3$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.2$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.1$	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2

Table 7: The results with respect to $WA_A - PFVNV$ when the parameter δ varies from 0.1 and 1

δ	Ranking order	Best alternative
$\delta = 1$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.9$	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2
$\delta = 0.8$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.7$	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2
$\delta = 0.6$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.5$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.4$	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2
$\delta = 0.3$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.2$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.1$	$A_3 \prec A_1 \prec A_2 \prec A_4$	A_4

Table 8: The results with respect to $WG_A - PFVNV$ when the parameter δ varies from 0.1 and 1

6.3.2 Comparison with respect to parameter δ

We investigate the effect of the parameter δ on the ranking of alternatives with respect to $WA_A - PFVNV$ and $WG_A - PFVNV$. The results are summarized in Table 7 and Table 8. From these results, it is concluded that the ranking of the alternative is $A_3 \prec A_1 \prec A_4 \prec A_2$ when δ equals to 0.1 and 1 with respect to $WA_A - PFVNV$ while $A_1 \prec A_3 \prec A_2 \prec A_4$ when δ equals to 0.6 and 0.8 for $WG_A - PFVNV$. When δ equals to 0.1 and 1 with respect to $WA_A - PFVNV$, we conclude that A_2 is the best alternative, while by using $WG_A - PFVNV$, the best alternative is A_4 when δ equals to 0.6 and 0.8. The graphical representation of the behaviour of the alternatives with the δ variation is shown in Figure 6.

6.4 Time complexity of the proposed MCDM method

In this sub-section we analysis the complexity of the MCDM method proposed in Sub-section 6.1. Actually we evaluate the time complexity that depends on the number of times of multiplication, exponential, summation

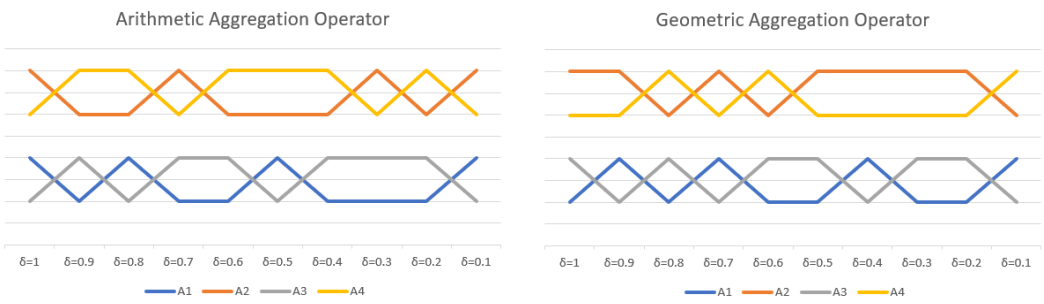

Figure 6: Effect of the parameter δ to the solution

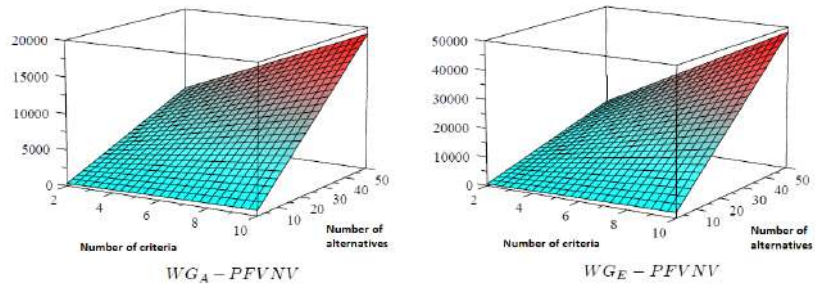

Figure 7: Time complexity of the proposed MCDM method

as in[4] and.[8] Consider a MCDM problem with n alternatives and k criteria. In Step 2 we need k operations, in Step 3 we need $3kn$ operations, in Step 4 we need $6kn$ operations, in Step 5 we need $3n\,(6k+2)$ operations if we use the aggregation operator $WG_A - PFVNV$ and we need $n(75k+21)$ operations if we use the aggregation operator $WG_E - PFVNV$. In Step 6 we need $3n$ operations, in Step 7 we need $98n$ operations and in Step 8 we need $5n$ operations. So the time complexity T_{nk} of the MCDM method is

$$T_{nk}^{A} = k + 27kn + 112n$$

for the aggregation operator $WG_A - PFVNV$ and

$$T_{nk}^{E} = k + 84kn + 127n$$

for the aggregation operator $WG_E - PFVNV$. Clearly the bi-variate functions $g_A, g_E : [2,\infty)^2 \to \mathbb{R}$ defined by $g_A(x,y) = x + 27xy + 112y$ and $g_E(x,y) = x + 84xy + 127y$ both take absolute minimum at point $(2,2)$. Figure 7 illustrates the change of the time complexity with respect to the change in the numbers of the alternatives and the criteria for $WG_A - PFVNV$ and $WG_E - PFVNV$.

7 Conclusion

The main aim of this study is to introduce the concept of PFVNS constructed by considering PFVs rather than numbers, inspired by IFVNSs. Thus, a PFVNS is an extension of IFVNSs. PFVNSs are used to express uncertainty in a more extended fuzzy environment. Therefore, larger information can be kept while the data is converted to a FS. In this way, information loss is prevented. In this study, some set operations between PFVNSs are proposed. We also introduce Pythagorean fuzzy t-norms and t-conorms with motivation from intuitionistic fuzzy t-norms and t-conorms. We show that some Pythagorean fuzzy t-norms and t-conorms are expressed via continuous Archimedean t-norms and continuous Archimedean t-conorms on $[0,1]$. Then we define some algebraic operations between PFVNVs by utilizing continuous Archimedean t-norm and continuous Archimedean t-conorms on $[0,1]$. By way of these algebraic operations, some weighted aggregation operators are proposed. Input values represented by PFVNVs are transformed to a single output value by using weighted aggregation operators. Also, we introduce a method that converts neutrosophic fuzzy values to PFVNVs. By defining a simplified neutrosophic valued modified fuzzy cross-entropy measure we manage to rank output values represented by PFVNVs with the help of a score function in simplified neutrosophic environment. Next a MCDM method is given to see practicability of the proposed theory. The proposed method is made use of to solve a MCDM problem adapted from the literature in Pythagorean fuzzy valued neutrosophic environment. A comparison analysis and a complexity analysis are also provided. In the future, different kind of aggregation operators and cross-entropy measures can be considered. The applications of the proposed theory can be extended to some decision making problems such as face recognition systems, classification and medical diagnosis.

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References

[1] Abdullah, L., Fuzzy multi criteria decision making and its applications: a brief review of category. Procedia-Social and Behavioral Sciences, 97, 131-136, 2013.

[2] Atanassov, K. T., Intuitionistic fuzzy sets. In Intuitionistic fuzzy sets (pp. 1-137). Physica, Heidelberg, 1999.

[3] Beliakov, G., Bustince, H., Goswami, D.P., Mukherjee, U.K., Pal, N.R., On averaging operators for Atanassov's intuitionistic fuzzy sets. Information Sciences. 181, 1116-1124, 2011.

[4] Chang, D. Y., Applications of the extent analysis method on fuzzy AHP. European journal of operational research, 95(3), 649-655, 1996.

[5] Cover, T. M., and Thomas, J. A., Elements of information theory 2nd edition (wiley series in telecommunications and signal processing), 2006.

[6] Deschrijver, G., Cornelis, C., Kerre, E. E., On the representation of intuitionistic fuzzy t-norms and t-conorms. IEEE transactions on fuzzy systems, 12(1), 45-61, 2004.

[7] Garg, H., Arora, R., Generalized Maclaurin symmetric mean aggregation operators based on Archimedean t-norm of the intuitionistic fuzzy soft set information. Artif Intell Rev 54:3173-3213, 2021.

[8] Junior, F. R. L., Osiro, L., and Carpinetti, L. C. R., A comparison between Fuzzy AHP and Fuzzy TOPSIS methods to supplier selection. Applied soft computing, 21, 194-209, 2014.

[9] Kahraman, C., (Ed.) Fuzzy multi-criteria decision making: theory and applications with recent developments (Vol. 16). Springer Science and Business Media, 2008.

[10] Klement, E. P., Mesiar, R., Pap, E., Triangular norms, Kluwer Academic Publishers, Dordrecht, 2002.

[11] Klement, E. P., Mesiar, R., Pap, E., Triangular norms. Position paper III: continuous t-norms, Fuzzy Sets and Systems, 145(3) 439-454, 2004.

[12] Klir, G. Yuan, B., Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall, Upper Saddle River, NJ, 1995.

[13] Kullback, S., Leibler, R. A., On information and sufficiency. Annals of the Institute of Statistical Mathematics. 4: 99–111, 1951.

[14] Kullback, S., Information theory and statistics. Wiley, New York, 1959.

[15] Lin, J., Divergence measures based on Shannon entropy, IEEE Trans. Information Theory 37(1) 145-151, 1991.

[16] Menger, K., Statistical metrics. Proceedings of the National Academy of Sciences of the United States of America, 28(12), 535, 1942.

[17] Naz, S., and Akram, M., Novel decision-making approach based on hesitant fuzzy sets and graph theory. Computational and Applied Mathematics, 38(1), 1-26, 2019.

[18] Peng, J. J., Wang, J. Q., Wang, J., Zhang, H. Y., Chen, X. H., Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems. International journal of systems science, 47(10), 2342-2358, 2016.

[19] Peng, X., Yang, Y., Some results for Pythagorean fuzzy sets. International Journal of Intelligent Systems, 30(11), 1133-1160, 2015.

[20] Peng, X., Yuan, H., Yang, Y. Pythagorean fuzzy information measures and their applications. International Journal of Intelligent Systems, 32(10), 991-1029, 2017.

[21] Schweizer, B. and Sklar, A., Probabilistic metric spaces, North-Holland, New York, 1983.

[22] Shang, X. G., Jiang, W. S., A note on fuzzy information measures. Pattern Recognition Letters, 18(5), 425-432, 1997.

[23] Shannon, C., A mathematical theory of communication. Bell Syst., Tech. J. 27:379-423, 1948.

[24] Shannon, C., Weaver, W., The mathematical theory of communication. Urbana: University of Illinois Press, 1949.

[25] Singh, A., Joshi, D. K., Kumar, S., A novel construction method of intuitionistic fuzzy set from fuzzy set and its application in multi-criteria decision-making problem. In Advanced Computing and Communication Technologies (pp. 67-75). Springer, Singapore, 2019.

[26] Smarandache, F., A unifying field of logics. Neutrosophy: Neutrosophic probability, set and logic. American Research Press, Rehoboth, 1998.

[27] Sun, B., Tong, S., Ma, W., Wang, T., and Jiang, C., An approach to MCGDM based on multi-granulation Pythagorean fuzzy rough set over two universes and its application to medical decision problem. Artificial Intelligence Review, 55, 1887–1913, 2022.

[28] Ünver, M., Türkarslan, E., Olgun, M., Ye, J., Intuitionistic fuzzy-valued neutrosophic multi-sets and numerical applications to classification. Complex and Intelligent Systems, 1-19, 2022.

[29] Ünver, M., Olgun, M., and Türkarslan, E., Cosine and cotangent similarity measures based on Choquet integral for Spherical fuzzy sets and applications to pattern recognition. Journal of Computational and Cognitive Engineering, 1(1) 21-31, 2022.

[30] Vlachos, I.K., Sergiadis, G.D., Intuitionistic fuzzy information - Applications to pattern recognition, Pattern Recognition Letters 28 (2007) 197-206, 2007.

[31] Wei, P., Ye, J., Improved intuitionistic fuzzy cross-entropy and its application to pattern recognitions. In 2010 IEEE International Conference on Intelligent Systems and Knowledge Engineering (pp. 114-116). IEEE, 2010.

[32] Wu, X. H., Wang, J. Q., Peng, J. J., Chen, X., H. Cross-entropy and prioritized aggregation operator with simplified neutrosophic sets and their application in multi-criteria decision-making problems. International Journal of Fuzzy Systems, 18(6), 1104-1116, 2016.

[33] Yager, R. R., Pythagorean membership grades in multicriteria decision making. IEEE Transactions on Fuzzy Systems, 22(4), 958-965, 2013a.

[34] Yager, R. R., Pythagorean fuzzy subsets. In 2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS) (pp. 57-61). IEEE, 2013b.

[35] Yager, R. R., and Abbasov, A. M., Pythagorean membership grades, complex numbers, and decision making. International Journal of Intelligent Systems, 28(5), 436-452, 2013.

[36] Ye, S., Ye, J., Dice similarity measure between single valued neutrosophic multisets and its application in medical diagnosis. Neutrosophic Sets Syst. 6:48-53, 2014.

[37] Ye, J., A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. Journal of Intelligent and Fuzzy Systems, 26(5), 2459-2466, 2014.

[38] Ye, J., Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. International Journal of General Systems, 42(4), 386-394, 2013.

[39] Ye, J., Single-Valued Neutrosophic Clustering Algorithms Based on Similarity Measures. Journal of Classification, 34(1), 2017.

[40] Ye, J., Türkarslan, E., Ünver, M., Olgun, M., Algebraic and Einstein weighted operators of neutrosophic enthalpy values for multi-criteria decision making in neutrosophic multi-valued set settings. Granular Computing, 1-9, 2021.

[41] Yeni, F. B., and Özçelik, G., Interval-valued Atanassov intuitionistic Fuzzy CODAS method for multi criteria group decision making problems. Group Decision and Negotiation, 28(2), 433-452, 2019.

[42] Zadeh, L. A., Fuzzy sets. Information and control. 8(3):338-353, 1965.

[43] Zadeh, L. A., Probability measures of fuzzy events. Journal of mathematical analysis and applications, 23(2), 421-427, 1968.

Evaluation of the alternatives of introducing electric vehicles in developing countries using Type-2 neutrosophic numbers based RAFSI model

Ilgin Gokasar [a], Muhammet Deveci [b,c,*], Mehtap Isik [d], Tugrul Daim [e,h,**], Aws A. Zaidan [f], Florentin Smarandache [g]

[a] Department of Civil Engineering, Bogazici University, 34342 Bebek, Istanbul, Turkey
[b] Department of Industrial Engineering, Turkish Naval Academy, National Defence University, 34940 Tuzla, Istanbul, Turkey
[c] Royal School of Mines, Imperial College London, London SW7 2AZ, UK
[d] Department of International Trade, Bogazici University, 34342 Bebek, Istanbul, Turkey
[e] Portland State University, Portland, OR, USA
[f] SP Jain School of Global Management, Lidcombe, Sydney, NSW 2141, Australia
[g] Mathematics and Science Division, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, United States
[h] Chaoyang University of Technology, Taichung City, Taiwan

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ABSTRACT

This study focuses on implementing electric vehicles (EVs) in developing countries where energy production is mainly based on fossil fuels. Although for these countries the environmental short-run benefits of the EVs cannot offset the short-run costs, it may still be the best option to implement the EVs as soon as possible. Hence, it is necessary to evaluate the alternatives to introducing EVs to the market due to the environmental concerns that created an opportunity for some developing countries to catch up with the international competition. Therefore, we develop a case scenario to explore the decision-making process in implementing the EVs with three alternatives and twelve criteria. We solve the decision-making problem by using Type-2 neutrosophic numbers (T2NNs) based on the RAFSI (Ranking of Alternatives through Functional mapping of criterion sub-intervals into a Single Interval) method. The proposed model combines the advantages of the RAFSI technique, and it applies T2NNs to address the uncertainties. The results show that the alternatives that may suspend the implementation of the EVs are inferior. Direct implementation of EVs is prioritized. The policy implications of the results are discussed in the study.

1. Introduction

The environmental benefits of EVs are obvious in various aspects. Different types of EVs are proposed to be implemented for the sake of environmental protection (Yuan and Cai, 2021). The EVs market is getting bigger and expanding to supply different tastes (Tchetchik et al., 2020; Benzidia et al., 2021) The increasing number of patents show that EVs are the future of the automotive industry (Yuan and Li, 2020; Hain et al., 2022; de Paulo et al., 2020). However, the extra demand for electricity created by the increased use of EVs should be a concern for decision-makers. Unsurprisingly, under some circumstances, the cost of using EVs can exceed its benefits in the short run time.

A recent study (Sahin, 2022) put forward an interesting case analysis in a developing country, Turkey. The results show that in the total traffic demand, there is optimal use rate of electrical Autonomous Vehicles (AVs) which is between 30 % and 50 %. Also, the increase in the number of these electrical AVs can affect the occurrence of traffic accidents. The major reason behind this appears to be the resource of electricity supply in the country, in addition to the infrastructural challenges. As electricity is primarily produced from fossil fuels in Turkey, the environmental costs of using electrical AVs exceed its benefits.

The emergence of specific technologies may differ across countries (Li et al., 2021). Moreover, the transformation of the existing system is a challenging issue and may fail because of various reasons, from

macroeconomic conditions to the national industry's international position. Technological strategies must be determined over micro and macro economies as well as the stakeholders' needs (Mohamad and Songthaveephol, 2020; Wu et al., 2021; Ball et al., 2021; Ye et al., 2021). Although the economic and technical conditions are appropriate, society's acceptance of the new technology may also be an obstacle (Adu-Gyamfi et al., 2022; Featherman et al., 2021). Some policy mixes in both macro and micro levels that can cure at least some of the problems and prepare the existing system for the transformation period (Raven and Walrave, 2020; Deuten et al., 2020). Even some major changes in culture and consumer behavior that are required can be promoted by the policy makers (Köhler et al., 2020; Jiang et al., 2022; Moon et al., 2021; Jaiswal et al., 2021; Huang et al., 2021a). The company-wide and system-level changes must be planned by respecting the existing traditions (Wesseling et al., 2020; Singh et al., 2021; Llopis-Albert et al., 2021b; Babar and Ali, 2021). All the evidence from the literature clue in the difficulties of policymaking in implementing EVs regarding its economy-wide impacts as well as the technology diffusion in society. In this study, we aim to explore the decision-making process, and to reach some policy implications in implementing EVs when environment-friendly energy production is an additional concern in a country. Different scenarios that simulate real-life conditions and reveal the bottlenecks of the transition problems are possible to consider analyzing this strategic decision-making process (Kishita et al., 2020; Krawinkler et al., 2022). The major concern that we focus on is the structure of the energy market, and the rate of change in fossil fuel consumption as the use of EVs increases in traffic.

The infrastructural investments are expensive and constitute a significant financial constraint in both the transportation sector and energy sector. The impact of transformation on the automotive industry is more obvious, and it may be required to support the stakeholders financially (Llopis-Albert et al., 2021a; Gao and Zhang, 2022). While implementing EV technology requires a lot of investment, it has a strong impact on the whole economy (Tamba et al., 2022). This is especially true in terms of how the transformation process affects the energy market most directly where as the demand for fossil fuels for vehicles decreases, the demand for electricity and hydrogen increases. Fossil fuel consumption is expected to be reduced with the implementation of EVs in transportation. It is more likely that the implementation of EVs achieves successful climate actions in a shorter time if electricity production does not require fossil fuels. Investments in renewable energy resources in developed countries have long since started this process (Pereira et al., 2022). However, fossil fuels are still the major sources to produce electricity in most of the developing world. For example, hydrogen demand is expected to be doubled by 2040 (Park et al., 2022), yet the paradigm shift that comes along with the new EV technologies can be an opportunity for those developing countries who are trying to catch up with the global competition in the automotive industry (Altenburg et al., 2022). In that respect, it is not about whether to implement EVs or not, but it is about how to implement the EVs so that the utilization can be optimized. Therefore, it is vital to offer decision-makers in developing nations an understanding of alternate methods for the utilization of electric vehicles (EVs). Fuzzy based decision making models have been successfully integrated into various problems (Zhou et al., 2022; Gopal and Panchal, 2021).

In this study, a Type-2 neutrosophic numbers (T2NNs) based methods RAFSI method is applied to choose the best alternative among the three alternatives. The T2NNs are used to address incomplete, indeterminate, and inconsistent information, which is why T2NNs are effective at handling uncertain information. RAFSI provides flexible decision-making and simulation of different risk levels to effectively control the robustness of the results. The analysis results indicate that the improved model based on T2NNs is a powerful, robust, and applicable tool that can be applied to solve decision-making problems.

The rest of this study is structured as follows. Section 2 provides a literature review. Section 3 defines the problem, alternatives, and criteria. The methodology is explained in Section 4. The case study, results, discussion, and policy implications are provided in Sections 5, 6, and 7, respectively. Finally, the conclusion of the study is explained in Section 8.

2. Literature review

According to a study regarding driven factors of renewable energy development, the best way to promote renewable energy is through high feed-in-tariff (FIT) pricing, simple transmission access, and minimal transmission adjustments (Alagappan et al., 2011). From a different perspective, energy itself is known as a "strategic commodity". Therefore, especially in developing countries, sustainable energy is a trend-topic, and renewable energy transition from fossil fuel combustion as a primary energy supply is a long-term must (Sen and Ganguly, 2017). Moreover, incrementation in investment in the region's development of renewable energy will be boosted by economic growth (Xu et al., 2019). In addition, to advance the growth of renewable energy power generation, it will be beneficial to support the development of renewable energy technologies that will lower development, maintenance costs, and increase reliability, applicability, and energy conversion efficiency (Dinçer et al., 2022). Altogether, it can be concluded that encouraging renewable energy development could create a better future not only for the electric vehicle industry but also for a sustainable economy and environment.

Encouraging the use of electric vehicles could be done in various ways. Benefits gained from it could exceed its costs. Lifting barriers in front of EV adoption is the key to boosting the EV population. Local and national governments may take action to increase the rate of transition to EVs. Incentive policies such as operation subsidy, purchase subsidy, access to high occupancy vehicle lanes, and free parking could be the points in a written list (Kwon et al., 2018). Other than the governmental assistance, the flexibility provided by regulations could result in good feedback. Madina et al. (2016) state that accesses to private home charging, in particular, is anticipated to be a significant factor in the adoption of electric vehicles as long as owners can cover high yearly mileage, charge their EVs more cheaply at night, and receive incentives for EV purchases. To sum up, accomplishing proper ways to encourage the use of electric vehicles could increase the civilization of the country in many parts.

However, creating financial support results in different consequences. First of all, fuel tax directly affects usage and its capital could become a possible source for subsidies. The average gallon of gasoline in the United States costs forty cents more in state and federal taxes than it would without those taxes. This amounts to $8 billion in annual tax receipts (EIA, 2008). Fuel taxes may not originally have been developed for environmental purposes, but their impact is certainly ecological (Parry and Small, 2005). Another outcome is that implementing a carbon tax plays a significant role in climate change actions. If fiscal revenues aren't redistributed, carbon taxes will impose a higher cost on polluters than either command-and-control policies or emissions trading systems with an initial free allocation of permits (Keohane et al., 2017). Carbon taxes have been a popular idea among economists and international organizations (EEA, 1996; OECD, 1996). In the future, vehicle tax reform for internal combustion engines could be a new course of action. Although there may be some discomfort for consumers, the transition could be more sustainable financially. For example, Norway started this procedure years ago. In 2007, the travel part of the tax was replaced by the potential intensity of the vehicle's CO_2 emissions (Ciccone, 2018).

3. Problem definition

The latest climate events and their social and economic effects enforce an ultimate change in the traditional way of modern life. The transportation sector is carrying some burden of this transformation as well. From pollution control to energy efficiency, various concerns are

being closely watched. Clean and efficient alternatives are expected to be developed. Using electrical vehicles (EVs) is highly encouraged especially because of their positive impacts on the environment. However, EVs require significant infrastructural investments. In this study, three alternatives to the EV adaptation process are analyzed using decision-making with 4 aspects and 12 criteria. The decision hierarchy of the decision-making problem is illustrated in Fig. 1.

3.1. Definition of alternatives

The definition of alternatives is explained in the following:

A_1: Encouraging renewable energy development: Energy is the world's most valuable asset. Today, most energy is provided by fossil fuels, yet the consequences of climate change cannot be disregarded (Stern and Stern, 2007). Economic damage caused by climate change could equal the whole cost of World Wars I and II. It also addresses environmental concerns. Renewable energy production could be the light that humanity is looking for to thrive on our only planet.

A_2: Encouraging the use of electric vehicles: The widespread availability of electric vehicles is a trend that can be seen in both developed and developing countries. The source of the electricity that will be utilized in EVs is the most contentious aspect of this discussion because it is the most important question. Aside from that, electric vehicles have significantly fewer greenhouse gas emissions than vehicles powered by internal combustion engines. The promotion of the use of electric vehicles will make a significant contribution to the accomplishment of the goals of sustainable development. The adoption of electric mobility across the globe is helping to lessen humankind's negative impact on the environment, lessen reliance on fossil fuels, and broaden the range of available energy sources for transportation (Onat and Kucukvar, 2022).

A_3: Regulation of the existing system through taxation and penalties: Because the primary goal of changing the existing system is to reduce GHG emissions and break free from the energy-dependent position caused by fossil fuel combustion, governments could enact new legislation to regulate the existing system through taxation and penalties. From the Kyoto Protocol in 1997 to the Paris Agreement in 2015, this approach has been legally popular and active. According to Goulder (2020), interactions between policy efforts and the financial system have a significant impact on policy reforms' political viability, distributional equality, and cost-effectiveness.

3.2. Definition of criteria

(1) Social and Political Aspect:

C_1: Social acceptance (benefit): The usefulness of any novelty intended for society is proportional to its acceptance by society. The impact of alternatives on society and people's reactions to them are critical variables in determining their efficacy. The user's comfort with the use of the technology also has a considerable impact on whether they intend to employ auto systems or renewable energy sources (Park et al.,2018).

C_2: Reduced political support (cost): Decision-makers must win public support, and the simplest way to do so is to increase their level of comfort. Aside from reducing GHG emissions, the EU's energy crisis caused by the Russia-Ukraine war needs alternate energy sources and new taxes. For example, the results of a survey done in Switzerland on changing support for renewable energy sources are particularly fascinating. It also revealed the biggest tendencies toward higher levels of support since the war, with slightly under a third of respondents increasing their support (Steffen and Patt, 2022).

C_3: Enhanced social inclusiveness (benefit): The social inclusivity of the alternatives is an essential problem because the usage areas of these alternatives may be limited. These advances necessitate conscious people to reach their full potential. The key concerns have been identified as familial factors that impact purchasing decisions, a lack of trust in EVs because they use a developing technology, and related societal difficulties because most consumers are not early adopters (Goel et al., 2021).

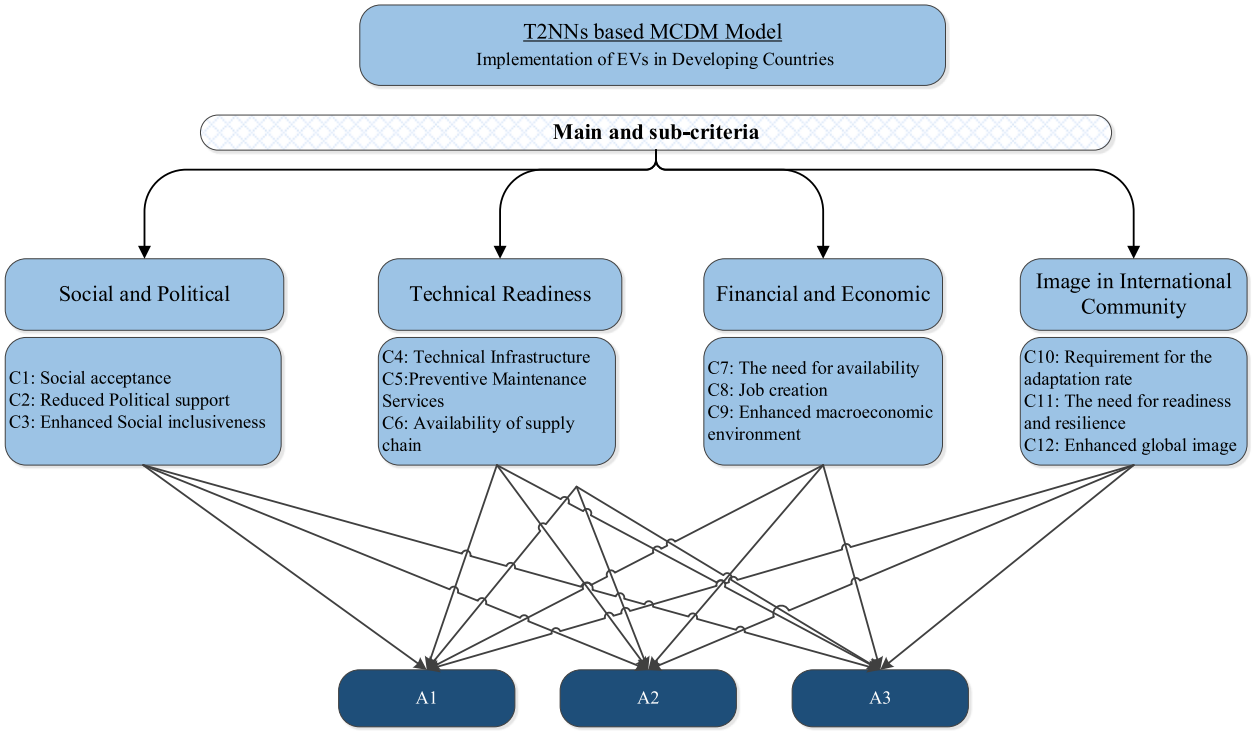


Fig. 1. The decision hierarchy of the decision-making problem.

(2) Technical Readiness Aspect:

C_4: Need for existence of technical infrastructure (cost): The use of electricity as a source of power is necessary for electric vehicles. They will require specialized infrastructure to function properly. Although there has been an increase in investment in infrastructure as a result of this market's expansion over time, there is still not enough public infrastructure to meet the demand. At this time, existing gas stations will house fast-charging infrastructure with high power of up to 350 kW. This will be the case for Ireland, the United Kingdom, the United States, and Germany, as well as the Tesla Supercharger network (Funke et al., 2019).

C_5: Need for existence of preventive maintenance services (cost): Because of the way they are built, renewable energy systems like solar panels and electric vehicles need to have regular maintenance performed for them to be used for extended periods. In contrast to the framework that has been established for electric vehicle services, renewable energy facilities already have a well-established service infrastructure in place. As was to be anticipated, maintaining the considered framework in a working state for a longer time, and consequently increasing the device's accessibility, results in a greater amount of energy given, and consequently, a higher income (Rinaldi et al., 2017).

C_6: Availability of supply chain of physical capital to implement the alternative (benefit): The growing demand for EVs and renewable energy necessitates the supply chain of items such as batteries, cells, technologies, and so on. The most common battery type is Lithium-Ion, and according to economic theories, if manufacturers are unable to make enough batteries to fulfill demand, the EV business will suffer. Because of major technological advancements, changes in battery structural design, changes in the supply chain and manufacturing process, as well as network relationships, the transition to electromobility creates uncertainty in the automotive supply chain (Marcos et al., 2021)

(3) Financial and Economic Aspect:

C_7: The need for availability of financial support (cost): When the project is completed, financial support is the most important factor. To make aspirations a reality, both renewable energy sources and electric vehicles must be economically sustainable. Penalties and taxation are two methods that governments could use to raise cash. Environmental taxes, for example, are used to alter the overall price of producing power from fossil or non-fossil sources (Abdmouleh et al., 2015).

C_8: Job creation (benefit): The range of the EV and renewable energy industries creates a huge number of job opportunities. For instance, the maintenance and repair of EVs differ from those of vehicles with internal combustion engines. Although additional jobs are projected to be produced in the household power in-dustry, as stated in the preceding section, employment growth estimates are provided for both the battery manufacturing and charging network businesses (Becker et al., 2009).

C_9: Enhanced macroeconomic environment of the country (benefit): Every country has priorities that are directly tied to its geographical location and political status. Some EU members could redistribute resources to increase renewable energy sources, while other countries must prioritize public welfare. Furthermore, countries with economic insecurity face greater dangers than economically stable countries. Because of the two-way causal relationship between economic growth (GDP per capita) and renewable energy, per capita income influences the development and consumption of renewable energy, and encouraging economic development is beneficial to the growth of renewable energy (Shakouri and Khoshnevis Yazdi, 2017).

(4) Image in International Community Aspect

C_{10} Requirement for the adaptation rate of the industries to the regulations (cost): To mitigate the effects of climate change, industrialized and developing countries with the ability to change are compelled to act. The latter differs from the former in that emerging countries' growth may be hampered due to the limited use of ecologically unfriendly resources. However, the benefits of environmentally friendly technologies (EFT) may outweigh their costs. Annual forages planted during the fallow season produced more energy than they consumed, resulting in a lower energy ratio than in conventional cropping systems (Deng et al., 2021).

C_{11}: The need for readiness and resilience to climate change (cost): Governments are being forced to take climate action in transportation. Such agreements are mainly signed by developing countries. The extent to which current infrastructure is robust and prepared for disasters is critical in the decision-making process. Infrastructure could be harmed if enough precautions are not taken and no plans are made to prepare for future disasters. Infrastructure must be available if the new transportation system is to be integrated. The growing number of EVs may cause concerns in the future due to oversupply; however, countries may be able to tackle these issues by investing in the industry with less money than projected. Payback guarantee issues could be resolved by integrating renewable energy sources with EV charging stations (Alghoul et al., 2018).

C_{12}: Enhanced global image of the government (benefit): In today's global environment, a government is no longer the sole decision-maker for international community-related acts. Global images are increasingly crucial for governments as nations become more intertwined through trade and financial difficulties. Governments take steps to strengthen their international power, comfort, and reputation. Climate change measures have been one of the most important topics related to reputation in recent years. Furthermore, making a decision may affect other countries due to agreements. As a result, governments must use greater caution when executing movements. Despite officials at the Ministry of Environment seeing the national benefit of forest preservation and carbon trading mechanisms when compared to the Ministries of Foreign Relations (MRE) and Science and Technology (MS&T), this ministry's role in universal agreements is negligible (Lahsen, 2013).

4. Methodology

Zadeh (1965) introduced the fuzzy set theory to address the uncertainties in knowledge. Afterward, Atanassov (1986) generalized the fuzzy sets theory to the intuitionistic fuzzy sets that represent the membership and non-membership functions. Later, the neutrosophic sets were presented by Smarandache as an extension of fuzzy sets in (Smarandache, 1998). Neutrosophic sets have been successfully applied to decision problems (Abdel-Basset et al., 2022; Mishra et al., 2021; Donbosco and Ganesan, 2022; Broumi et al., 2023). The membership functions are illustrated in Fig. 2. In this section, some of the fundamental notations and operations are provided on Type-1 neutrosophic and Type-2 neutrosophic sets, and the steps of the proposed decision-making model are given as follows:

4.1. Type-1 neutrosophic set

Definition 1 (Smarandache, 1998). Let F be a universe of discourse. A neutrosophic set $\wp$ in F is represented by truth $\pi_{\wp}(f)$, indeterminacy $\zeta_{\wp}(f)$, and falsity $\varphi_{\wp}(f)$ membership functions. These functions are expressed by:

$$\wp = \{\langle f,\ \pi_{\wp}(f),\ \zeta_{\wp}(f),\ \varphi_{\wp}(f)\rangle \mid f \in F\}. \tag{1}$$

where these functions present real subsets of $\left[{}^{-}0, 1^{+}\right]$, $\pi_{\wp}(f)$, $\zeta_{\wp}(f)$, and $\varphi_{\wp}(f) : F \rightarrow \left[{}^{-}0, 1^{+}\right]$. The sum of three membership functions must satisfy:

$${}^{-}0 \leq \pi_{\wp}(f) + \zeta_{\wp}(f) + \varphi_{\wp}(f) \leq 3^{+}, \quad \forall_f \in F \tag{2}$$

4.2. Type-2 neutrosophic set

The Type-2 neutrosophic numbers (T2NNs) were presented by Abdel-Basset et al. (2019), and extend the neutrosophic set developed by Smarandache (1998). The fundamental definitions and operations of T2NNs are defined by:

Definition 2 (Abdel-Basset et al., 2019). Let F be a universe of discourse and $N[0,1]$ be the set on $S[0,1]$. A T2NNs $\widehat{Z}$ in F is expressed by $\widehat{Z} = \{\langle f,\ \pi_{\hat{Z}}(f),\ \zeta_{\hat{Z}}(f),\ \varphi_{\hat{Z}}(f)\rangle \mid f \in F\}$, where $\pi_{\hat{Z}}(f) : F \to N[0,\ 1]$, $\zeta_{\hat{Z}}(f) : F \to N[0,\ 1]$, and $\varphi_{\hat{Z}}(f) : F \to N[0,\ 1]$.

A T2NNS $\breve{N}$ are represented by a truth $\pi_{\hat{Z}}(f)$, indeterminacy $\zeta_{\hat{Z}}(f)$, and falsity $\varphi_{\hat{Z}}(f)$memberships and defined by $\pi_{\hat{Z}}(x) = (\pi_{\pi_{\hat{Z}}}(f),\ \pi_{\zeta_{\hat{Z}}}(f),\ \pi_{\varphi_{\hat{Z}}}(f))$, $\zeta_{\hat{Z}}(x) = (\zeta_{\pi_{\hat{Z}}}(f),\ \zeta_{\zeta_{\hat{Z}}}(f),\ \zeta_{\varphi_{\hat{Z}}}(f))$, and $\varphi_{\hat{Z}}(x) = (\varphi_{\pi_{\hat{Z}}}(f),\ \varphi_{\zeta_{\hat{Z}}}(f),\ \varphi_{\varphi_{\hat{Z}}}(f))$ belonging to the truth, indeterminacy, and falsity memberships of f in $\widehat{Z}$, and for every $f \in F$: $0 \le \pi_{\hat{Z}}(f)^3 + \zeta_{\hat{Z}}(f)^3 + \varphi_{\hat{Z}}(f)^3 \le 3$.

A Type-2 neutrosophic number (T2NN) can be represented by $\widehat{Z} = \langle(\pi_{\pi_{\hat{Z}}}(f),\ \pi_{\zeta_{\hat{Z}}}(f),\ \pi_{\varphi_{\hat{Z}}}(f)),(\zeta_{\pi_{\hat{Z}}}(f),\ \zeta_{\zeta_{\hat{Z}}}(f),\ \zeta_{\varphi_{\hat{Z}}}(f)),$
$(\varphi_{\pi_{\hat{Z}}}(f),\ \varphi_{\zeta_{\hat{Z}}}(f),\ \varphi_{\varphi_{\hat{Z}}}(f))\rangle$.

Definition 3 (Abdel-Basset et al., 2019). Let two T2NNs $\widehat{Z}_1$ and $\widehat{Z}_2$ be expressed by:

$$\widehat{Z}_1 = \left\langle \left(\pi_{\pi_{\hat{Z}_1}}(f),\ \pi_{\zeta_{\hat{Z}_1}}(f),\ \pi_{\varphi_{\hat{Z}_1}}(f)\right), \left(\zeta_{\pi_{\hat{Z}_1}}(f),\ \zeta_{\zeta_{\hat{Z}_1}}(f),\ \zeta_{\varphi_{\hat{Z}_1}}(f)\right), \left(\varphi_{\pi_{\hat{Z}_1}}(f),\ \varphi_{\zeta_{\hat{Z}_1}}(f),\ \varphi_{\varphi_{\hat{Z}_1}}(f)\right)\right\rangle,$$

and

$$\widehat{Z}_2 = \left\langle \left(\pi_{\pi_{\hat{Z}_2}}(f),\ \pi_{\zeta_{\hat{Z}_2}}(f),\ \pi_{\varphi_{\hat{Z}_2}}(f)\right), \left(\zeta_{\pi_{\hat{Z}_2}}(f),\ \zeta_{\zeta_{\hat{Z}_2}}(f),\ \zeta_{\varphi_{\hat{Z}_2}}(f)\right), \left(\varphi_{\pi_{\hat{Z}_2}}(f),\ \varphi_{\zeta_{\hat{Z}_2}}(f),\ \varphi_{\varphi_{\hat{Z}_2}}(f)\right)\right\rangle$$

Some basic operations can be expressed by:

(a) Addition "$\oplus$"

$$\begin{aligned}\widehat{Z}_1 \oplus \widehat{Z}_2 = \Big\langle & \left(\pi_{\pi_{\hat{Z}_1}}(f) + \pi_{\pi_{\hat{Z}_2}}(f) - \pi_{\pi_{\hat{Z}_1}}(f)\cdot\pi_{\pi_{\hat{Z}_2}}(f)\right),\ \left(\pi_{\zeta_{\hat{Z}_1}}(f) + \pi_{\zeta_{\hat{Z}_2}}(f) - \pi_{\zeta_{\hat{Z}_1}}(f)\cdot\pi_{\zeta_{\hat{Z}_2}}(f)\right) \\ & , \left(\pi_{\varphi_{\hat{Z}_1}}(f) + \pi_{\varphi_{\hat{Z}_2}}(f) - \pi_{\varphi_{\hat{Z}_1}}(f)\cdot\pi_{\varphi_{\hat{Z}_2}}(f)\right), \\ & \left(\zeta_{\pi_{\hat{Z}_1}}(f)\cdot\zeta_{\pi_{\hat{Z}_2}}(f),\ \zeta_{\zeta_{\hat{Z}_1}}(f)\cdot\zeta_{\zeta_{\hat{Z}_2}}(f),\ \zeta_{\varphi_{\hat{Z}_1}}(f)\cdot\zeta_{\varphi_{\hat{Z}_2}}(f)\right), \\ & \left(\varphi_{\pi_{\hat{Z}_1}}(f)\cdot\varphi_{\pi_{\hat{Z}_2}}(f),\ \varphi_{\zeta_{\hat{Z}_1}}(f)\cdot\varphi_{\zeta_{\hat{Z}_2}}(f),\ \varphi_{\varphi_{\hat{Z}_1}}(f)\cdot\varphi_{\varphi_{\hat{Z}_2}}(f)\right)\Big\rangle,\end{aligned} \tag{3}$$

(b) Multiplication "$\otimes$"

$$\begin{aligned}\widehat{Z}_1 \otimes \widehat{Z}_2 = \Big\langle & \left(\pi_{\pi_{\hat{Z}_1}}(f) + \pi_{\pi_{\hat{Z}_2}}(f),\ \pi_{\zeta_{\hat{Z}_1}}(f)\cdot\pi_{\zeta_{\hat{Z}_2}}(f),\ \pi_{\varphi_{\hat{Z}_1}}(f)\cdot\pi_{\varphi_{\hat{Z}_2}}(f)\right), \\ & \left(\varphi_{\pi_{\hat{Z}_1}}(f) + \varphi_{\pi_{\hat{Z}_2}}(f) - \varphi_{\pi_{\hat{Z}_1}}(f)\cdot\varphi_{\pi_{\hat{Z}_2}}(f),\ \zeta_{\zeta_{\hat{Z}_1}}(f) + \zeta_{\zeta_{\hat{Z}_2}}(f)\right. \\ & \left. - \zeta_{\zeta_{\hat{Z}_1}}(f)\cdot\zeta_{\zeta_{\hat{Z}_2}}(f),\ \zeta_{\varphi_{\hat{Z}_1}}(f) + \zeta_{\varphi_{\hat{Z}_2}}(f) - \zeta_{\varphi_{\hat{Z}_1}}(f)\cdot\zeta_{\varphi_{\hat{Z}_2}}(f)\right), \\ & \left(\varphi_{\pi_{\hat{Z}_1}}(f) + \varphi_{\pi_{\hat{Z}_2}}(f) - \varphi_{\pi_{\hat{Z}_1}}(f)\cdot\varphi_{\pi_{\hat{Z}_2}}(f),\ \varphi_{\zeta_{\hat{Z}_1}}(f) + \varphi_{\zeta_{\hat{Z}_2}}(f)\right. \\ & \left. - \varphi_{\zeta_{\hat{Z}_1}}(f)\cdot\varphi_{\zeta_{\hat{Z}_2}}(f),\ \varphi_{\varphi_{\hat{Z}_1}}(f) + \varphi_{\varphi_{\hat{Z}_2}}(f) - \varphi_{\varphi_{\hat{Z}_1}}(f)\cdot\varphi_{\varphi_{\hat{Z}_2}}(f)\right)\Big\rangle,\end{aligned} \tag{4}$$

(c) Scalar multiplication

$$\begin{aligned}\Omega\widehat{Z} = \Big\langle & \left(1-(1-\pi_{\pi_{\hat{Z}}}(f))^{\Omega},\ 1-(1-\pi_{\zeta_{\hat{Z}}}(f))^{\Omega},\ 1-(1-\pi_{\varphi_{\hat{Z}}}(f))^{\Omega}\right), \\ & \left((\zeta_{\pi_{\hat{Z}}}(f))^{\Omega},\ (\zeta_{\zeta_{\hat{Z}}}(f))^{\Omega},\ (\zeta_{\varphi_{\hat{Z}}}(f))^{\Omega}\right), \\ & \left((\varphi_{\pi_{\hat{Z}}}(f))^{\Omega},\ (\varphi_{\zeta_{\hat{Z}}}(f))^{\Omega},\ (\varphi_{\tau_{\hat{Z}}}(f))^{\Omega}\right)\Big\rangle,\end{aligned} \tag{5}$$

where $\Omega > 0$.

(d) Power

$$\begin{aligned}\widehat{Z}^{\Omega} = \Big\langle & \left((\pi_{\pi_{\hat{Z}}}(f))^{\Omega},\ (\pi_{\zeta_{\hat{Z}}}(f))^{\Omega},\ (\pi_{\varphi_{\hat{Z}}}(f))^{\Omega}\right), \\ & \left(1-(1-(\zeta_{\pi_{\hat{Z}}}(f))^{\Omega},\ 1-(1-\zeta_{\zeta_{\hat{Z}}}(f))^{\Omega},\ 1-(1-\zeta_{\varphi_{\hat{Z}}}(f))^{\Omega}\right), \\ & \left(1-(1-\varphi_{\pi_{\hat{Z}}}(f))^{\Omega},\ 1-(1-\varphi_{\zeta_{\hat{Z}}}(f))^{\Omega},\ 1-(1-\varphi_{\tau_{\hat{Z}}}(f))^{\Omega}\right)\Big\rangle.\end{aligned} \tag{6}$$

where $\Omega > 0$.

Definition 4 (Abdel-Basset et al., 2019). The score function $H(\widehat{Z})$ of $\widehat{Z}$ is can be expressed by:

$$\begin{aligned}H(\widehat{Z}) = \frac{1}{12}\langle 8 + (\pi_{\pi_{\hat{Z}}}(f) + 2\,(\pi_{\zeta_{\hat{Z}}}(f)) + \pi_{\varphi_{\hat{Z}}}(f)) - (\zeta_{\pi_{\hat{Z}}}(f) + 2(\zeta_{\zeta_{\hat{Z}}}(f)) + \zeta_{\varphi_{\hat{Z}}}(f)) \\ -(\varphi_{\pi_{\hat{Z}}}(f) + 2(\varphi_{\zeta_{\hat{Z}}}(f)) + \varphi_{\varphi_{\hat{Z}}}(f))\rangle.\end{aligned} \tag{6}$$

The accuracy function $K(\widehat{Z})$ of $\widehat{Z}$ is expressed by:

$$K(\widehat{Z}) = \frac{1}{4}\langle(\pi_{\pi_{\hat{Z}}}(f) + 2\ (\pi_{\zeta_{\hat{Z}}}(f)) + \pi_{\varphi_{\hat{Z}}}(f)) - (\varphi_{\pi_{\hat{Z}}}(f) + 2(\varphi_{\zeta_{\hat{Z}}}(f)) + \varphi_{\varphi_{\hat{Z}}}(f))\rangle \tag{7}$$

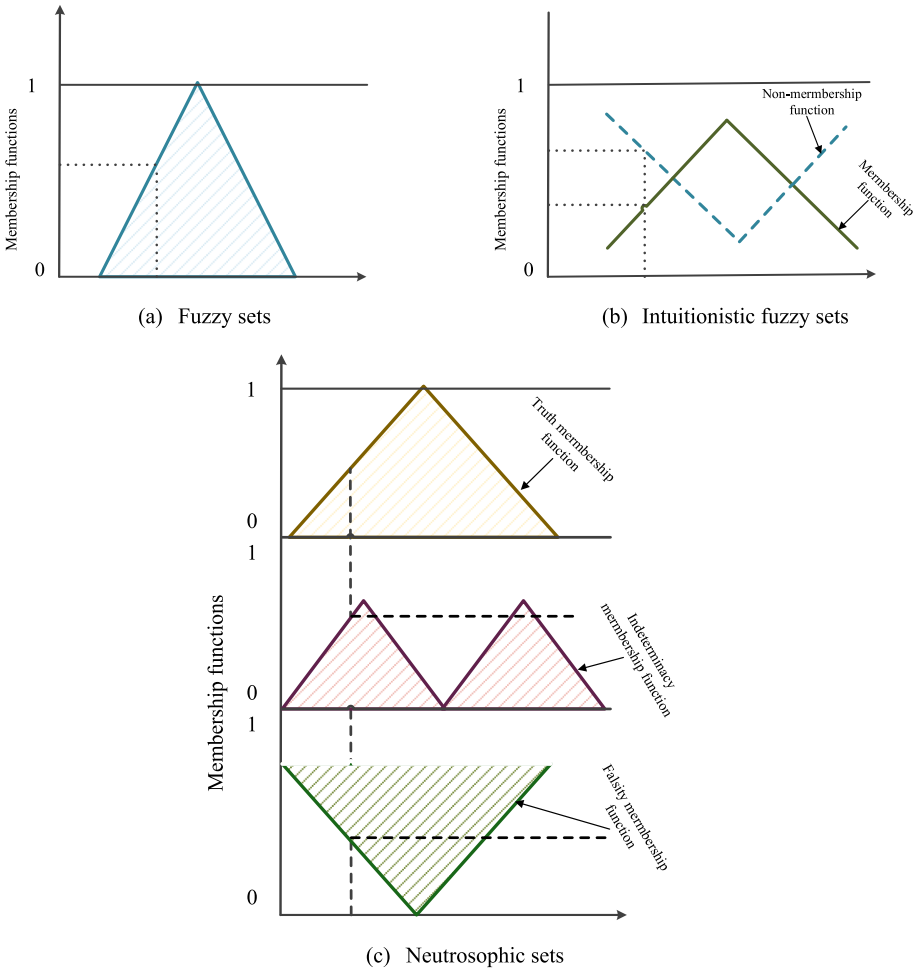
The relations between $H(\widehat{Z})$ and $K(\widehat{Z})$ can be defined as follows:

1) If $H(\widehat{Z}_1) \rangle H(\widehat{Z}_2)$, then $\breve{N}_1$ is greater than $\widehat{Z}_2$, denoted by $\widehat{Z}_1 > \widehat{Z}_2$.
2) If $H(\widehat{Z}_1) = H(\widehat{Z}_2)$, and $K(\widehat{Z}_1) \rangle K(\widehat{Z}_2)$, then $\breve{N}_1$ is superior to $\widehat{Z}_2$, denoted by $\breve{N}_1 > \breve{N}_2$.
3) If $H(\widehat{Z}_1) = H(\widehat{Z}_2)$, and $K(\widehat{Z}_1) = K(\widehat{Z}_2)$, then $\widehat{Z}_1$ is equal to $\widehat{Z}_2$, denoted by $\widehat{Z}_1 = \widehat{Z}_2$.

4.3. T2NNs based RAFSI model

The RAFSI model (Žižović et al., 2020) was introduced to efficiently process group uncertain and unspecified information. This section provides a T2NNs based RAFSI model for choosing the most suitable alternatives for imposing electric vehicles. The RAFSI model has been successfully integrated into a variety of decision making problems (Alosta et al., 2021). The stages of the proposed T2NNs based RAFSI model are presented in Fig. 3.

The evaluation criteria, alternatives, and the group of experts are defined to construct the proposed model. The set $\wp_i = (\wp_1, \wp_2, \ldots, \wp_n)$ having $i = 1, 2, \ldots, n$ alternatives is assessed by $m$the decision criteria of the set $\varsigma_j = (\varsigma_1, \varsigma_2, \ldots, \varsigma_m)$ having $j = 1, 2, \ldots, m$ criteria using the group



(a) Fuzzy sets (b) Intuitionistic fuzzy sets

(c) Neutrosophic sets

Fig. 2. Some fuzzy sets and their membership functions.

of experts $\eta_l = (\eta_1, \eta_2, \ldots, \eta_e)(\ell = 1, 2, \ldots, e)$. The linguistic terms are determined.

Step 1. Create the initial decision matrix $\widehat{\mathrm{X}} = \left(\chi_{ij}\right)_{mxn}$ with T2NNs. Experts provides their opinions for each alternative regarding each decision criterion using linguistic terms.

$$\widehat{\mathrm{X}} = \left(\chi_{ij}\right)_{nxm} = \begin{array}{c} \\ \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_m \end{array} \begin{array}{c} \begin{array}{cccc} \wp_1 & \wp_2 & \cdots\cdots & \wp_n \end{array} \\ \begin{pmatrix} \chi_{11} & \chi_{12} & \cdots\cdots & \chi_{1m} \\ \chi_{21} & \chi_{22} & \cdots\cdots & \chi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{1n} & \chi_{2n} & \cdots\cdots & \chi_{nm} \end{pmatrix} \end{array} \tag{8}$$

where $1 \le i \le n$ and $1 \le j \le m$.

Afterward, the linguistic terms in the initial matrix into the T2NNs decision matrix are converted.

Step 2. Calculate the score values $G = \left(g_{ij}\right)_{mxn}$ using T2NNs based on the initial matrix with the help of Eq. (6):

$$G = \left(g_{ij}\right)_{nxm} = \begin{array}{c} \\ \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_m \end{array} \begin{array}{c} \begin{array}{cccc} \wp_1 & \wp_2 & \cdots\cdots & \wp_n \end{array} \\ \begin{pmatrix} g_{11} & g_{12} & \cdots\cdots & g_{1m} \\ g_{21} & g_{22} & \cdots\cdots & g_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ g_{1n} & g_{2n} & \cdots\cdots & g_{nm} \end{pmatrix} \end{array} \tag{9}$$

where $1 \le i \le n$ and $1 \le j \le m$.

Step 3. Find the weights of the criteria by applying Eq. (6) with the help of experts opinions in terms of each criterion. The criteria weights are represented by $w_j = (j = 1, 2, \ldots, m)$.

Step 4. Find the ideal and anti-ideal values using γ_{ij} and Eq. (10). The experts provide two values α_{I_j} and α_{A_j}, where α_{I_j} denotes the ideal value of ς_j, and α_{A_j} is the anti-ideal value of ς_j. It can be seen that $\alpha_{I_j} > \alpha_{A_j}$ for benefit (max), and $\alpha_{I_j} < \alpha_{A_j}$ for cost (min) criteria.

$$\varsigma_j \in \begin{cases} \left(\alpha_{A_j}, \alpha_{I_j}\right), for \quad benefit \quad criteria \\ \left(\alpha_{I_j}, \alpha_{A_j}\right), for \quad cost \quad criteria \end{cases} \tag{10}$$

Step 5. Build the standardized decision matrix $T = \left(\tau_{ij}\right)_{nxm}$ using Eqs. (11)–(13). In order to make all the criteria of the initial decision

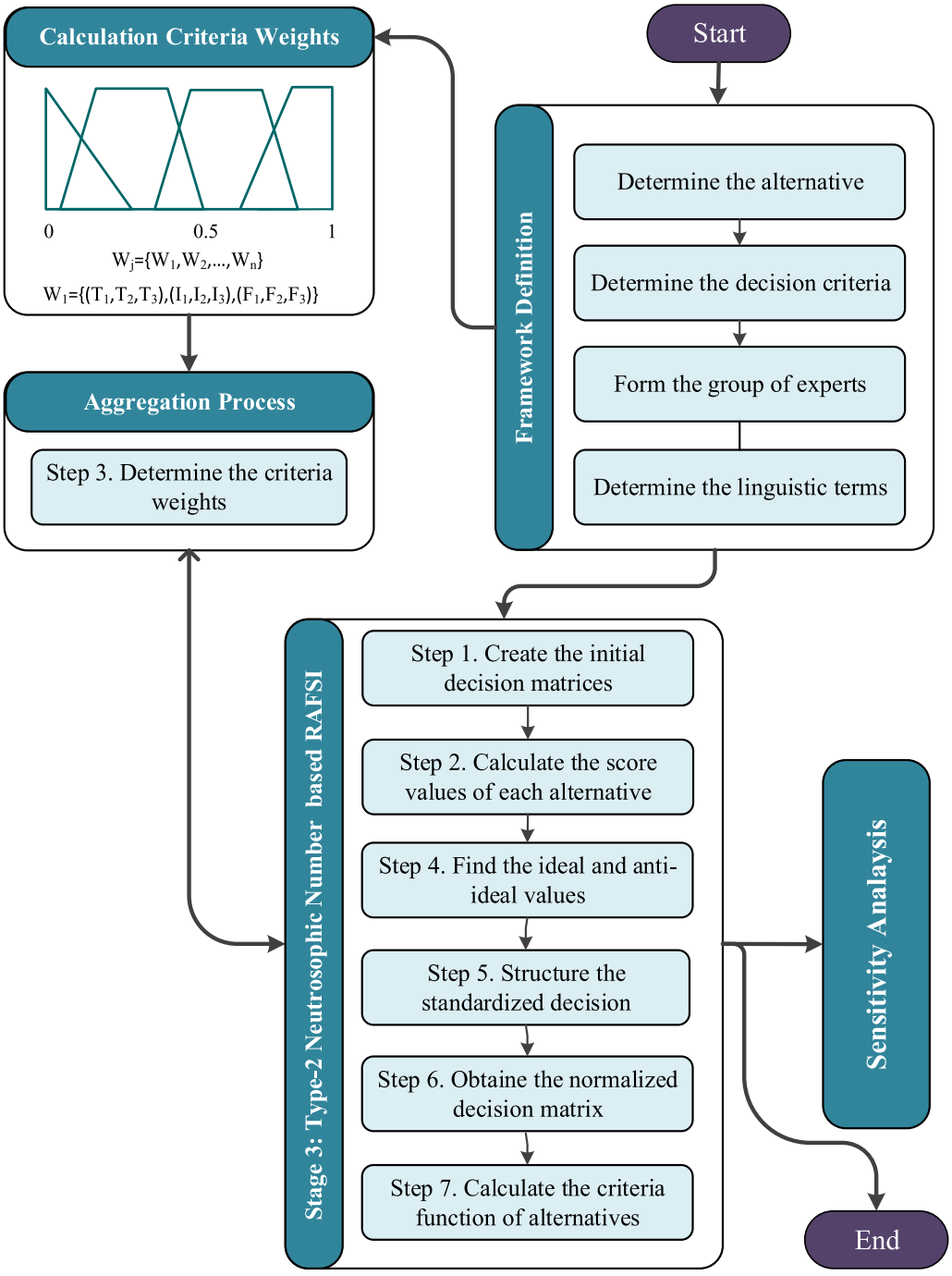


Fig. 3. The flowchart of the proposed T2NNs based RAFSI model.

matrix or to transfer them to the criteria range $[\mathfrak{F}_1, \mathfrak{F}_{2\lambda}]$, we are creating many sequences from the interval λ with $\lambda - 1$ points added between the highest and lowest values of the criteria interval. Later, the sub-intervals into the criteria interval are mapped with the help of functions Φ_1, Φ_2, Φ_3 and as illustrated in Fig. 4.

$$\mathfrak{F}_1 < \mathfrak{F}_2 \leq \mathfrak{F}_3 < \mathfrak{F}_4 \leq \mathfrak{F}_5 < \mathfrak{F}_6 \ldots \leq \mathfrak{F}_{2\lambda-1} \leq \mathfrak{F}_{2\lambda} \tag{11}$$

where $\Phi_d(x)$ denotes a function. It maps sub-intervals into the criteria interval $[\mathfrak{F}_1, \mathfrak{F}_{2\lambda}]$ using Eq. (12).

$$\Phi_d(x) = \frac{\mathfrak{F}_{2\lambda} - \mathfrak{F}_1}{\alpha_{I_j} - \alpha_{A_j}} \alpha_{ij} + \frac{\alpha_{I_j} \cdot \mathfrak{F}_1 - \alpha_{A_j} \cdot \mathfrak{F}_{2\lambda}}{\alpha_{I_j} - \alpha_{A_j}} \tag{12}$$

where $\mathfrak{F}_{2\lambda}$ and $\mathfrak{F}_1$ denotes the relations. α_{ij} is the value of the i-th alternative for the j-th criterion from the initial decision matrix.

$$T = (\tau_{ij})_{nxm} = \begin{matrix} \\ \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_m \end{matrix} \begin{matrix} \begin{matrix} \wp_1 & \wp_2 & \ldots\ldots & \wp_n \end{matrix} \\ \begin{pmatrix} \tau_{11} & \tau_{12} & \ldots\ldots & \tau_{1m} \\ \tau_{21} & \tau_{22} & \ldots\ldots & \tau_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{1n} & \tau_{2n} & \ldots\ldots & \tau_{nm} \end{pmatrix} \end{matrix} \tag{13}$$

Step 6. Find the normalized decision matrix with the help of Eqs. (14)–(17).

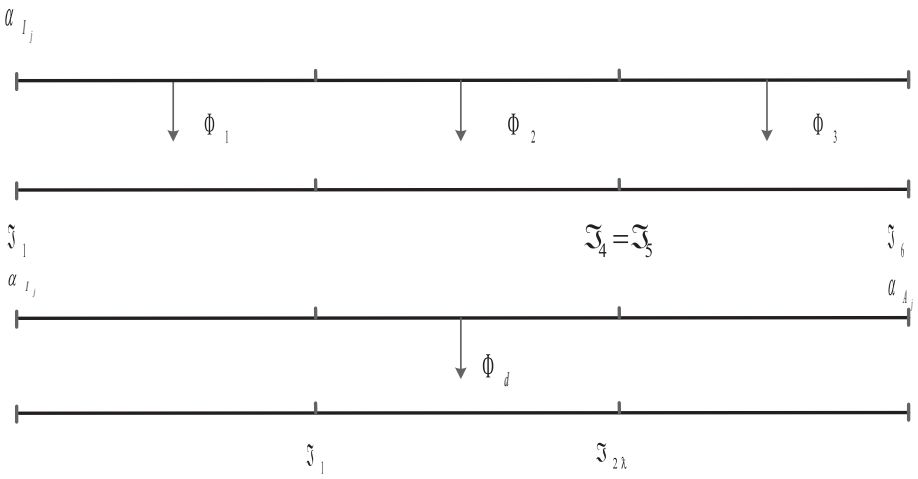


Fig. 4. Mapping of sub-intervals into the criteria interval.

$$\psi_{ij} = \begin{cases} \dfrac{\tau_{ij}}{2\beta}, & if \quad j \in max \\ \dfrac{\delta}{2\tau_{ij}}, & if \quad j \in min \end{cases} \tag{14}$$

β and δ represent the arithmetic and harmonic means, respectively. The β and δ values are found using Eqs. (15)–(16) for *min* and *max* sequence of the elements $\mathfrak{I}_{2\lambda}$ and $\mathfrak{I}_1$.

$$\beta = \frac{\mathfrak{I}_1 + \mathfrak{I}_{2\lambda}}{2} \tag{15}$$

$$\delta = \frac{2}{\frac{1}{\mathfrak{I}_1} + \frac{1}{\mathfrak{I}_{2\lambda}}} \tag{16}$$

Afterward, the values are normalized with the help of Eq. (17).

$$\aleph = \left(\varepsilon_{ij}\right)_{nxm} = \begin{matrix} \\ \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_m \end{matrix} \begin{matrix} \begin{matrix} \wp_1 & \wp_2 & \cdots\cdots & \wp_n \end{matrix} \\ \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots\cdots & \varepsilon_{1m} \\ \varepsilon_{21} & \varepsilon_{22} & \cdots\cdots & \varepsilon_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{1n} & \varepsilon_{2n} & \cdots\cdots & \varepsilon_{nm} \end{pmatrix} \end{matrix} \tag{17}$$

where $\varepsilon_{ij} \in [0, 1]$ is the elements of $\aleph$.

Step 7. Obtain the criteria function of each alternative θ_i using Eq. (18).

$$\theta_i = w_1\varepsilon_{i1} + w_2\varepsilon_{i2} + \cdots + w_j\varepsilon_{ij} = \sum_{j=1}^{m} \omega_j\varepsilon_{ij} \tag{18}$$

Then, rank the alternatives in decreasing order according to the values of θ_i.

5. Case study

To simulate the decision-making process in implementing EVs where the electricity is mostly produced by fossil fuels, we consider a scenario that includes three alternatives: One alternative is to focus on the renewable energy development first, before starting to finance the implementation of EVs; another alternative is to focus on the imposing the EVs by directing the available financial resources to the process; a third alternative is to maintain the status quo for a while and to regulate the existing system before transforming the whole system. In choosing the alternative to implement EVs, we build up a set of 12 criteria based on literature and experts' opinions. Different outcomes may occur in different countries because of the importance of the 12 criteria. The case scenario is tested in Turkey through questionnaires by Turkish experts from academia and the transportation sector. The novel method that we used to solve the decision-making problem is tested against two other already existing fuzzy methods.

Table 1
The list of the alternatives and evaluation criteria for imposing electric vehicles in developing countries.

Main-criteria	Sub-criteria	Types
Social and Political (MC_1)		
C_1	Social acceptance	Benefit
C_2	Reduced political support	Cost
C_3	Enhanced social inclusiveness	Benefit
Technical Readiness Aspect (MC_2)		
C_4	Need for the existence of technical infrastructure	Cost
C_5	Need for the existence of preventive maintenance services.	Cost
C_6	Availability of supply chain of physical capital to implement the alternative	Benefit
Financial and Economic Aspect (MC_3)		
C_7	The need for availability of the financial support	Cost
C_8	Job creation	Benefit
C_9	The enhanced macroeconomic environment of the country	Benefit
Image in International Community Aspect (MC_4)		
C_{10}	Requirement for the adaptation rate of the industries to the regulations	Cost
C_{11}	The need for Readiness and resilience to climate change	Cost
C_{12}	Enhanced global image of the government	Benefit

Table 2
The linguistic scale for evaluation ratings of the criteria (Abdel-Basset et al., 2019).

Linguistic variables/terms	T2NN
Weakly important (WI)	<(0.20, 0.30, 0.20), (0.60, 0.70, 0.80), (0.45, 0.75, 0.75)>
Equal important (EI)	<(0.40, 0.30, 0.25), (0.45, 0.55, 0.40), (0.45, 0.60, 0.55)>
Strong important (SI)	<(0.50, 0.55, 0.55), (0.40, 0.45, 0.55), (0.35, 0.40, 0.35)>
Very strongly important (VSI)	<(0.80, 0.75, 0.70), (0.20, 0.15, 0.30), (0.15, 0.10, 0.20)>
Absolutely important (AI)	<(0.90, 0.85, 0.95), (0.10, 0.15, 0.10), (0.05, 0.05, 0.10)>

Table 3
The linguistic scale for evaluation ratings of alternatives (Abdel-Basset et al., 2019).

Linguistic variable	Type-2 neutrosophic number
Very Bad (VB)	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>
Bad (B)	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>
Medium Bad (MB)	<(0.50, 0.30, 0.50), (0.50, 0.35, 0.45), (0.45, 0.30, 0.60)>
Medium (M)	<(0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45)>
Medium Good (MG)	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>
Good (G)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>
Very Good (VG)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)>

5.1. Proposed methodology results

Three alternatives are defined under four main and twelve sub-criteria, and these criteria are used to select an alternative to imposing electric vehicles in developing countries with the help of six experts. The list of the alternatives and criteria is reported in Table 1.

The linguistic ratings for the evaluation of the alternatives and weighting of the criteria are given in Tables 2 and 3.

The twelve criteria are evaluated by six experts using the linguistic terms given in Table 2. The expert opinions of each alternative regarding the criteria using linguistic terms given in Table 3 are collected. The evaluation results are reported in Tables 4 and 5.

Step 1. The linguistic assessments of alternatives are transformed into the corresponding Type-2 neutrosophic numbers using Table 5 with the help of Table 3 and structured the initial decision matrix.
Step 2. The score values of each alternative concerning are obtained using Eqs. (8) and (8), and are presented in Table 6.

Step 3. Applying Eq. (6), the weight coefficient of the criteria is found with the help of Table 4. The weights of each criterion are illustrated in Fig. 5 and presented in Table 7.

Step 4. The ideal α_{A_j} and anti-ideal α_{I_j} values for each criterion are defined using Eq. (10) and Table 6 and are listed in Table 8.

Step 5. The standardized normalized decision matrix is obtained using Eqs. (11)–(13) and Tables 6 and 8. The values of the matrix are presented in Table 9.

Step 6. The normalized matrix of each alternative is found using Eqs. (14)–(17) and Table 9, and are presented in Table 10.

Step 7. The overall values of each alternative are obtained using Eq. (18) and Table 10. These values are given in Table 11. The ranking of the alternative by θ_i in decreasing order is listed in Table 11 as

Table 6
The score values of each alternative regarding criteria.

Alternatives	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.878	0.834	0.832	0.800	0.780	0.888
A_2	0.893	0.847	0.855	0.742	0.758	0.887
A_3	0.819	0.742	0.857	0.890	0.896	0.816
Type	Max	Min	Max	Min	Min	Max

Alternatives	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
A_1	0.750	0.886	0.895	0.805	0.742	0.879
A_2	0.800	0.893	0.896	0.778	0.780	0.889
A_3	0.778	0.769	0.874	0.733	0.758	0.831
Type	Min	Max	Max	Min	Min	Max

Table 4
The linguistic assessments of the criteria regarding criteria by experts.

Experts	Criteria											
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
E_1	VSI	VSI	AI	AI	SI	SI	VSI	EI	AI	EI	VSI	SI
E_2	EI	VSI	SI	VSI	SI	AI	AI	SI	VSI	VSI	AI	SI
E_3	VSI	VSI	EI	AI	EI	AI	VSI	VSI	AI	VSI	VSI	SI
E_4	AI	VSI	WI	VSI	EI	AI	AI	VSI	VSI	SI	VSI	EI
E_5	VSI	EI	EI	VSI	VSI	AI	VSI	VSI	AI	VSI	SI	SI
E_6	SI	AI	EI	VSI	VSI	AI	AI	EI	SI	SI	VSI	SI

Table 5
The linguistic assessments of the alternatives regarding criteria by experts.

Alternatives	Experts	Criteria											
		C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
A_1	E_1	G	B	VG	B	VB	VG	B	G	VG	B	VB	VG
	E_2	G	MG	B	B	M	MG	VB	G	MG	B	VB	G
	E_3	G	MB	M	MG	MB	G	VB	G	G	MB	B	VG
	E_4	G	M	M	M	B	MG	B	VG	VG	B	VB	VG
	E_5	G	MB	M	MB	M	MG	VB	G	VG	B	VB	VG
	E_6	MG	G	MB	VB	VB	VG	VB	MG	MB	MG	VB	VG
A_2	E_1	G	VB	G	VB	VB	VG	VB	VG	VG	VB	MG	VG
	E_2	VG	G	MG	VB	VB	MG	MG	G	MG	MB	B	G
	E_3	G	G	MB	B	B	G	B	VG	VG	VB	B	G
	E_4	VG	MB	M	VB	VB	VG	VB	VG	VG	VB	VB	VG
	E_5	VG	MB	MG	VB	MB	G	B	VG	G	VB	B	VG
	E_6	G	G	G	VB	VB	B	G	MG	B	G	VB	VG
A_3	E_1	VG	VB	M	G	G	M	G	B	G	VB	B	M
	E_2	VB	VB	VG	VG	VG	VB	B	B	G	VB	MB	M
	E_3	B	B	M	G	G	B	B	B	MG	VB	VB	MG
	E_4	MB	VB	M	VG	VG	VB	VB	VB	VG	VB	VB	M
	E_5	M	VB	M	B	MB	MG	VB	M	MG	VB	VB	MG
	E_6	B	VB	VG	G	VG	VG	VB	VB	B	VB	VB	MG

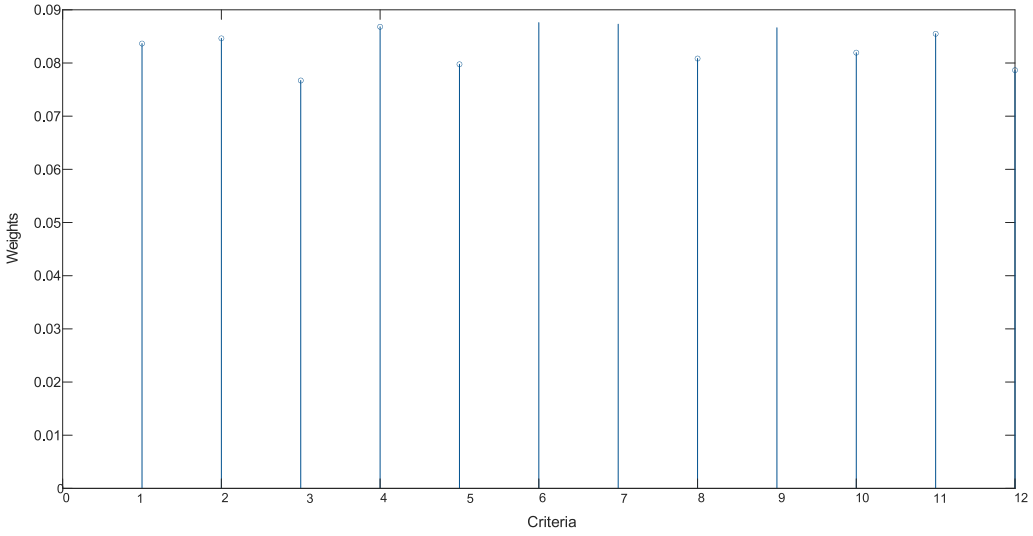


Fig. 5. The weights of each criterion.

Table 7
Weighting coefficients of each criterion.

	Aggregated T2NNs of the criteria	Score values	Normalized values
C_1	<(0.68, 0.65, 0.63), (0, 0, 0), (0, 0, 0)>	0.885	0.084
C_2	<(0.73, 0.68, 0.66), (0, 0, 0), (0, 0, 0)>	0.895	0.085
C_3	<(0.47, 0.43, 0.41), (0, 0, 0), (0, 0, 0)>	0.811	0.077
C_4	<(0.78, 0.75, 0.75), (0, 0, 0), (0, 0, 0)>	0.918	0.087
C_5	<(0.56, 0.53, 0.5), (0, 0, 0), (0, 0, 0)>	0.843	0.080
C_6	<(0.78, 0.76, 0.81), (0, 0, 0), (0, 0, 0)>	0.926	0.088
C_7	<(0.79, 0.76, 0.78), (0, 0, 0), (0, 0, 0)>	0.923	0.087
C_8	<(0.61, 0.56, 0.52), (0, 0, 0), (0, 0, 0)>	0.855	0.081
C_9	<(0.76, 0.74, 0.76), (0, 0, 0), (0, 0, 0)>	0.916	0.087
C_{10}	<(0.62, 0.6, 0.57), (0, 0, 0), (0, 0, 0)>	0.867	0.082
C_{11}	<(0.74, 0.71, 0.7), (0, 0, 0), (0, 0, 0)>	0.904	0.085
C_{12}	<(0.48, 0.51, 0.5), (0, 0, 0), (0, 0, 0)>	0.832	0.079

ranked in $\wp_2 > \wp_1 > \wp_3$. It can be seen that $\wp_2$ is the most suitable alternative.

5.2. Validation of the results

In this section, the results of the proposed T2NNs based RAFSI model are tested and validated by comparing it with T2NNs based TOPSIS (Abdel-Basset et al., 2019) and T2NN based CoCoSo (Deveci et al., 2022). The ranking results of each methodology are shown in Fig. 6, and are reported in Table 12. The ranking of alternatives obtained from each methodology is $A_2 > A_1 > A_3$. Each T2NNs based model obtained A_2 to be the most suitable alternative while A_3 is the less suitable alternative for the three methodologies.

6. Results and discussion

All the methods verified the same solution that the implementation of EVs directly is best to start immediately although it may cause a higher consumption of fossil fuels in the short run. The second alternative has the highest rank, while the first alternative is the second-best

Table 9
The standardized normalized decision matrix.

Alternatives	C_1	C_2	C_3	C_4	C_5	C_6
A_1	4.882	4.342	2.578	3.488	3.009	5.418
A_2	5.633	4.679	3.762	2.041	2.457	5.365
A_3	1.933	2.041	3.863	5.757	5.889	1.807
Type	Max	Min	Max	Min	Min	Max

Alternatives	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
A_1	2.249	5.638	5.750	3.620	3.082	4.959
A_2	3.491	5.825	5.783	2.942	5.019	5.452
A_3	2.942	2.733	4.678	1.833	3.914	2.551
Type	Min	Max	Max	Min	Min	Max

Table 10
The normalized matrix.

Alternatives	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.697	0.197	0.368	0.246	0.285	0.774
A_2	0.805	0.183	0.537	0.420	0.349	0.766
A_3	0.276	0.420	0.552	0.149	0.146	0.258

Alternatives	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
A_1	0.381	0.805	0.821	0.237	0.278	0.708
A_2	0.245	0.832	0.826	0.291	0.171	0.779
A_3	0.291	0.390	0.668	0.468	0.219	0.364

Table 11
The overall values of each alternative.

Alternatives	θ_i	Rank
A_1	0.484	2
A_2	0.515	1
A_3	0.349	3

Table 8
The set of ideal and anti-ideal values.

Alternatives	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
Ideal	0.9	0.7	0.9	0.7	0.7	0.9	0.7	0.9	0.9	0.7	0.7	0.9
Anti-ideal	0.8	0.9	0.8	0.9	0.9	0.8	0.9	0.7	0.8	0.9	0.8	0.8

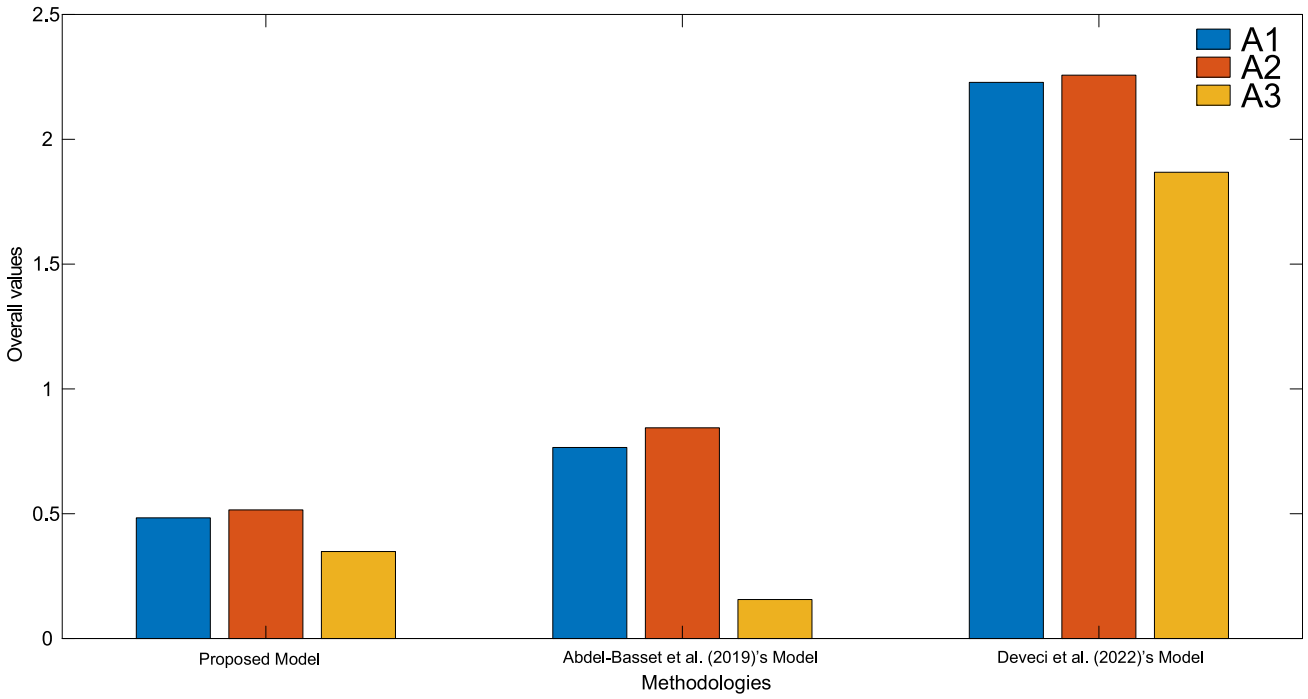


Fig. 6. The comparison of the proposed model with existing T2NN based MCDM models.

Table 12
The ranking results of comparative analysis.

	Proposed model		Abdel-Basset et al. (2019)		Deveci et al. (2022)	
	Overall values	Rank	Overall values	Rank	Overall values	Rank
A_1	0.484	2	0.766	2	2.228	2
A_2	0.515	1	0.844	1	2.257	1
A_3	0.349	3	0.156	3	1.868	3

option, and the third one is the least favorable.

The results put forward that even in the case where energy production is highly dependent on fossil fuels, the investments should be directed to the implementation of EVs at the first hand. Although environmental concerns are the prior reasons behind the support given to the development of EVs, the benefits can be agreed to be utilized in the long run.

The investments that are made to improve renewable energy production are highly important not just for the transportation industry but for the whole economy. It is inevitable to accelerate renewable energy production for all the countries, taken the depleting fossil fuel resources of the world. However, the results of the model imply that it should be already an ongoing process while societies are adapting themselves to the new environment-friendly technologies socially, culturally, and economically.

The regulations can be temporary solutions in transition periods to encourage stakeholders to relinquish traditional systems and internalize environment-friendly technologies. Yet, the results hint at the direct implementation of the new technology is significantly prioritized for the model.

As a result, the findings of the model suggest that the direct imposition of EVs is prioritized over the other alternatives that may temporarily suspend the implementation process in developing countries. Even if the short-run costs may exceed the short-run benefits, the environmental concerns are serious enough so that the expected long-run benefits override the short-run losses.

7. Policy implications

The enormous increase in energy demand since the beginning of industrialization caused the Western world finally to search for non-fossil energy sources. After all, the reallocation trials of the fossil fuel resources of the world had led to drastic consequences such as the world wars or the Cold War period. The recent urges to take climate action accelerated the efforts in producing energy from renewables. Unfortunately, the developing world is not ready to invest in renewable energy resources, yet. We are all in the same boat, and the countries, whether developed or not, have to take climate action. However, the implementation of environment-friendly solutions, such as the use of EVs, can create some additional problems due to the lack of required infrastructure. In this study, we analyze a decision-making process for implementing EVs in a developing country in which energy production is mainly from fossil fuels. If it is a fact that the environmental benefits of the new technology can be only utilized in the long run, could it be still in the best interest of a country to implement it directly? The results of the model give a significant positive answer to that question.

The results imply that alternatives that may improve the short-run environmental benefits of the use of EVs are seen as the delaying of the implementation process. Renewable energy production is highly important, and must be done but it should be prioritized over the direct implementation process.

The implementation of EVs is already a challenging process. It requires a gentle planning procedure that includes macro-and microeconomic concerns. The financial investments, which are economy-wide, are supposed to be supported by the government. Studies show that manufacturers of EVs as well as the other stakeholders in the automotive industry are better to be financially supported at the first stages of the transition period (Gao and Zhang, 2022) However, excessive subsidies may cause uncontrollable imperfections and market failures (Ye et al., 2021). In other words, careful industrial policies have to be designed and then practiced. Moreover, the consumers' preferences may tend to stick to traditional vehicles depending on the lifestyle and demography of the country (Huang et al., 2021b; Bas et al., 2021). Therefore, first, consumer tastes should be tested and then strong promotions should be provided to the consumers to convince them to use EVs. Even in some

cases, traditional vehicles can be adapted to new technologies, such as e-rickshaws in India (Singh et al., 2021). This kind of hybrid solution that brings traditional and new technology together can be much more easily internalized. Yet, it may take some time for consumers to be convinced about the benefits of EVs. From both the consumer and producer sides, there are various obstacles to solve. Industry-wide transformation of the business models may be required.

The implementation of EVs also transforms a lot of other industries other than automotive, such as the transportation networks and the logistics sector (Monios and Bergqvist, 2020). Similarly, the vertical markets are also directly affected by the transition. The production of power systems to charge EVs, or hydrogen production, requires huge investments that cannot be financed by the government's budgets. Therefore, it may take some time before authorities to come up with an appropriate financing system such as a public-private partnership (Sheng et al., 2020).

All in all, the implementation process of EVs is not easy and takes time. The results that we obtained in this study show that the experts are aware of the process. In a developing country that is ready to use the opportunity window opened by a newly introduced technology, suspension of the process is not a favorable option against all the odds. Indeed, the short-run environmental benefits of EVs can be increased through further use of circular economy concepts such as recycling power batteries (Chen et al., 2022). Moreover, the economic and environmental long-run benefits are both so strong that it is worth taking the short-run losses.

8. Conclusion

In this study, we explore the decision-making process for implementing EVs in a developing country where energy is mainly produced from fossil fuels. A scenario is developed to analyze the decision-making process, where three alternatives are considered. The alternatives are to focus on the timing of implementation of the EVs considering the short-run environmental costs can exceed the benefits of EVs. The results of the model show that even though fossil fuel use may increase in the short-run period, the direct implementation of the new system is prioritized. The policy implications of the results are discussed in this study. The long-run benefit expectations are so strong that the short-run losses can be tolerated. Moreover, the implementation process is already challenging, and it requires a specific amount of time and financial investment. Renewable energy production should be invested but it shouldn't slow down the direct implementation of EVs.

Here, we consider only the timing of the implementation process of EVs taking the probable short-run losses into account. When electricity is produced by fossil fuels, such losses are inevitable. The policy-making process regarding the adaptation process is a fertile research area. How the technological acceptance pace of society can be evaluated in the process is also an interesting dimension to be included in a model. Also, different agent-based models can be considered in the decision-making process (Mehdizadeh et al., 2022). Another future work would be contributing to the strategic management literature by studying how the innovation policy mixes can be used for the implementation of EVs (Phirouzabadi et al., 2022). The limitations of the study are that an expert cannot evaluate an alternative equally in terms of all criteria. Therefore, the criteria can be grouped under the main criteria, and the decision-making model can be run by giving weight to the experts for these groups.

CRediT authorship contribution statement

Ilgin Gokasar: Conceptualization, Data curation, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Visualization. **Muhammet Deveci:** Methodology, Software, Validation, Formal analysis, Writing – original draft, Writing – review & editing, Visualization. **Mehtap Isik:** Conceptualization, Data curation, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Visualization. **Tugrul Daim:** Supervision, Validation, Writing – original draft, Writing – review & editing. **Aws A. Zaidan:** Supervision, Validation, Writing – original draft, Writing – review & editing. **Florentin Smarandache:** Supervision, Validation, Writing – original draft, Writing – review & editing.

Data availability

The data that has been used is confidential.

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References

Abdel-Basset, M., Saleh, M., Gamal, A., Smarandache, F., 2019. An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Appl. Soft Comput. 77, 438–452.
Abdel-Basset, M., Mostafa, N.N., Sallam, K.M., Elgendi, I., Munasinghe, K., 2022. Enhanced COVID-19 X-ray image preprocessing schema using type-2 neutrosophic set. Appl. Soft Comput. 123, 108948.
Abdmouleh, Z., Alammari, R.A., Gastli, A., 2015. Review of policies encouraging renewable energy integration & best practices. Renew. Sust. Energ. Rev. 45, 249–262.
Adu-Gyamfi, G., Song, H., Asamoah, A.N., Li, L., Nketiah, E., Obuobi, B., Cudjoe, D., 2022. Towards sustainable vehicular transport: empirical assessment of battery swap technology adoption in China. Technol. Forecast. Soc. Chang. 184, 121995.
Alagappan, L., Orans, R., Woo, C.K., 2011. What drives renewable energy development? Energy Policy 39 (9), 5099–5104.
Alghoul, M.A., Hammadi, F.Y., Amin, N., Asim, N., 2018. The role of existing infrastructure of fuel stations in deploying solar charging systems, electric vehicles and solar energy: a preliminary analysis. Technol. Forecast. Soc. Chang. 137, 317–326.
Alosta, A., Elmansuri, O., Badi, I., 2021. Resolving a location selection problem by means of an integrated AHP-RAFSI approach. Rep. Mech. Eng. 2 (1), 135–142.
Altenburg, T., Corrocher, N., Malerba, F., 2022. China's leapfrogging in electromobility. A story of green transformation driving catch-up and competitive advantage. Technol. Forecast. Soc. Chang. 183, 121914.
Atanassov, K., 1986. Intuitionistic fuzzy sets. fuzzy sets and systems 20 (1), 87–96.
Babar, A.H.K., Ali, Y., 2021. Enhancement of electric vehicles' market competitiveness using fuzzy quality function deployment. Technol. Forecast. Soc. Chang. 167, 120738.
Ball, C.S., Vögele, S., Grajewski, M., Kuckshinrichs, W., 2021. E-mobility from a multi-actor point of view: uncertainties and their impacts. Technol. Forecast. Soc. Chang. 170, 120925.
Bas, J., Cirillo, C., Cherchi, E., 2021. Classification of potential electric vehicle purchasers: a machine learning approach. Technol. Forecast. Soc. Chang. 168, 120759.
Becker, T.A., Sidhu, I., Tenderich, B., 2009. Electric Vehicles in the United States: A New Model With Forecasts to 2030. Center for Entrepreneurship and Technology, University of California, Berkeley, p. 24.
Benzidia, S., Luca, R.M., Boiko, S., 2021. Disruptive innovation, business models, and encroachment strategies: Buyer's perspective on electric and hybrid vehicle technology. Technol. Forecast. Soc. Chang. 165, 120520.
Broumi, S., Mohanaselvi, S., Witczak, T., Talea, M., Bakali, A., Smarandache, F., 2023. Complex fermatean neutrosophic graph and application to decision making. Decis. Mak.: Appl. Manag. Eng. 6 (1), 474–501.
Chen, J., Zhang, W., Gong, B., Zhang, X., Li, H., 2022. Optimal policy for the recycling of electric vehicle retired power batteries. Technol. Forecast. Soc. Chang. 183, 121930.
Ciccone, A., 2018. Environmental effects of a vehicle tax reform: empirical evidence from Norway. Transp. Policy 69, 141–157.
de Paulo, A.F., Nunes, B., Porto, G., 2020. Emerging green technologies for vehicle propulsion systems. Technol. Forecast. Soc. Chang. 159, 120054.
Deng, M., Wang, L., Vaghefinazari, B., Xu, W., Feiler, C., Lamaka, S.V., Snihirova, D., 2021. High-energy and durable aqueous magnesium batteries: recent advances and perspectives. Energy Storage Mater. 43, 238–247.
Deuten, S., Vilchez, J.J.G., Thiel, C., 2020. Analysis and testing of electric car incentive scenarios in the Netherlands and Norway. Technol. Forecast. Soc. Chang. 151, 119847.
Deveci, M., Pamucar, D., Gokasar, I., Delen, D., Wu, Q., Simic, V., 2022. An analytics approach to decision alternative prioritization for zero-emission zone logistics. J. Bus. Res. 146, 554–570.
Dinçer, H., Yüksel, S., Martínez, L., 2022. Collaboration enhanced hybrid fuzzy decision-making approach to analyze the renewable energy investment projects. Energy Rep. 8, 377–389.
Donbosco, J.S.M., Ganesan, D., 2022. The energy of rough neutrosophic matrix and its application to MCDM problem for selecting the best building construction site. Decis. Mak. Appl. Manag. Eng. 5 (2), 30–45.

EEA, 1996. Environmental Taxes Implementation and Environmental Effectiveness. Retrieved on April 2023 from. https://www.eca.europa.eu/.
EIA, 2008 April. Federal Financial Interventions and Subsidies in Energy Markets 2007. https://www.eia.gov/analysis/requests/2008/subsidy2/pdf/subsidy08.pdf.
Featherman, M., Jia, S.J., Califf, C.B., Hajli, N., 2021. The impact of new technologies on consumers beliefs: reducing the perceived risks of electric vehicle adoption. Technol. Forecast. Soc. Chang. 169, 120847.
Funke, S.Á., Sprei, F., Gnann, T., Plötz, P., 2019. How much charging infrastructure do electric vehicles need? A review of the evidence and international comparison. Transp. Res. Part D: Transp. Environ. 77, 224–242.
Gao, J., Zhang, T., 2022. Effects of public funding on the commercial diffusion of on-site hydrogen production technology: a system dynamics perspective. Technol. Forecast. Soc. Chang. 175, 121380.
Goel, P., Sharma, N., Mathiyazhagan, K., Vimal, K.E.K., 2021. Government is trying but consumers are not buying: a barrier analysis for electric vehicle sales in India. Sustain. Prod. Consum. 28, 71–90.
Gopal, N., Panchal, D., 2021. A structured framework for reliability and risk evaluation in the milk process industry under fuzzy environment. Facta Univ. Ser.: Mech. Eng. 19 (2), 307–333.
Goulder, L.H., 2020. Timing is everything: how economists can better address the urgency of stronger climate policy. Rev. Environ. Econ. Policy.
Hain, D.S., Jurowetzki, R., Buchmann, T., Wolf, P., 2022. A text-embedding-based approach to measuring patent-to-patent technological similarity. Technol. Forecast. Soc. Chang. 177, 121559.
Huang, Y., Qian, L., Soopramanien, D., Tyfield, D., 2021a. Buy, lease, or share? Consumer preferences for innovative business models in the market for electric vehicles. Technol. Forecast. Soc. Chang. 166, 120639.
Huang, Y., Qian, L., Tyfield, D., Soopramanien, D., 2021b. On the heterogeneity in consumer preferences for electric vehicles across generations and cities in China. Technol. Forecast. Soc. Chang. 167, 120687.
Jaiswal, D., Kaushal, V., Kant, R., Singh, P.K., 2021. Consumer adoption intention for electric vehicles: insights and evidence from indian sustainable transportation. Technol. Forecast. Soc. Chang. 173, 121089.
Jiang, H.D., Xue, M.M., Liang, Q.M., Masui, T., Ren, Z.Y., 2022. How do demand-side policies contribute to the electrification and decarburization of private transportation in China? A CGE-based analysis. Technol. Forecast. Soc. Chang. 175, 121322.
Keohane, N., Petsonk, A., Hanafi, A., 2017. Toward a club of carbon markets. Clim. Chang. 144, 81–95.
Kishita, Y., Mizuno, Y., Fukushige, S., Umeda, Y., 2020. Scenario structuring methodology for computer-aided scenario design: an application to envisioning sustainable futures. Technol. Forecast. Soc. Chang. 160, 120207.
Köhler, J., Turnheim, B., Hodson, M., 2020. Low carbon transitions pathways in mobility: applying the MLP in a combined case study and simulation bridging analysis of passenger transport in the Netherlands. Technol. Forecast. Soc. Chang. 151, 119314.
Krawinkler, A., Breitenecker, R.J., Maresch, D., 2022. Heuristic decision-making in the green energy context: bringing together simple rules and data-driven mathematical optimization. Technol. Forecast. Soc. Chang. 180, 121695.
Kwon, Y., Son, S., Jang, K., 2018. Evaluation of incentive policies for electric vehicles: an experimental study on Jeju Island. Transp. Res. A Policy Pract. 116, 404–412.
Lahsen, M., 2013. Climategate: the role of the social sciences. Clim. Chang. 119 (3-4), 547–558.
Li, M., Porter, A.L., Suominen, A., Burmaoglu, S., Carley, S., 2021. An exploratory perspective to measure the emergence degree for a specific technology based on the philosophy of swarm intelligence. Technol. Forecast. Soc. Chang. 166, 120621.
Llopis-Albert, C., Palacios-Marqués, D., Simón-Moya, V., 2021b. Fuzzy set qualitative comparative analysis (fsQCA) applied to the adaptation of the automobile industry to meet the emission standards of climate change policies via the deployment of electric vehicles (EVs). Technol. Forecast. Soc. Chang. 169, 120843.
Llopis-Albert, C., Rubio, F., Valero, F., 2021a. Impact of digital transformation on the automotive industry. Technol. Forecast. Soc. Chang. 162, 120343.
Madina, C., Zamora, I., Zabala, E., 2016. Methodology for assessing electric vehicle charging infrastructure business models. Energy Policy 89, 284–293.
Marcos, J.T., Scheller, C., Godina, R., Spengler, T.S., Carvalho, H., 2021. Sources of uncertainty in the closed-loop supply chain of lithium-ion batteries for electric vehicles. Clean. Logist. Supply Chain 1, 100006.
Mehdizadeh, M., Nordfjaern, T., Klöckner, C.A., 2022. A systematic review of the agent-based modelling/simulation paradigm in mobility transition. Technol. Forecast. Soc. Chang. 184, 122011.
Mishra, A.R., Rani, P., Prajapati, R.S., 2021. Multi-criteria weighted aggregated sum product assessment method for sustainable biomass crop selection problem using single-valued neutrosophic sets. Appl. Soft Comput. 113, 108038.
Mohamad, M., Songthaveephol, V., 2020. Clash of titans: the challenges of socio-technical transitions in the electrical vehicle technologies–the case study of Thai automotive industry. Technol. Forecast. Soc. Chang. 153, 119772.
Monios, J., Bergqvist, R., 2020. Logistics and the networked society: a conceptual framework for smart network business models using electric autonomous vehicles (EAVs). Technol. Forecast. Soc. Chang. 151, 119824.
Moon, H., Park, S.Y., Woo, J., 2021. Staying on convention or leapfrogging to eco-innovation?: identifying early adopters of hydrogen-powered vehicles. Technol. Forecast. Soc. Chang. 171, 120995.
OECD, 1996. Economic/Fiscal Instruments: Taxation (i.e., Carbon/Energy). Report retrieved on April, 2023 from. https://oecd.org/env.
Onat, N.C., Kucukvar, M., 2022. A systematic review on sustainability assessment of electric vehicles: knowledge gaps and future perspectives. Environ. Impact Assess. Rev. 97, 106867.
Park, C., Lim, S., Shin, J., Lee, C.Y., 2022. How much hydrogen should be supplied in the transportation market? Focusing on hydrogen fuel cell vehicle demand in South Korea: hydrogen demand and fuel cell vehicles in South Korea. Technol. Forecast. Soc. Chang. 181, 121750.
Parry, I.W.H., Small, K.A., 2005. Does Britain or the United States have the right gasoline tax? Am. Econ. Rev. 95 (4), 1276–1289.
Pereira, G.I., Niesten, E., Pinkse, J., 2022. Sustainable energy systems in the making: a study on business model adaptation in incumbent utilities. Technol. Forecast. Soc. Chang. 174, 121207.
Phirouzabadi, A.M., Blackmore, K., Savage, D., Juniper, J., 2022. Modelling and simulating a multi-modal and multi-dimensional technology interaction framework: the case of vehicle powertrain technologies in the US market. Technol. Forecast. Soc. Chang. 175, 121412.
Raven, R., Walrave, B., 2020. Overcoming transformational failures through policy mixes in the dynamics of technological innovation systems. Technol. Forecast. Soc. Chang. 153, 119297.
Rinaldi, G., Thies, P.R., Walker, R., Johanning, L., 2017. A decision support model to optimise the operation and maintenance strategies of an offshore renewable energy farm. Ocean Eng. 145, 250–262.
Sahin, Rahmi, 2022. Evaluation of the Environmental Effects of Autonomous Vehicles in Traffic Incident Scenarios on Uninterrupted Facilities. M.S. Thesis.
Sen, S., Ganguly, S., 2017. Opportunities, barriers and issues with renewable energy development–a discussion. Renew. Sust. Energ. Rev. 69, 1170–1181.
Shakouri, B., Khoshnevis Yazdi, S., 2017. Causality between renewable energy, energy consumption, and economic growth. Energy Sources, Part B: Econ. Plan. Policy 12 (9), 838–845.
Sheng, M., Sreenivasan, A.V., Sharp, B., Wilson, D., Ranjitkar, P., 2020. Economic analysis of dynamic inductive power transfer roadway charging system under public-private partnership–evidence from New Zealand. Technol. Forecast. Soc. Chang. 154, 119958.
Singh, R., Mishra, S., Tripathi, K., 2021. Analysing acceptability of E-rickshaw as a public transport innovation in Delhi: a responsible innovation perspective. Technol. Forecast. Soc. Chang. 170, 120908.
Smarandache, F., 1998. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis.
Steffen, B., Patt, A., 2022. A historical turning point? Early evidence on how the Russia-Ukraine war changes public support for clean energy policies. Energy Res. Soc. Sci. 91, 102758.
Stern, N., Stern, N.H., 2007. The Economics of Climate Change: The Stern Review. Cambridge University Press.
Tamba, M., Krause, J., Weitzel, M., Ioan, R., Duboz, L., Grosso, M., Vandyck, T., 2022. Economy-wide impacts of road transport electrification in the EU. Technol. Forecast. Soc. Chang. 182, 121803.
Tchetchik, A., Zvi, L.I., Kaplan, S., Blass, V., 2020. The joint effects of driving hedonism and trialability on the choice between internal combustion engine, hybrid, and electric vehicles. Technol. Forecast. Soc. Chang. 151, 119815.
Wesseling, J.H., Bidmon, C., Bohnsack, R., 2020. Business model design spaces in socio-technical transitions: the case of electric driving in the Netherlands. Technol. Forecast. Soc. Chang. 154, 119950.
Wu, Z., Shao, Q., Su, Y., Zhang, D., 2021. A socio-technical transition path for new energy vehicles in China: a multi-level perspective. Technol. Forecast. Soc. Chang. 172, 121007.
Xu, X., Wei, Z., Ji, Q., Wang, C., Gao, G., 2019. Global renewable energy development: influencing factors, trend predictions and countermeasures. Res. Policy 63, 101470.
Ye, R.K., Gao, Z.F., Fang, K., Liu, K.L., Chen, J.W., 2021. Moving from subsidy stimulation to endogenous development: a system dynamics analysis of China's NEVs in the post-subsidy era. Technol. Forecast. Soc. Chang. 168, 120757.
Yuan, X., Cai, Y., 2021. Forecasting the development trend of low emission vehicle technologies: based on patent data. Technol. Forecast. Soc. Chang. 166, 120651.
Yuan, X., Li, X., 2020. A network analytic method for measuring patent thickets: a case of FCEV technology. Technol. Forecast. Soc. Chang. 156, 120038.
Zadeh, L.A., 1965. Fuzzy sets. Inf. Control. 8 (1965), 338–353.
Zhou, B., Chen, J., Wu, Q., Pamučar, D., Wang, W., Zhou, L., 2022. Risk priority evaluation of power transformer parts based on hybrid FMEA framework under hesitant fuzzy environment. Facta Univ. Ser.: Mech. Eng. 20 (2), 399–420.
Žižović, M., Pamučar, D., Albijanić, M., Chatterjee, P., Pribićević, I., 2020. Eliminating rank reversal problem using a new multi-attribute model—The RAFSI method. Mathematics 8 (6), 1015.

Investigation of a neutrosophic group

A. Elrawy,[a] **Florentin Smarandache,**[b] **Ayat A. Temraz**[a]

[a] *Department of Mathematics, Faculty of Science, SVU, Qena 83523, Egypt.*

[b] *University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA.*

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Abstract. We use a neutrosophic set, instead of an intuitionistic fuzzy because the neutrosophic set is more general, and it allows for independent and partial independent components $\mu(\chi), \gamma(\chi), \zeta(\chi)$, while in an intuitionistic fuzzy set, all components are totally dependent. In this article, we present and demonstrate the concept of neutrosophic invariant subgroups. We delve into the exploration of this notion to establish and study the neutrosophic quotient group. Further, we give the concept of a neutrosophic normal subgroup as a novel concept.

Keywords: Neutrosophic set, invariant sub-groups, normal sub-group.

1. Introduction

In dealing with many uncertainties that have emerged in a variety of real-life domains, such as sociology, economics, medical research, and the environment, traditional mathematical tools may not be suitable. Despite being well-known and frequently helpful methods for describing uncertainty. Therefore, Zadeh [21] initially introduced the notion of a fuzzy set as an alternative to classical sets, aiming to address the limitations in handling uncertainties. According to this definition, a fuzzy set is a function that assigns membership values graded over a unit interval. However, it has been recognized that this definition falls short when considering both membership and non-membership degrees. In order to tackle this inherent ambiguity, Atanassov [1] introduced a new theory known as intuitionistic fuzzy theory, which extends the concept of fuzzy sets. Nevertheless, the application of intuitionistic fuzzy sets has encountered certain challenges. To overcome issues related to ambiguity and inconsistency in information, Smarandache [17,16] proposed the concept of a neutrosophic set (NS). This novel approach aims to provide a solution to the problems arising from uncertain and inconsistent data.

Rosenfeld [13], Mukherjee [10], Biswas [2], Fathi et al. [5], Marashdeh et al. [8], Sharma [15], Smarandache et al. [6,7], Elrawy et al. [4], and several others extended the classical group theory to the fuzzy set, intuitionistic fuzzy set, and neutrosophic set. This is supported by multiple authors who applied the theory of fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets to various algebraic structures.

The intersection and union relations of neutrosophic sets has been discussed so far from three main perspectives. The first definitions are offered by Smarandache [16,18,19] and are signified by $\cap_1$ and $\cup_1$. The sec-

ond definitions are given in [20] and are signified by $\cap_2$ and $\cup_2$. The third definitions are offered in [22] and are signified by $\cap_3$ and $\cup_3$. In addition, Vildan et al. [3] presented a new approach to a neutrosophic sub-group that based on the second viewpoint. Moreover, El Rawy et al. [4] have established and studied another approach of a neutrosophic sub-group and level sub-group based on the first viewpoint. We have improved the definition of the neutrosophic subgroup in [3]. Since in paper [3] the authors use $\min / \min / \max$ and $\geqslant / \geqslant / \leqslant$ while we use $\min / \max / \max$ and $\geqslant / \leqslant / \leqslant$, respectively. Our approach is better since the components: μ is considered of positive quality, while γ and ζ of negative quality, so γ and ζ should have the same operation $\max / \max$ and $\leqslant / \leqslant$, respectively.

Motivated by some of the above work, the goal of this paper is to examine the neutrosophic sub-group and normal sub-group through the study and investigation of various relevant properties and theorems. In the next section, we afford some basic, important definitions for this study. In section 3, the notion of neutrosophic invariant sub-groups is presented and studied. In section 4, we define left and right neutrosophic coset. Also, neutrosophic normal sub-group is established and studied. In the last section, we draw some conclusions.

2. Basic concepts

We discuss a few concepts and results that we utilize in the following section in this section.

Definition 2.1 *[17,20] Let E be the universe set. An NS $\mathcal{N}$ on E is defined as:*

$$\mathcal{N} = \{< \chi, \mu(\chi), \gamma(\chi), \zeta(\chi) >: \chi \in E\},$$

with $\mu, \gamma, \zeta : E \to]0, 1[$. Also, an NS $\mathcal{N}$ on E is called single valued neutrosophic set for any $\chi \in E$ one has $0 \leqslant \mu(\chi) + \gamma(\chi) + \zeta(\chi) \leqslant 3$ and of course $\mu(\chi), \gamma(\chi), \zeta(\chi) \in [0, 1]$.

Definition 2.2 *[16,18,19] Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two NSs on E. Then we define the follows:*

1. $\mathcal{N}_1 \cup_1 \mathcal{N}_2 = \{< \chi, max(\mu_1(\chi), \mu_2(\chi)), min(\gamma_1(\chi), \gamma_2(\chi)), min(\zeta_1(\chi), \zeta_2(\chi)) >: \chi \in E\}$,
2. $\mathcal{N}_1 \cap_1 \mathcal{N}_2 = \{< \chi, min(\mu_1(\chi), \mu_2(\chi)), max(\gamma_1(\chi), \gamma_2(\chi)), max(\zeta_1(\chi), \zeta_2(\chi)) >: \chi \in E\}$.

Definition 2.3 *[4] Let $\mathcal{G}$ be a group. A neutrosophic subset $\Upsilon = \{< \chi, \mu(\chi), \gamma(\chi), \zeta(\chi) >: \chi \in \mathcal{G}\}$ of $\mathcal{G}$ is called a neutrosophic sub-group of $\mathcal{G}$ if the following conditions are satisfied:*

$$(i) \begin{cases} \mu(\chi\varsigma) \geqslant \min(\mu(\chi), \mu(\varsigma)) \\ \gamma(\chi\varsigma) \leqslant \max(\gamma(\chi), \gamma(\varsigma)) \\ \zeta(\chi\varsigma) \leqslant \max(\zeta(\chi), \zeta(\varsigma)) \end{cases}$$

$$(ii) \begin{cases} \mu\left(\chi^{-1}\right) \geqslant \mu(\chi) \\ \gamma\left(\chi^{-1}\right) \leqslant \gamma(\chi) \\ \zeta\left(\chi^{-1}\right) \leqslant \zeta(\chi) \end{cases}$$

where $\chi, \varsigma \in \mathcal{G}$.

In what follows $\mathcal{S}(\mathcal{G})$ will denote a family of all neutrosophic sets in $\mathcal{G}$.

3. Neutrosophic invariant sub-groups

In this section, we define and study the concept of neutrosophic invariant sub-groups.

It is useful to present the next definition before introduce the notion of neutrosophic invariant sub-groups.

Definition 3.1 *Let $\mathcal{G}$ be a group with a binary operation $\star$, then we define $\Upsilon_1 * \Upsilon_2$ as follows*

$$(\mu_1 * \mu_2)(u) = \begin{cases} \sup_{v\star w=u} min(\mu_1(v), \mu_2(w)) & \text{if } v \star w = u \\ 0 & \text{if } v \star w \neq u \end{cases}$$

$$(\gamma_1 * \gamma_2)(u) = \begin{cases} \inf_{v\star w=u} max(\gamma_1(v), \gamma_2(w)) & \text{if } v \star w = u \\ 0 & \text{if } v \star w \neq u \end{cases}$$

$$(\zeta_1 * \zeta_2)(u) = \begin{cases} \inf_{v\star w=u} max(\zeta_1(v), \zeta_2(w)) & \text{if } v \star w = u \\ 0 & \text{if } v \star w \neq u \end{cases}$$

where $u, v, w \in \mathcal{G}$,
$\Upsilon_1 = \{< \chi, \mu_1(\chi), \gamma_1(\chi), \zeta_1(\chi) >: \chi \in \mathcal{G}\}$ *and* $\Upsilon_2 = \{< \chi, \mu_2(\chi), \gamma_2(\chi), \zeta_2(\chi) >: \chi \in \mathcal{G}\}$.

Definition 3.2 *Let Υ be a neutrosophic sub-group of a group $\mathcal{G}$, then we call Υ a neutrosophic invariant sub-group if*

$$\mu(\chi \star \varsigma) = \mu(\varsigma \star \chi),$$

$$\gamma(\chi \star \varsigma) = \gamma(\varsigma \star \chi),$$

$$\zeta(\chi \star \varsigma) = \zeta(\varsigma \star \chi),$$

where $\chi,\ \varsigma \in \mathcal{G}$.

Note that, when $\mathcal{G}$ is abelian group, then we find every neutrosophic sub-group of a group $\mathcal{G}$ is a neutrosophic invariant sub-group.

Example 3.3 Assume that $\mathcal{G} = Z_3$ is a group under a binary operation $\otimes_3$, Define a neutrosophic sub-group $\Upsilon = \{< 0, 0.7, 0.1, 0.2 >, < 1, 0.6, 0.3, 0.5 >, < 2, 0.6, 0.3, 0.5 >\}$ of Z_3. Since Z_3 is abelian group, the Υ is a neutrosophic invariant sub-group.

Proposition 3.4 *Presume Υ_1 is a neutrosophic invariant sub-group,*
(1) $\Upsilon_1 * \Upsilon_2 = \Upsilon_2 * \Upsilon_1$, *for any* $\Upsilon_2 \in \mathcal{S}(\mathcal{G})$,
(2) *When Υ_2 is a neutrosophic sub-group, then also $\Upsilon_2 * \Upsilon_1$ is a neutrosophic sub-group.*

Proof. (1) It is obvious from Definition 3.2.
(2) Suppose that Υ_2 is a neutrosophic sub-group, then we prove $\Upsilon_2 * \Upsilon_1$ is a neutrosophic sub-group by check all axioms of Definition 2.3 as follows

$$(i)\begin{cases}(\mu_2*\mu_1)(up)=\sup\limits_{v_1v_2\star w_1w_2=up} min(\mu_1(v_1v_2),\mu_2(w_1w_2))\\ \geq \sup\limits_{v_1\star v_2\star w_1\star w_2=up} min(\mu_1(v_1)\mu_1(v_2),\mu_2(w_1)\mu(w_2))\\ =\mu_2(u)*\mu_1(p).\\ (\gamma_2*\gamma_1)(up)=\inf\limits_{v_1v_2\star w_1w_2=up} max(\mu_1(v_1v_2),\mu_2(w_1w_2))\\ \leq \inf\limits_{v_1\star v_2\star w_1\star w_2=up} max(\gamma_1(v_1)\gamma_1(v_2),\gamma_2(w_1)\gamma(w_2))\\ =\gamma_2(u)*\gamma_1(p).\\ (\zeta_2*\zeta_1)(up)=\inf\limits_{v_1v_2\star w_1w_2=up} max(\zeta_1(v_1v_2),\zeta_2(w_1w_2))\\ \leq \inf\limits_{v_1\star v_2\star w_1\star w_2=up} max(\zeta_1(v_1)\zeta_1(v_2),\zeta_2(w_1)\zeta(w_2))\\ =\zeta_2(u)*\zeta_1(p).\end{cases}$$

$$(ii)\begin{cases}(\mu_2*\mu_1)(u^{-1})=\sup\limits_{v\star w=u^{-1}} min(\mu_2(v),\mu_1(w))\\ =\sup\limits_{w^{-1}\star v^{-1}=u} min(\mu_2((v^{-1})^{-1}),\mu_1((w^{-1})^{-1}))\\ \geq \sup\limits_{w^{-1}\star v^{-1}=u} min(\mu_2(v^{-1}),\mu_1(w^{-1}))\\ =(\mu_2*\mu_1)(u^{-1}).\\ (\gamma_2*\gamma_1)(u^{-1})=\inf\limits_{v\star w=u^{-1}} max(\mu_2(v),\mu_1(w))\\ =\inf\limits_{w^{-1}\star v^{-1}=u} max(\gamma_2((v^{-1})^{-1}),\gamma_1((w^{-1})^{-1}))\\ \leq \inf\limits_{w^{-1}\star v^{-1}=u} max(\gamma_2(v^{-1}),\gamma_1(w^{-1}))\\ =(\gamma_2*\gamma_1)(u^{-1}).\\ (\zeta_2*\zeta_1)(u^{-1})=\inf\limits_{v\star w=u^{-1}} max(\zeta_2(v),\zeta_1(w))\\ =\inf\limits_{w^{-1}\star v^{-1}=u} max(\zeta_2((v^{-1})^{-1}),\zeta_1((w^{-1})^{-1}))\\ \leq \inf\limits_{w^{-1}\star v^{-1}=u} max(\zeta_2(v^{-1}),\zeta_1(w^{-1}))\\ =(\zeta_2*\zeta_1)(u^{-1}).\end{cases}$$

In what follows, suppose that $e \in \mathcal{G}$ is an identity, and Υ is non-empty set belong to $\mathcal{S}(\mathcal{G})$. Also, we define the set $\mathcal{G}_\Upsilon$ as follows

$$\mathcal{G}_\Upsilon = \{u \in \mathcal{G} : \ \mu(u) = \mu(e), \ \gamma(u) = \gamma(e), \ \zeta(u) = \zeta(e)\}.$$

Proposition 3.5 *Presume Υ is a neutrosophic invariant sub-group of $\mathcal{G}$, then $\mathcal{G}_\Upsilon$ is an invariant sub-group of $\mathcal{G}$.*

Proof. Assume that $\mathcal{G}_\Upsilon$ is a non-empty set. Let $u \in \mathcal{G}_\Upsilon$ and $v \in \mathcal{G}$, then we have

$$\begin{aligned}\mu(vuv^{-1}) &= \mu((vu)v^{-1}) \\ &= \mu((uv)v^{-1}) \\ &= \mu(u(vv^{-1})) = \mu(u) = \mu(e).\end{aligned}$$

Similarly, we find $\gamma(vuv^{-1}) = \gamma(e)$ and $\zeta(vuv^{-1}) = \zeta(e)$. Therefore, $vuv^{-1} \in \mathcal{G}_\Upsilon$.

Next, we investigate a definition of neutrosophic invariant sub-group to define the neutrosophic quotient group.

Definition 3.6 Assume that Υ is a neutrosophic invariant sub-group of $\mathcal{G}$. Then the quotient $\mathcal{G}/\mathcal{G}_\Upsilon$ is called the neutrosophic quotient group of $\mathcal{G}$.

Now, by using the definition of a homomorphism between two groups $\mathcal{F} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$, we have the following definition.

Definition 3.7 Let $\Upsilon_1 \in \mathcal{S}(\mathcal{G}_1)$ and $\Upsilon_2 \in \mathcal{S}(\mathcal{G}_2)$. Then we define $\mathcal{F}(\Upsilon_1)$ in $\mathcal{S}(\mathcal{G}_2)$ as follows

$$\begin{cases} (\mathcal{F}(\mu_1))(\varsigma) = \begin{cases} \sup_{\mathcal{F}(\chi)=\varsigma} \mu_1(\chi) & \text{when } \mathcal{F}^{-1}(\varsigma) \neq \Phi \\ 0 & \text{when } \mathcal{F}^{-1}(\varsigma) = \Phi \end{cases} \\ (\mathcal{F}(\gamma_1))(\varsigma) = \begin{cases} \inf_{\mathcal{F}(\chi)=\varsigma} \gamma_1(\chi) & \text{when } \mathcal{F}^{-1}(\varsigma) \neq \Phi \\ 0 & \text{when } \mathcal{F}^{-1}(\varsigma) = \Phi \end{cases} \\ (\mathcal{F}(\zeta_1))(\varsigma) = \begin{cases} \inf_{\mathcal{F}(\chi)=\varsigma} \zeta_1(\chi) & \text{when } \mathcal{F}^{-1}(\varsigma) \neq \Phi \\ 0 & \text{when } \mathcal{F}^{-1}(\varsigma) = \Phi \end{cases} \end{cases}$$

and we define $\mathcal{F}^{-1}(\Upsilon_2)$ by the following $\mathcal{F}^{-1}(\Upsilon_2)(\chi) = \Upsilon_2(\mathcal{F}(\chi))$, where $\chi \in \mathcal{G}_1$ as a fuzzy set [12].

Again, we suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are groups and $\mathcal{F} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ is an onto group homomorphism, then we get the following:
When Υ_1 is a neutrosophic sub-group of $\mathcal{G}_1$, then also $\mathcal{F}(\Upsilon_1)$ is a neutrosophic sub-group of $\mathcal{G}_2$ for all $v \in \mathcal{G}_2$.

$$
\begin{aligned}
(\mathcal{F}(\mu_1))(v^{-1}) &= \sup_{\mathcal{F}(w)=v^{-1}} \mu_1(w) \\
&= \sup_{\mathcal{F}(w^{-1})=v} \mu_1((w^{-1})^{-1}) \\
&\geq \sup_{\mathcal{F}(w^{-1})=v} \mu_1(w^{-1}) \\
&= \sup_{\mathcal{F}(u)=v} \mu_1(u) = (\mathcal{F}(\mu_1))(v). \\
(\mathcal{F}(\gamma_1))(v^{-1}) &= \inf_{\mathcal{F}(w)=v^{-1}} \gamma_1(w) \\
&= \inf_{\mathcal{F}(w^{-1})=v} \gamma_1((w^{-1})^{-1}) \\
&\leq \inf_{\mathcal{F}(w^{-1})=v} \gamma_1(w^{-1}) \\
&= \inf_{\mathcal{F}(u)=v} \gamma_1(u) = (\mathcal{F}(\gamma_1))(v). \\
(\mathcal{F}(\zeta_1))(v^{-1}) &= \inf_{\mathcal{F}(w)=v^{-1}} \zeta_1(w) \\
&= \inf_{\mathcal{F}(w^{-1})=v} \zeta_1((w^{-1})^{-1}) \\
&\leq \inf_{\mathcal{F}(w^{-1})=v} \zeta_1(w^{-1}) \\
&= \inf_{\mathcal{F}(u)=v} \zeta_1(u) = (\mathcal{F}(\zeta_1))(v).
\end{aligned}
$$

The same way we have $\mathcal{F}^{-1}(\Upsilon_2)$ in $\mathcal{G}_1$, when Υ_2 is a neutrosophic sub-group of $\mathcal{G}_2$.

Proposition 3.8 *Assume that $\ell : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}_{\Upsilon_1}$ is a natural map. When Υ_1 is a neutrosophic invariant group of $\mathcal{G}$, then we get $\ell^{-1}(\ell(\Upsilon_2)) = \mathcal{G}_{\Upsilon_1} * \Upsilon_2$*

Proof. Suppose that $u \in \mathcal{G}$, then we get

$$
\begin{aligned}
\ell^{-1}(\ell(\mu_2)) &= (\ell(\mu_2))(\ell(u)) \\
&= \sup_{\ell(v)=\ell(u)} \mu_2(v) \\
&= \sup_{uv^{-1}\in\mathcal{G}_{\Upsilon_1}} \mu_2(v). \\
\ell^{-1}(\ell(\gamma_2)) &= (\ell(\gamma_2))(\ell(u)) \\
&= \inf_{\ell(v)=\ell(u)} \gamma_2(v) \\
&= \inf_{uv^{-1}\in\mathcal{G}_{\Upsilon_1}} \gamma_2(v). \\
\ell^{-1}(\ell(\zeta_2)) &= (\ell(\zeta_2))(\ell(u)) \\
&= \inf_{\ell(v)=\ell(u)} \zeta_2(v) \\
&= \inf_{uv^{-1}\in\mathcal{G}_{\Upsilon_1}} \zeta_2(v).
\end{aligned}
$$

Also we have

$$
\begin{aligned}
(\mathcal{G}_{\mu_1} * \mu_2)(u) &= \sup_{v\star w=u} min(\mathcal{G}_{\mu_1}(w), \mu_2(v)) \\
&= \sup_{w=uv^{-1}\in\mathcal{G}_{\mu_1}} \mu_2(v). \\
(\mathcal{G}_{\gamma_1} * \gamma_2)(u) &= \inf_{v\star w=u} min(\mathcal{G}_{\gamma_1}(w), \gamma_2(v)) \\
&= \inf_{w=uv^{-1}\in\mathcal{G}_{\gamma_1}} \gamma_2(v). \\
(\mathcal{G}_{\zeta_1} * \zeta_2)(u) &= \inf_{v\star w=u} min(\mathcal{G}_{\zeta_1}(w), \zeta_2(v)) \\
&= \inf_{w=uv^{-1}\in\mathcal{G}_{\zeta_1}} \zeta_2(v).
\end{aligned}
$$

Therefore, the proposition is proved.

4. Neutrosophic normal sub-group

In this section, we investigate the concept of a neutrosophic sub-group [4] to define left and right neutrosophic cosets. Also, we define and study the neutro-

sophic normal sub-group.
Firstly, it is useful to define and study the right and left neutrosophic cosets.

Definition 4.1 Let Υ be a neutrosophic set of $\mathcal{G}$ and we define the functions ${}_{\chi}\Gamma : \mathcal{G} \longrightarrow \mathcal{G}$ and $\Gamma_{\chi} : \mathcal{G} \longrightarrow \mathcal{G}$ where ${}_{\chi}\Gamma(\varsigma) = \chi\varsigma$ and $\Gamma_{\chi}(\varsigma) = \varsigma\chi$, receptively. A neutrosophic right (left) coset is defined as follows

$$\begin{cases} \mu\chi = \Gamma_{\chi}(\mu) \ (\chi\mu =_{\chi} \Gamma(\mu)) \\ \gamma\chi = \Gamma_{\chi}(\gamma) \ (\chi\gamma =_{\chi} \Gamma(\gamma)) \\ \zeta\chi = \Gamma_{\chi}(\zeta) \ (\chi\zeta =_{\chi} \Gamma(\zeta)) \end{cases}$$

where $\chi, \varsigma \in \mathcal{G}$.

Remark 4.2 Clearly, in the case of neutrosophic right and left cosets, we find the following:

$$\begin{cases} (\mu\chi)(\varsigma) = \mu(\varsigma\chi^{-1}) & \text{and } (\chi\mu)(\varsigma) = \mu(\chi^{-1}\varsigma) \\ (\gamma\chi)(\varsigma) = \gamma(\varsigma\chi^{-1}) & \text{and } (\chi\gamma)(\varsigma) = \gamma(\chi^{-1}\varsigma) \\ (\zeta\chi)(\varsigma) = \zeta(\varsigma\chi^{-1}) & \text{and } (\chi\zeta)(\varsigma) = \zeta(\chi^{-1}\varsigma) \end{cases}$$

for each $\chi, \varsigma \in \mathcal{G}$.

Proposition 4.3 *Let Υ be a neutrosophic set of $\mathcal{G}$, then the next axioms are equivalent:*

$$(i) \begin{cases} \mu(\chi\varsigma\chi^{-1}) \geq \mu(\varsigma) \\ \gamma(\chi\varsigma\chi^{-1}) \leq \gamma(\varsigma) \\ \zeta(\chi\varsigma\chi^{-1}) \leq \zeta(\varsigma) \end{cases}$$

$$(ii) \begin{cases} \mu(\chi\varsigma\chi^{-1}) = \mu(\varsigma) \\ \gamma(\chi\varsigma\chi^{-1}) = \gamma(\varsigma) \\ \zeta(\chi\varsigma\chi^{-1}) = \zeta(\varsigma) \end{cases}$$

$$(iii) \begin{cases} \mu(\chi\varsigma) = \mu(\varsigma\chi) \\ \gamma(\chi\varsigma) = \gamma(\varsigma\chi) \\ \zeta(\chi\varsigma) = \zeta(\varsigma\chi) \end{cases}$$

$$(iv) \begin{cases} \chi\mu = \mu\chi \\ \chi\gamma = \gamma\chi \\ \chi\zeta = \zeta\chi \end{cases}$$

$$(v) \begin{cases} \chi\mu\chi^{-1} = \mu \\ \chi\gamma\chi^{-1} = \gamma \\ \chi\zeta\chi^{-1} = \zeta \end{cases}$$

for each $\chi, \varsigma \in \mathcal{G}$.

Proof. From Definition 2.3 and Remark 4.2, the proof is straightforward.

Proposition 4.4 *Let Υ be a neutrosophic sub-group of $\mathcal{G}$, then $\chi\Upsilon\chi^{-1}$ is a neutrosophic sub-group of $\mathcal{G}$ $\forall$ $\chi \in \mathcal{G}$.*

Proof. Suppose that Υ be a neutrosophic sub-group of $\mathcal{G}$ and $\chi \in \mathcal{G}$, then we check the axioms of Definition 2.3 as follows:

$$(i) \begin{cases} \kappa\mu\kappa^{-1}(\chi\varsigma) = \mu(\kappa^{-1}(\chi\varsigma)\kappa) \\ \qquad\qquad = \mu(\kappa^{-1}(\chi\kappa\kappa^{-1}\varsigma)\kappa) \\ \qquad\qquad = \mu((\kappa^{-1}\chi\kappa)(\kappa^{-1}\varsigma\kappa)) \\ \qquad\qquad \geqslant \min(\mu(\kappa^{-1}\chi\kappa), \mu(\kappa^{-1}\varsigma\kappa)). \\ \kappa\gamma\kappa^{-1}(\chi\varsigma) = \gamma(\kappa^{-1}(\chi\varsigma)\kappa) \\ \qquad\qquad = \gamma(\kappa^{-1}(\chi\kappa\kappa^{-1}\varsigma)\kappa) \\ \qquad\qquad = \gamma((\kappa^{-1}\chi\kappa)(\kappa^{-1}\varsigma\kappa)) \\ \qquad\qquad \leqslant \max(\gamma(\kappa^{-1}\chi\kappa), \gamma(\kappa^{-1}\varsigma\kappa)). \\ \kappa\zeta\kappa^{-1}(\chi\varsigma) = \zeta(\kappa^{-1}(\chi\varsigma)\kappa) \\ \qquad\qquad = \zeta(\kappa^{-1}(\chi\kappa\kappa^{-1}\varsigma)\kappa) \\ \qquad\qquad = \zeta((\kappa^{-1}\chi\kappa)(\kappa^{-1}\varsigma\kappa)) \\ \qquad\qquad \leqslant \max(\zeta(\kappa^{-1}\chi\kappa), \zeta(\kappa^{-1}\varsigma\kappa)). \end{cases}$$

$$(ii)\begin{cases}\kappa\mu\kappa^{-1}\left(\chi^{-1}\right)=\mu(\kappa^{-1}\left(\chi^{-1}\right)\kappa)\\ \qquad\qquad\geqslant\mu(\kappa^{-1}\chi\kappa).\\ \kappa\gamma\kappa^{-1}\left(\chi^{-1}\right)=\gamma(\kappa^{-1}\left(\chi^{-1}\right)\kappa)\\ \qquad\qquad\leqslant\gamma(\kappa^{-1}\chi\kappa).\\ \kappa\zeta\kappa^{-1}\left(\chi^{-1}\right)=\zeta(\kappa^{-1}\left(\chi^{-1}\right)\kappa)\\ \qquad\qquad\leqslant\zeta(\kappa^{-1}\chi\kappa).\end{cases}$$

where $\kappa,\chi,\varsigma\in\mathcal{G}$.

Definition 4.5 When a neutrosophic sub-group of Υ satisfies one of the axioms in Proposition 4.3, then Υ is called a neutrosophic normal sub-group.

Example 4.6 Consider $\mathcal{X}=\{\pm1,\pm i\}$ is a group under usual multiplication and $\mathcal{A}=\{\pm1,\pm i\}$ sub-group of $\mathcal{X}$. Now, we define a neutrosophic sub-group $\Upsilon=\{<1,0.2,0.5,0.3>,<-1,0.2,0.5,0.3>\}$ of $\mathcal{A}$. Also, it is easy to show any axioms in Proposition 4.3. Therefore, Υ neutrosophic normal sub-group.

Proposition 4.7 *Any intersection of two neutrosophic normal sub-groups of $\mathcal{G}$ is also a neutrosophic normal sub-group of $\mathcal{G}$.*

Proof. Suppose that Υ_1 and Υ_2 are any two neutrosophic normal sub-group of $\mathcal{G}$, then firstly we find $\Upsilon_1\cap_1\Upsilon_2$ is a neutrosophic sub-group of $\mathcal{G}$ [4]. Secondly, we show $\Upsilon_1\cap_1\Upsilon_2$ is a neutrosophic normal sub-group of $\mathcal{G}$. Let $\chi,\varsigma\in\mathcal{G}$, then

$$\begin{cases}(i)\quad(\mu_1\cap_1\mu_2)(\chi\varsigma\chi^{-1})=\mu_1(\chi\varsigma\chi^{-1})\wedge\mu_2(\chi\varsigma\chi^{-1})\\ \qquad\qquad\geqslant\mu_1(\varsigma)\wedge\mu_2(\varsigma)\\ \qquad\qquad\geqslant(\mu_1\cap_1\mu_2)(\varsigma).\\ (ii)\quad(\gamma_1\cap_1\gamma_2)(\chi\varsigma\chi^{-1})=\gamma_1(\chi\varsigma\chi^{-1})\vee\gamma_2(\chi\varsigma\chi^{-1})\\ \qquad\qquad\leqslant\gamma_1(\varsigma)\vee\gamma_2(\varsigma)\\ \qquad\qquad\leqslant(\gamma_1\cap_1\gamma_2)(\varsigma).\\ (iii)\quad(\zeta_1\cap_1\zeta_2)(\chi\varsigma\chi^{-1})=\zeta_1(\chi\varsigma\chi^{-1})\vee\zeta_2(\chi\varsigma\chi^{-1})\\ \qquad\qquad\leqslant\zeta_1(\varsigma)\vee\zeta_2(\varsigma)\\ \qquad\qquad\leqslant(\zeta_1\cap_1\zeta_2)(\varsigma).\end{cases}$$

Therefore, $\Upsilon_1\cap_1\Upsilon_2$ is a neutrosophic normal sub-group of $\mathcal{G}$ and the proposition is claim.

Proposition 4.8 *Assume that $\mathcal{F}:\mathcal{G}\longrightarrow\mathcal{G}'$ is a group homomorphism, then*

(1) *$\mathcal{F}^{-1}(\Upsilon)$ is a neutrosophic normal sub-group of $\mathcal{G}$ when Υ is a neutrosophic normal sub-group of $\mathcal{G}'$.*
(2) *$\mathcal{F}(\Upsilon)$ is a neutrosophic normal sub-group of $\mathcal{G}'$ when Υ is a neutrosophic normal sub-group of $\mathcal{G}$ and $\mathcal{F}$ is an onto.*

Proof. (1) Let Υ be a neutrosophic normal sub-group of $\mathcal{G}'$. Firstly, we explaine that $\mathcal{F}^{-1}(\Upsilon)$ is a neutrosophic sub-group as follows

$$(i)\begin{cases}\mathcal{F}^{-1}(\mu(\chi\varsigma)) = \mu(\mathcal{F}(\chi\varsigma)) \\ \qquad\qquad = \mu(\mathcal{F}(\chi)\mathcal{F}(\varsigma)) \\ \qquad\qquad \geqslant \min(\mu(\mathcal{F}(\chi)), \mu(\mathcal{F}(\varsigma))) \\ \qquad\qquad = \mathcal{F}^{-1}\min(\mu(\chi), \mu(\varsigma)). \\ \mathcal{F}^{-1}(\gamma(\chi\varsigma)) = \gamma(\mathcal{F}(\chi\varsigma)) \\ \qquad\qquad = \gamma(\mathcal{F}(\chi)\mathcal{F}(\varsigma)) \\ \qquad\qquad \leqslant \max(\gamma(\mathcal{F}(\chi)), \gamma(\mathcal{F}(\varsigma))) \\ \qquad\qquad = \mathcal{F}^{-1}\max(\gamma(\chi), \gamma(\varsigma)). \\ \mathcal{F}^{-1}(\zeta(\chi\varsigma)) = \zeta(\mathcal{F}(\chi\varsigma)) \\ \qquad\qquad = \zeta(\mathcal{F}(\chi)\mathcal{F}(\varsigma)) \\ \qquad\qquad \leqslant \max(\zeta(\mathcal{F}(\chi)), \zeta(\mathcal{F}(\varsigma))) \\ \qquad\qquad = \mathcal{F}^{-1}\max(\zeta(\chi), \zeta(\varsigma)).\end{cases}$$

$$(ii)\begin{cases}\mathcal{F}^{-1}(\mu\left(\chi^{-1}\right)) = \mu\mathcal{F}\left(\chi^{-1}\right) \\ \qquad\qquad = \mu((\mathcal{F}(\chi))^{-1}) \\ \qquad\qquad \geqslant \mu(\mathcal{F}(\chi)) \\ \qquad\qquad = \mathcal{F}^{-1}(\mu(\chi)). \\ \mathcal{F}^{-1}(\gamma\left(\chi^{-1}\right)) = \gamma\mathcal{F}\left(\chi^{-1}\right) \\ \qquad\qquad = \gamma((\mathcal{F}(\chi))^{-1}) \\ \qquad\qquad \leqslant \gamma(\mathcal{F}(\chi)) \\ \qquad\qquad = \mathcal{F}^{-1}(\gamma(\chi)). \\ \mathcal{F}^{-1}(\zeta\left(\chi^{-1}\right)) = \zeta\mathcal{F}\left(\chi^{-1}\right) \\ \qquad\qquad = \zeta((\mathcal{F}(\chi))^{-1}) \\ \qquad\qquad \leqslant \zeta(\mathcal{F}(\chi)) \\ \qquad\qquad = \mathcal{F}^{-1}(\zeta(\chi)).\end{cases}$$

where $\chi, \varsigma \in \mathcal{G}$. Secondly, we show $\mathcal{F}^{-1}(\Upsilon)$ is a neutrosophic normal sub-group

$$\begin{cases}\mathcal{F}^{-1}(\mu)(\chi\varsigma\chi^{-1}) = \mu(\mathcal{F}(\chi\varsigma\chi^{-1})) \\ \qquad\qquad = \mu(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi^{-1})) \\ \qquad\qquad = \mu(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi)^{-1}) \\ \qquad\qquad = \mu(\mathcal{F}(\varsigma)) \\ \qquad\qquad = \mathcal{F}^{-1}(\mu)(\varsigma). \\ \mathcal{F}^{-1}(\gamma)(\chi\varsigma\chi^{-1}) = \gamma(\mathcal{F}(\chi\varsigma\chi^{-1})) \\ \qquad\qquad = \gamma(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi^{-1})) \\ \qquad\qquad = \gamma(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi)^{-1}) \\ \qquad\qquad = \gamma(\mathcal{F}(\varsigma)) \\ \qquad\qquad = \mathcal{F}^{-1}(\gamma)(\varsigma). \\ \mathcal{F}^{-1}(\zeta)(\chi\varsigma\chi^{-1}) = \zeta(\mathcal{F}(\chi\varsigma\chi^{-1})) \\ \qquad\qquad = \zeta(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi^{-1})) \\ \qquad\qquad = \zeta(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi)^{-1}) \\ \qquad\qquad = \zeta(\mathcal{F}(\varsigma)) \\ \qquad\qquad = \mathcal{F}^{-1}(\zeta)(\varsigma).\end{cases}$$

Thus, $\mathcal{F}^{-1}(\Upsilon)$ is a neutrosophic normal sub-group of $\mathcal{G}$.

(2) Suppose that Υ is a neutrosophic normal sub-group of $\mathcal{G}$ and $\mathcal{F}$ is an onto. Firstly, we explaine that $\mathcal{F}(\Upsilon)$ is a neutrosophic sub-group as follows

$$(i)\begin{cases}\mathcal{F}(\mu(\chi\varsigma)) \geqslant \mathcal{F}(\min(\mu(\chi), \mu(\varsigma))) \\ \qquad\qquad = \min(\mathcal{F}(\mu(\chi)), \mathcal{F}(\mu(\varsigma))). \\ \mathcal{F}(\gamma(\chi\varsigma)) \leqslant \mathcal{F}(\max(\gamma(\chi), \gamma(\varsigma))) \\ \qquad\qquad = \max(\mathcal{F}(\gamma(\chi)), \mathcal{F}(\gamma(\varsigma))). \\ \mathcal{F}(\zeta(\chi\varsigma)) \leqslant \mathcal{F}(\max(\zeta(\chi), \zeta(\varsigma))) \\ \qquad\qquad = \max(\mathcal{F}(\zeta(\chi)), \mathcal{F}(\zeta(\varsigma))).\end{cases}$$

$$(ii)\begin{cases}\mathcal{F}(\mu\left(\chi^{-1}\right)) \geqslant \mathcal{F}(\mu(\chi)) \\ \mathcal{F}(\gamma\left(\chi^{-1}\right)) \leqslant \mathcal{F}(\gamma(\chi)) \\ \mathcal{F}(\zeta\left(\chi^{-1}\right)) \leqslant \mathcal{F}(\zeta(\chi))\end{cases}$$

where $\chi, \varsigma \in \mathcal{G}'$. Secondly, we show $\mathcal{F}(\Upsilon)$ is a neutro-

sophic normal sub-group

$$\begin{cases} \mathcal{F}(\mu)(\chi\varsigma\chi^{-1}) = \sup\limits_{\mathcal{F}(\varrho)=\chi\varsigma\chi^{-1}}(\mu(\varrho)) \\ \qquad\qquad = \sup\limits_{\mathcal{F}(o)=\chi,\mathcal{F}(\varrho)=\varsigma}(\mu(o\varrho o^{-1})) \\ \qquad\qquad = \sup\limits_{\mathcal{F}(\varrho)=\varsigma}(\mu(\varrho)) \\ \qquad\qquad = \mathcal{F}(\mu)(\varsigma) \\ \mathcal{F}(\gamma)(\chi\varsigma\chi^{-1}) = \inf\limits_{\mathcal{F}(\varrho)=\chi\varsigma\chi^{-1}}(\gamma(\varrho)) \\ \qquad\qquad = \inf\limits_{\mathcal{F}(o)=\chi,\mathcal{F}(\varrho)=\varsigma}(\gamma(o\varrho o^{-1})) \\ \qquad\qquad = \inf\limits_{\mathcal{F}(\varrho)=\varsigma}(\gamma(\varrho)) \\ \qquad\qquad = \mathcal{F}(\gamma)(\varsigma) \\ \mathcal{F}(\zeta)(\chi\varsigma\chi^{-1}) = \inf\limits_{\mathcal{F}(\varrho)=\chi\varsigma\chi^{-1}}(\zeta(\varrho)) \\ \qquad\qquad = \inf\limits_{\mathcal{F}(o)=\chi,\mathcal{F}(\varrho)=\varsigma}(\zeta(o\varrho o^{-1})) \\ \qquad\qquad = \inf\limits_{\mathcal{F}(\varrho)=\varsigma}(\zeta(\varrho)) \\ \qquad\qquad = \mathcal{F}(\zeta)(\varsigma) \end{cases}$$

since $\mathcal{F}$ is onto homomorphism and $o, \varrho \in \mathcal{G}$.

Definition 4.9 Assume that Υ is a neutrosophic sub-group of $\mathcal{G}$ and $\mathcal{H} \subset \mathcal{G}$. then we call $\mathcal{H}$ the normalizer of Υ if $h\Upsilon h^{-1} = \Upsilon \; \forall h \in \mathcal{H}$.

Proposition 4.10 *A normalizer of a neutrosophic sub-group of $\mathcal{G}$ is a sub-group of $\mathcal{G}$.*

5. Conclusions

Here, we have studied some concepts from the point of view of the definition of the neutrosophic group, which was introduced by El Rawy et al. [4]. The concept of neutrosophic invariant sub-groups has been introduced. Also, we have investigated this to define the neutrosophic quotient group. Furthermore, we have defined the neutrosophic normal sub-group. In each part, several related theorems have been constructed, and these are illustrated. The study and development of NS theory will have a new approach opened up by these concepts.

Data Availability Statement

No data were used to support the study.

Declarations

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Not applicable.

References

[1] Atanassov, K. (1986). Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20, 87 - 96.

[2] Biswas, R. (1989). Intuitionistic fuzzy subgroups. In Mathematical Forum (Vol. 10, pp. 37-46).

[3] Çetkin, V., & Aygün, H. (2015). An approach to neutrosophic subgroup and its fundamental properties. Journal of Intelligent & Fuzzy Systems, 29(5), 1941-1947.

[4] Elrawy, A., Saleem, M. A., & Mohamed, A. (2022). Another approach to neutrosophic group. In press Journal of Mathematics.

[5] Fathi, M., & Salleh, A. R. (2009). Intuitionistic fuzzy groups. Asian Journal of Algebra, 2(1), 1-10.

[6] Vasantha Kandasamy, W. B., & Smarandache, F. (2006). Some neutrosophic algebraic structures and neutrosophic N-algebraic structures. arXiv Mathematics e-prints, math-0603581.

[7] Vasantha, W. B., & Smarandache, F. Basic neutrosophic algebraic structures and their application to fuzzy and neutrosophic models; Hexis: Phoenix, AZ, USA, 2004. ISBN 978-1-931233-87-X.

[8] Marashdeh, M. F., & Salleh, A. R. (2010). The Intuitionistic Fuzzy Normal Subgroup. Int. J. Fuzzy Log. Intell. Syst., 10(1), 82-88.

[9] Mashour, A. S., Ghanim, M. H., & Sidky, F. I. (1990). Normal fuzzy subgroups. Information Sciences, 20, 53-59.

[10] Mukherjee, N. P., & Bhattacharya, P. (1986). Fuzzy groups: some group-theoretic analogs. Information sciences, 39(3), 247-267.

[11] Oyebo, Y. T. (2012). Neutrosophic groups and sub-groups. Mathematical Combinatorics, Vol. 3/2012: international book series, 1.

[12] PU, P., & MING, L. Y. (1980). Fuzzy topology. II: Product and quotient spaces.

[13] Rosenfeld, A. (1971). Fuzzy groups. Journal of mathematical analysis and applications, 35(3), 512-517.

[14] Salama, A. A., & Alblowi, S. A. (2012). Neutrosophic set and neutrosophic topological spaces. IOSR Journal of Mathematics (IOSR-JM), 3(4).

[15] Sharma, P. K. (2011). Intuitionistic fuzzy groups. IFRSA Int. J. Data Warehous Min, 1(1), 86-94.

[16] Smarandache, F. (1998). Neutrosophy, Neutrosophic Probability, Set and logic, Amer.Res. Press, Rehoboth, USA, 105, http://fs.gallup.unm.edu/eBOOK- neutrosophics 4.pdf (fourth version).

[17] Smarandache, F. (2005). Neutrosophic set-a generalization of the intuitionistic fuzzy set. International journal of pure and applied mathematics, 24(3), 287.

[18] Smarandache, F. (2006, May). Neutrosophic set-a generalization of the intuitionistic fuzzy set. In 2006 IEEE international conference on granular computing (pp. 38-42). IEEE.

[19] Smarandache, F. (2017). Smarandache, F. (2017). Neutrosophic perspectives: triplets, duplets, multisets, hybrid operators, modal logic, hedge algebras and applications. Pons Editions, Bruxelles, 323, 115-119.

[20] Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). Single valued neutrosophic sets. Multisp. Multistruct. 4, 410-413.

[21] Zadeh, L. A. (1996). Fuzzy sets. In Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh (pp. 394-432).

[22] Zhang, X., Bo, C., Smarandache, F., & Dai, J. (2018). New inclusion relation of neutrosophic sets with applications and related lattice structure. International Journal of Machine Learning and Cybernetics, 9, 1753-1763.

Valued-inverse Dombi neutrosophic graph and application

Mohammad Hamidi,[1] **Florentin Smarandache**[2]

1 Department of Mathematics, University of Payame Noor, Tehran, Iran
2 Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA

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Abstract: Utilizing two ideas of neutrosophic subsets (NS) and triangular norms, we introduce a new type of graph as valued-inverse Dombi neutrosophic graphs. The valued-inverse Dombi neutrosophic graphs are a generalization of inverse neutrosophic graphs and are dual to Dombi neutrosophic graphs. We present the concepts of (complete) strong valued-inverse Dombi neutrosophic graphs and analyze the complement of (complete) strong valued-inverse Dombi neutrosophic graphs and self-valued complemented valued-inverse Dombi neutrosophic graphs. Since the valued-inverse Dombi neutrosophic graphs depend on real values, solving the non-equation and the concept of homomorphism play a prominent role in determining the complete, strong, complementarity and self-complementarity of valued-inverse Dombi neutrosophic graphs. We introduce the truth membership order, indeterminacy membership order, falsity membership order, truth membership size, indeterminacy membership size and indeterminacy membership size of any given valued-inverse Dombi neutrosophic graph, which play a major role in the application of valued inverse Dombi neutrosophic graphs in complex networks. An application of a valued-inverse Dombi neutrosophic graph is also described in this study.

Keywords: neutrosophic graph; Dombi triangular operator; inverse fuzzy graph; Dombi fuzzy graph; valued-Dombi inverse fuzzy graph
Mathematics Subject Classification: 03E72, 05C72

1. Introduction

The fuzzy graph theory, as a generalization of graph theory, is a symmetric and binary fuzzy relation on the fuzzy subset and is introduced and extended by Kaufmann and Rosenfeld [13, 18]. Later on, some researchers extended this concept, such as the complement of fuzzy graphs, the notions of regular fuzzy graphs, the notation of strong fuzzy graphs, totally regular fuzzy graphs and degree and total degree of a vertex in some fuzzy graphs, the novel concepts of M-strong fuzzy graphs and some types

on domination number based on fuzzy graphs. Smarandache has presented the central concept idea of the neutrosophic set by extending the idea of fuzzy subsets [21]. A neutrosophic subset considers vague, indeterminate and inconsistent information of any real-life problem. A neutrosophic subset has three independent membership grades: truth, indeterminate and false. The bipolar fuzzy graph that is introduced by Ghorai and Pal is based on two ideas of fuzzy subset and graph structure and is used to design real-life problems [9]. The theory of single-valued neutrosophic graphs is a generalization of fuzzy graphs and has some essential applications in the real world. M. Akram has collected and has introduced a book on the scope of single-valued neutrosophic graphs that presents readers with fundamental concepts, including single-valued neutrosophic, neutrosophic graph structures, bipolar neutrosophic graphs, domination in bipolar neutrosophic graphs, bipolar neutrosophic planar graphs, interval-valued neutrosophic graphs, interval-valued neutrosophic graph structures, rough neutrosophic digraphs, neutrosophic rough digraphs, neutrosophic soft graphs, and intuitionistic neutrosophic soft graphs [2]. Recently, many researchers have more actively worked on neutrosophic graph theory, for instance, certain properties of the single-valued neutrosophic graph with application in food and agriculture organization [22], bipolar neutrosophic graph structures [6], a note on different types of product of neutrosophic graphs [15], generalized neutrosophic planar graphs and its application [16] and regularity of Pythagorean neutrosophic graphs with an illustration in MCDM [5]. The theory of single-valued neutrosophic hypergraphs as a generalization of single-valued neutrosophic graphs is a novel concept in the scope of single-valued neutrosophics and hypergraphs, which is introduced by M. Akram [1]. There is some applied research in the scope of single-valued neutrosophic hypergraphs, which are valuable in the real world, such as the implementation of single-valued neutrosophic soft hypergraphs on the human nervous system [3], and algorithms for the computation of regular single-valued neutrosophic soft hypergraphs applied to supranational Asian bodies [4]. A study on generalized graphs representations of complex neutrosophic information [19] and a novel decision-making method based on bipolar neutrosophic information [23] was conducted. The combination of triangular norms and fuzzy graphs is a novel concept for the generalization of fuzzy graphs and has applications of triangular norms in the real world. Recently, Ashraf et. al introduced the concept of Dombi fuzzy graphs and investigated some of their properties [7]. Later, Borzooei et al. presented the concept of inverse fuzzy graphs with application and analyzed the difference between fuzzy graphs and inverse fuzzy graphs [8]. Hamidi et al. applied the notation of topological spaces and fuzzy metrizable as the class of metrizable-topological spaces and introduced the concept of KM-fuzzy metric graph and KM-fuzzy metric hypergraph [11, 12]. Pal et al. introduced the combination of triangular norms and neutrosophic graphs as a generalization of the combination of triangular norms and fuzzy graphs, as some operations on Dombi neutrosophic graph, such as union, intersection composition, cartesian product, box dot product, homomorphic product, and modular product [14].

Motivation and advantage: One of the most essential applications of mathematics in the real world is to model a real problem based on the concepts of mathematics to analyze it according to the logic governing the problem. Undoubtedly, graph theory is one of the essential branches of mathematics in this modeling, which makes it easy to solve problems. Due to the fact that real problems are based on weight and real labels, weighted graphs can be essential. One of the essential areas of mathematics is fuzzy graphs, which can be effective in this field. Due to the strong connections of the elements in the real world to each other and the various properties and characteristics of objects, fuzzy graphs alone cannot analyze all the problems, and therefore, this limitation is considered in this field. Therefore, the

concept of inverse single-valued neutrosophic graphs can overcome this limitation and cover different characteristics. As an example, consider a social network whose members are humans. In this network, every human being has a different temperament, and therefore, good, bad, kindness, anger, and similar (the truth value of each person represents his positive influence on others, falsity value represents his negative influence on others and indeterminacy value represents uncertainty in his influence) traits can have a significant impact on this network. Our most essential motivation in this research is based on the fact that in a mixed network that has several essential characteristics and some properties that act opposite to other properties, we model them based on inverse single-valued neutrosophic graphs. In this modeling, we deal with the reverse effect of each characteristic and examine its effect. Regarding these points, we try to combine the notations of triangular norms and neutrosophic subsets to construct an extension of graphs. Thus, we apply the notation of the Dombi triangular norm and inverse fuzzy graph to introduce the Dombi inverse neutrosophic graph. The prominent motivation for this work is a type of generalization of crisp graphs to weighted graphs, which are essential in the real world. Indeed, for any given greater than or equal to one nonnegative real number, a valued Dombi inverse neutrosophic graph is introduced. Additionally, we investigated the properties of valued Dombi inverse neutrosophic graphs and an application of the Dombi neutrosophic graph is also described in this manuscript.

2. Preliminaries

In this section, we recall some definitions and results, which we need use in what follows.

Definition 2.1. [17] *A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a t-norm, if for all $x, y, z, w \in [0,1]$:*

(*i*) $T(1,x) = x$,
(*ii*) $T(x,y) = T(y,x)$,
(*iii*) $T(T(x,y),z) = T(x,T(y,z))$,
(*iii*) *if $w \leq x$ and $y \leq z$, then $T(w,y) \leq T(x,z)$.*

For any $x, y \in [0,1]$, the t-norm $T_{Do}(x,y) = \dfrac{xy}{x+y-xy}$ is called the Dombi t-norm as an especial of Hamacher family of t-norms.

Definition 2.2. [20] *Let V be a universal set. A neutrosophic subset (NS) X in V is defined by $X = \{(x, T_X(x), I_X(x), F_X(x)) \mid x \in V, 0 \leq T_X(x) + I_X(x) + F_X(x) \leq 3\}$, which $X : V \to [0,1] \times [0,1] \times [0,1]$ is characterized by a truth-membership function T_X, an indeterminacy-membership function I_X and a falsity-membership function F_X.*

Definition 2.3. [2] *A single-valued neutrosophic graph on a nonempty X is a pair $G = (A, B)$, where A is single-valued neutrosophic set in X and B single-valued neutrosophic relation on X, such that $T_B(xy) \leq \min\{T_A(x), T_A(y)\}, I_B(xy) \leq \min\{I_A(x), I_A(y)\}, F_B(xy) \leq \max\{F_A(x), F_A(y)\}$ for all $x, y \in X$. A is called single-valued neutrosophic vertex set of G and B is called single-valued neutrosophic edge set of G, respectively.*

Definition 2.4. [10] *Let $k \in \mathbb{R}^{>0}, G^* = (V, E)$ be a simple graph, $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$ be fuzzy subsets. Then $G = (\sigma, \mu, T_{Do})$ is called a k-Dombi inverse fuzzy graph, if for any $xy \in E$, get $k\mu(xy) \geq T_D(\sigma(x), \sigma(y))$. If $k = 1$, then $G = (\sigma, \mu, T_{Do})$ is called a Dombi inverse fuzzy graph.*

Definition 2.5. [14] *Let $G^* = (V, E)$ be a simple graph, $\sigma = \{(x, T_V(x), I_V(x), F_V(x)) \mid x \in V\}$ and $\mu = \{(xy, T_E(xy), I_E(xy), F_E(xy)) \mid xy \in E\}$ be two NS. Then $G = (\sigma, \mu, T_{Do})$ is called a Dombi neutrosophic graph (DNG), if for any $xy \in E$, get $T_E(xy) \leq T_{Do}(T_V(x), T_V(y)), I_E(xy) \leq T_{Do}(I_V(x), I_V(y))$ and $F_E(xy) \geq T_{Do}(F_V(x), F_V(y))$.*

3. Dombi inverse neutrosophic graph

In this section, for any $k, l, m \in \mathbb{R}^{>0}$, introduce the concept of (k, l, m)-Dombi inverse neutrosophic graph, via Dombi triangular norm and inverse neutrosophic graph.

Definition 3.1. *Let $k, l, m \in \mathbb{R}^{>0}, G^* = (V, E)$ be a simple graph, $\sigma = \{(x, T_V(x), I_X(x), F_V(x)) \mid x \in V\}$ and $\mu = \{(xy, T_E(xy), I_E(xy), F_E(xy)) \mid xy \in E\}$ be two NS. Then*

(i) $G = (\sigma, \mu, T_{Do})$ is called a (k, l, m)-Dombi inverse neutrosophic graph ((k, l, m)-DING), if for any $xy \in E$, get $kT_E(xy) \geq T_{Do}(T_V(x), T_V(y)), lI_E(xy) \geq T_{Do}(I_V(x), I_V(y))$ and $mF_E(xy) \leq T_{Do}(F_V(x), F_V(y))$,

(ii) $G = (\sigma, \mu, T_{Do})$ is called a strong (k, l, m)-Dombi inverse neutrosophic graph (strong (k, l, m)-DING), if for any $xy \in E$, $kT_E(xy) = T_{Do}(T_V(x), T_V(y)), lI_E(xy) = T_{Do}(I_V(x), I_V(y))$ and $mF_E(xy) = T_{Do}(F_V(x), F_V(y))$,

(iii) $G = (\sigma, \mu, T_{Do})$ is called a complete (k, l, m)-Dombi neutrosophic graph (complete (k, l, m)-DING), if for any $x, y \in V$, $kT_E(xy) = T_{Do}(T_V(x), T_V(y)), lI_E(xy) = T_{Do}(I_V(x), I_V(y))$ and $mF_E(xy) = T_{Do}(F_V(x), F_V(y))$.

In the above definition, if $(k, l, m) = (1, 1, 1)$, we will call $G = (\sigma, \mu, T_{Do})$ is a (complete) strong Dombi inverse neutrosophic graph.

Example 3.2. *Let $V = \{a, b, c, d\}$. Then $G = (\sigma, \mu, T_{Do})$ is a $(4, 3, 2)$-DING as shown in Figure 1.*

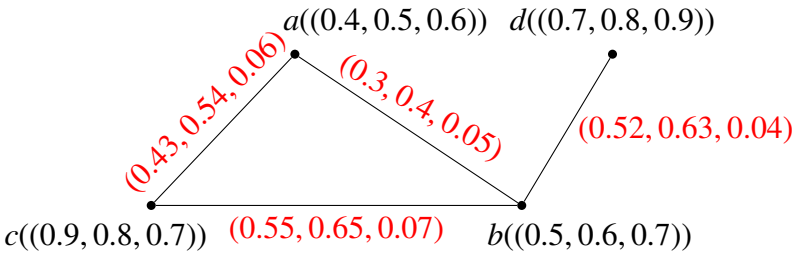

Figure 1. $(4, 3, 2)$-DING $G = (\sigma, \mu, T_{Do})$.

Theorem 3.3. *Let $G = (\sigma, \mu, T_{Do})$ be a (k, l, m)-DING and $k, k', l, l', m, m' \in \mathbb{R}^{>0}$. If $k' \geq k, l' \geq l, m \geq m'$, then $G = (\sigma, \mu, T_{Do})$ is a (k', l', m')-DING.*

Proof. Let $k, k', l, l', m, m' \in \mathbb{R}^{>0}$. Since $G = (\sigma, \mu, T_{Do})$ is a (k, l, m)-DING and $k' \geq k, l' \geq l, m \geq m'$, we get that

$$k'T_E(xy) \geq kT_E(xy) \geq T_{Do}(T_V(x), T_V(y)), l'I_E(xy) \geq lI_E(xy) \geq I_{Do}(I_V(x), I_V(y))$$
$$\text{and } m'F_E(xy) \leq mF_E(xy) \leq T_{Do}(F_V(x), F_V(y)),$$

and so $G = (\sigma, \mu, T_{Do})$ is a (k', l', m')-DING. □

Proposition 3.4. *Let $G = (\sigma, \mu, T_{Do})$ be a DNG and a (k, l, m)-DING. Then*

(i) $k \geq 1, l \geq 1$ *and* $0 < m \leq 1$.
(ii) $G = (\sigma, \mu, T_{Do})$ *is strong DNG and strong* (k, l, m)*-DING if and only if* $(k, l, m) = (1, 1, 1)$.

Proof. *(i)* Let $x, y \in V$. Then

$$\frac{1}{k}T_{Do}(T_V(x), T_V(y)) \leq T_E(xy) \leq T_{Do}(T_V(x), T_V(x)), \frac{1}{l}T_{Do}(I_V(x), I_V(y)) \leq I_E(xy) \leq T_{Do}(I_V(x), I_V(x)),$$
$$\text{and } \frac{1}{m}T_{Do}(F_V(x), F_V(y)) \geq F_E(xy) \geq T_{Do}(F_V(x), F_V(x)).$$

It follows that

$$\frac{1}{k}T_{Do}(T_V(x), T_V(y)) \leq T_{Do}(T_V(x), T_V(x)), \frac{1}{l}T_{Do}(I_V(x), I_V(y)) \leq T_{Do}(I_V(x), I_V(x)),$$
$$\text{and } \frac{1}{m}T_{Do}(F_V(x), F_V(y)) \geq T_{Do}(I_V(x), I_V(x)).$$

Hence $k \geq 1, l \geq 1$ and $m \leq 1$, because of $0 < x, y \leq 1$.
(ii) It is clear by *(i)*. □

Definition 3.5. *Let $G = (\sigma, \mu, T_{Do})$ be a (k, l, m)-DING and $k, k,' l, l', m, m' \in \mathbb{R}^{>0}$. Then $G^{(c,(k',l',m'))} = (\sigma^c, \mu^c, T_{Do})$ is called a (k', l', m')-complement of (k, l, m)-DING of G, if $\sigma^c = \sigma$ and for all $x, y \in V$,*

$$T_E^c(xy) = \begin{cases} k'T_E(xy) - T_{Do}(T_V(x), T_V(y)) & \text{if } xy \in E \\ T_{Do}(T_V(x), T_V(y)) & \text{if } xy \notin E, \end{cases},$$

$$I_E^c(xy) = \begin{cases} l'I_E(xy) - T_{Do}(I_V(x), I_V(y)) & \text{if } xy \in E \\ T_{Do}(I_V(x), I_V(y)) & \text{if } xy \notin E, \end{cases}$$

and $F_E^c(xy) = \begin{cases} F_E(xy) - m'T_{Do}(F_V(x), F_V(y)) & \text{if } xy \in E \\ T_{Do}(F_V(x), F_V(y)) & \text{if } xy \notin E, \end{cases}$, *where* $\mu^c = (T_E^c, I_E^c, F_E^c)$ *and* $\sigma^c = (T_V^c, I_V^c, F_V^c)$.

Theorem 3.6. *Let $G = (\sigma, \mu, T_{Do})$ be a (k, l, m)-DING and $k, l, m, k', l', m' \in \mathbb{R}^{>0}$. Then*

(i) $G^{(c,(k',l',m'))} = (\sigma^c, \mu^c, T_{Do})$ *is a* $(\frac{k}{k'-k}, \frac{l}{l'-l}, \frac{m}{(1-mm')})$*-DING.*
(ii) $G^{(c,(2k,2l,\frac{1-m}{m}))} = (\sigma^c, \mu^c, T_{Do})$ *is a DING.*
(iii) If $G = (\sigma, \mu, T_{Do})$ is strong, then $G^{(c,(2k,2l,\frac{1-m}{m}))} = (\sigma^c, \mu^c, T_{Do})$ is strong.

Proof. *(i)* Let $xy \in E$. Then

$$\begin{aligned} T_E^c(xy) &= k'T_E(xy) - T_{Do}(T_V(x), T_V(y)) \geq \frac{k'}{k}T_{Do}(T_V(x), T_V(y)) - T_{Do}(T_V(x), T_V(y)) \\ &\geq (\frac{k'}{k} - 1)T_{Do}(T_V(x), T_V(y)), \\ I_E^c(xy) &= l'I_E(xy) - T_{Do}(I_V(x), I_V(y)) \geq \frac{l'}{l}T_{Do}(I_V(x), I_V(y)) - T_{Do}(I_V(x), I_V(y)) \end{aligned}$$

$$\geq (\frac{l}{l}-1)T_{Do}(I_V(x),I_V(y)) \text{ and}$$
$$F_E^c(xy) = F_E(xy)-m'T_{Do}(F_V(x),F_V(y)) \leq (1-mm')F_E(xy)$$
$$\leq \frac{(1-mm')}{m}T_{Do}(F_V(x),F_V(y)).$$

It follows that

$$\frac{k}{k'-k}T_E^c(xy) \geq T_{Do}(T_V(x),T_V(y)), \frac{l}{l'-l}I_E^c(xy) \geq T_{Do}(I_V(x),I_V(y)),$$
$$\frac{m}{(1-mm')}F_E^c(xy) \leq T_{Do}(F_V(x),F_V(y)),$$

and so $G^{(c,(k',l',m'))} = (\sigma^c,\mu^c,T_{Do})$ is a $(\frac{k}{k'-k},\frac{l}{l'-l},\frac{m}{(1-mm')})$-DING.

(*ii*) Since $2k = k', 2l = l', m' = \frac{1-m}{m}$, by (*i*), $G^{(c,(2k,2l,\frac{1-m}{m}))} = (\sigma^c,\mu^c,T_{Do})$ is a DING.

(*iii*) It is clear by (*i*), (*ii*). □

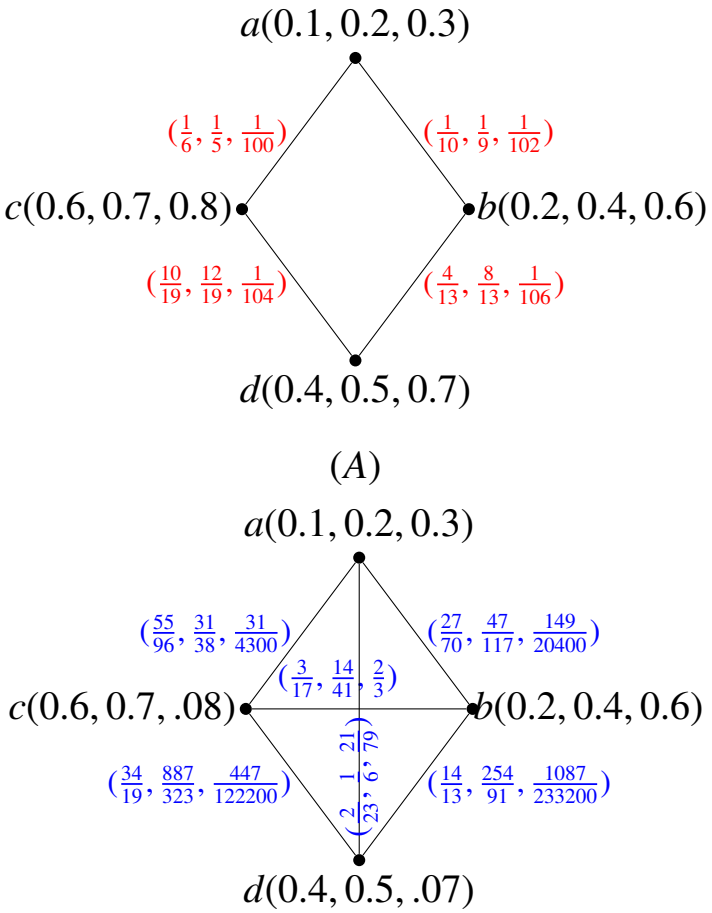

Figure 2. (*A*) $G = (\sigma,\mu,T_{Do})$, (*B*) $G^{(c,(4,5,\frac{1}{100}))} = (\sigma^c,\mu^c,T_{Do})$.

Corollary 3.7. *Let $G = (\sigma, \mu, T_{Do})$ be a (k, l, m)-DING and $k, l, m \in \mathbb{R}^{>0}$. Then*

(i) $G^{(c,(k,l,m))} = (\sigma^c, \mu^c, T_{Do})$ *is't a DING.*
(ii) $G^{(c,(k',l',m'))} = (\sigma^c, \mu^c, T_{Do})$ *is't a DING, which* $k' < k, l' < l, m' > \frac{1}{m}$.

Example 3.8. *Let $V = \{a, b, c, d\}$. Then $G = (\sigma, \mu, T_{Do})$ is a $(2, 3, 4)$-DING as shown in Figure 2 (A). Then $G^{(c,(4,5,\frac{1}{100}))} = (\sigma^c, \mu^c, T_{Do})$ is the $(4, 5, \frac{1}{100})$-complement of $(2, 3, 4)$-DING G in Figure 2 (B).*

Definition 3.9. *Let $k, k', k'', m, m', m'', l, l', l'' \in \mathbb{R}^{>0}, G = (\sigma, \mu, T_{Do})$ and $G' = (\sigma', \mu', T_{Do})$ be (k, l, m)-DING and (k', l', m')-DING on simple graph $G^* = (V, E)$ and $G'^* = (V', E')$, respectively.*

(i) A homomorphism $h : G \to G'$ is a map $h : V \to V'$, which for all $x \in V$, $T_V(x) \geq T'_V(h(x)), I_V(x) \geq I'_V(h(x)), F_V(x) \geq F'_V(h(x))$ and for all $xy \in E$, $kT_E(xy) \geq k'T'_E(h(x)h(y))$, $lI_E(xy) \geq l'T'_E(h(x)h(y)), mF_E(xy) \geq m'F'(h(x)h(y))$.
(ii) A bijective mapping $h : V \to V'$ is called an isomorphism, if for any $xy \in E$ and $x \in V$, $T_V(x) = T'_V(h(x)), I_V(x) = I'_V(h(x)), F_V(x) = F'_V(h(x))$ and for all $xy \in E$, $kT_E(xy) = k'T'_E(h(x)h(y)), lI_E(xy) = l'I'_E(h(x)h(y))$ and $mF_E(xy) = m'F'(h(x)h(y))$. We will say that $G = (\sigma, \mu, T_{Do})$ and $G' = (\sigma', \mu', T_{Do})$ are isomorphic and will denote it by $G \cong G'$.
(iii) G is called a self (k'', l'', m'')-complemented (k, l, m)-DING, if $G \cong G^{(c,(k'',l'',m''))}$.

Example 3.10. *(i) Let $V = \{a, b, c\}$. Then $G_1 = (\sigma_1, \mu_1, T_{Do}) \cong G_2 = (\sigma_2, \mu_2, T_{Do})$, where G_1 is a $(2, 3, 4)$-DING and G_2 is a $(4, 6, 8)$-DING as shown in Figure 3.*

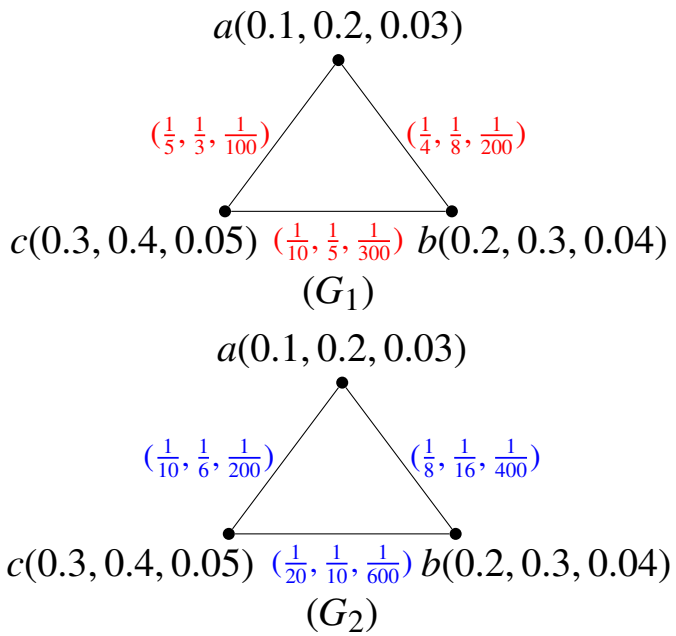

Figure 3. $(2, 3, 4)$-DING G_1 , $(4, 6, 8)$-DING G_2.

(ii) (i) Consider the $(4, 5, 1)$-DING $G = (\sigma, \mu, T_{Do})$ and $(8, 9, 0.5)$-DING $G^c = (\sigma^c, \mu^c, T_{Do})$ in Figures 4 (A) and (B), respectively. Then G is a self $(8, 9, 0.5)$-complemented $(4, 5, 1)$-DING.

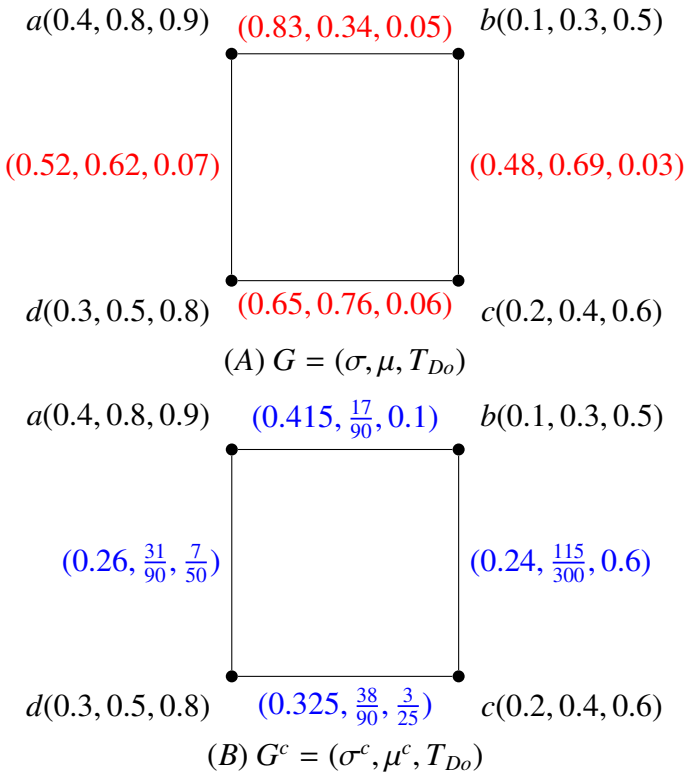

Figure 4. $(4, 5, 1)$-DING G and $(8, 9, 0.5)$-DING G^c.

Theorem 3.11. *Let* $k_1, l_1, m_1, k_2, l_2, m_2, k_1', l', m_1', k_2', l_2', m_2' \in \mathbb{R}^{>0}, G_1 = (\sigma_1, \mu_1, T_{Do}) \cong G_2 = (\sigma_2, \mu_2, T_{Do})$, *where* G_1 *and* G_2 *are* (k_1, l_1, m_1)*-DING and* (k_2, l_2, m_2)*-DING, respectively. Then*

(*i*) *If* $k_1'k_2 = k_1k_2', l_1'l_2 = l_1l_2'$ *and* $m_1m_1' = m_2m_2'$, *then* $G_1^{(c,(k_1',l_1',m_1'))} \cong G_2^{(c,(k_2',l_2',m_2'))}$.
(*ii*) $G_1^{(c,(k_1,l_1,m_1))} \cong G_2^{(c,(k_1,l_1,m_1))}$.
(*iii*) $G_1^{(c,(\frac{k_1k_2'}{k_1}, \frac{l_1l_2'}{k_1}, \frac{m_1m_1'}{m_2'}))} \cong G_2^{(c,(k_2,l_2,m_2))}$.

Proof. (*i*) Let $G_1 \cong G_2$, so there exists $h : V_1 \to V_2$, such that for any $x_1 \in V_1$ and for all $x_1y_1 \in E_1$, $T_{1_V}(x_1) = T_{2_V}(h(x_1)), I_{1_V}(x_1) = I_{2_V}(h(x_1)), F_{1_V}(x_1) = F_{2_V}(h(x_1))$ and $k_1T_{1_E}(x_1y_1) = k_2T_{2_E}(h(x_1)h(y_1)), l_1I_{1_E}(x_1y_1) = l_2I_{2_E}(h(x_1)h(y_1)), m_1F_{1_E}(x_1y_1) = m_2F_{2_E}(h(x_1)h(y_1))$. It's clear that $T^c_{1_V}(x_1) = T^c_{12_V}(h(x_1)), I^c_{1_V}(x_1) = I^c_{12_V}(h(x_1)), F^c_{1_V}(x_1) = F^c_{12_V}(h(x_1))$ for all $x_1 \in V_1$. Let $x_1y_1 \in E_1, T_{1_E}(x_1y_1) \neq 0, I_{1_E}(x_1y_1) \neq 0$ and $F_{1_E}(x_1y_1) \neq 0$. By Theorem 3.6, $G^{(c,(k',l',m'))} = (\sigma^c, \mu^c, T_{Do})$ is a $(\frac{k}{k'-k}, \frac{l}{l'-l}, \frac{m}{(1-mm')})$-DING. Thus,

$$\frac{k_1}{k_1'-k_1}T^c_{1_E}(x_1y_1) = (\frac{k_1}{k_1'-k_1})k_1'T_{1_E}(x_1y_1) - (\frac{k_1}{k_1'-k_1})T_{Do}(T_{1_V}(x_1), T_{1_V}(y_1))$$
$$= (\frac{k_1}{k_1'-k_1})\frac{k_1'k_2}{k_1}T_{2_E}(h(x_1)h(y_1)) - (\frac{k_1}{k_1'-k_1})T_{Do}(T_{2_V}(h(x_1)), T_{2_V}(h(y_1))).$$

Thus,

$$(\frac{k_1}{k_1'-k_1})T^c_{1_E}(x_1y_1) = (\frac{k_2}{k_2'-k_2})T^c_{2_E}(h(x_1)h(y_1))$$
$$\Leftrightarrow \quad (\frac{k_1}{k_1'-k_1})\frac{k_1'k_2}{k_1}T_{2_E}(h(x_1)h(y_1)) - (\frac{k_1}{k_1'-k_1})T_{Do}(T_{2_V}(h(x_1)),T_{2_V}(h(y_1))) = (\frac{k_2}{k_2'-k_2})T^c_{2_E}(h(x_1)h(y_1))$$
$$\Leftrightarrow \quad (\frac{k_1}{k_1'-k_1})\frac{k_1'k_2}{k_1}T_{2_E}(h(x_1)h(y_1)) - (\frac{k_1}{k_1'-k_1})T_{Do}(T_{2_V}(h(x_1)),T_{2_V}(h(y_1)))$$
$$= \quad ((\frac{k_2}{k_2'-k_2})k_2'T_{2_E}(h(x_1)h(y_1)) - (\frac{k_2}{k_2'-k_2})T_{Do}(T_{2_V}(h(x_1)),T_{2_V}(h(y_1))) \Leftrightarrow k_1'k_2 = k_1k_2'.$$

Let $x_1y_1 \in E$ and $T_{1_E}(x_1y_1) = 0$. Thus,

$$\begin{aligned} T^c_{1_E}(x_1y_1) &= T_{Do}(T_{1_V}(x_1),T_{1_V}(y_1)) \\ &= T_{Do}(T_{2_V}(h(x_1)),T_{2_V}(h(y_1))) = T^c_{2_E}(h(x_1)h(y_1)). \end{aligned}$$

In a similar way, $(\frac{l_1}{l_1'-l_1})I^c_{1_E}(x_1y_1) = (\frac{l_2}{l_2'-l_2})I^c_{2_E}(h(x_1)h(y_1))$ if and only if $l_1'l_2 = l_1l_2'$. In addition,

$$\begin{aligned} \frac{m_1}{1-m_1m_1'}F^c_{1_E}(x_1y_1) &= \frac{m_1}{1-m_1m_1'}F_{1_E}(x_1y_1) - (\frac{m_1}{1-m_1m_1'})m_1'T_{Do}(F_{1_V}(x_1),F_{1_V}(y_1)) \\ &= (\frac{m_1}{1-m_1m_1'})\frac{m_2}{m_1}F_{2_E}(h(x_1)h(y_1)) - (\frac{m_1}{1-m_1m_1'})m_1'T_{Do}(F_{2_V}(h(x_1)),F_{2_V}(h(y_1))). \end{aligned}$$

Thus,

$$(\frac{m_1}{1-m_1m_1'})F^c_{1_E}(x_1y_1) = (\frac{m_2}{1-m_2m_2'})F^c_{2_E}(h(x_1)h(y_1))$$
$$\Leftrightarrow \quad (\frac{m_1}{1-m_1m_1'})\frac{m_2}{m_1}F_{2_E}(h(x_1)h(y_1)) - (\frac{m_1}{1-m_1m_1'})m_1'T_{Do}(F_{2_V}(h(x_1)),F_{2_V}(h(y_1)))$$
$$= \quad ((\frac{m_2}{1-m_2m_2'})F_{2_E}(h(x_1)h(y_1)) - (\frac{m_2}{1-m_2m_2'})m_2'T_{Do}(F_{2_V}(h(x_1)),F_{2_V}(h(y_1))) \Leftrightarrow m_1m_1' = m_2m_2'.$$

Let $x_1y_1 \in E$ and $F_{1_E}(x_1y_1) = 0$. Thus,

$$\begin{aligned} F^c_{1_E}(x_1y_1) &= T_{Do}(F_{1_V}(x_1),F_{1_V}(y_1)) \\ &= T_{Do}(F_{2_V}(h(x_1)),F_{2_V}(h(y_1))) = F^c_{2_E}(h(x_1)h(y_1)). \end{aligned}$$

(ii), (iii) are obtained from (i). □

Theorem 3.12. *Let $G = (\sigma, \mu, T_{Do})$ be a self (k', l', m')-complemented of strong (k, l, m)-DING on $G^* = (V, E)$. Then*

(i) $\sum\limits_{\substack{x\neq y \\ xy\in E}} T_E(xy) = (\frac{k'^2-kk'}{k^2})\sum\limits_{x\neq y} T_{Do}(T_V(x),T_V(y))$.

(ii) $\sum\limits_{\substack{x\neq y \\ xy\in E}} I_E(xy) = (\frac{l'^2-ll'}{k^2})\sum\limits_{x\neq y} T_{Do}(I_V(x),I_V(y))$.

(iii) $\sum\limits_{\substack{x\neq y\\ xy\in E}} F_E(xy) = (\frac{m'-mm'^2}{m^2})\sum\limits_{x\neq y} T_{Do}(F_V(x),F_V(y))$.

(iv) $k'>k, l'>l$ *and* $mm'<1$.

Proof. $(i),(ii),(iii)$ Let $G=(\sigma,\mu,T_{Do})$ be a self (k',l',m')-complemented (k,l,m)-DING on $G^*=(V,E)$. Since $G\cong G^{(c,(k',l',m'))}$, there exists an isomorphism $f:V\to V$ such that for any $xy\in E$,

$$k'T_E^c(f(x)f(y))=kT_E(xy), l'I_E^c(f(x)f(y))=lI_E(xy), m'F_E^c(f(x)f(y))=mF_E(xy),$$

and for any $x\in V$,

$$T_V^c(f(x))=T_V(x), I_V^c(f(x))=I_V(x), F_V^c(f(x))=F_V(x).$$

It follows that

$$\begin{aligned} & kT_E(xy)=k'T_E^c(f(x)f(y))=k'(k'T_E(f(x)f(y))-T_{Do}(T_V(f(x)),T_V(f(y))) \\ = \; & k'(\frac{k'}{k}T_{Do}(T_V(f(x)),T_V(f(y))-T_{Do}(T_V(f(x)),T_V(f(y)))=k'(\frac{k'}{k}-1)T_{Do}(T_V(f(x)),T_V(f(y)). \end{aligned}$$

Hence $T_E(xy)=(\frac{k'^2-kk'}{k^2})T_{Do}(T_V(f(x)),T_V(f(y))$. In a similar way, one can see that $I_E(xy)=(\frac{k'^2-kk'}{k^2})T_{Do}(I_V(f(x)),I_V(f(y))$ and so for any $xy\in E$, we have

$$\sum_{\substack{x\neq y\\ xy\in E}} T_E(xy)=(\frac{k'^2-kk'}{k^2})\sum_{x\neq y} T_{Do}(T_V(x),T_V(y)) \text{ and } \sum_{\substack{x\neq y\\ xy\in E}} I_E(xy)=(\frac{l'^2-ll'}{k^2})\sum_{x\neq y} T_{Do}(I_V(x),I_V(y)).$$

In addition,

$$\begin{aligned} & mF_E(xy)=m'F_E^c(f(x)f(y))=m'(F_E(f(x)f(y))-m'T_{Do}(F_V(f(x)),F_V(f(y))) \\ = \; & m'(\frac{1}{m}T_{Do}(F_V(f(x)),F_V(f(y))-m'T_{Do}(F_V(f(x)),F_V(f(y)))=m'(\frac{1-mm'}{m})T_{Do}(F_V(f(x)),F_V(f(y)). \end{aligned}$$

Thus, $F_E(xy)=(\frac{m'-mm'^2}{m^2})T_{Do}(F_V(f(x)),F_V(f(y))$ and so

$$\sum_{\substack{x\neq y\\ xy\in E}} F_E(xy)=(\frac{m'-mm'^2}{m^2})\sum_{x\neq y} T_{Do}(F_V(x),F_V(y)).$$

(iv) Since $\frac{k'^2-kk'}{k^2}>0$, $\frac{l'^2-ll'}{l^2}>0$ and $\frac{m'-mm'^2}{m^2}>0$, we get that $k'>k, l'>l$ and $mm'<1$. □

Let $G=(\sigma,\mu,T_{Do})$ be a (k,l,m)-DING on $G^*=(V,E), k',l',m'\in\mathbb{R}^{>0}, \alpha=\frac{k'}{k'^2-k}, \beta=\frac{l'}{l'^2-l}, \gamma=\frac{m'-mm'^2}{m^2}, \alpha'=\frac{1}{k-1}, \beta'=\frac{1}{l-1}$ and $\gamma'=\frac{1-m^2}{m}$. Thus, we have the following theorem.

Theorem 3.13. *Let $G=(\sigma,\mu,T_{Do})$ be a (k,l,m)-DING on $G^*=(V,E), k',l',m'\in\mathbb{R}^{>0}$.*

(*i*) *If* $\mu(xy) = (\alpha(T_{Do}(T_V(x), T_V(y))), \beta(T_{Do}(I_V(x), I_V(y))), \gamma(T_{Do}(F_V(x), F_V(y))))$, *then* $k' > \sqrt{k}, l' > \sqrt{l}, m' < \frac{1}{m}$ *and* G *is a self* (k', l', m')*-complemented* (k, l, m)*-DING.*

(*ii*) *If* $\mu(xy) = (\alpha'(T_{Do}(T_V(x), T_V(y))), \beta'(T_{Do}(I_V(x), I_V(y))), \gamma'(T_{Do}(F_V(x), F_V(y))))$, *then* $k > 1, l > 1, m < 1$ *and* G *is a self* (k, l, m)*-complemented* (k, l, m)*-DING.*

Proof. (*i*) Let $T_E(xy) = (\frac{k'}{k'^2 - k})(T_{Do}(T_V(x), T_V(y)))$ and $f : V \underset{x \mapsto x}{\to} V$. Then

$$\begin{aligned} k'T_E^c(xy) &= k'(k'T_E(xy) - T_{Do}(T_V(x), T_V(y))) \\ &= k'(k'(\frac{k'}{k'^2 - k})(T_{Do}(T_V(x), T_V(y))) - T_{Do}(T_V(x), T_V(y)))) \\ &= (\frac{kk'}{k'^2 - k})T_{Do}(T_V(x), T_V(y))) = kT_E(xy). \end{aligned}$$

It follows that G is a self k'-complement DING. It is similar to equation $l'I_E^c(xy) = lTI_E(xy)$ if and only if $I_E(xy) = \frac{l'}{l'^2 - l}T_{Do}(I_V(x), I_V(y)))$. In addition, $F_E(xy) = (\frac{k'}{k'^2 - k})(T_{Do}(F_V(x), F_V(y)))$, implies that

$$\begin{aligned} m'F_E^c(xy) &= m'(F_E(xy) - m'T_{Do}(F_V(x), F_V(y))) \\ &= m'((\frac{1}{m})(T_{Do}(F_V(x), F_V(y))) - m'T_{Do}(F_V(x), F_V(y)))) \\ &= (\frac{m' - mm'^2}{m^2})T_{Do}(F_V(x), F_V(y))) = mF_E(xy). \end{aligned}$$

(*ii*) Consider $(k', l', m') = (k, l, m)$, then by (*i*), G is a self (k, l, m)-complemented (k, l, m)-DING. □

Theorem 3.14. *Let* $G = (\sigma, \mu, T_{Do})$ *be a* $(1, 1, \frac{1}{4})$*-DING on* $G^* = (V, E)$. *Then* $G = (\sigma, \mu, T_{Do})$ *is a self* $(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, 2 + \sqrt{3})$*-complemented DING.*

Proof. Let $G = (\sigma, \mu, T_{Do})$ be a DING on $G^* = (V, E)$. If $G = (\sigma, \mu, T_D)$ is a self (k, l, m)-complemented Dombi inverse fuzzy graph, then

$$\begin{aligned} &kT_E^c(xy) = k(kT_E(xy) - T_E(xy)) = k^2T_E(xy) - kT_E(xy) = T_E(xy), \\ &kI_E^c(xy) = k(kI_E(xy) - I_E(xy)) = k^2I_E(xy) - kI_E(xy) = I_E(xy), \\ &kF_E^c(xy) = k(F_E(xy) - kT_{Do}(F_V(x), F_V(y))) = kF_E(xy) - k^2T_{Do}(F_V(x), F_V(y)) = \frac{1}{4}F_E(xy), \end{aligned}$$

because of, $T_E(xy) = T_{Do}(T_V(x), T_V(y)), I_E(xy) = T_{Do}(I_V(x), I_V(y))$ and $\frac{1}{4}F_E(xy) = T_{Do}(F_V(x), F_V(y))$. It follows that, $k = l = \frac{1+\sqrt{5}}{2}$ and $m = 2 + \sqrt{3}$. □

Corollary 3.15. *Let* $G = (\sigma, \mu, T_{Do})$ *be a DING on* $G^* = (V, E)$. *Then* $G = (\sigma, \mu, T_{Do})$ *is't a self complement DING.*

Proof. Let $G = (\sigma, \mu, T_{Do})$ be a self complemented DING on $G^* = (V, E)$ and $T_E(xy) \neq 0, I_E(xy) \neq 0$ and $F_E(xy) \neq 0$ for any $x, y \in V$. By Theorem 3.14, $k = l = \frac{1+\sqrt{5}}{2}$. In addition, $kF_E^c(xy) = F_E(xy)$, implies that $m = \frac{1 + \sqrt{3}i}{2} \notin \mathbb{R}^{>0}$, which is a contradiction. □

Theorem 3.16. *Let $G = (\sigma, \mu, T_{Do})$ be a (k, l, m)-DING. Then $G^{(c,(c,(k',l',m')))} \cong G$ if and only if $\mu(xy) = (\delta T_{Do}(T_V(x), T_V(y)), \theta T_{Do}(I_V(x), I_V(y)), \vartheta T_{Do}(F_V(x), F_V(y))$, where $\delta = \frac{k'^2 + k'}{k'^3 - k}, \theta = \frac{l'^2 + l'}{l'^3 - l}$ and $\vartheta = \frac{2m'^2}{m' - m}$.*

Proof. Let $xy \in E$ and $x \in V$. Then $(T_V^c)^c(x) = T_V(x), (I_V^c)^c(x) = I_V(x), (F_V^c)^c(x) = F_V(x)$. In addition, $k'((T_E^c)^c(xy)) = kT_E(xy)$ if and only if

$$\begin{aligned} & k'\Big(k'(T_E^c(xy)) - T_{Do}(T_V(x), T_V(y))\Big) = kT_E(xy) \\ \iff \quad & k'\Big(k'(k'T_E(xy) - T_{Do}(T_V(x), T_V(y)))) - T_{Do}(T_V(x), T_V(y))\Big) = kT_E(xy) \\ \iff \quad & T_E(xy) = (\frac{k'^2 + k'}{k'^3 - k})(T_{Do}(T_V(x), T_V(y))). \end{aligned}$$

In a similar way, $l'((I_E^c)^c(xy)) = lI_E(xy)$ if and only if $I_E(xy) = (\frac{l'^2 + l'}{l'^3 - l})(T_{Do}(I_V(x), I_V(y)))$. Moreover, $m'((F_E^c)^c(xy)) = mF_E(xy)$ if and only if

$$\begin{aligned} & m'\Big(F_E^c(xy)) - m'T_{Do}(F_V(x), F_V(y))\Big) = mF_E(xy) \\ \iff \quad & m'\Big((F_E(xy) - m'T_{Do}(F_V(x), F_V(y)))) - m'T_{Do}(F_V(x), F_V(y))\Big) = mF_E(xy) \\ \iff \quad & F_E(xy) = (\frac{2m'^2}{m' - m})(T_{Do}(F_V(x), F_V(y))). \end{aligned}$$

It follows that $G^{(c,(c,(k',l',m')))} \cong G$ if and only if $\mu(xy) = (\delta T_{Do}(T_V(x), T_V(y)), \theta T_{Do}(I_V(x), I_V(y)), \vartheta T_{Do}(F_V(x), F_V(y))$, where $\delta = \frac{k'^2 + k'}{k'^3 - k}, \theta = \frac{l'^2 + l'}{l'^3 - l}$ and $\vartheta = \frac{2m'^2}{m' - m}$. □

Let $n \in \mathbb{N}$. Then from now on, will denote $G^{(c,(c,(k',l',m')))}$ by $G^{(c^2,(k',l',m'))}$, $G^{(c,c,(c,(k',l',m')))}$ by $G^{(c^3,(k',l',m'))}$ and $G^{(\underbrace{c, c, \ldots, c}_{(n-1)\text{-times}},(c,(k',l',m')))}$ by $G^{(c^n,(k',l',m'))}$. Additionally consider $\delta_n = \frac{k'(k'^n - 1)}{((k')^{n+1} - k)(k' - 1)}), \theta_n = \frac{l'(l'^n - 1)}{((l')^{n+1} - l)(l' - 1)})$ and $\vartheta_n = \frac{nm'^2}{m' - m}$.

Corollary 3.17. *Let $G = (\sigma, \mu, T_{Do})$ be a (k, l, m)-DING and $k \in \mathbb{R}^{>0}$. Then*

(i) $G^{(c^n,(k',l',m'))} \cong G \Leftrightarrow \mu(xy) = (\delta_n(T_{Do}(T_V(x), T_V(y)), \theta_n(T_{Do}(I_V(x), I_V(y)), \vartheta_n(T_{Do}(F_V(x), F_V(y)))$.

(ii) $G^{(c^n,(k,l,2m))} \cong G \Leftrightarrow \mu(xy) = (a(T_{Do}(T_V(x), T_V(y))), b(T_{Do}(I_V(x), I_V(y))), (c(T_{Do}(F_V(x), F_V(y)))$, *where $a = \frac{1}{k - 1}, b = \frac{1}{l - 1}$ and $c = 4nm$.*

(ii) $G^{(c^n,(k,l,m))} \ncong G$

Theorem 3.18. *Let $G = (\sigma, \mu, T_{Do})$ be a complete (k, l, m)-DING and $k, l, m \in \mathbb{R}^{>0}$. Then*

(i) $G^{(c,(k',l',m'))} = (\sigma^c, \mu^c, T_{Do})$ *is a complete* $(\frac{k}{k' - k}, \frac{l}{l' - l}, \frac{1 - mm'}{m})$*-DING.*

(ii) $G^{(c^n,(k',l',m'))}$ *is a complete* $(\frac{k}{k'^n - k(\sum\limits_{0 \le i \le n-1} k'^i)}, \frac{j}{j'^n - j(\sum\limits_{0 \le i \le n-1} j'^i)}, \frac{1 - nmm'}{m})$*-DING.*

Proof. (i) Let $xy \in E$. Then

$$\begin{aligned} T_E^c(xy) &= k'T_E(xy) - T_{Do}(T_V(x), T_V(y)) = \frac{k'}{k}T_{Do}(T_V(x), T_V(y)) - T_{Do}(T_V(x), T_V(y)) \\ &= (\frac{k'-k}{k})T_{Do}(T_V(x), T_V(y)). \end{aligned}$$

In a similar way, $I_E^c(xy) = (\frac{l'-l}{l})T_{Do}(I_V(x), I_V(y))$. Moreover,

$$\begin{aligned} F_E^c(xy) &= F_E(xy) - m'T_{Do}(F_V(x), F_V(y)) = \frac{1}{m}T_{Do}(F_V(x), F_V(y)) - m'T_{Do}(F_V(x), F_V(y)) \\ &= (\frac{1-mm'}{m})T_{Do}(T_V(x), T_V(y)). \end{aligned}$$

(ii) It is obtained from (i) and by induction. □

Corollary 3.19. *Let $n \in \mathbb{N}, G = (\sigma, \mu, T_{Do})$ be a complete (k, l, m)-DING and $k, l, m \in \mathbb{R}^{>0}$. Then $G^{(c^n,(k,l,m))}$ can't be a complete (k, l, m)-DING.*

Definition 3.20. *Let $G = (\sigma, \mu, T_{Do})$ be a (k, l, m)-DING on a simple graph $G^* = (V, E)$. Then $O_T(G) = \sum\limits_{x\in V} T_V(x)$ (as truth membership order of G), $O_I(G) = \sum\limits_{x\in V} I_V(x)$ (as indeterminacy membership order of G), $O_F(G) = \sum\limits_{x\in V} F_V(x)$ (as falsity membership order of G), $S_T(G) = k \sum\limits_{\substack{xy\in E \\ x\neq y}} T_E(xy)$(as truth membership size of G), $S_I(G) = l \sum\limits_{\substack{xy\in E \\ x\neq y}} I_E(xy)$(as indeterminacy membership size of G) and $S_F(G) = m \sum\limits_{\substack{xy\in E \\ x\neq y}} F_E(xy)$(as indeterminacy membership size of G).*

Theorem 3.21. *Let $G = (\sigma, \mu, T_{Do})$ be a (k, l, m)-DING and $G' = (\sigma', \mu', T_{Do})$ be a (k', l', m') on $G^* = (V, E)$ and $G'^* = (V', E')$, respectively. Then*

(i) $O_T(G) = O_T(G')$, $O_I(G) = O_T(G')$ and $O_F(G) = O_F(G')$.
(ii) $S_T(G) = S_T(G')$, $S_I(G) = S_I(G')$ and $S_F(G) = S_F(G')$.

Proof. It is clear by definition. □

Example 3.22. *Let $V = \{a, b, c\}$. Then $G_1 = (\sigma_1, \mu_1, T_{Do}) \cong G_2 = (\sigma_2, \mu_2, T_{Do})$ are $(5, 5, 0.2)$-DING and G_2 and $(4, 6, 0.4)$-DING, respectively as shown in Figure 5. Computations show that $O_T(G_1) = O_T(G_2)$, $O_I(G_1) = O_T(G_2), O_F(G_1) = O_F(G_2), S_T(G_1) = S_T(G_2)$, $S_I(G_1) = S_I(G_2)$ and $S_F(G_1) = S_F(G_2)$, while $G_1 \ncong G_2$.*

Example 3.22, shows that the converse of the Theorem 3.24, is not necessarily true.

Definition 3.23. *Let $G = (\sigma, \mu, T_{Do})$ be a (k, l, m)-DING on a simple graph $G^* = (V, E)$ and $x \in V$. Then $d_T(x, G) = \sum\limits_{xy\in E} T_E(xy)$ (as truth membership value of x), $d_I(x, G) = \sum\limits_{xy\in E} I_E(xy)$ (as indeterminacy membership value of x), $d_F(x, G) = \sum\limits_{xy\in E} F_E(xy)$ (as falsity membership value of x), $d_T(xy, G) = T_V(x) + T_V(y)$ (as truth membership value of xy), $d_I(xy, G) = I_V(x) + I_V(y)$ (as indeterminacy membership value of xy) and $d_F(xy, G) = F_V(x) + F_V(y)$ (as falsity membership value of xy).*

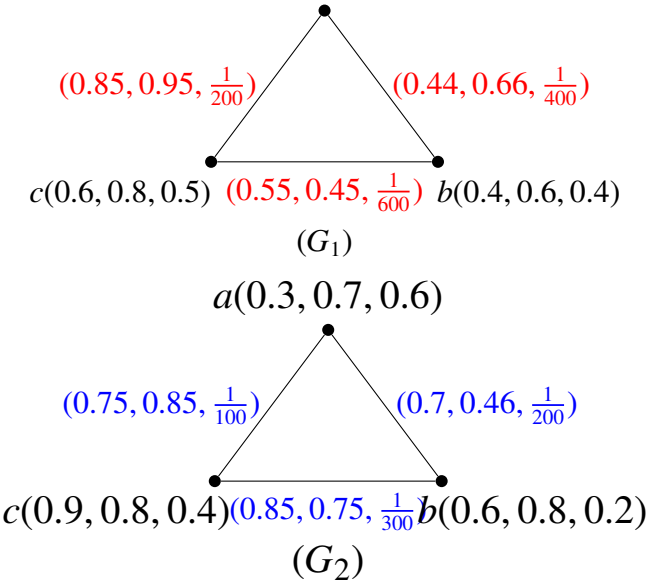

Figure 5. $(5, 5, 0.2)$-DING G_1 , $(4, 5, 0.4)$-DING G_2.

Theorem 3.24. *Let $G = (\sigma, \mu, T_{Do})$ be a (k, l, m)-DING and $G' = (\sigma', \mu', T_{Do})$ be a (k', l', m') on $G^* = (V, E)$ and $G'^* = (V', E')$, respectively. Then*

(i) $d_T(x, G) = \frac{k'}{k} d_T(x, G')$, $d_T(x, G) = \frac{l'}{l} d_T(x, G')$ *and* $d_T(x, G) = \frac{m'}{m} d_T(x, G')$.
(ii) $S_T(G) = 2k d_T(x, G)$, $S_I(G) = 2l d_I(x, G)$ *and* $S_F(G) = 2m d_F(x, G)$.

Proof. It is clear by definition. □

4. An application of valued inverse Dombi neutrosophic graph

In this section, we apply the concept of valued inverse Dombi neutrosophic graph in the real world. In mathematics, a network is called a graph, objects are called vertices (or nodes), and the connections are called edges. For example, when we represent the social network of a collection of people as a graph, the vertices are the people under consideration, while the edges list all the friendships between them. **Social network:** Establishing social relations between people can have an impactive on a person's life. In today's developing world, social relations are one of the basic principles for the durability and sustainability of individual life. According to the experts in sociology, social relations can be defined as communication between two people or several groups, that is, social relations are the communication and interdependence of people towards each other. These relationships can be positive and constructive, like friendship, or negative and destructive, like enmity. In the unique sense of social communication, it refers to communication that leads to the transfer of meaning or message among a group of people. Therefore, social communication requires several elements such as intelligence (as a truth membership degree), incuriosity (as an indeterministic membership degree) and deception (as a falsity membership degree). Now, we consider a complex network, which we investigated as the following algorithm:

step1: In Table 1, we introduce a group of people with some characteristics of intelligence, incuriosity, and deception in social personality that are supposed to work in a social network(for instance, Benjamin$(0.6, 0.7, 0.1)$ means that, based on psychological and scientific research, Benjamin uses 60 percent of his talent, 70 percent is curious and 10 percent is deceitful).

step2: In Figure 6 (the vertex set represents the people and the edge set represents the communication between them), we there are essential factors that can create communication between people in the group such as mutual respect (as a truth membership degree), responsibility (as an indeterministic membership degree) and insincere (as a falsity membership degree). In this step, we compute the relation between of vertices with impact $(1, 1, 10)$. For instance, $T_E(BO) = \frac{0.6 \times 0.7}{0.6 + 0.7 - 0.42} = \frac{21}{44}, I_E(BO) = \frac{0.7 \times 0.8}{0.7 + 0.8 - 0.56} = \frac{28}{47}, F_E(BO) = \frac{1}{10}\frac{0.1 \times 0.2}{0.1 + 0.2 - 0.02} = \frac{1}{140}$.

Table 1. Social personality.

	Intelligence	Incuriosity	Deception
Benjamin	0.6	0.7	0.1
Oliver	0.7	0.8	0.2
James	0.8	0.9	0.3
William	0.4	0.5	0.4
Lucas	0.5	0.6	0.5

step 3: By definition 3.23, we want to identify the best one in terms of the most mutual respect, the most responsibility and the least insincere and to identify the best co-worker in terms of the most intelligence and mutual respect, incuriosity, and responsibility, and the least deception and Insincere. Thus, computations show these aims in Tables 2 and 3. For instance, $d_T(W, G) = \frac{4}{11} + \frac{6}{19} \simeq 0.7, d_T(B, G) = \frac{21}{44} + \frac{6}{19} \simeq 0.8$ and so $d_T(BW) = 0.7 + 0.8 = 1.5$.

Table 2. Best candidate in term of mutual respect, responsibility, and insincere.

$d_T(x, G)$	$d_I(x, G)$	$d_F(x, G)$
(O, 1.5)	(O, 1.8)	(B, 0.02)
(J, 1)	(J, 1.2)	(L, 0.02)
(B, 0.8)	(B, 1)	(J, 0.03)
(W, 0.7)	(W, 0.9)	(W, 0.03)
(L, 0.4)	(L, 0.5)	(O, 0.04)

Table 3. Best candidate in term of co-worker intelligence and mutual respect, co-worker incuriosity and responsibility and co-worker deception and insincere.

$d_T(xy, G)$	$d_I(xy, G)$	$d_F(xy, G)$
(OJ, 2.5)	(OJ, 3)	(WB, 0.05)
(BO, 2.3)	(BO, 2.8)	(BO, 0.06)
(LO, 1.9)	(LO, 2.3)	(JW, 0.06)
(WJ, 1.7)	(JW, 2.1)	(LO, 0.06)
(BW, 1.5)	(BW, 1.9)	(JO, 0.07)

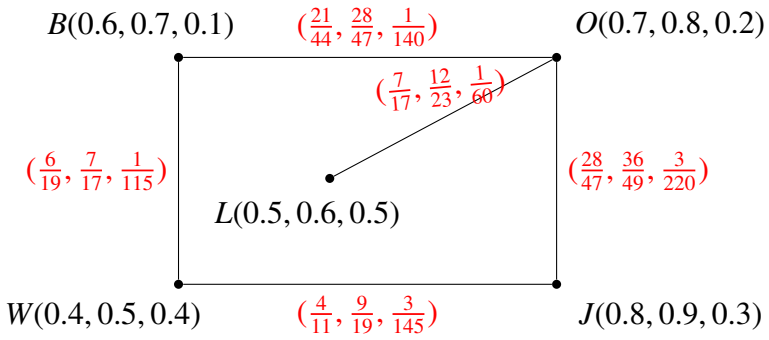

Figure 6. DING $G = (\sigma, \mu, T_{Do})$.

step 4: From Table 2, we find that the best candidate In term of mutual respect and responsibility is Oliver, and the best candidate In term of insincere is Benjamin. Additionally for any $x, y, d_T(JO) \geq d_T(xy), d_I(JO) \geq d_I(xy)$, while $d_F(JO) \leq d_T(xy)$.

5. Conclusions

The current paper has introduced a novel concept of valued-inverse Dombi neutrosophic graph and (complete) strong valued-inverse Dombi neutrosophic graph as a generalization of the inverse neutrosophic graph. This study investigates some conditions such that neutrosophic subsets be valued-inverse Dombi neutrosophic graphs and Dombi neutrosophic graphs.The complement of valued-inverse Dombi neutrosophic graphs is introduced and concerning the notation of isomorphism theorems, we can prove that the complement of a valued-inverse Dombi neutrosophic graph is a valued-inverse Dombi neutrosophic graph. The complement of valued-inverse Dombi neutrosophic graphs is extended by induction and the isomorphic complement of valued-inverse Dombi neutrosophic graphs is analyzed. Finally, we apply these concepts to the real world and design a real problem in the social network. Also

(*i*) we show that for any given (k, l, m)-DING, with what changes of parameters is a valued-inverse Dombi neutrosophic graph.
(*ii*) We show that for any given (k, l, m)-DING, conditions in parameters, is a strong valued-inverse Dombi neutrosophic graph and strong (k, l, m)-DING.
(*iii*) The solving of non-equations plays a prominent role in finding the ordered triple for a value of complement of valued-inverse Dombi neutrosophic graphs.
(*iv*) We find the conditions on ordered triples and how a complement of strong valued-inverse Dombi neutrosophic graph is strong.
(*v*) It is introduced as the notation of homomorphism (isomorphism), and it proves how the complement of two isomorphic valued-inverse Dombi neutrosophic graphs are isomorphic.
(*vi*) It introduces the concept of self-valued-inverse Dombi of strong (k, l, m)-DING and investigates the relation between the components of its neutrosophic vertices and neutrosophic edges.
(*vii*) It presents the truth membership order, indeterminacy membership order, falsity membership order, truth membership size, indeterminacy membership size, and indeterminacy membership size of any given (k, l, m)-DING.
(*viii*) The truth membership order, indeterminacy membership order, falsity membership order, truth

membership size, indeterminacy membership size, and indeterminacy membership size of any given (k, l, m)-DING play the prominent role in the application of valued-inverse Dombi neutrosophic graphs in complex networks.

References

1. M. Akram, S. Shahzadi, A. B. Saeid, Single valued neutrosophic Hypergraphs, *TWMS J. Appl. Eng. Math.*, **8** (2018), 122–135.
2. M. Akram, *Single-Valued Neutrosophic Graphs*, Singapore: Springer, 2018. https://doi.org/10.1007/978-981-13-3522-8
3. M. Akram, H. Saba Nawaz, Implementation of single-valued neutrosophic soft hypergraphs on human nervous system, *Artif. Intell. Rev.*, **56** (2022), 1387–1425.
4. M. Akram, H. Saba Nawaz, Algorithms for the computation of regular single-valued neutrosophic soft hypergraphs applied to supranational asian bodies, *J. Appl. Math. Comput.* , **68** (2022), 4479–4506.
5. D. Ajay, P. Chellamani, G. Rajchakit, N. Boonsatit, P. Hammachukiattikul, Regularity of Pythagorean neutrosophic graphs with an illustration in MCDM, *AIMS Mathematics*, **7** (2022), 9424–9442. https://doi.org/10.3934/math.2022523
6. M. Akram, M. H Sarwar, W. A. Dudek, Bipolar Neutrosophic Graph Structures, In: *Graphs for the Analysis of Bipolar Fuzzy Information*, Singapore: Springer, 2020, 393–446. https://doi.org/10.1007/978-981-15-8756-6_10
7. S. Ashraf, S. Naz, E. E. Kerre, Dombi Fuzzy Graphs, *Fuzzy Inf. Eng.*, **10** (2018), 58–79. https://doi.org/10.1080/16168658.2018.1509520
8. R. A. Borzooei, R. Almallah, Y. B. Jun, H. Ghaznavi, Inverse Fuzzy Graphs with Applications, *New Math. Nat. Comput.*, **16** (2020), 397–418. https://doi.org/10.1142/S1793005720500246
9. G. Ghorai, M. Pal, Certain types of product bipolar fuzzy graphs, *Int. J. Appl. Comput. Math.*, **3** (2017), 605–619.
10. S. Jahanpanah, M. Hamidi, Valued-Inverse Dombi Fuzzy Graph, 2023, Submitted.
11. M. Hamidi, S. Jahanpanah, A. Radfar, Extended graphs based on KM-fuzzy metric spaces, *Iran. J. Fuzzy Syst.*, **17** (2020), 81–95.
12. M. Hamidi, S. Jahanpanah, A. Radfar, On KM-Fuzzy Metric Hypergraphs, *Fuzzy Inf. Eng.*, **12** (2020), 300–321. https://doi.org/10.1080/16168658.2020.1867419

13. A. Kaufmann, *Introduction to the theory of fuzzy subsets*, New York: Academic Press, 1975.

14. T. S. Lakhwani, K. Mohanta, A. Dey, S. P. Mondal, A. Pal, Some operations on Dombi neutrosophic graph, *J. Ambient Intell. Human. Comput.*, **13** (2022), 425–443. https://doi.org/10.1007/s12652-021-02909-3

15. K. Mohanta, A. Dey, A. Pal, A note on different types of product of neutrosophic graphs, *Complex. Intell. Syst.*, **7** (2021), 857–871. https://doi.org/10.1007/s40747-020-00238-0

16. R. Mahapatra, S. Samanta, M. Pal, Generalized neutrosophic planar graphs and its application, *J. Appl. Math. Comput.*, **65** (2021), 693–712. https://doi.org/10.1007/s12190-020-01411-x

17. H. T. Nguyen, C. L. Walker, E. A. Walker, *A First Course in Fuzzy Logic*, 4 Eds., Boca Raton: CRC Press, 2018.

18. A. Rosenfeld, Fuzzy graphs, In: *Fuzzy sets and their applications to cognitive and decision processes*, New York: Academic Press, 1975, 77–95. https://doi.org/10.1016/B978-0-12-775260-0.50008-6

19. S. Siddique, U. Ahmad, M. Akram, A study on generalized graphs representations of complex neutrosophic information, *J. Appl. Math. Comput.*, **65** (2021), 481–514. https://doi.org/10.1007/s12190-020-01400-0

20. S. Sahin, A. Kargın, M. Yucel, Hausdorff Measures on Generalized Set Valued Neutrosophic Quadruple Numbers and Decision Making Applications for Adequacy of Online Education, *Neutrosophic Sets Syst.*, **40** (2021), 86–116.

21. F. Smarandache, Neutrosophic set a generalization of the intuitionistic fuzzy set, *Int. J. Pure. Appl. Math.*, **24** (2005), 287–297.

22. S. Zeng, M. Shoaib, S. Ali, F. Smarandache, H. Rashmanlou, F. Mofidnakhaei, Certain Properties of Single-Valued Neutrosophic Graph With Application in Food and Agriculture Organization, *Int. J. Comput. Intell.*, **14** (2021), 1516–1540. https://doi.org/10.2991/ijcis.d.210413.001

23. J. Zhan, M. Akram, M. Sitara, Novel decision-making method based on bipolar neutrosophic information, *Soft Comput.*, **23** (2019), 9955–9977.

A label-guided weighted semi-supervised neutrosophic clustering algorithm

Dan Zhang[a], Yingcang Ma[a,*], Hengdong Zhu[b] and Florentin Smarandache[c]
[a]*School of Science, Xi'an Polytechnic University, Xi'an, China*
[b]*Department of Public Basic Courses, Hunan Institute Of Traffic Engineering, Hengyang, Hunan*
[c]*Mathematics and Science Division, Gallup Campus, University of New Mexico, Gallup, NM, USA*

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Abstract. The traditional neutrosophic clustering method only performs cluster analysis on the data itself, and often ignores the supervision information of data. In order to solve the above problems, a label-guided weighted semi-supervised neutrosophic clustering algorithm is proposed in the paper. On the one hand, the paired constraint information is used to construct the supervision weight coefficient and the distance measurement learning is combined to re-measure the degree of membership of the data and the cluster center; On the other hand, by minimizing the sum of squares of error between membership matrix and label matrix, the purpose of clustering results guided by label information is realized. Experiments on various data sets and comparisons with other clustering algorithms show that the new clustering algorithm can make full use of supervisory information and improve the accuracy of clustering.

Keywords: Semi-supervised clustering, label information, neutrosophic set, clustering

1. Introduction

Clustering plays a crucial role in image processing [1–3], data mining [4], machine learning [5] and other fields. Traditional clustering algorithms include hard clustering and fuzzy clustering. For example, the k-means [6] algorithm is susceptible to an initial value and cannot process data sets with overlapping regions. The fuzzy c-means(FCM) [7] algorithm uses membership matrix to get the membership degree of data points belonging to each cluster, which can well divide overlapping regions, however the FCM is too sensitive to outliers and noise points. Literature [8] proposed a robust sparse fuzzy k-means algorithm (RSFKM), which introduced robust functions to deal with outliers and noise points, so as to enhance the robustness and sparsity of the FCM algorithm. Wang [9] proposed an extreme clustering method, which overcomes the shortcomings of the peak clustering algorithm on the basis of retaining the advantages of the peak clustering algorithm, and has strong robustness to unevenly distributed data sets. Tong [10] proposed a clustering algorithm based on density peaks, which can automatically determine the number of clusters. Chang [11] proposed a novel compound rank- k projection (CRP) algorithm for bilinear analysis. The CRP algorithm deals with matrices directly, thus preserving the correlations within the matrix and decreasing the computation complexity. Bui [12] proposed the F-Mapper algorithm, based on the foundation of the Mapper algorithm, to solve the problem of automating when dividing cover intervals with an arbitrary percentage of overlap. Bui [13] proposed a new method called the shape fuzzy c-means (SFCM) algorithm, and the SFCM algorithm can not only exhibit the same clustering ability as the FCM but also reveal

some relationships through visualizing the global shape of data supplied by the Mapper. These algorithms have effectively improved the performance of the FCM, but they have some limitations for fuzzy clustering of uncertain problems.

In order to solve the problems of fuzzy clustering, the neutrosophic set theory proposed by Smarandache in 1993 [14] is welcomed by scholars, and the neutrosophic clustering algorithm based on neutrosophic theory [15–17] is widely used in various fields. Guo [18] proposed the neutrosophic c-means clustering algorithm (NCM) based on the neutrosophic set framework and the FCM, which can be effectively distinguish sample points, boundary points and outliers in clusters. The membership degree of the NCM is not affected by noise, which effectively solves the problem that the FCM cannot detect not typical data points. Ye [19] proposed a single-valued neutrosophic minimum spanning tree clustering algorithm by defining a generalized single-valued neutrosophic set distance measure, which shows great superiority in the clustering of single-valued neutrosophic observation data. Kandasamy [20] proposed a double-valued neutrosophic minimum spanning tree clustering algorithm to cluster data represented by doublel-valued neutrosophic information. Thong [21] proposed an image fuzzy clustering algorithm (FC-PFS) based on image fuzzy sets. The algorithm needs to calculate three matrices, so it has high computational complexity for high-dimensional data. Li [22] proposed a single-value neutrosophic clustering algorithm based on Tsallis entropy maximization based on image fuzzy set clustering and single-value neutrosophic set. The algorithm showed satisfactory results in image segmentation. According to the density characteristics of data points, Rashno [23] proposed a neutrosophic clustering algorithm based on data uncertainty. This method deals with the membership degree of boundary points and outliers more accurately. Akbulut [24] proposed the kernel neutrosophic c-means clustering(KNCM), which uses nonlinear transformation to map the inseparable nonlinear data in the low-dimensional space to the separable high-dimensional linear feature space for clustering. It has good clustering effect.

The above algorithms do not rely on any prior information, and mainly performs cluster analysis on a datasets itself, and ignores a small volume of labeled information on the dataset, which belongs to unsupervised learning. On the contrary, supervised learning requires a large volume of label information. However, in real applications, some datasets often contain only a small volume of label information, which cannot support supervised learning. Semi-supervised learning [25], as a learning method that combines supervised learning and unsupervised learning, can use labeled data and unlabeled data at the same time, and rationally use more or less labeled information in the data.

Semi-supervised clustering is a combination of supervised information and clustering algorithms, which can guide the data clustering process more effectively. Studies have found that high-quality supervision information can not only guide unlabeled samples to correctly cluster to improve the accuracy of the clustering results, but also speed up the convergence of clustering [26, 27]. Scholars have proposed various semi-supervised clustering algorithms. Fariba [28] used the entropy and relative entropy divergence measure to replace the fuzzy measure of the unsupervised segment and the geometric distance measure of the semi-supervised segment respectively, emphasizing the explicit combination of unsupervised and semi-supervised segments. Zhuang [29] explicitly incorporated the labeled information into the graph learning method by forcing the edge weight between different types of labeled samples to zero. Nie [30] proposed using elastic embedding constraints on the predictive label matrix to better explore the manifold structure and effectively learn normal data under Gaussian distribution. Zhu [31] improved the clustering effect by combining the label information matrix and the manifold regular term, and proposed an adaptive semi-supervised neighborhood clustering algorithm. Chen [32] proposed a semi-supervised deep model that not only addresses the problem in multimodal sensor data, but also addresses the imbalanced distribution of labeled data across categories at the same time.

At present, most of the neutrosophic clustering algorithms are unsupervised neutrosophic clustering algorithms [33–35], which cannot adjust the membership matrix by using labeled information to obtain a membership matrix more in line with the real situation, thus affecting the clustering effect. To solve this problem, a label-guided weighted semi-supervised intermediate-intelligence clustering algorithm (LG-WSSNCM) is proposed in this paper. Semi-supervised learning is introduced into the neutrosophic clustering algorithm, and supervised information is used to optimize the neutrosophic clustering algorithm to improve the clustering performance.

2. Related work

2.1. Neutrosophic clustering method

Neutrosophic c-means (NCM) [10] defines the degree of membership of deterministic and uncertain clusters, so that it can handle the noise or outliers contained in the data set itself, and proposes solving the following convex optimization problems:

$$J_{NCM}(T, I, F, C) = \sum_{i=1}^{N}\sum_{j=1}^{C}\left(\varpi_1 T_{ij}\right)^m \left\|x_i - c_j\right\|_2^2 + \sum_{i=1}^{N}\left(\varpi_2 I_i\right)^m \left\|x_i - c_{i\max}\right\|_2^2 + \delta^2 \sum_{i=1}^{N}\left(\varpi_3 F_i\right)^m, \tag{1}$$

where m is a constant. T_{ij}, I_i, and F_i are true membership, uncertain membership and false membership respectively. x_i is the i-th data point, c_j is the j-th cluster center. ϖ_1, ϖ_2 and ϖ_3 are constants, which belong to [0,1]. Define $0 < T_{ij}, I_i, F_i < 1$ to satisfy the following constraints:

$$\sum_{j=1}^{C} T_{ij} + I_i + F_i = 1. \tag{2}$$

For each data x_i, use the cluster center with the largest and second largest T_{ij} values to calculate $c_{i\max}$:

$$c_{i\max} = \frac{c_{pi} + c_{qi}}{2}, \tag{3}$$

$$\begin{cases} p_i = \arg\max_{j=1,2,\ldots C}(T_{ij}), \\ q_i = \arg\max_{j \neq p_i \cap j=1,2,\ldots C}(T_{ij}) \end{cases}. \tag{4}$$

δ^2 is defined as follows:

$$\delta^2 = \lambda \frac{\sum_{i=1}^{N}\sum_{j=1}^{C}\left\|x_i - c_j\right\|_2^2}{NC}. \tag{5}$$

2.2. Supervision information

Supervision information in the clustering process usually refers to the constraint information of the sample. In semi-supervised clustering, it generally contains in two types:

(1) Sample label information: usually refers to the information with labeled data in the supervised learning algorithm. In most cases, the labeled data set is far less than the unlabeled data set, and its main function is auxiliary clustering.

(2) Paired constraint information: usually refers to the information about whether the data is in the same cluster. Literature [36, 37] proposes two commonly used paired constraint information: $must-link$ (ml) and $cannot-link$ (cl). For points x_i and x_j, if they belong to cluster C_m and C_n, the paired constraint information is defined as: if $m = n$, then x_i and x_j belong to the same cluster, then $(x_i, x_j) \in must-link$; otherwise, if $m \neq n$, then it means x_i and x_j do not belong to the same cluster, then $(x_i, x_j) \notin cannot-link$. Paired constraint information has the properties of transitivity and consistency.

3. Label-guided weighted semi-supervised neutrosophic clustering algorithm

3.1. Define supervision information

First, giving the definition of the supervision information used in this paper. Let $X \in R^{N\times d}$ be a set of n objects, the number of clusters is c. Assuming that the label information of l data points is known, the data point x_i belongs to the j-th cluster, where $i = 1, 2, ..., N$ and $j = 1, 2, ..., C$, define $y_{ij} = 1$ in the label vector, and the rest take the value 0, the definition $Y = \begin{cases} Y_l \\ Y_u \end{cases} \in R^{N\times C}$ of the label matrix can be obtained, and Y_u is the label matrix corresponding to the unlabeled data, and it is a zero matrix.

According to the label matrix, the paired constraint information can be obtained, that is, whether the data x_i and the center c_j are of the same cluster. It is defined as follows:

$$Pcl = \begin{cases} 1, & x_i \in Y_l \cap y_{ij} = 1; \\ 0, & others; \\ -1, & x_i \in Y_l \cap y_{ij} = 0. \end{cases} \tag{6}$$

where $Pcl \in R^{N\times C}$, the value of Pcl_{ij} is 1 or -1 stands for $(x_i, c_j) \in must-link$ or $(x_i, c_j) \notin cannot-link$. Therefore, the definition of weight matrix A is as follows:

$$A = \begin{cases} a, & Pcl_{ij} = 1; \\ 1, & Pcl_{ij} = 0; \\ 1/a, & Pcl_{ij} = -1. \end{cases} \tag{7}$$

where a is a constant and $0 < a < 1$.

3.2. Proposed method

It can be seen from model (1) that the distance reflects the degree of membership. A smaller distance $\left\|x_i - c_j\right\|_2^2$ corresponds to a larger degree of membership, indicating that the data point x_i is easy to be classified into the cluster corresponding to c_j; on the contrary, a larger distance $\left\|x_i - c_j\right\|_2^2$ corresponds to a smaller degree of membership, indicating that the data point x_i is not easy to be divided into the cluster corresponding to c_j. Therefore, if the paired constraint information of x_i and c_j is known, a new distance measurement function is constructed through the weight coefficient A_{ij}. Specifically, if $\left(x_i, c_j\right) \in must-link$ is known, the new distance $A_{ij}\left\|x_i - c_j\right\|_2^2$ is made smaller by applying a small weighting coefficient, thereby enlarging the value of membership; on the contrary, $\left(x_i, c_j\right) \notin cannot-link$, by applying a large weight coefficient, makes the new distance $A_{ij}\left\|x_i - c_j\right\|_2^2$ larger, thereby reducing the value of the degree of membership.

Different from the traditional least squares method $\|T - Y\|_F^2$, in order to ensure the consistency of the labeled data's label matrix and the clustering result, and to avoid the membership degree $T_u \to 0$ of the unlabeled data, the label matrix is used to construct diagonal matrix U to impose the least squares constraint on the labeled data. The diagonal elements $U_{ii} = 1$ correspond to the l labeled data, and the value of the diagonal elements of unlabeled data is 0.

From the above analysis, let $m = 2$, our optimization model is as follows:

$$\begin{aligned} J(T, I, F, C) &= \sum_{i=1}^{N}\sum_{j=1}^{C}\left(\varpi_1 T_{ij}\right)^2 A_{ij}\left\|x_i - c_j\right\|_2^2 \\ &+\alpha tr\left[(T-Y)^T U (T-Y)\right] \\ &+\sum_{i=1}^{N}\left(\varpi_2 I_i\right)^2\left\|x_i - c_{i\max}\right\|_2^2 \\ &+\delta^2\sum_{i=1}^{N}\left(\varpi_3 F_i\right)^2, \\ s.t. \quad &\sum_{j=1}^{C} T_{ij} + I_i + F_i = 1. \end{aligned} \tag{8}$$

Where T_{ij}, I_i, and F_i are the membership value of determined data points, boundary data points and noise data points respectively; Y and U are label matrices; A_{ij} is weight coefficient; ϖ_1, ϖ_2, ϖ_3 and $\alpha > 0$ are constants.

Equation (8) is solved by alternating optimization method to construct its Lagrangian function:

$$\begin{aligned} L(T, I, F, C, \lambda) &= \sum_{i=1}^{N}\sum_{j=1}^{C}\left(\varpi_1 T_{ij}\right)^2 A_{ij}\left\|x_i - c_j\right\|_2^2 \\ &+\alpha tr\left[(T-Y)^T U (T-Y)\right] \\ &+\sum_{i=1}^{N}\left(\varpi_2 I_i\right)^2\left\|x_i - c_{i\max}\right\|_2^2 \\ &+\delta^2\sum_{i=1}^{N}\left(\varpi_3 F_i\right)^2 \\ &-\sum_{i=1}^{N}\lambda_i\left(\sum_{j=1}^{C} T_{ij} + I_i + F_i - 1\right). \end{aligned} \tag{9}$$

Taking the derivative of (9), we get

$$\frac{\partial L}{\partial T_{ij}} = 2\varpi_1^2 T_{ij} A_{ij}\left\|x_i - c_j\right\|_2^2 + 2\alpha U_{ii}\left(T_{ij} - Y_{ij}\right) - \lambda_i, \tag{10}$$

$$\frac{\partial L}{\partial I_i} = 2\varpi_2^2 I_i\left\|x_i - c_{i\max}\right\|_2^2 - \lambda_i, \tag{11}$$

$$\frac{\partial L}{\partial F_i} = 2\delta^2\varpi_3^2 F_i - \lambda_i, \tag{12}$$

$$\frac{\partial L}{\partial c_j} = -2\sum_{i=1}^{N}\left(\varpi_1 T_{ij}\right)^2 A_{ij}\left(x_i - c_j\right). \tag{13}$$

By considering $\frac{\partial L}{\partial T_{ij}} = 0$, $\frac{\partial L}{\partial I_i} = 0$, $\frac{\partial L}{\partial F_i} = 0$ and $\frac{\partial L}{\partial c_j} = 0$, we get:

$$T_{ij} = \frac{2\alpha U_{ii} Y_{ij} + \lambda_i}{2\varpi_1^2 A_{ij}\left\|x_i - c_j\right\|_2^2 + 2\alpha U_{ii}} \tag{14}$$

$$I_i = \frac{\lambda_i}{2\varpi_2^2\left\|x_i - c_{i\max}\right\|_2^2} \tag{15}$$

$$F_i = \frac{\lambda_i}{2\delta^2\varpi_3^2} \tag{16}$$

$$c_j = \frac{\sum_{i=1}^{N}\left(\varpi_1 T_{ij}\right)^2 A_{ij} x_i}{\sum_{i=1}^{N}\left(\varpi_1 T_{ij}\right)^2 A_{ij}} \tag{17}$$

λ can be computed by replacing T_{ij}, I_i, F_i into the constraints in (8),

$$\sum_{j=1}^{C} \frac{2\alpha U_{ii} Y_{ij} + \lambda_i}{2\varpi_1^2 A_{ij} \left\| x_i - c_j \right\|_2^2 + 2\alpha U_{ii}} + \frac{\lambda_i}{2\varpi_2^2 \left\| x_i - c_{i\max} \right\|_2^2} + \frac{\lambda_i}{2\delta^2 \varpi_3^2} = 1 \tag{18}$$

$$\lambda_i = \frac{\left(1 - \sum_{j=1}^{c} \frac{2\alpha U_{ii} Y_{ij}}{2\varpi_1^2 A_{ij} \left\| x_i - c_j \right\|_2^2 + 2\alpha U_{ii}}\right)}{\left(\sum_{j=1}^{c} \frac{1}{2\varpi_1^2 A_{ij} \left\| x_i - c_j \right\|_2^2 + 2\alpha U_{ii}} + \frac{1}{2\varpi_2^2 \left\| x_i - c_{i\max} \right\|_2^2} + \frac{1}{2\delta^2 \varpi_3^2}\right)} \tag{19}$$

LG-WSSNCM Algorithm:

inputs: dataset X, label matrix Y, U, parameter $\varpi_1, \varpi_2, \varpi_3, \alpha$
initialization: T, I, F, δ^2
while not converge do
1. Calculate T by Equation (14)
2. Calculate A by Equation (20)
3. Calculate I by Equation (15)
4. Calculate F by Equation (16)
5. Calculate c_j by Equation (17)
6. Calculate λ_i by Equation (19)
7. Check convergence conditions $\left\| T^{k+1} - T^k \right\|_F^2 \leq \varepsilon$
end while
outputs: Optimal solution T, I, F, c_j

The time complexity of LG-WSSNCM is mainly composed of two parts: (1) To construct the supervision matrix and the weight matrix, the coefficients need to be determined according to the information of each sample point, and the time complexity is O(n^2); (2) LG -WSSNCM flow is mainly the alternating iteration of T, I, F, the time complexity of iterative update for T is O($n^2\,cd$), the time complexity of iterative update for I is O($n^2 d$), and the time complexity of iterative update for F is O($n^2 d$), where n is the number of data X, c is the number of clusters, and d is the dimension of X. If the number of iterations of the algorithm is t, the time complexity is O(tn^2cd+tn^2d). Therefore, the overall time complexity of LG-WSSNCM is O(tn^2cd+tn^2d).

3.3. The update strategy of the weight matrix

In the neutrosophic clustering method, the clustering results are obtained by dividing the membership matrix, that is, the most valuable (maximum) T_{ij} is corresponding to the j-th cluster, indicating that x_i belongs to the j-th cluster. Assuming that the number of clusters is c, it is obvious that when $T_{ij} = 1/C$ is the membership matrix is the most difficult to divide. In order to avoid the above situation and make full use of the paired constraint information, in the iterative optimization process, the obtained membership matrix is compared with $1/C$, and the weight coefficients are dynamically adjusted. The definition formula is given as follows:

$$A_{ij} = \begin{cases} a_1, & Pcl_{ij} = 1 \cap T_{ij} > \frac{1}{C}; \\ a_2, & Pcl_{ij} = 1 \cap T_{ij} \leq \frac{1}{C}; \\ 1, & others; \\ \frac{1}{a_1}, & Pcl_{ij} = -1 \cap T_{ij} > \frac{1}{C}; \\ \frac{1}{a_2}, & Pcl_{ij} = -1 \cap T_{ij} \leq \frac{1}{C}. \end{cases} \tag{20}$$

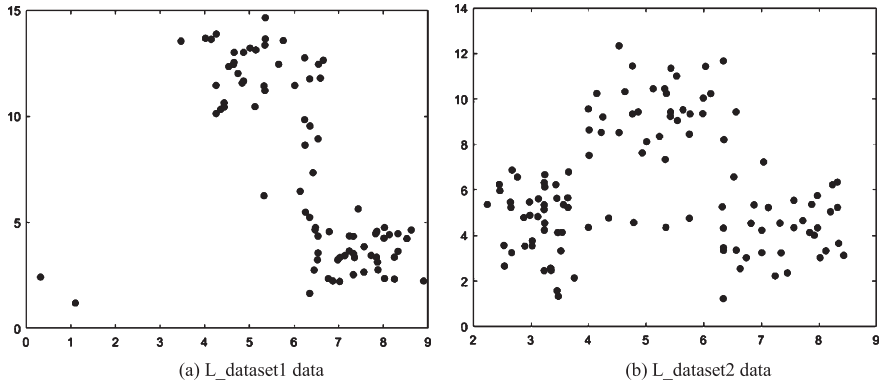
Where $0 < a_2 < a_1 < 1$, the degree of adjustment of weight coefficient is also different for different membership values. Specifically, if $(x_i, c_j) \in must - link$ is known, the corresponding membership T_{ij} is amplified by giving a smaller weighting coefficient A_{ij}. In order to carry out supervised learning with the help of paired constraint information more reasonably, considering the comparative relationship between membership degree and $1/C$, assuming $T_{ij} > 1/C$, it means that T_{ij} itself has certain distinguishing ability, and the weight coefficient a_1 does not need to be too small. If $T_{ij} \leq 1/C$, it indicates that the value of T_{ij} has a large deviation,

Table 1
Parameter selection of the method in this paper

	w_1	w_2	w_3	a_1	a_2	α
Iris	0.1	0.8	0.9	0.6	0.003	5
Zoo	0.3	0.4	0.55	0.6	0.003	5
USPSdata_20	0.3	0.4	0.45	0.6	0.001	25
MnistData_05	0.1	0.8	0.9	0.6	0.001	25
Yale32x32	0.3	0.4	0.45	0.6	0.001	25
ORL_32x32	0.1	0.8	0.9	0.6	0.001	25
COIL20data_25	0.1	0.8	0.9	0.6	0.001	25
caltech101_silhouettes_28	0.2	0.4	0.45	0.6	0.001	25

Table 2
Four artificial data sets

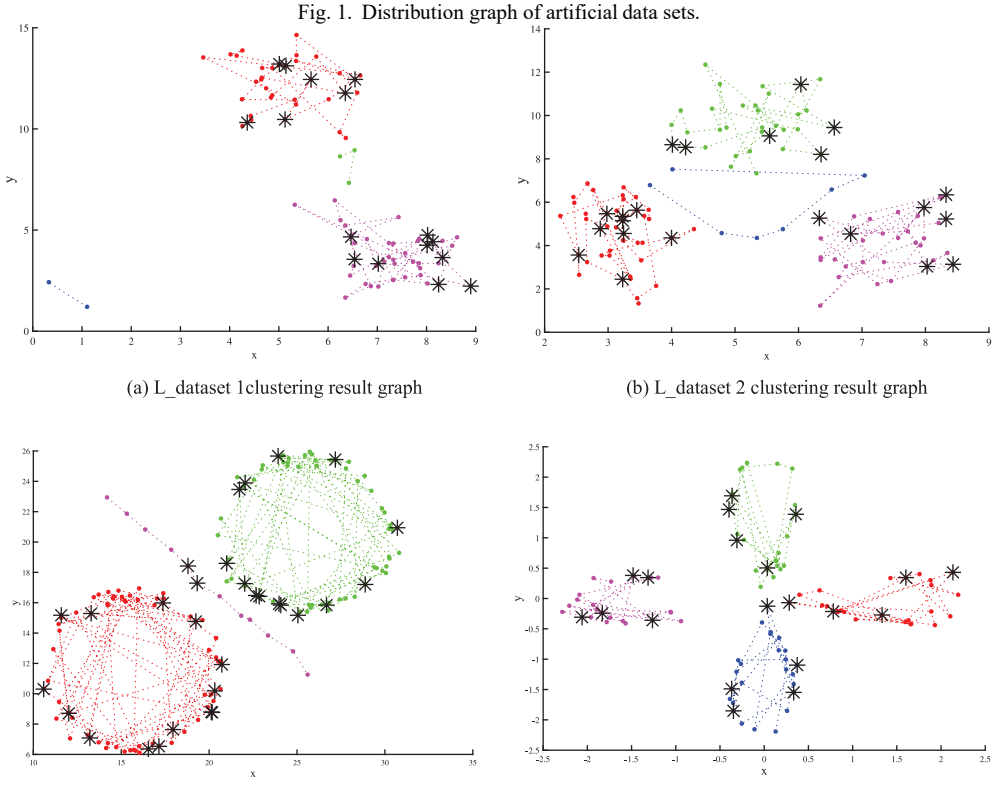
Dataset	No. of instances	No. of feature	No. of classes
L_dataset1	85	2	2
L_dataset2	115	2	3
L_dataset3	152	2	3
Cdata04	100	2	4



(a) L_dataset1 data (b) L_dataset2 data

(c) L_dataset3 data (d) Cdata04 data

Fig. 1. Distribution graph of artificial data sets.



(a) L_dataset 1clustering result graph (b) L_dataset 2 clustering result graph

(c) L_dataset 3 clustering result graph (d) Cdata04 clustering result graph

Fig. 2. Experimental results on artificial data sets.

and a small weight coefficient a_2 is needed to minimize the deviation as much as possible.

4. Experiments and results

4.1. Parameters tuning

In the following experiments, the LG-WSSNCM algorithm will be compared with the SSLRR [29], SEE [30] and CSFS [38] respectively. Accuracy(ACC) index and Rand Index(RI) are used to evaluate the clustering performance. Experimental data sets contain artificial data sets and real data sets. The artificial data sets select Cdata04 and other artificial data sets to illustrate the effectiveness of LG-WSSNCM algorithm. Iris, ZOO, USPS, Mnist, Yale, Coil20, ORL and caltech101_silhouettes_28 are used for comparison experiments.

In the parameter setting, SSLRR: $\sigma = 0.1$, $\gamma = 1$; SEE: $\lambda = 0.1$; CSFS: $k = 10$, $\alpha = 0.1$. The parameters of the LG-WSSNCM algorithm are shown in Table 1.

4.2. Experiment and analysis of artificial data sets

The detailed information of the datasets is shown in Table 2. Distribution graph of the four data sets

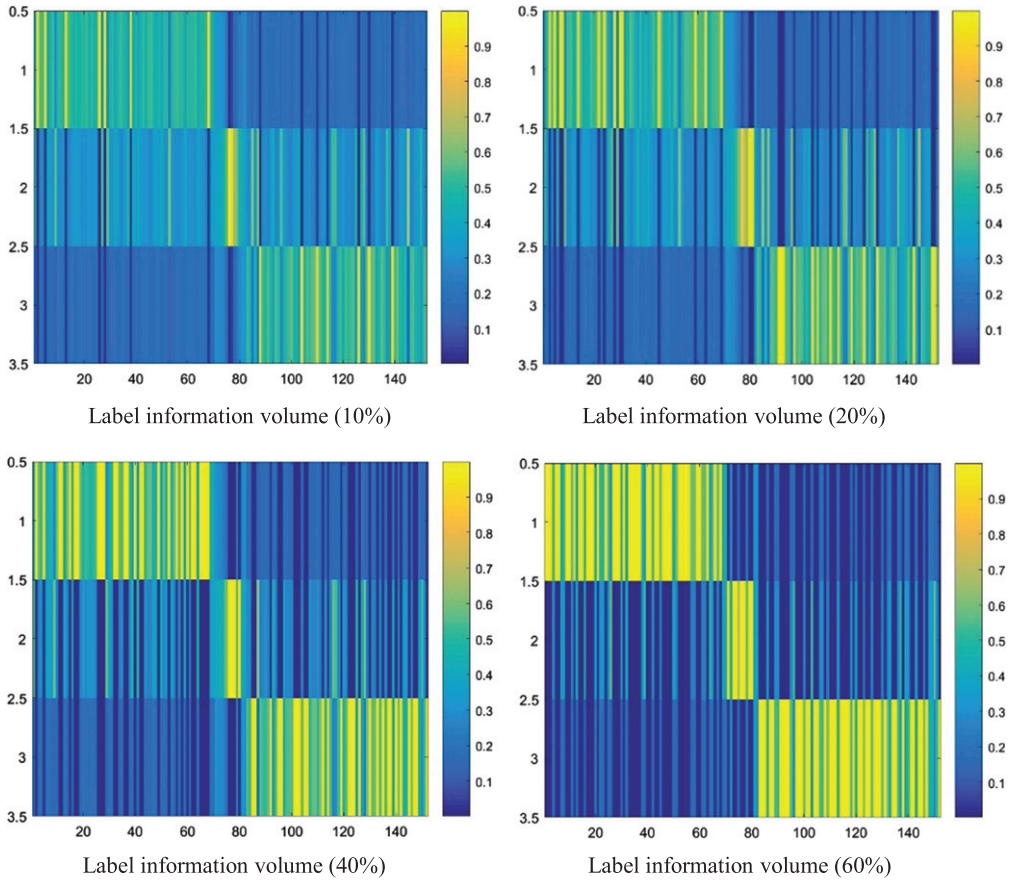
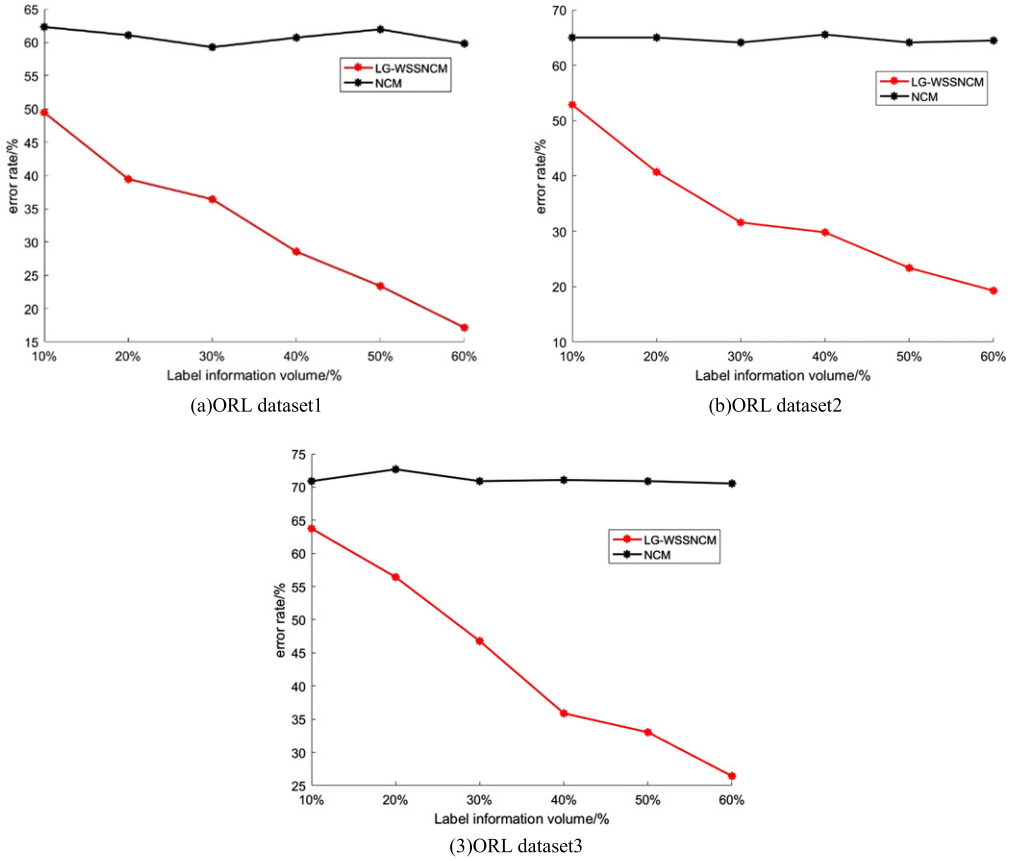


Fig. 3. Influence of the volume of labeled information on the membership matrix based on L_dataset3.

Fig. 4. The ORL face data sets.



(a)ORL dataset1 (b)ORL dataset2

(3)ORL dataset3

Fig. 5. The ORL face data of 1, 2, 3 recognition error rate of LG-WSSNCM and NCM algorithms.

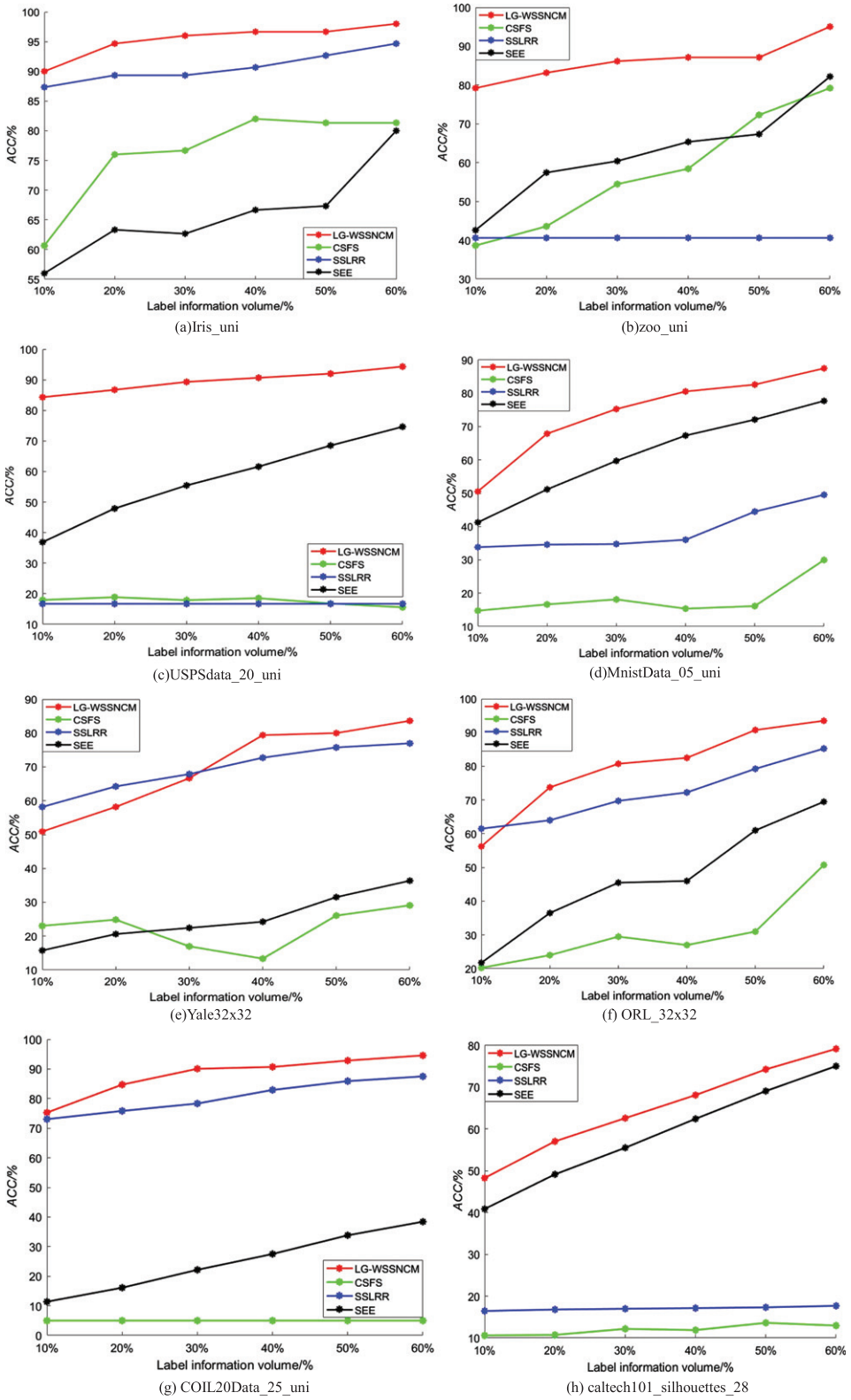
are shown in Fig. 1. In the experiment, the value of label information is 20%. Random extraction of label information will result in some data sets not being extracted. Therefore, for fairness, we extract the same proportion of label information from each class of each data set to ensure that the label information contains the information of each class. Label matrix Y is constructed according to the data point category information of each selected data set. Finally, the paired constraint information of each data set can be obtained through Equation (6).

Experimental results are shown in Fig. 2 on four datasets, where Fig. 2(a), (b), (c) and (d) represent the original data graph, and (e), (f), (g) and (h) are clustering result graph, the black "*" in the figure indicates the known label information of the points. It can be seen clearly that the clustering results satisfy the $tr(T\text{-}Y)$ in the model. At the same time, it can be seen that LG-WSSNCM has obtained accurate clustering results, and found boundary points and noise points. On the one hand, LG-WSSNCM uses paired constraint information to construct supervision weight coefficients to adjust the membership matrix. On the other hand, LG-WSSNCM uses label information to correctly guide the clustering process to obtain correct clustering results.

From the experiments, it can be seen that LG-WSSNCM can correctly guide the clustering process, to improve the clustering effect.

Table 3
UCI real data set

Datasets	No. of instances	No. of feature	No. of class
Iris	150	4	3
zoo	101	16	7
USPSdata_20	1854	265	10
MnistData_05	3495	784	10
Yale32x32	165	1024	15
ORL_32x32	400	1024	40
COIL20data_25	1440	1024	20
caltech101_silhouettes_28	8671	784	101



(a)Iris_uni (b)zoo_uni

(c)USPSdata_20_uni (d)MnistData_05_uni

(e)Yale32x32 (f) ORL_32x32

(g) COIL20Data_25_uni (h) caltech101_silhouettes_28

Fig. 6. Evaluation results of UCI real data set on clustering algorithm performance based on ACC.

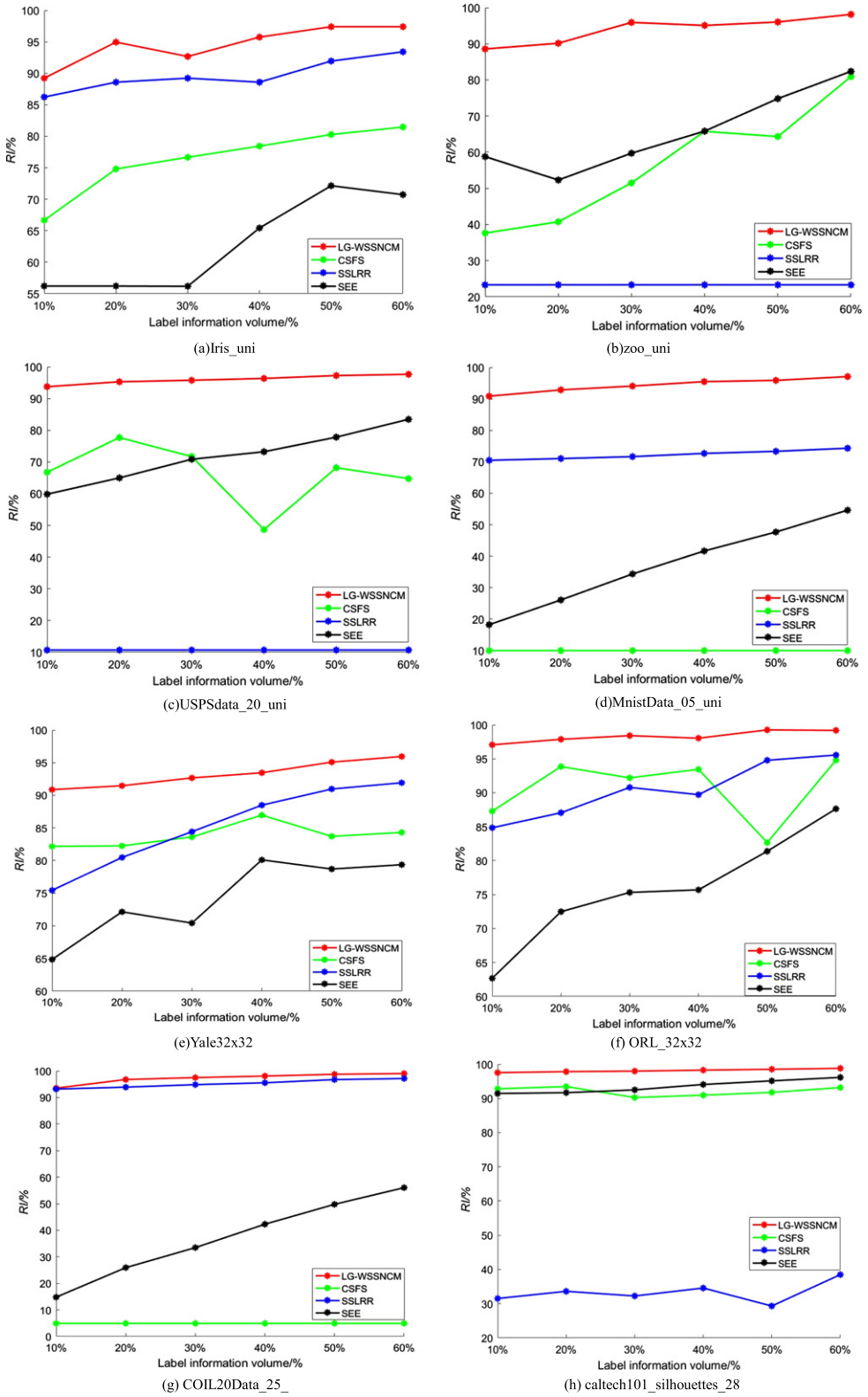


Fig. 7. Evaluation results of UCI real data set on clustering algorithm performance based on RI.

4.3. *Exploring the influence of the volume of labeled information on the distinguishability of the membership matrix T*

In this part, we use a visualized artificial dataset to explore the influence of label information on the clustering performance of LG-WSSNCM. The distinguishability of T under different volumes of label information is showed in Fig. 3. It can be found that as the volume of label information increases, the degree of membership of T becomes more distinguishable, that is, the value of the corresponding cluster of T increases.

4.4. *Face data set*

The ORL face database contains 400 face pictures of 40 people. The resolution size of each face image is 112×92. In order to reduce the storage space of the algorithm, the pixel value is standardized to [0, 1]. In this experiment, five images are randomly selected from the ORL face database. LG-WSSNCM and NCM on the ORL face were compared, as shown in Fig. 4.

In Fig. 5, the error rate is shown on ORL face. It can be found that the results of the NCM algorithm fluctuate slightly, however, it is limited by the inability to use the known label information to improve the clustering effect, so the error rate is high. While the LG-WSSNCM can improve the clustering accuracy with the help of label information, and its error rate continues to decrease as the volume of known information increases.

4.5. *Experiment and analysis on UCI data set*

In order to verify the effect of the algorithm on real data sets and determine whether the algorithm has practical significance, the eight data sets in the UCI database are used for comparison experiments, and their detailed information are shown in Table 3. The label information volume of data used in the experiment ranges from [10%, 60%].

According to the semi-supervised learning theory, as the volume of label information increases, the clustering accuracy should also improve. In Fig. 6, when the volume of label information gradually increases, the ACC of LG-WSSNCM gradually increases. However, the CSFS algorithm is descending, and the ACC of SSLRR is not changed. The NMI also has the same curve trend in Fig. 7.

In conclusion, the LG-WSSNCM can reasonably use label information to improve the clustering effect, and the clustering performance will improve with the increase of the volume of label information of the data, and finally get good clustering results. The experimental results show that the semi-supervised neutrosophic clustering algorithm proposed in this paper is reasonable and effective.

5. Conclusion

The LG-WSSNCM proposed in this paper can make full use of supervised information. On the one hand, supervised weight coefficient is constructed to adjust the membership degree of data points and clustering centers through paired constraint information, and on the other hand, label information is used to guide clustering results. Experimental results on a variety of data sets verify that the LG-WSSNCM is reasonable and effective. In the future, we will consider applying the semi-supervised neutrosophic clustering algorithm to image processing, and reduce the influence of parameters through reasonable methods.

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References

[1] L.H. Son, DPFCM, a novel distributed picture fuzzy clustering method on picture fuzzy sets, *Expert Systems with Applications* **42**(1) (2015), 51–66.

[2] J. Shan, H.D. Cheng and Y. Wang, A novel segmentation method for breast ultrasound images based on neutrosophic l-means clustering, *Medical Physics* **39**(9) (2012), 5669.

[3] Y. Guo, R. Xia, A. Şengür and K. Polat, A novel image segmentation approach based on neutrosophic c-means clustering and indeterminacy filtering, *Neural Comput & Applic* **28**(10) (1997), 3009–3019.

[4] M.C. Massi, F. Ieva and E. Lettieri, Data mining application to healthcare fraud detection: a two-step unsupervised clustering method for outlier detection with administrative databases, *BMC Medical Informatics and Decision Making* **20**(1) (2020), 160.

[5] K. Jain, M.N. Mutry and P.J. Flynn, Data clustering in a review, *Acm Computing Surveys* **31**(3) (1999), 264–323.

[6] S.J. Nanda, I. Gulati and R. Chauhan, A k-means galactic swarm optimization based clustering algorithm with otsu's entropy for brain tumor detection, *Applied Artificial Intelligence* **33**(2) (2019), 152–170.

[7] J.C. Bezdek, R. Ehrlich and W. Full, FCM.: the fuzzy c-means clustering algorithm, *Computers and Geoences* **10**(2–3) (1984), 191–203.

[8] J.L. Xu, J.W. Han and K. Xiong, Robust and sparse fuzzy k-means clustering, *Proceedings of the 25th International Joint Conference on Artificial Intelligence* (2016), 2224–2230.

[9] S. Wang, Q. Li, C. Zhao, et al., Extreme clustering – a clustering method via density extreme points, *Information Sciences* **542** (2021), 24–39.

[10] W. Tong, S. Liu, X.Z. Gao. A density-peak-based clustering algorithm of automatically determining the number of clusters, *Neurocomputing* **458**(8) (2020), 655–666.

[11] X. Chang, F. Nie, S. Yang, et al., Compound rank-k projections for bilinear analysis, *IEEE Transactions on Neural Networks & Learning Systems* **27**(7) (2016), 1502–1513.

[12] Q.T. Bui, B. Vo, H.A. Do, et al., F-mapper: a fuzzy mapper clustering algorithm, *Knowledge-Based Systems* **189** (2019), 105107.

[13] Q.T. Bui, B. Vo, V. Snasel, et al., SFCM: a fuzzy clustering algorithm of extracting the shape information of data, *IEEE Transactions on Fuzzy Systems* **29**(1) (2020),75–89.

[14] F. Smarandache, Neutrosophy, a new branch of pilosophy, *Multiple Valued Logic* **8**(3) (2002), 297.

[15] J. Ye, Single-valued neutrosophic clustering algorithms based on similarity measures, *Journal of Classification* **34**(1) (2017), 148–162.

[16] Y. Guo, A. Şengür, NECM: neutrosophic evidential c-means clustering algorithm, *Neural Computing and Applications* **26**(3) (2014), 561–571.

[17] H.V. Long, M. Ali, L.H. Son, et al., A novel approach for fuzzy clustering based on neutrosophic association matrix, *Computers & Industrial Engineering* **127** (2019), 687–697.

[18] Y. Guo and A. Sengur, NCM: neutrosophic c-means clustering algorithm, *Pattern Recognition* **48**(8) (2015), 2710–2724.

[19] J. Ye, Single-valued neutrosophic minimum spanning tree and its clustering method, *Journal of Intelligent Systems* **23**(3) (2014), 311–324.

[20] I. Kandasamy, Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm, *Journal of Intelligent Systems* **27**(2) (2018), 163–182.

[21] P.H. Thong and L.H. Son, Picture fuzzy clustering: a new computational intelligence method, *Soft Computing* **20**(9) (2016), 3549–3562.

[22] Q. Li, Y. Ma and F. Smarandache, et al., Single-valued neutrosophic clustering algorithm based on tsallis entropy maximization, *Axioms* **7**(3) (2018), 57.

[23] E. Rashno, B. Minaei-Bidgoli and Y Guo, An effective clustering method based on data indeterminacy in neutrosophic set domain, *Engineering Applications of Artificial Intelligence* **89** (2020), 103411.

[24] Y. Akbulut, A. Şengür, Y.Guo, et al., KNCM: Kernel neutrosophic c-means clustering, *Applied Soft Computing* **52** (2017), 714–724.

[25] C.J. Merz, D.C.S. Clair and W.E. Bond, Semi-supervised adaptive resonance theory, *International Joint Conference on Neural Networks* (1992), 851–856.

[26] Y. Liu and X. Zhang, Semi-supervised spectral clustering based on density adaptive neighborhood similarity graphs, *Computer Application Research* **37**(9) (2020), 2604–2609.

[27] S. Qiu, F. Nie, X. Xu and C. Qing, Accelerating flexible mani-fold embedding foer scalable semi-supervised learning, *IEEE Transactions on Circuits & Systems for Video Technology* **29**(9) (2018), 2786–2795.

[28] S. Fariba, R.K. Mohammad and S. Arash, SMKFC-ER: Semi-supervised multiple kernel fuzzy clustering based on entropy and relative entropy, *Information Sciences* **547** (2021), 667–688.

[29] L. Zhuang, Z. Zhou, S. Gao, J. Yin, Z. Lin and Y. Ma, Label information guided graph construction for semi-supervised learning, *IEEE Transactions on Image Processing* **26**(9) (2017), 4182–4192.

[30] F. Nie, H. Wang, H. Huang and C. Ding, Adaptive loss minimization for semi-supervised elastic embedding, *Proceedings of the 23th International Joint Conference on Artificial Intelligence* (2013), 1565–1571.

[31] H. Zhu, Y. Ma and X. Dai, Adaptive semi-supervised neighborhood clustering algorithm, *Journal of Shandong University(Engineering Science)* **51**(4) (2021), 24–34.

[32] K. Chen, L. Yao, D. Zhang, et al., A semisupervised recurrent convolutional attention model for human activity recognition, *IEEE Transactions on Neural Networks and Learning Systems* **31**(5) (2020), 1747–1756.

[33] B. Prasanth and N. Nagamalleswararao, Optimal kernel based neutrosophic soft sets clustering for image segmentation based on pareto optimal algorithm, *International Journal of Recent Technology and Engineering* **8**(4) (2019), 582–591.

[34] H.V. Long, M. Ali, L.H. Son, M. Khan and D.N. Tu, A novel approach for fuzzy clustering based on neutrosophic association matrix, *Computers & Industrial Engineering* **127**(2018), 687–697.

[35] L. Madan, K. Lakhwinder and G. Savita, Automatic segmentation of tumors in B-Mode breast ultrasound images using information gain based neutrosophic clustering, *Journal of X-ray Science and Technology* **26**(2) (2018), 209–225.

[36] K. Wagstaff and C. Cardie, Clustering with instance level constraints, *Proceedings of 17th International Conference on Machine Learning* (2000), 1097–1103.

[37] K. Wagstaff, C. Cardie, S. Roger and S. Schroedl, Constrained k-means clustering with background knowledge, *Proceedings of 18th International Conference on Machine Learning* (2001), 577–584.

[38] X. Chang, F. Nie, Y. Yang and H. Huang, A convex formulation for semi-supervised multi-label feature selection, *AAAI'14 Proceedings of the 28th AAAI Conference on Artificial Intelligence* (2014), 1171–1177.

A hybrid time series forecasting method based on neutrosophic logic with applications in financial issues

Seyyed Ahmad Edalatpanah [a], Farnaz Sheikh Hassani [a], Florentin Smarandache [b], Ali Sorourkhah [a], Dragan Pamucar [c,d,e,*], Bing Cui [f]

[a] *Ayandegan Institute of Higher Education, Tonekabon, Iran*
[b] *University of New Mexico, United States*
[c] *Department of Computer Science and Mathematics, Lebanese American University, Byblos 1102 2801, Lebanon*
[d] *College of Engineering, Yuan Ze University, Taoyuan City 320315, Taiwan*
[e] *Department of Operations Research and Statistics, Faculty of Organizational Sciences, University of Belgrade, Belgrade, Serbia*
[f] *Alliance Manchester Business School, The University of Manchester, Manchester, UK*

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ABSTRACT

Rising market demands, economic pressures, and technological advancements have spurred researchers to seek ways to enhance business environments and scientific productivity. Predictive science, crucial in this context, has gained prominence due to the rapid progress in information technology and forecasting algorithms. Time series forecasting, widely used in fields like engineering, economics, tourism, and energy, has inherent limitations with classical statistical methods, leading researchers to explore artificial intelligence and fuzzy logic for more ac-curate predictions. However, despite extensive efforts to improve accuracy, challenges persist. The research introduces a model aimed at surpassing existing methods in time series forecasting accuracy. This approach combines meta-heuristic optimization algorithms and neutrosophic logic to enhance precision in uncertain and complex environments, promising improved forecasting outcomes. The study shows that the performance of the neutrosophic time series modeling approach is highly dependent on the optimal selection of the universe of discourse and its corresponding intervals. This study selects the quantum optimization algorithm (QOA), genetic algorithm (GA), and particle swarm optimization (PSO) to address this weakness. These optimization algorithms improve the performance of the NTS modeling approach by selecting the global universe of discourse and cor-responding intervals from the list of locally optimal solutions. The proposed hybrid model (i.e., NTS-QOA model) is verified and validated with datasets of university enrollment of Alabama (USA), Taiwan futures exchange (TAIFEX) index, and Taiwan Stock Exchange Corporation (TSEC) weighted index. Various experimental results signified the efficiency of the proposed model over existing benchmark models in terms of average forecasting error rate (AFER). This value using the proposed NTS QOA, NTS GA, and NTS PSO method on the university dataset is 0.166, 0.167, 0.164, on the TAIFEX dataset, is 0.081, 0.081, and 0.081, and on the TSEC dataset is 0.09, 0.09, and 0.09, respectively.

1. Introduction

Researchers have been driven by the urgent demands of the market, economy, and production industries, as well as the need to keep up with advancing technology to seek out ways to enhance the business environment and scientific production (Shemshad and Karim, 2023). Among these efforts, forecasting science is particularly essential (Taghipourian et al., 2021). Forecasting involves predicting future events based on scientific and logical principles and rules, making it a crucial tool in many contexts. We require information and data and practical and rational methods for data analysis to achieve accurate predictions (Shoaee and Gholi Keshmarzi, 2023).

Forecasting is essential for effective planning, regardless of the circumstances or time horizon. With the rapid advancement of information technology and forecasting algorithms, relying solely on existing methods may no longer suffice (Kuranga et al., 2023). In today's world,

informed decision-making requires accurate forecasting, and time series forecasting is a widely studied method with applications across various fields such as engineering (Abbasimehr and Khodizadeh Nahari, 2020), economics (Udoka, 2020), (Fakhrehosseini and Kaviani, 2023), tourism (Dong et al., 2023), and energy (Guo et al., 2023). Precise analysis and forecasting of data provide valuable insights to researchers. While classical statistical methods are commonly used for predicting raw statistical data, they may have unavoidable inaccuracies (Etemadi and Khashei, 2023), (Casal-Guisande et al., 2023). Hence, researchers have turned to prediction methods based on artificial intelligence and fuzzy logic for more precise outputs.

Time series forecasting involves selecting the optimal model parameters and settings to achieve accurate predictions (Abdolhosseini, 2023). Time series data often contain inherent uncertainty and complexity, which classical statistical methods may struggle to address effectively (ADEBISI and Smarandache, 2022). Metaheuristic algorithms are well-suited to handle such complex and uncertain environments (Espinosa et al., 2023). Metaheuristic algorithms, such as the Quantum Optimization Algorithm (QOA), Genetic Algorithm (GA), and Particle Swarm Optimization (PSO), are designed to search for optimal solutions in complex and high-dimensional spaces. Traditional statistical methods and optimization techniques may find local optima and struggle to explore the entire solution space (Eyo et al., 2021). Metaheuristic algorithms are designed to escape local optima and search for global optima, which is crucial for identifying the best combination of parameters for the NTS model. This global optimization aspect is particularly important in improving forecasting accuracy. Metaheuristic algorithms automate the parameter selection process, reducing the need for manual tuning and experimentation. This automation improves the efficiency of the forecasting process, making it more practical for real-world applications.

The study employs GA, PSO, and QOA for specific reasons, and each of these algorithms has advantages over other optimization techniques in the context of the research objectives: GA is a widely recognized and powerful optimization algorithm that is particularly effective in searching for global optima in complex, high-dimensional search spaces. It is known for its ability to explore a wide range of potential solutions, making it suitable for problems with numerous parameters. GA is also highly adaptable and can be customized to various problem domains (Sohail, 2023). GA global optimization capabilities are valuable in ensuring the chosen parameters are well-suited to the forecasting task. PSO is another popular optimization technique, particularly for continuous parameter optimization. It is known for its simplicity and ease of implementation. It is often computationally efficient and can quickly converge to optimal solutions (Daneshvar et al., 2023). The advantage of PSO lies in its efficiency and its ability to handle continuous search spaces (Etebari et al., 2021). QOA is a relatively newer optimization algorithm inspired by quantum computing principles. QOA has shown promise in handling optimization problems by leveraging quantum principles, which can lead to potentially superior solutions for certain problems. QOA introduces a cutting-edge dimension to the research, showcasing the potential of quantum-inspired optimization techniques in improving the forecasting model's accuracy.

While other optimization algorithms could be considered, the use of GA, PSO, and QOA aligns well with the research objectives, offering a combination of global search capabilities, efficiency, and potential advancements in optimization for the specific problem of parameter selection in Neutrosophic Time Series (NTS) modeling.

Despite significant efforts to improve existing models' prediction abilities, enhancing accuracy remains complex and challenging (Hajirahimi and Khashei, 2023). Over the past few decades, fuzzy sets have been utilized to predict time series (Abbaspour Ghadim Bonab, 2022). However, their limitations lie in the fact that they only express a phenomenon's membership level, neglecting numerous uncertain concepts (Qiu et al., 2023). Generalized fuzzy systems such as intuitive, interval, and Pythagorean fuzzy were developed to overcome this issue (Martin and Edalatpanah, 2023). While these tools have reasonable efficiency compared to fuzzy systems, they cannot measure the concept of indeterminism (Saberhoseini et al., 2022). Our world is full of different phenomena - some are true, some are false, and others cannot be easily classified. Therefore, Smarandache introduced neutrosophic logic, currently considered the most comprehensive measurement tool in uncertain and complex environments.

Neutrosophic sets extend the representation of uncertainty beyond what traditional fuzzy sets offer. While traditional fuzzy sets express membership and non-membership degrees, neutrosophic sets add a third component: indeterminacy. This three-degree membership concept allows for a more comprehensive and nuanced representation of uncertainty, which is particularly useful when dealing with complex and ambiguous data in time series forecasting (Veeramani et al., 2022). In many real-world situations, data is not just a matter of being true or false; there is often an indeterminate or uncertain aspect. Neutrosophic sets excel at capturing and managing this indeterminacy, a common feature in time series data.

The neutrosophic theory is particularly effective in handling complex and uncertain scenarios. It can represent situations where a phenomenon is partly true, partly false, and somewhat indeterminate. These situations are often the case in time series data, where some values may be imprecise, uncertain, or fluctuating. Other generalizations of fuzzy sets, such as interval-valued or intuitionistic fuzzy sets, do not provide as comprehensive a framework for dealing with this aspect of uncertainty (Ihsan et al., 2023).

The challenge in earlier studies, which this research aims to address, is the limited capability of existing time series forecasting models, particularly those based on traditional statistical methods or fuzzy logic, to effectively capture and model uncertainty and complexity in time series data.

Earlier studies often relied on fuzzy logic-based time series forecasting models (Hieu et al., 2023), primarily dealing with the membership degree of elements in a set. For instance, traditional Fuzzy Time Series (FTS) models could handle concepts like "very likely" or "unlikely." However, they struggled to represent and handle indeterminate data points or had multiple degrees of membership. As a result, these models couldn't effectively capture complex patterns in data where elements may be partly true, partly false, or entirely uncertain (Pramanik et al., 2023).

Prior research sometimes focused on time series classification (Zarghami et al., 2023), where the goal was to assign a category or label to a time series based on historical data. While these approaches were effective for classification tasks, they didn't necessarily address the challenge of accurate time series prediction or forecasting, where the goal is to predict future values or trends. Earlier studies often lacked real-world applications or demonstrations of the proposed models' effectiveness in practical scenarios, limiting the ability to assess the models' impact on real decision-making and planning.

The examples provided highlight the challenge that earlier studies faced in effectively modeling and addressing uncertainty, indeterminacy, and complexity in time series data, as well as the limitation of their real-world applicability. This research seeks to overcome these limitations by introducing neutrosophic logic and employing advanced optimization algorithms, thus advancing the field of time series forecasting.

The current study offers several practical advantages over existing literature in time series forecasting. The study demonstrates that the proposed NTS-QOA model outperforms existing benchmark models regarding Average Forecasting Error Rate (AFER). It means that the model provides more accurate predictions, which can be highly beneficial in real-world applications where precise forecasting is crucial for decision-making. Integrating neutrosophic logic allows for a more comprehensive representation of uncertainty in time series data. Considering indeterminacy also goes beyond fuzzy logic's binary membership/non-membership representation. This characteristic is valuable when data is inherently uncertain and ambiguous.

The study applies the proposed approach to various datasets, including university enrollment data, financial indices, and stock exchange data. This versatility suggests that the model can be applied to a wide range of domains, making it a practical choice for different industries and research areas. Incorporating meta-heuristic optimization algorithms (QOA, GA, and PSO) enables the model to automatically select the most suitable universe of discourse and intervals for forecasting. It improves accuracy and streamlines forecasting, making it more practical for real-world applications. The enhanced accuracy of the forecasting model can provide decision-makers with more reliable insights. Accurate predictions are crucial for strategic planning, resource allocation, and risk management in business and other fields. The study's approach can serve as a valuable tool for informed decision-making.

The study's contributions are as follows.

A. The study acknowledges the limitations of previous time series forecasting models based on fuzzy logic and their inability to capture indeterminacy effectively.
B. The study introduces a novel approach that combines neutrosophic logic with time series forecasting. This integration allows for a more comprehensive representation of uncertainty in time series data, including true, indeterminate, and false components, which goes beyond the capabilities of traditional fuzzy logic.
C. The research incorporates three meta-heuristic optimization algorithms: the QOA, GA, and PSO. These algorithms enhance the accuracy of the NTS modeling approach by selecting the optimal universe of discourse and intervals for better predictions.
D. The proposed hybrid model, NTS-QOA, outperforms existing benchmark models in terms of AFER on various datasets, including university enrollment, financial indices, and stock exchange data. The study demonstrates that the NTS-QOA model offers higher precision in forecasting.

1.1. Related concepts

1.1.1. Time series

Time series refers to the collection of data and events that occur over a specific period, often presented as a sequence of regular observations, such as weekly oil or stock prices. The primary objectives of time series analysis include data description, modeling, forecasting (the most critical goal), and control. Typically, time series are represented as follows:

$X(t), T = 1, 2, 3, \ldots$

Where X(t) represents the random variable X at time t.

1.1.2. Fuzzy set theory

Uncertainty is a common aspect of events that occur around us. Classic approaches remove uncertainty from the real world and are not always applicable (Li et al., 2023). Therefore, Scientists have extensively utilized fuzzy sets to address uncertainty-related problems.

Definition 1. (Zadeh, 1965): Suppose X is given, then the fuzzy set $\tilde{A}$ in X is a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$, where $\mu\tilde{A}(x)$ is the membership function of the fuzzy set $\tilde{A}$. The membership function maps each member of X to a degree of membership that is a value between 0 and 1.

Despite the success of fuzzy sets in handling uncertainties arising from partial belonging or ambiguity within a set, they fall short in modeling all the uncertainty states prevalent in real-life problems, particularly those involving insufficient information. In light of this, Atanassov (1986) proposed a generalized version of fuzzy sets, known as intuitionistic fuzzy sets.

Definition 2. (Atanassov, 1986): An intuitionistic fuzzy set in U is a set like A in which two degrees are attributed to each member $u \in U$; one is the "degree of membership," and the other is the "degree of non-membership."

If the membership function is $\mu(A):X \rightarrow [0,1]$ and the non-membership function is $\nu(A): X \rightarrow [0,1]$, then the indeterminacy value of A is $1 - \mu(A) - \nu(A)$, and condition $0 \leq \mu(A) + \nu(A) \leq 1$ is always satisfied.

These types of sets not only have the degree of membership and non-membership but also consider the degree of indeterminacy and inconsistency. Accordingly, the function "Truth Membership" T(x), "Indeterminacy Membership" I(x), and the Falsehood Membership of F(x) are considered.

Definition 3. (Smarandache, 1998): A neutrosophic set is mapped as $A = \{<x, T_A(x), I_A(x), F_A(x)> | x \in X\}$, $T_A: X \rightarrow [0,1]$, $I_A: X \rightarrow [0,1]$, $F_A: X \rightarrow [0,1]$, and condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ is always satisfied.

1.1.3. Fuzzy time series

Unlike traditional time series that work with real numbers, Fuzzy Time Series (FTS) have become famous for predicting dynamic and nonlinear data sets in various fields, including stock markets (Chen and Chen, 2015). Over the years, several models for forecasting time series have been introduced in the scientific world, such as the famous Box-Jenkins model and ARIMA. However, these classical models face challenges in predicting linguistic variables and require vast data for accurate predictions. To overcome these challenges, Song & Chissom (Sun et al., 2015) proposed the FTS model, which consists of six steps: defining and dividing the reference set, defining fuzzy sets for observations, fuzzing observations, expressing fuzzy relations, making predictions, and defuzzifying observation results. These models have proven effective in solving forecasting problems in various fields, especially in accurately predicting stock prices.

1.1.4. Meta-heuristic algorithms

Experts utilize approximate algorithms to tackle intricate optimization problems. Such algorithms efficiently produce nearly optimal solutions within a short time frame and can be grouped into two types: Heuristic and algorithms. While heuristic algorithms may get trapped in local optima and cannot resolve diverse problems, meta-heuristic algorithms were designed to overcome these limitations (Sharifi et al., 2022). Essentially, meta-heuristic algorithms are probabilistic optimization algorithms that offer solutions not limited to local optima and can be implemented for a broad range of problems (Abdi et al., 2021).

1.1.5. Genetic algorithms (GA)

During the 1970s, John Holland, a scientist from the University of Michigan, proposed utilizing GA for engineering optimization. Other scientists, including Goldberg and Dejong, later expanded on this algorithm's application. The GA draws inspiration from the natural processes of living organisms and can be classified as a numerical method, direct search, and random search. The algorithm is based on iteration and borrows from principles observed in natural evolution to enhance solutions using the information within a population effectively.

1.1.6. Particle swarm optimization (PSO)

For the first time in 1995, Kennedy and Eberhart proposed their technique by modeling the movement of birds in the air, discovering the logical relationship between the change of direction and speed of birds, and using the knowledge of physics. These scientists realized the dependence of these movements on each other in their research and found that the movement of a bird is caused by the information it receives from the birds around it. Therefore, they completed the presented method and named it swarm movement. The cumulative movement algorithm of particles does not use operators such as intersection and mutation. As a result, it does not need to use strings of numbers and its coding stage. Therefore, it is much simpler than algorithms such as genetics. This algorithm divides the solution space into multi-piece paths

using a pseudo-probability function formed by the movement of individual particles in space. The motion of a group of particles consists of two main components: the deterministic component and the probabilistic component. Each particle is interested in moving toward the current best solution, x*, or the best solution obtained so far, g*.

1.1.7. Quantum optimization algorithm (QOA

In quantum computing, the entanglement of quantum particles is essential to quantum mechanics. This unique property of quantum has no parallel in classical mechanics. Therefore, if the energy between two particles was high at any time in the past, those two particles can be considered intertwined (Melkikh, 2017). But none of these two particles are involved in the emission of other particles. Entanglement is commonly used in quantum communication, cryptography, and quantum computing (Melkikh, 2015). Schrödinger proposed the concept of "entanglement" in quantum mechanics (Melkikh, 2017) (Melkikh, 2015). Singh et al. (2018) proposed the QOA based on this quantum mechanics.

2. Literature review

Researchers have argued that traditional methods are less effective in predicting electricity production (EP) time series due to their nonstationary and nonlinearity. Therefore, they have proposed a novel combination prediction model based on wavelet transform (WT), long short-term memory (LSTM), and stacked autoencoder (SAE). Empirical results have shown that the combination model, specifically SAE-LSTM, outperforms benchmark models. Furthermore, the results suggest that WT-SAE-LSTM is superior to EMD, EEMD-SAE-LSTM, and SAE (Qiao et al., 2022).

A research paper presents a new hybrid forecasting model for short-term wind power forecasting. The model combines variational mode decomposition (VMD), two deep learning models - long short-term memory neural networks (LSTM) and deep belief networks (DBN) optimized with PSO - and a nonlinear weighted combination method based on PSO-DBN. The VMD technique initially decomposes the original wind power series to extract local features. Then, LSTM and PSO-DBN are used to build prediction models for the sub-series. Finally, these multiple sub-series forecasting models are integrated by a nonlinear weighted combination method based on PSO-DBN to form a hybrid model for short-term wind power forecasting. The study results demonstrate that the proposed method outperforms existing methods in effectiveness (Duan et al., 2022).

A research paper introduced a new hybrid model to predict carbon prices. The model is based on various methods such as Improved Complete Ensemble Empirical Mode Decomposition with Adaptive Noise (ICEEMDAN), Multiscale Fuzzy Entropy (MFE), Complete Ensemble Empirical Mode Decomposition (CEEMD), Improved Random Forest by Salp Swarm Algorithm (SSARF), Improved Backpropagation by Cuckoo Search (CSBP), Improved Extreme Learning Machine by Whale Optimization Algorithm (WOAELM), and Error Correction (EC). The proposed model outperformed other commonly used international carbon financial price forecasting models regarding forecast accuracy, with lower MAPE, MAE, and RMSE values. The results suggest that this model can provide a theoretical and data basis for carbon pricing and help formulate carbon reduction policies in China (Yang et al., 2023).

A study proposed an approach to cryptocurrency price prediction and portfolio allocation by considering it an MCDM problem. The researchers argued that the traditional time series forecasting approach is insufficient to tackle this field's dynamic challenges. The study employed the Prophet Forecasting Model (PFM) for time series forecasting to address this issue. The study extended Cluster analysis for asset allocation to improve the Multiple Criteria Decision Analysis (CLUS-MCDA) algorithm. The new algorithm incorporated additional features such as Density-Based Spatial Clustering of Applications with Noise (DBSCAN) and the Višekriterijumsko kompromisno rangiranje (VIKOR) methods. This approach provides a predictive and analytical framework that guides investors in making informed decisions (Maghsoodi, 2023).

The study's authors stated that accurately predicting water quality parameters is challenging due to water quality data's non-smooth and non-linear nature. Additionally, they noted that water quality parameters have a strong coupling relationship, further complicating the prediction process. The authors combined machine learning and intelligent optimization algorithms to overcome these challenges. They established a Back Propagation Neural Network (BPNN) model using the Artificial Bee Colony (ABC) algorithm. They incorporated three adaptive evolutionary strategies - dynamic adaptive factors, probability selection, and gradient initialization - to form the Adaptive Evolutionary Artificial Bee Colony (AEABC) algorithm (Chen et al., 2023).

A group of researchers have developed self-normalization (SN) to detect and estimate a single change point in the linear trend of a nonstationary time series. They have also combined the SN-based change-point test with the NOT algorithm to estimate multiple change points. The researchers used this technique to analyze the trajectory of the cumulative COVID-19 cases and deaths in 30 major countries. They have found some interesting patterns that could affect different countries' effectiveness of pandemic responses. Additionally, the researchers have designed a simple two-stage forecasting scheme for COVID-19 based on the change-point detection algorithm and a flexible extrapolation function. They have demonstrated its promising performance in predicting cumulative deaths in the United States (Jiang et al., 2023).

An academic paper used two methods to estimate coal prices: a time series method and a radial basis function (RBF) neural network method. To simulate the time series method, the authors used Monte Carlo simulation. Combining the two approaches could approximate coal prices with acceptable precision. The results showed that the combination of Brownian motion with mean return (BMMR) and RBF NN model (CBRN) was effective in reducing errors caused by various factors, such as economic and political conditions and differences in data (Sohrabi et al., 2023).

Researchers have pointed out that while conventional statistical and machine learning methods have been extensively used for forecasting, these methods have limitations in identifying nonlinear relations and modeling sequential data. To address this issue, these researchers employed a deep learning algorithm to predict air pollutants and meteorological features such as wind speed, wind direction, and air temperature in the Istanbul metropolitan area. To evaluate the efficacy of the GA approach, the researchers compared the prediction results of deep learning algorithms with default hyperparameters and random search algorithms. The proposed method outperformed the other configurations, with the mean squared error (MSE) reduced by 13.38% and 55.30% for testing performance, respectively. The experimental results indicated that GA is promising and applicable in hyperparameter optimization of deep neural network models (Erden, 2023).

It has been argued that time series analysis is a complex task due to the influence of various external factors like economic fluctuations and weather conditions. Due to this, accurate predictions are difficult to make. A machine-learning-based technique has been proposed to improve the precision and computation performance of time series forecasting models. This technique involves a multi-population PSO-based nonlinear time series predictive model that breaks the task into three sub-tasks. These sub-tasks are observation window optimization, predictive model induction task, and forecasting horizon prediction. The results showed that the proposed technique effectively induced a forecasting model with improved predictive and computation performance. Compared to benchmark techniques, the proposed technique performed better on all datasets (Kuranga et al., 2023).

A research paper introduced a computational forecasting model that simplifies determining the appropriate interval length and order of FTS—the model utilized PSO to identify the optimal interval length for partitioning the universe of discourse. Additionally, a dynamic order

approach was implemented to choose the order of FTS in the proposed model. During the training phase, a sequence of orders was established based on forecast accuracy, which was then utilized for forecasting by specific rules. The model was tested on various actual time series, including the benchmark data set of enrolments of Alabama University, the Taiwan stock exchange capitalization-weighted stock index, and West Texas Intermediate crude oil prices. The experimental results demonstrated that the proposed model outperforms existing models regarding forecasting accuracy (Goyal and Bisht, 2023).

In this study, we use numerical examples that have been conducted in previous studies. Song & Chissom (Song and Chissom, 1994) used FTS and its models for the first time in 1993. In general, the difference between the crisp time series and the FTS introduced by them is that the data value in the crisp ones is a number, while in the fuzzy ones, these values are in the form of linguistic values or fuzzy sets. Following this, they predicted the number of enrollments at the University of Alabama based on 20-year data using a time-independent model. Again, Song & Chissom (Song and Chissom, 1993) 1994 used the University of Alabama enrollment prediction based on a time-independent model and three-layer neural network method to de-fuzzify the output, lowering the average error.

In 1996, Chen et al. (Chen, 1996) modified Song and Chisom's method and claimed to have developed a better strategy than Song and Chisom's method. In 2001, Hurang (Huarng, 2001a) proposed an approach to improve prediction by integrating problem-specific heuristic knowledge and Chen's model. The predictive ability of these early studies on FTS drew the attention of researchers to this field. As a result, FTSs have been used as a powerful forecasting tool in a wide range of domains.

In 2007, Liu (2007) introduced an improved FTS model by considering the trapezoidal fuzzy number for the predicted value. In 2008, Chen et al. (CHENG et al., 2008) integrated the FCM clustering algorithm with FTS to effectively segment time series datasets. In 2011, Qiu et al. (2011) proposed a weighting method for fuzzy relations in the FTS modeling approach, which can improve prediction accuracy. In 2013, Singh and Borah (2013) combined a new data discretization approach with the FTS modeling approach to solve the problem of determining the length of effective intervals. Similarly, in 2018, Gupta et al. (2018) proposed a new hybrid model by integrating an automatic clustering approach with an FTS modeling approach. Bas et al. (2018) proposed applying an artificial neural network to determine fuzzy relations in the FTS modeling approach. In 2019, Chen et al. (2019) offered an FTS based on distance ratios and PSO technique to improve the forecasting ability of FTS. In 2020, Gao and Duru (2020) introduced a parsimonious time series, and Vovan and Lethithu (2022) proposed an FTS model based on improved fuzzy functions.

Studies show that the prediction accuracy of the FTS modeling approach mainly depends on two factors (Singh, 2017): determining the effective length of intervals and establishing fuzzy logic relationships (which are called decision rules). However, determining effective decision rules relies more on the appropriate selection of intervals. In the FTS modeling approach, researchers use different optimization algorithms to solve the problem of choosing the proper distances. For the right choice of distances, researchers mainly use GA (Chen and Chung, 2006), PSO (Kuo et al., 2010), Simulated Annealing (SA) (Lee et al., 2008), Geese Movement Based Optimization Algorithm (GMBOA) (Singh and Dhiman, 2018), Harmony Search Algorithm (Jiang et al., 2017) and QOA (Singh et al., 2018).

Time series data sets usually contain insufficient or incomplete information. In the case of the fuzzy-based modeling approach (i.e., the FTS modeling approach), the inherited uncertainty in the time series data set is characterized only by using the membership function. Intuitionistic Fuzzy Sets (IFS) theory has been introduced (Atanassov, 1986) to overcome this problem. In IFS, this lack of information is represented by both membership and non-membership functions.

This concept can help display time series data sets regarding membership and non-membership functions. Considering that many phenomena in the real world have a true part, a false part, and a part that cannot be commented on, Smarandache (1998) presented the concept of neutrosophic sets from a philosophical point of view, which can represent uncertainties in terms of determining the three degrees of membership, namely true function (Tf), indeterminate function (If) and falsehood function (Ff). A recent application of NS theory can be found in the classification of stock index datasets in terms of Tf, If, and Ff (Singh and Rabadiya, 2018). Except for this work, no other work has been reported in the literature that can provide a prediction model for time series data sets exclusively based on the NS theory. Accordingly, we were motivated to share our research on developing a time series prediction model utilizing NS theory and optimization algorithms.

3. Methodology and implementation

In this section, the proposed model for forecasting time series is introduced. Our proposed model is a combination of neutrosophic and optimization algorithms. The optimization algorithms we use are the QOA, GA, and PSO algorithms. First, the experiment uses the University of Alabama enrollment data set. Each step of the proposed model is explained below.

3.1. Algorithm 1 (NTS-QOA)

The proposed NTS-QOA algorithm for forecasting time series is as follows.

1 Gathering input data from the databases
2 Determining the number of intervals (n) and dividing the data in the database by n
3. Formulating the optimization function and specifying the objective function
4. Setting parameters related to the QOA optimization algorithm
5. Implementing the optimization algorithm on the data to minimize the prediction error
6. Updating the set of new intervals to improve the forecasting process
7 Converting crisp data to neutrosophic data
8 Calculating Entropy on the neutrosophic data set
9 Calculating the Neutrosophic Entropy Relationship (NER) parameter based on the entropy values obtained from the neutrosophic data set
10 Calculating Neutrosophic Entropy Relationship Group (NERG) based on the obtained NER values
11. Implementing the deneutrosophication operator to obtain predicted time series values
12. Evaluating the efficiency of the proposed model

To explain Algorithm 1, consider Table 1.

Let E_{min} and E_{max} be the minimum and maximum values of the time series data set. According to E_{min}, in the Universe of discourse, U_0 can be

Table 1
The university of Alabama enrollment dataset (Song and Chissom, 1994).

Row	Year	Enrollment	Row	Year	Enrollment
1	1971	13,055	12	1982	15,433
2	1972	13,563	13	1983	15,497
3	1973	13,867	14	1984	15,145
4	1974	14,696	15	1985	15,163
5	1975	15,460	16	1986	15,984
6	1976	15,311	17	1987	16,859
7	1977	15,603	18	1988	18,150
8	1978	15,861	19	1989	18,970
9	1979	16,807	20	1990	19,328
10	1980	16,919	21	1991	19,337
11	1981	16,388	22	1992	18,876

defined as $U_0 = [E_{min}\text{-}A_N, E_{max} + A_P]$, where A_N and A_P are two adjustment factors.

From Table 3, we have $E_{min} = 13{,}055$ and $E_{max} = 19{,}337$. At first, we assume $A_N = 3555$ and $A_P = 10{,}66$. Now the universe set U_0 is defined as $U_0 =$ (Zadeh, 1965), where min $(U_0) = 9500$ and max $(U_0) = 30{,}000$.

In the next step, to divide the reference set U_0 into n equal lengths of intervals, the following equation is used:

$$\mathbf{a_i} = \left[\mathbf{min(U_0)} + \mathbf{(i\text{-}1)}\frac{\mathbf{max(U_0)\text{-}min(U_0)}}{\mathbf{j}}, \mathbf{min(U_0)} + \mathbf{i}\frac{\mathbf{max(U_0)\text{-}min(U_0)}}{\mathbf{j}}\right] \tag{1}$$

for i = 1, 2, …, n, and j is the number of intervals.

Using Eqn. (1), the set of intervals resulting from the reference set U_0 can be defined as follows:

$$a_1 = [L_{B1}, U_{B1}], a_2 = [L_{B2}, U_{B2}], \ldots, a_n = [L_{Bn}, U_{Bn}]$$

where, L_{Bi}, $U_{Bi} \in a_i$ and $a_i \in U$.

Assuming j = 5 in Eqn. (1), the reference set $U_0 =$ (Zadeh, 1965) equal to 5 lengths of the reference set as $a_1 =$ [9500, 13,600], $a_2 =$ [13, 600, 17,700], …, $a_5 =$ (Ihsan et al., 2023; Zadeh, 1965) is divided. In this set of intervals, we can assume $U_{B1} = 13{,}600 \in a_1$, $U_{B2} = 17{,}700 \in a_2$, …, $U_{B5} = 30{,}000 \in a_5$.

An optimization problem can be formulated based on the objective function for the third step. As the primary goal of this study is to improve prediction accuracy by selecting optimal intervals from the reference, set U_0, the model's performance should be determined by adopting the appropriate evaluation parameter. This parameter can also be used as an objective function. In the literature, the Average Forecasting Error Rate (AFER) is chosen by most researchers (Singh, 2016). Clearly, if the proposed model chooses optimal intervals, it will minimize AFER. Now, based on AFER, we can formulate the objective function as follows:

$$\textbf{Minimize AFER} = \frac{\sum_{i=1}^{N}|\mathbf{A_i}\text{-}\mathbf{F_i}|\mathbf{A_i}}{\mathbf{N}} \tag{2}$$

s.to

$$\begin{aligned}
\mathbf{U_{B1}} &\le \mathbf{max}\ (\mathbf{U_0})\\
\mathbf{U_{B2}} &\le \mathbf{max}\ (\mathbf{U_0})\\
&\ldots\\
\mathbf{U_{Bn}} &\le \mathbf{max}\ (\mathbf{U_0})\\
\mathbf{U_{B1}} < \mathbf{U_{B2}} &< \mathbf{U_{B3}} < \ldots < \mathbf{U_{Bn}}
\end{aligned}$$

where each F_i and A_i represent, respectively, the predicted and actual value of a particular year/day i and N is the total number of predicted years/days.

We have the set of upper bounds $U_{B1} = 13{,}600$, $U_{B2} = 17{,}700$, …, $U_{B7} = 30{,}000$. In terms of the set of upper bounds, the following constraints to minimize AFER - as defined in Eqn. (2) - are defined as follows:

Table 2
Search factors and their corresponding calculation values.

Search factors	a	b	$\lvert\varphi\rvert = \sqrt{a^2 + b^2}$	Q_{ij}	Q_{hi}	Final value
q_1	0.5	0.37	0.65	0.06	0.06	0.06
q_2	0.7	0.49	10.9	0.53	0.53	0.53
q_3	0.8	0.16	0.89	0.98	0.96	0.98
q_4	0.0	0.90	0.90	0.77	0.01	0.70
q_5	0.5	0.30	0.60	0.18	0.59	0.34

$$\begin{aligned}
13{,}600 &\le 30{,}000\\
17{,}700 &\le 30{,}000\\
&\vdots\\
30{,}000 &\le 30{,}000\\
13{,}600 < 17{,}700 &< 20{,}800 < \ldots < 30{,}000
\end{aligned}$$

In step 4, the sub-steps of using QOA to minimize AFER are as follows.

- Sorting n number of intervals in the two-dimensional search space

The upper bounds of the set of intervals $a_1 = [L_{B1}, U_{B1}], a_2 = [L_{B2}, U_{B2}], \ldots, a_n = [L_{Bn}, U_{Bn}]$ that is, U_{B1}, U_{B2}, …, U_{Bn} are arranged in ascending order as $U_{B1} < U_{B2} < \ldots < U_{Bn}$. Before starting the iteration process, it is assumed that this set of upper bounds is the best set of upper bounds (i.e., the set of upper bounds that gives the minimum AFER value).

For the set $U_0 =$ (Zadeh, 1965), the set of intervals is defined as $a_1 =$ [9500,13,600], $a_2 =$ [13,600, 17,700], …, $a_5 =$ (Ihsan et al., 2023; Zadeh, 1965). This set of ranges consists of upper bounds of 13,600, 17, 700, …, and 30,000. These upper limits are now arranged in ascending order as $13{,}600 < 17{,}700 < \ldots < 30{,}000$. At first, it is assumed that this set of upper bounds creates the optimal result in terms of AFER.

- Initialization of the search factors q_i (i = 1, 2, 3, …, n) for each of the upper bounds of U_{Bi}

$$\begin{aligned}
\mathbf{q_1} &= |\boldsymbol{\varphi}| \times \mathbf{Q_{1j}} + \mathbf{(1\text{-}|\varphi|)} \times \mathbf{Q_{h1}} \in \mathbf{U_{B1}}\\
\mathbf{q_2} &= |\boldsymbol{\varphi}| \times \mathbf{Q_{2j}} + \mathbf{(1\text{-}|\varphi|)} \times \mathbf{Q_{h2}} \in \mathbf{U_{B2}}\\
&\vdots\\
\mathbf{q_n} &= |\boldsymbol{\varphi}| \times \mathbf{Q_{nj}} + \mathbf{(1\text{-}|\varphi|)} \times \mathbf{Q_{hn}} \in \mathbf{U_{Bn}}
\end{aligned} \tag{3}$$

For the set of upper bounds 13,600, 17,700, …, 30,000, search factors are initialized $q_1, q_2, \ldots q_5$, respectively. In Eqn. (3), the values of wave functions Q_{ij} and Q_{hi} are assumed between the range of random numbers [0, 1]. The true value of $\varphi = a + bi$ can be obtained as $|\varphi| = \sqrt{a^2 + b^2}$, which is assumed to be $a \in [0, 1]$ and $b \in [0, 1]$. Based on these assumptions, the search factors $q_1, q_2, \ldots, q_5$ can be defined according to Eqn. (4) as follows:

$$\begin{aligned}
\mathbf{q_1} &= |\boldsymbol{\varphi}| \times \mathbf{Q_{1j}} + \mathbf{(1\text{-}|\varphi|)} \times \mathbf{Q_{h1}} \in \mathbf{13{,}600}\\
\mathbf{q_2} &= |\boldsymbol{\varphi}| \times \mathbf{Q_{2j}} + \mathbf{(1\text{-}|\varphi|)} \times \mathbf{Q_{h2}} \in \mathbf{17{,}700}\\
&\vdots\\
\mathbf{q_5} &= |\boldsymbol{\varphi}| \times \mathbf{Q_{5j}} + \mathbf{(1\text{-}|\varphi|)} \times \mathbf{Q_{h5}} \in \mathbf{30{,}000}
\end{aligned} \tag{4}$$

For Eqn. (4), the search factors and their corresponding calculation values are shown in Table 2.

- Obtaining the current position of each search agent q_i in terms of position $\vec{P}(x_i)$.

$$\vec{\mathbf{P}}(\mathbf{x_1}) = \frac{\mathbf{1}}{\mathbf{L_1}}\mathbf{e^{-2/L_1}} \in \mathbf{q_1}\,\vec{\mathbf{P}}(\mathbf{x_2}) = \frac{\mathbf{1}}{\mathbf{L_2}}\mathbf{e^{-2/L_2}} \in \mathbf{q_2} : \vec{\mathbf{P}}(\mathbf{x_n}) = \frac{\mathbf{1}}{\mathbf{L_n}}\mathbf{e^{-2/L_n}} \in \mathbf{q_n} \tag{5}$$

Here, each $\vec{P}(x_i)$ and L_i represent, respectively, the current position and search domain of each search agent q_i, where i = 1, 2, 3, …, n.

According to Eqn. (5), the current positions of search agents q_1‘ q_2‘ … ‘ q_5 can be calculated as follows:

$$\begin{aligned}
\vec{\boldsymbol{P}}(\boldsymbol{x_1}) = \frac{\mathbf{1}}{\boldsymbol{L_1}}\boldsymbol{e}^{-2/L_1} &= \frac{\mathbf{1}}{\mathbf{13{,}600}}\boldsymbol{e}^{-2/13{,}600} = \mathbf{74} \times \mathbf{10^{-6}} \in \boldsymbol{q_1}\,\vec{\boldsymbol{P}}(\boldsymbol{x_2}) = \frac{\mathbf{1}}{\boldsymbol{L_2}}\boldsymbol{e}^{-2/L_2}\\
&= \frac{\mathbf{1}}{\mathbf{17{,}700}}\boldsymbol{e}^{-2/17{,}700} = \mathbf{56} \times \mathbf{10^{-6}} \in \boldsymbol{q_2} : \vec{\boldsymbol{P}}(\boldsymbol{x_5}) = \frac{\mathbf{1}}{\boldsymbol{L_5}}\boldsymbol{e}^{-2/L_5} = \frac{\mathbf{1}}{\mathbf{30{,}000}}\boldsymbol{e}^{-2/30{,}000}\\
&= \mathbf{33} \times \mathbf{10^{-6}} \in \boldsymbol{q_5}
\end{aligned} \tag{6}$$

Table 3
NTS display from the University of Alabama Enrollment dataset.

Year	Enrollment	NTS	Year	Enrollment	NTS
1971	13,055	13055	1982	15,603	15433
		< 0.2, 08, 0.82 >			< 0.1244, 0.8756, 0.8844 >
1972	13,563	13563	1983	15,861	15497
		< 0.0852, 0.9148, 0.9188 >			< 0.1258, 0.8742, 0.8832 >
1973	13,867	13867	1984	16,807	15145
		< 0.0916, 0.9084, 0.913 >			< 0.1184, 0.8816, 0.8895 >
1974	14,696	14696	1985	16,919	15163
		< 0.109, 0.891, 0.8977 >			< 0.1188, 0.8812, 0.8892 >
1975	15,460	15460	1986	15,497	15984
		< 0.125, 0.875, 0.8839 >			< 0.136, 0.864, 0.8747 >
1976	15,311	15311	1987	18,970	16859
		< 0.1219, 0.8781, 0.8866 >			< 0.1543, 0.8457, 0.8596 >
1977	16,388	15603	1988	19,328	18150
		< 0.128, 0.872, 0.8814 >			< 0.1814, 0.8186, 0.8385 >
1978	15,433	15861	1989	19,337	18970
		< 0.1334, 0.8666, 0.8768 >			< 0.1986, 0.8014, 0.8256 >
1979	15,984	16807	1990	15,145	19328
		< 0.1532, 0.8468, 0.8605 >			< 0.2061, 0.7939, 0.8202 >
1980	16,859	16919	1991	15,163	19337
		< 0.1556, 0.8444, 0.8586 >			< 0.2063, 0.7937, 0.8201 >
1981	18,150	16388	1992	18,876	18876
		< 0.1444, 0.8556, 0.8677 >			< 0.1966, 0.8034, 0.8271 >

In Eqn. (5), each search range L_i shows the current position of the search agent q_i. Therefore, for the search domains L_1‘ L_2‘ … ‘ L_5, we have considered the initial values of the upper bounds U_{B1}‘ U_{B2}‘ … ‘ U_{B5}, respectively, in Eqn. (6).

- Calculating the displacement of each search factor q_i between 0 and 1

$$\begin{aligned} \boldsymbol{D(q_1)} &= \int_0^1 \left|\vec{\boldsymbol{P}}(\boldsymbol{x_1})\right| \boldsymbol{dx} = \boldsymbol{e^{-2/L_1} D(q_2)} = \int_0^1 \left|\vec{\boldsymbol{P}}(\boldsymbol{x_2})\right| \boldsymbol{dx} \\ &= \boldsymbol{e^{-2/L_2}} : \boldsymbol{D(q_n)} = \int_0^1 \left|\vec{\boldsymbol{P}}(\boldsymbol{x_n})\right| dx \\ &= \boldsymbol{e^{-2/L_n}} \end{aligned} \tag{7}$$

The displacement of each search factor $q_1, q_2, \ldots, q_5$ can be calculated using Eqn. (7) as follows:

$$D(q_1) = \int_0^1 \left|\vec{P}(x_1)\right| dx = e^{-2/L_1} = e^{-2/13,600} = 0.99 D(q_2)$$

$$= \int_0^1 \left|\vec{P}(x_2)\right| dx = e^{-2/L_2} = e^{-2/17,700}$$

$$= 0.99 : D(x_5) = \int_0^1 \left|\vec{P}(x_5)\right| dx = e^{-2/L_5}$$

$$= e^{-2/30,000} = 0.99$$

- Generating a random position y_i for each search factor q_i

$$\begin{aligned} \boldsymbol{y_1} &= \left|\boldsymbol{q_1} \pm \frac{\boldsymbol{L_1}}{\boldsymbol{2}} \boldsymbol{ln}\ (\boldsymbol{1/v})\right|, \boldsymbol{v} \in [\boldsymbol{0}, \boldsymbol{1}] \\ \boldsymbol{y_2} &= \left|\boldsymbol{q_2} \pm \frac{\boldsymbol{L_2}}{\boldsymbol{2}} \boldsymbol{ln}\ (\boldsymbol{1/v})\right|, \boldsymbol{v} \in [\boldsymbol{0}, \boldsymbol{1}] : \\ \boldsymbol{y_n} &= \left|\boldsymbol{q_n} \pm \frac{\boldsymbol{L_n}}{\boldsymbol{2}} \boldsymbol{ln}\ (\boldsymbol{1/v})\right|, \boldsymbol{v} \in [\boldsymbol{0}, \boldsymbol{1}] \end{aligned} \tag{8}$$

where v is a random number with a value between [0, 1].

The random position for each search factor $q_1, q_2, \ldots, q_5$ is created using Eqn. (8) as follows:

$$y_1 = q_1 - \frac{L_1}{2} ln(1/v) = 0.06 - \frac{13,600}{2} ln(1/0.65) = 1272.5$$

$$y_2 = q_2 \frac{L_2}{2} ln(1/v) = 0.53 - \frac{17,700}{2} ln(1/0.73) = 1209 :$$

$$y_5 = q_5 - \frac{L_5}{2} ln(1/v) = 0.34 - \frac{30,000}{2} ln(1/0.55) = 3894.6$$

Here, the operator "−" was used instead of "±" (see Eqn. (8)). The value of search factors $q_1, q_2, \ldots, q_5$ can be obtained from Table 2.

- Calculating the search range of each search agent q_i according to the upper limit $U_{Bi} \in U$ and the random position corresponding to y_i

$$\mathbf{L_1} = \mathbf{2} \bullet |\mathbf{U_{B1}} - \mathbf{y_1}| \in \vec{\mathbf{P}}(\mathbf{x_1}) \mathbf{L_2} = \mathbf{2} \bullet |\mathbf{U_{B2}} - \mathbf{y_2}| \in \vec{\mathbf{P}}(\mathbf{x_2}) : \mathbf{L_n} = \mathbf{2} \bullet |\mathbf{U_{Bn}} - \mathbf{y_n}| \in \vec{\mathbf{P}}(\mathbf{x_n}) \tag{9}$$

For each search factor $q_1, q_2, \ldots, q_5$, the search domain corresponding to them, that is, $L_1, L_2 \ldots, L_5$, can be calculated using Eqn. (9) as follows:

$$L_1 = 2 \bullet |U_{B1} - y_1| = 2 \bullet |13,600 - 1272.5| = 24,655 \in \vec{P}(x_1) L_2$$

$$= 2 \bullet |U_{B2} - y_2| = 2 \bullet |17,700 - 1209| = 32,982 \in \vec{P}(x_2) : L_5$$

$$= 2 \bullet |U_{B5} - y_5| \in \vec{P}(x_5) = 2 \bullet |30,000 - 3894.6| = 52,210.8$$

$$\in \vec{P}(x_5)$$

- Obtaining the final position of each search factor q_i

$$\begin{aligned} \boldsymbol{y_{f1}} &= \boldsymbol{q_1} \pm \boldsymbol{\alpha} \bullet |\boldsymbol{U_{B1}} - \boldsymbol{y_1}| \bullet \boldsymbol{ln}(\boldsymbol{1/v}) \\ \boldsymbol{y_{f2}} &= \boldsymbol{q_2} \pm \boldsymbol{\alpha} \bullet |\boldsymbol{U_{B2}} - \boldsymbol{y_2}| \bullet \boldsymbol{ln}(\boldsymbol{1/v}) : \\ \boldsymbol{y_{fn}} &= \boldsymbol{q_n} \pm \boldsymbol{\alpha} \bullet |\boldsymbol{U_{Bn}} - \boldsymbol{y_n}| \bullet \boldsymbol{ln}\ (\boldsymbol{1/v}) \end{aligned} \tag{10}$$

Here, each y_{fi} represents the corresponding final position for the search factor q_i, and α represents the tuning parameter of the algorithm.

The final position for each search factor $q_1, q_2, \ldots, q_5$ is calculated using Eqn. (10) as follows:

$$\begin{aligned} y_{f1} &= q_1 - \alpha \bullet |U_{B1} - y_1| \bullet ln(1/v) = 0.36 - 0.3 \cdot |13,600 - 1272.5| ln(1/0.65) = -691.53 \\ y_{f2} &= q_2 - \alpha \bullet |U_{B2} - y_2| \bullet ln(1/v) = 0.26 - 0.3 \cdot |17,700 - 1272.5| ln(1/0.73) = -673.32 : \\ y_{f5} &= q_5 - \alpha \bullet |U_{B5} - y_5| \bullet ln(1/v) = 0.06 - 0.3 \cdot |30,000 - 3894.6| ln(1/0.55) = -2237.9 \end{aligned}$$

Here, the operator "−" is used instead of "±" and the value of 0.3 is

considered for α.

- Updating each search factor q_i according to the upper bound $U_{Bi} \in U$ and the previous corresponding position $\overrightarrow{P}(x_1)$.

$$\overrightarrow{\boldsymbol{P}}(\boldsymbol{x_1}+\mathbf{1}) = \overrightarrow{\boldsymbol{P}}(\boldsymbol{x_1}) + \boldsymbol{y_{f1}} \in \boldsymbol{x_1}\, \overrightarrow{\boldsymbol{P}}(\boldsymbol{x_2}+\mathbf{1}) = \overrightarrow{\boldsymbol{P}}(\boldsymbol{x_2}) + \boldsymbol{y_{f2}} \in \boldsymbol{x_2} : \overrightarrow{\boldsymbol{P}}(\boldsymbol{x_n}+\mathbf{1}) = \overrightarrow{\boldsymbol{P}}(\boldsymbol{x_n}) + \boldsymbol{y_{fn}} \in \boldsymbol{x_n} \tag{11}$$

Here $\overrightarrow{P}(x_i+1)$ and $\overrightarrow{P}(x_i)$ represent the new and previous positions of each search factor q_i in the search space, respectively.

For each search factor $q_1, q_2, \ldots, q_5$, the position is updated based on Eqn. (11):

$$\overrightarrow{P}(x_1+1) = \overrightarrow{P}(x_1) y_{f1} = 13,600\text{-}691.53 = 12,908.47$$

$$\in x_1\, \overrightarrow{P}(x_2+1) = \overrightarrow{P}(x_2) + y_{f2} = 17,700\text{-}673.32$$

$$= 17,026.68 \in x_2 : \overrightarrow{P}(x_5+1) = \overrightarrow{P}(x_5) + y_{f5}$$

$$= 30,000\text{-}2273.9 = 27,762.1 \in x_5$$

In the next step, Updating the set of new intervals to improve the forecasting process is done as follows:

The set of new intervals can be defined as follows:

$$a_1(\text{new}) = \left|L_{B1}\, \overrightarrow{P}(x_1+1)\right|, a_2(\text{new}) = \left|\overrightarrow{P}(x_1+1), \overrightarrow{P}(x_2+1)\right|, \ldots, a_n(\text{new}) = \left|\overrightarrow{P}(x_{n\text{-}z}+1), \overrightarrow{P}(x_n+1)\right|$$

where the new reference set U_{1st} can be defined as $U_{1st} = |\overrightarrow{P}(x_n+1)|$. Here, L_{B1} and $\overrightarrow{P}(x_n+1)$ represent the lower and upper bounds of the new reference set U_{1st}, respectively. It should be noted that the reference set obtained after the ith iteration has a U_i representation, where i = 1, 2, 3, …, n.

The new set of intervals can be defined as follows:

$$a_1(\text{new}) = [9500, 12,325.34], a_2(\text{new}) = [12,908.47, 16432.13], \ldots, a_5(\text{new}) = [12,338.1, 27,002.11]$$

For this new set of intervals, their reference set can be defined as $U_{1st} = [9500, 27002.11]$.

In step 7, crisp data is converted to neutrosophic data.

The time series data set in the NTS data set can be displayed as follows:

$$\mathbb{N}_{t_i} = \frac{\mathbf{t_i}}{< \mathbf{T_f}(\mathbf{t_i}), \mathbf{I_f}(\mathbf{t_i}), \mathbf{F_f}(\mathbf{t_i}) >} \tag{12}$$

where $T_f, I_f, F_f : U_{1st} \rightarrow [0,1]$ and $\forall t_i \in U_{1st}$, $t_i \equiv t_i(T_f(t_i), I_f(t_i), F_f(t_i))t_i))$ $T_f(t_i), I_f(t_i), F_f \in \mathbb{N}_{t_i}$.

The values of $T_f(t_i)$,$I_f(t_i)$and$F_f(t_i)$ can be defined according to the new discourse $U_{1st} = [L_{B1}\, \overrightarrow{P}(x_n+1)]$ using the following equations:

$$\mathbf{T_f}(\mathbf{t_i}) = \frac{\mathbf{t_i}\text{-}\mathbf{L_{B1}}}{\overrightarrow{\mathbf{P}}(\mathbf{x_n}+\mathbf{1})\text{-}\mathbf{L_{B1}}} \tag{13}$$

$$\mathbf{F_f}(\mathbf{t_i}) = \mathbf{1}\text{-}\mathbf{T_f}(\mathbf{t_i}) \tag{14}$$

$$\mathbf{I_f}(\mathbf{t_i}) = \sqrt{\mathbf{T_f}(\mathbf{t_i})^2 + \mathbf{F_f}(\mathbf{t_i})^2} \tag{15}$$

For example, consider the enrollment value for 1971, which is 13,055. Now, neutrosophication can be done using Eqns. 13–15 as follows:

$$T_f(13,055) = \frac{t_i\text{-}L_B}{U_B(\text{new})\text{-}L_B} = \frac{13,055\text{-}9500}{27,002.11\text{-}9500} = 0.2$$

$$F_f(13,055) = 1\text{-}T_f(t_i) = 1\text{-}0.2 = 0.8$$

$$I_f(13,055) = \sqrt{T_f(t_i)^2 + F_f(t_i)^2} = \sqrt{0.2^2 + 0.8^2} = 0.82$$

Now based on $T_f(13,055)$, $F_f(13,055)$, and $I_f(13,055)$, the registration value of "13,055″ in NTS is displayed as follows:

$$\mathbb{N}_{13,055} = \frac{t_i}{< T_f(t_i), I_f(t_i), F_f(t_i) >} = \frac{13,055}{< 0.2, 0.8, 0.82 >}$$

A complete representation of the University of Alabama enrollment dataset in the NTS dataset is shown in Table 3.

In the next step, the entropy is calculated on the neutrosophic data set as follows:

$$\mathbf{E}_{\mathbb{N}}(\mathbf{t_i}) = \mathbf{1}\text{-}\frac{\mathbf{1}}{\mathbf{n}} \sum_{\mathbf{t_i} \in \mathbf{U_{1st}}} \left(\mathbf{T_f}(\mathbf{t_i}) + \mathbf{I_f}(\mathbf{t_i}) + \mathbf{F_f}(\mathbf{t_i})\right) \times \mathbf{E_1 E_2 E_3} \tag{16}$$

where $E_1 = |T_f(t_i)\text{-}T_f^c(t_i)|$,$E_2 = |I_f(t_i)\text{-}I_f^c(t_i)|$ and $E_3 = |F_f(t_i)\text{-}F_f^c(t_i)|$, and $U_{1st} = |L_{B1}, \overrightarrow{P}(x_n+1)|$ is a new reference set.

For example, consider the NTS display for the register value "13,055." Now, its corresponding entropy value can be obtained using Eqn. (16):

$$E_{\mathbb{N}}(13,055) = 1\text{-}\frac{1}{3} \sum_{t_i \in a_i(\text{new})} (0.2 + 0.8 + 0.83) \times 0.6 \times 0.64 \times 0.6 = 0.6$$

where $E_1 = |0.2\text{-}0.8| = 0.6$, $E_2 = |0.82\text{-}0.18| = 0.64$, and $E_3 = |0.8\text{-}0.2| = 0.6$.

Thus, the entropy for the NTS data set can be given by the new reference set $U_{1st} = |L_{B1}, \overrightarrow{P}(x_n+1)|$, as shown in Table 4.

In the next step, the NER parameter is calculated based on the entropy values obtained from the neutrosophic data set.

NER can be established between two consecutive NTS data set entropy values. A NER represents the relationship between two or more consecutive entropies. For example, $E_{\mathbb{N}}(t_i)$ and $E_{\mathbb{N}}(t_{i+1})$ are two consecutive entropy values in the NTS dataset. NER between them can be set as $E_{\mathbb{N}}(t_i) \rightarrow E_{\mathbb{N}}(t_{i+1})$.

In Table 4, the entropy values for 1973 and 1974 are 0.67 and 0.71, respectively. So, we can set a NER between 0.67 and 0.71 as 0.67 → 0.71. Thus, we have obtained the NERs for the NTS data set of University Enrollment, shown in Table 5.

In this step, the NERG is created among NERs:

NERGs can be created between two or more NERs. In the set of NERs, relations with the same previous state can be grouped, called NERGs.

Here is an example of creating a NERG from the list of NERs in Table 5. In this table, two NERs have the same status as before: 0.71 → 0.73 and 0.71 → 0.72. NERG can be formed among these NERs as 0.71 → 0.72, 0.73. We have thus generated NERGs for the University of Alabama enrollment data set presented in Table 6.

Table 4
Entropy for NTS dataset.

Year	Entropy	Year	Entropy
1971	0.615	1982	0.7225
1972	0.64	1983	0.7225
1973	0.67	1984	0.715
1974	0.71	1985	0.7367
1975	0.7225	1986	0.7733
1976	0.7367	1987	0.7967
1977	0.745	1988	0.845
1978	0.7733	1989	0.865
1979	0.785	1990	0.8633
1980	0.7967	1991	0.8633
1981	0.75	1992	0.85

In the next step, the de-fuzzification operation is performed to obtain the predicted time series values according to the following stages.

- Getting the corresponding entropy value for a specific day \ year t_i as $E_{\mathbb{N}}(t_i)$.
- Finding the NERG for the corresponding $E_{\mathbb{N}}(t_i)$, which can be represented as:

$$\mathbf{E_{\mathbb{N}}(t_i) \rightarrow E_{\mathbb{N}}(t_{k1}) \rightarrow E_{\mathbb{N}}(t_{k2}) \rightarrow E_{\mathbb{N}}(t_{k3}), \ldots, E_{\mathbb{N}}(t_{kn})} \quad (17)$$

where $E_{\mathbb{N}}(t_i)$ is the previous state value of the NTS for day/year t_i, while $E_{\mathbb{N}}(t_{k1})$, $E_{\mathbb{N}}(t_{k2})$, …, $E_{\mathbb{N}}(t_{kn})$ respectively, are called the current state of NTS values for days \ years t_{k1}‘t_{k2}‘t_{k3}‘ … ‘ t_{kn}.

- Obtaining the optimal entropy available in the current state of NERG

$$\mathbf{E_{\mathbb{N}}(optimal) = \frac{1}{n} \sum_{i=1}^{n} E_{\mathbb{N}}(t_{kn})} \quad (18)$$

- Obtaining the real values of the time series corresponding to the previous state of NERG as t_i
- Applying the following de-fuzzification formula to calculate the predicted value for a day \ year t_{i+1}

$$\mathbf{Forecasting}\ (t_{i+1}) = \frac{\mathbf{E_{\mathbb{N}}(optimal)}}{\mathbf{E_{\mathbb{N}}(t_i)}} \quad (19)$$

Suppose we want to predict the registration of the year “1976". For this year, the corresponding entropy value is 0.73. NERG is then obtained for an entropy value of 0.73 as 0.73, 0.74, 0.77, where 0.73 is the previous state, and 0.74, 0.77 is the current NERG state. Now we have:

$$E_{\mathbb{N}}(\text{optimal}) = \frac{0.74 + 0.77}{2} = 0.75$$

$$\text{Forecasting}(1976) = \frac{0.75 \times 15,311}{0.73} = 15,730.4$$

The proposed model’s efficiency is evaluated in the algorithm’s last step.

Efficiency is evaluated using AFER, Eqn. (2). If the AFER value shows that there is more scope for improving the performance of the proposed model, then the procedure of the previous sub-steps is repeated.

After a certain number of repetitions, the calculation is stopped if the desired result is obtained. This study repeats the proposed NTS-QOA model 100 times for optimal solutions. The reference set and corresponding intervals that create the optimal AFER can be considered the best global solution.

Before applying QOA in the proposed NTS-QOA model, an initial assumption has been made for the reference set as $U_0 =$ (Zadeh, 1965) for the enrollment data set of the University of Alabama. In step 2 of the proposed model, $U_0 =$ (Zadeh, 1965) is divided into five equal lengths of intervals as $a_1 =$ [9500, 13,600], $a_2 =$ [13,600, 17,700], $a_3 =$ [17,700, 21, 800], $a_4 =$ [21,800, 25,900], and $a_5 =$ (Ihsan et al., 2023; Zadeh, 1965).

Table 5
NERs for the University of Alabama enrollment dataset.

NERs	
0.61 → 0.64	0.72 → 0.71
0.73 → 0.74	0.84 → 0.86
0.75 → 0.72	0.71 → 0.72
0.77 → 0.79	0.78 → 0.79
0.86 → 0.85	0.71 → 0.73
0.64 → 0.67	0.86 → 0.86
0.74 → 0.77	0.72 → 0.73
0.72 → 0.72	0.79 → 0.75
0.79 → 0.84	0.73 → 0.77
0.67 → 0.71	0.86 → 0.86
0.77 → 0.78	–

Table 6
NERGs for the dataset.

NERGs
0.6 → 0.64
0.67 → 0.71
0.71 → 0.72, 0.73
0.72 → 0.73, 0.72
0.73 → 0.74, 0.77
0.74 → 0.77
0.77 → 0.78, 0.79
0.78 → 0.79
0.75 → 0.72

In the search for the optimal reference set and its corresponding intervals, the upper bounds, displayed as $U_{B1} = 13,600 \in a_1$, $U_{B2} = 17,700 \in a_2$, $U_{B3} = 21,800 \in a_3$, $U_{B4} = 25,900 \in a_4$ and $U_{B5} = 30,000 \in a_5$ in step 2, are considered. Now, each upper limit in the form of $13,600 \in q_1$, $17,700 \in q_2$, $21,800 \in q_3$, $25,900 \in q_4$ and $30,000 \in q_5$ is assigned to each search agent. The initial Random Position (RM) of each search factor $q_{i\ (i=1,\ 2,\ \ldots,\ n)}$ for the reference set is shown as a column in Table 7. For ease of calculation, the upper bound given to each search factor in this column specialization is arranged such that $U_{B1} < U_{B2} < U_{B3} < U_{B4} < U_{B5}$.

To find the optimal solution, these five search factors are allowed to move to other positions, and their movement is recorded and considered in the first iteration. During the movement of any search factor from one place to another, the upper bounds belonging to the new array must always be set such that they always consist of the ascending sequence of $U_{B1} < U_{B2} < U_{B3} < U_{B4} < U_{B5}$ follow.

For the first iteration, the position of each search factor is updated based on the reference set $U_0 =$ (Zadeh, 1965). Each search factor’s new position is considered their local best position (LB). These best local positions for the first iteration are shown in Table 7. Using the best local positions of the search factor, we define a new set of intervals and a new reference set. The first iteration’s new reference set is $U_{1st} =$ [9500, 27, 762.10]. The intervals corresponding to this reference set are listed in Table 9. NERHs are made based on the U_{1st} of NERs. Finally, the first iteration ends with the de-fuzzification operation. The predicted university enrollment values obtained after the first iteration are shown in Table 10. After the first iteration, the proposed NTS-QOA model reaches an AFER of 0.167% (see Table 11).

For the 99th iteration, the timed position of each search factor is shown in Table 7. For iteration 99, the parameter values associated with the five search universes are listed in Table 8. Using the best local positions of the search factors, we define a new set of intervals and a new reference set. The new reference set resulting from iteration 99 is displayed as $U_{99th} =$ [9500, 25,579.6]. Based on the U_{99th}, the predicted university enrollment values are shown in Table 10. In this case, the proposed NTA-QOA model achieves an AFER of 0.167%.

Finally, after reaching 100 iterations, the computational process of the proposed NTS-QOA model is stopped. For the hundredth iteration, the timed position of each search factor is shown in Table 7. The parameter values corresponding to the five search factors are listed in Table 8. The new reference set resulting from the hundredth iteration is shown as $U_{100th} =$ [9500, 24,866.90]. The predicted university enrollment values after the hundredth iteration are shown in Table 10. After the hundredth iteration, the proposed NTS-QOA model reaches AFER of 0.166%.

Table 10 shows that the proposed NTS-QOA model reached AFERs of 0.167, 0.166, and 0.166, respectively, after the 1st, 99th, and 100th iterations. Hence, the predicted values of available university enrollment after the 100th iteration can be considered optimal. And the reference set and its corresponding intervals, through which these predicted values are obtained, can be regarded as the best global.

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Table 7
The positions of each search factor before and after iterations.

Position	Iteration	Reference set	q_1	q_2	q_3	q_4	q_5
RM	0	U_0 = [9500, 30,000]	13,600	17,700	21,800	25,900	30,000
LB	1	U_{1st} = [9500, 27762.10]	12908.47	17026.68	20648.80	23338.10	27762.10
LB	99	U_{99th} = [9500, 25579.60]	13599.97	16682.80	18341.70	25233.42	25579.60
GB	100	U_{100th} = [9500, 24866.90]	13303.33	15054.50	21633.69	23019.50	24866.90

*The abbreviations RM, LB, and GB in the Position column stand for random, local best, and global best, respectively.

Table 8
Selected parameter values corresponding to 5 search factors (iterations 1, 99, and 100).

Iteration	Factor	a	b	$\|\varphi\|$	Q_{ij}	Q_{hi}	q_n	$\vec{\mathbf{P}}(\mathbf{x_i})$	D (q_i)	v	y_i	L_i	α	y_{fi}	$\vec{\boldsymbol{P}}(\boldsymbol{x_i}+\mathbf{1})$
1	q_1	0.53	0.37	0.65	0.06	0.06	0.06	0.00	1.00	0.33	7570.43	12059.15	1.11	−2013.77	11586.23
	q_2	0.76	0.49	0.91	0.53	0.53	0.53	0.00	1.00	0.63	4098.65	27202.69	0.46	−1889.45	15810.55
	q_3	0.88	0.16	0.89	0.98	0.96	0.98	0.00	1.00	0.72	3567.18	36465.64	0.33	−1789.60	20010.40
	q_4	0.07	0.90	0.90	0.77	0.01	0.70	0.00	1.00	0.05	39202.96	26605.91	3.03	−12080.95	13819.05
	q_5	0.52	0.30	0.60	0.18	0.59	0.34	0.00	1.00	0.60	7694.11	44611.77	0.51	−3432.29	26567.71
99	q_1	0.20	0.24	0.31	0.62	0.62	0.62	0.00	1.00	0.63	3105.94	20988.11	0.46	−1437.64	12162.36
	q_2	0.14	0.72	0.73	0.40	0.46	0.42	0.00	1.00	0.91	878.97	33642.06	0.10	−501.01	17198.99
	q_3	0.71	0.40	0.81	0.01	0.07	0.03	0.00	1.00	0.63	5026.60	33546.80	0.46	−2320.53	19479.47
	q_4	0.59	0.45	0.74	0.93	0.09	0.71	0.00	1.00	0.01	55069.47	58338.95	4.25	−37212.46	11312.46
	q_5	0.38	0.55	0.66	0.11	0.90	0.38	0.00	1.00	0.32	17257.15	25485.69	1.15	−4397.82	25602.18
100	q_1	0.11	0.63	0.64	0.67	0.67	0.67	0.00	1.00	0.88	870.52	25458.97	0.13	−488.58	13111.42
	q_2	0.48	0.31	0.57	0.52	0.71	0.60	0.00	1.00	0.37	8711.95	17976.10	0.98	−2653.94	15046.06
	q_3	0.81	0.32	0.87	0.31	0.34	0.32	0.00	1.00	0.77	2894.84	37810.32	0.27	−1506.11	20293.89
	q_4	0.87	0.17	0.89	0.85	0.96	0.86	0.00	1.00	0.17	23092.56	5614.89	1.78	−1501.07	24398.93

Table 9
List of interval and reference sets corresponding to 5 search factors (Iterations 1, 99, and 100).

Iteration	a_1 (new)	a_2 (new)	a_3 (new)	a_4 (new)	a_5 (new)	Set
1	[9500, 12,908.47]	[12,908.47, 17,026.68]	[17,026.68, 20,648.80]	[20,648.80, 23,338.10]	[23,338.10, 27,762.10]	[9500, 27,762.10] ∈ U1st
99	[9500, 13,599.97]	[13,599.97, 16,682.80]	[16,682.80, 18,341.70]	[18,341.70, 25,233.42]	[25,233.42, 25,579.60]	[9500, 25,579.60] ∈ U_{99th}
100	[9500, 13,303.33]	[13,303.33, 15,054.50]	[15,054.50, 21,633.69]	[21,633.69, 23,019.50]	[23,019.50, 24,866.90]	[9500, 24,866.90] ∈ U100th

3.2. Algorithm 2 (NTS-GA)

The proposed NTS-GA model for forecasting time series.

1. Gathering input data from the desired databases
2. Determining the number of intervals (n) and dividing the data in the database by n
3. Setting parameters related to the GA optimization algorithm
4. Implementing the optimization algorithm on the data to minimize the prediction error
5. Updating the set of new intervals to improve the forecasting process
6 Converting crisp data to neutrosophic data
7 Calculating Entropy on the neutrosophic data set
8. Calculating the NER parameter based on the entropy values obtained from the neutrosophic data set
9. Calculating NERG based on the obtained NER values
10. Implementing deneutrosophicization operator to obtain predicted time series values
11. Evaluating the efficiency of the proposed model

In Algorithm 2, all the steps are followed as in Algorithm 1, with the difference that the genetic optimization algorithm is used to reduce the prediction error.

We divide the statistical population into sub-intervals (n = 5) and randomly select two parents from each sub-interval. We perform mutation and intersection operators on the parent differently to get new children from the above operators. Evaluator function (AFER amount obtained from new children): if the AFER result of the children is improved compared to the parent, we replace the parent with the obtained children. Otherwise, we will choose other parents. We continue this work until either the algorithm converges, that is, we reach the minimum value of AFER based on the desired value, or we terminate the algorithm's execution after several repetitions and no change in the mentioned parameter. The best result of AFER, its minimum value, is to save the desired effect and use the resulting parameters to predict the real data. Because the selections are local and random in the GA, the convergence result is less fast and accurate than the QOA and PSO algorithms.

3.3. Algorithm 3 (NTS-PSO)

The proposed NTS-PSO model for forecasting time series.

1 Gathering input data from the desired databases
2 Determining the number of intervals (n) and dividing the data in the database by n
3. Setting parameters related to the PSO optimization algorithm
4 Implementing the optimization algorithm on the data to minimize the prediction error
5 Updating the set of new intervals to improve the forecasting process
6. Converting crisp data to neutrosophic data
7. Calculating Entropy on the neutrosophic data set
8 Calculating the NER parameter based on the entropy values obtained from the neutrosophic data set
9 Calculating NERG based on the obtained NER values
10 Implementing deneutrosophicization operator to obtain predicted time series values
11 Evaluating the efficiency of the proposed model

Table 10
Predicted university enrollment values obtained after iterations 1, 99, and 100 using the proposed NTS-QOA model.

Iteration	Year	Real enrollment	Entropy	Optimal entropy	Predicted enrollment	AFER%
1	1971	13,055	0.86	0.88	13358.6	0.167402
	1972	13,563	0.9	0.905	13638.35	
	1973	13,867	0.91	0.93	14171.77	
	1974	14,696	0.95	0.96	14850.69	
	1975	15,460	0.97	0.97	15,460	
	1976	15,311	0.97	0.97	15,311	
	1977	15,603	0.97	0.97	15,603	
	1978	15,861	0.98	0.986667	15968.9	
	1979	16,807	0.99	0.988	16773.05	
	1980	16,919	0.99	0.988	16884.82	
	1981	16,388	0.99	0.988	16354.89	
	1982	15,433	0.97	0.97	15,433	
	1983	15,497	0.97	0.97	15,497	
	1984	15,145	0.96	0.966667	15250.17	
	1985	15,163	0.96	0.966667	15268.3	
	1986	15,984	0.98	0.986667	16092.73	
	1987	16,859	0.99	0.988	16824.94	
	1988	18,150	1	1	18,150	
	1989	18,970	1	1	18,970	
	1990	19,328	1	1	19,328	
	1991	19,337	1	1	19,337	
	1992	18,876	1	1	18,876	
99	1971	13,055	0.82	0.84	13373.41	0.16658
	1972	13,563	0.86	0.87	13720.71	
	1973	13,867	0.88	0.9	14182.16	
	1974	14,696	0.92	0.935	14935.61	
	1975	15,460	0.95	0.948	15427.45	
	1976	15,311	0.94	0.9475	15433.16	
	1977	15,603	0.95	0.948	15570.15	
	1978	15,861	0.96	0.973333	16081.29	
	1979	16,807	0.98	0.9825	16849.88	
	1980	16,919	0.98	0.9825	16962.16	
	1981	16,388	0.97	0.96	16219.05	
	1982	15,433	0.95	0.948	15400.51	
	1983	15,497	0.95	0.948	15464.37	
	1984	15,145	0.94	0.9475	15265.84	
	1985	15,163	0.94	0.9475	15283.98	
	1986	15,984	0.96	0.973333	16,206	
	1987	16,859	0.98	0.9825	16902.01	
	1988	18,150	1	1	18,150	
	1989	18,970	1	1	18,970	
	1990	19,328	1	1	19,328	
	1991	19,337	1	1	19,337	
	1992	18,876	1	1	18,876	
100	1971	13,055	0.82	0.84	13373.41	0.16658
	1972	13,563	0.86	0.87	13720.71	
	1973	13,867	0.88	0.9	14182.16	
	1974	14,696	0.92	0.935	14935.61	
	1975	15,460	0.95	0.948	15427.45	
	1976	15,311	0.94	0.9475	15433.16	
	1977	15,603	0.95	0.948	15570.15	
	1978	15,861	0.96	0.973333	16081.29	
	1979	16,807	0.98	0.9825	16849.88	
	1980	16,919	0.98	0.9825	16962.16	
	1981	16,388	0.97	0.96	16219.05	
	1982	15,433	0.95	0.948	15400.51	
	1983	15,497	0.95	0.948	15464.37	
	1984	15,145	0.94	0.9475	15265.84	
	1985	15,163	0.94	0.9475	15283.98	
	1986	15,984	0.96	0.973333	16,206	
	1987	16,859	0.98	0.9825	16902.01	
	1988	18,150	1	1	18,150	
	1989	18,970	1	1	18,970	
	1990	19,328	1	1	19,328	
	1991	19,337	1	1	19,337	
	1992	18,876	1	1	18,876	

Generally, algorithms based on cumulative particle methods are similar to basic evolutionary algorithms such as genetics. In cumulative methods such as the PSO algorithm, operators such as intersection and mutation are not used. These algorithms require coding, encoding, and decoding. As a result, they have better speed and accuracy results than the basic evolution algorithms. Generally, these algorithms divide a problem-solving space into several problem-solving subspaces. In each subspace, they seek to optimize the parameters related to the objective function and finally find the general solution by putting together the optimal local solutions. According to the local answers in the PSO algorithm, Local Best (x*) and the general optimal answers Global Best (g*) optimize its parameters according to the relationships in the main PSO formula.

Table 11
Predicted university enrollment values obtained after iterations 1, 99, and 100 using the proposed NTS-GA model.

Iteration	Year	Real enrollment	Entropy	Optimal entropy	Predicted enrollment	AFER%
1	1971	13,055	0.85	0.87	13362.18	0.167204
	1972	13,563	0.89	0.895	13639.2	
	1973	13,867	0.9	0.92	14175.16	
	1974	14,696	0.94	0.95	14852.34	
	1975	15,460	0.96	0.966667	15567.36	
	1976	15,311	0.96	0.966667	15417.33	
	1977	15,603	0.97	0.9725	15643.21	
	1978	15,861	0.97	0.9725	15901.88	
	1979	16,807	0.99	0.99	16,807	
	1980	16,919	0.99	0.99	16,919	
	1981	16,388	0.98	0.976667	16332.26	
	1982	15,433	0.96	0.966667	15540.17	
	1983	15,497	0.97	0.9725	15536.94	
	1984	15,145	0.96	0.966667	15250.17	
	1985	15,163	0.96	0.966667	15268.3	
	1986	15,984	0.98	0.976667	15929.63	
	1987	16,859	0.99	0.99	16,859	
	1988	18,150	1	1	18,150	
	1989	18,970	1	1	18,970	
	1990	19,328	1	1	19,328	
	1991	19,337	1	1	19,337	
	1992	18,876	1	1	18,876	
99	1971	13,055	0.84	0.86	13365.83	0.167001
	1972	13,563	0.88	0.89	13717.13	
	1973	13,867	0.9	0.92	14175.16	
	1974	14,696	0.94	0.95	14852.34	
	1975	15,460	0.96	0.96	15,460	
	1976	15,311	0.96	0.96	15,311	
	1977	15,603	0.97	0.98	15763.86	
	1978	15,861	0.97	0.98	16024.52	
	1979	16,807	0.99	0.99	16,807	
	1980	16,919	0.99	0.99	16,919	
	1981	16,388	0.98	0.97	16220.78	
	1982	15,433	0.96	0.96	15,433	
	1983	15,497	0.96	0.96	15,497	
	1984	15,145	0.95	0.956667	15251.28	
	1985	15,163	0.95	0.956667	15269.41	
	1986	15,984	0.97	0.98	16148.78	
	1987	16,859	0.99	0.99	16,859	
	1988	18,150	1	1	18,150	
	1989	18,970	1	1	18,970	
	1990	19,328	1	1	19,328	
	1991	19,337	1	1	19,337	
	1992	18,876	1	1	18,876	
100	1971	13,055	0.84	0.86	13365.83	0.167001
	1972	13,563	0.88	0.89	13717.13	
	1973	13,867	0.9	0.92	14175.16	
	1974	14,696	0.94	0.95	14852.34	
	1975	15,460	0.96	0.96	15,460	
	1976	15,311	0.96	0.96	15,311	
	1977	15,603	0.97	0.98	15763.86	
	1978	15,861	0.97	0.98	16024.52	
	1979	16,807	0.99	0.99	16,807	
	1980	16,919	0.99	0.99	16,919	
	1981	16,388	0.98	0.97	16220.78	
	1982	15,433	0.96	0.96	15,433	
	1983	15,497	0.96	0.96	15,497	
	1984	15,145	0.95	0.956667	15251.28	
	1985	15,163	0.95	0.956667	15269.41	
	1986	15,984	0.97	0.98	16148.78	
	1987	16,859	0.99	0.99	16,859	
	1988	18,150	1	1	18,150	
	1989	18,970	1	1	18,970	
	1990	19,328	1	1	19,328	
	1991	19,337	1	1	19,337	
	1992	18,876	1	1	18,876	

Table 12
Predicted university enrollment values obtained after iterations 1, 99, and 100 using the proposed NTS-PSO model.

Iteration	Year	Real enrollment	Entropy	Optimal entropy	Predicted enrollment	AFER%
1	1971	13,055	0.87	0.885	13280.08621	0.171761849
	1972	13,563	0.9	0.91	13713.7	
	1973	13,867	0.92	0.94	14168.45652	
	1974	14,696	0.96	0.97	14849.08333	
	1975	15,460	0.98	0.98	15,460	
	1976	15,311	0.97	0.9775	15429.38402	
	1977	15,603	0.98	0.98	15,603	
	1978	15,861	0.98	0.98	15,861	
	1979	16,807	1	0.996666667	16750.97667	
	1980	16,919	1	0.996666667	16862.60333	
	1981	16,388	0.99	0.992	16421.10707	
	1982	15,433	0.98	0.98	15,433	
	1983	15,497	0.98	0.98	15,497	
	1984	15,145	0.97	0.9775	15262.10052	
	1985	15,163	0.97	0.9775	15280.23969	
	1986	15,984	0.99	0.992	16016.29091	
	1987	16,859	1	0.996666667	16802.80333	
	1988	18,150	1	0.996666667	18089.5	
	1989	18,970	1	0.996666667	18906.76667	
	1990	19,328	0.99	0.992	19367.04646	
	1991	19,337	0.99	0.992	19376.06465	
	1992	18,876	1	0.996666667	18813.08	
99	1971	13,055	0.72	0.74	13417.63889	0.164124731
	1972	13,563	0.76	0.77	13741.46053	
	1973	13,867	0.78	0.805	14311.45513	
	1974	14,696	0.83	0.845	14961.59036	
	1975	15,460	0.86	0.86	15,460	
	1976	15,311	0.86	0.86	15,311	
	1977	15,603	0.87	0.875	15692.67241	
	1978	15,861	0.88	0.903333333	16281.55682	
	1979	16,807	0.91	0.915	16899.34615	
	1980	16,919	0.92	0.923333333	16980.30072	
	1981	16,388	0.9	0.88	16023.82222	
	1982	15,433	0.86	0.86	15,433	
	1983	15,497	0.86	0.86	15,497	
	1984	15,145	0.85	0.86	15323.17647	
	1985	15,163	0.85	0.86	15341.38824	
	1986	15,984	0.88	0.903333333	16407.81818	
	1987	16,859	0.92	0.923333333	16920.08333	
	1988	18,150	0.95	0.955	18245.52632	
	1989	18,970	0.96	0.965	19068.80208	
	1990	19,328	0.97	0.966666667	19261.58076	
	1991	19,337	0.97	0.966666667	19270.54983	
	1992	18,876	0.96	0.965	18974.3125	
100	1971	13,055	0.72	0.74	13417.63889	0.164124731
	1972	13,563	0.76	0.77	13741.46053	
	1973	13,867	0.78	0.805	14311.45513	
	1974	14,696	0.83	0.845	14961.59036	
	1975	15,460	0.86	0.86	15,460	
	1976	15,311	0.86	0.86	15,311	
	1977	15,603	0.87	0.875	15692.67241	
	1978	15,861	0.88	0.903333333	16281.55682	
	1979	16,807	0.91	0.915	16899.34615	
	1980	16,919	0.92	0.923333333	16980.30072	
	1981	16,388	0.9	0.88	16023.82222	
	1982	15,433	0.86	0.86	15,433	
	1983	15,497	0.86	0.86	15,497	
	1984	15,145	0.85	0.86	15323.17647	
	1985	15,163	0.85	0.86	15341.38824	
	1986	15,984	0.88	0.903333333	16407.81818	
	1987	16,859	0.92	0.923333333	16920.08333	
	1988	18,150	0.95	0.955	18245.52632	
	1989	18,970	0.96	0.965	19068.80208	
	1990	19,328	0.97	0.966666667	19261.58076	
	1991	19,337	0.97	0.966666667	19270.54983	
	1992	18,876	0.96	0.965	18974.3125	

$$\mathbf{v_i(t+1)} = \mathbf{v_i(t)} + \mathbf{c_1 r_1(t)}\ (\mathbf{p_{best}} - \mathbf{x_i(t)}) + \mathbf{c_2 r_2(t)}\ (\mathbf{g_{best}} - \mathbf{X_i(t)}) \tag{20}$$

$$\mathbf{X_i(t+1)} = \mathbf{X_i(t)} + \mathbf{v_i(t+1)} \tag{21}$$

r_1 and r_2 are random numbers in the interval [0, 1] generated in each run.

$$\mathbf{X(0)} = \mathbf{x_{min}} + \mathbf{R_i}(\mathbf{x_{max}} - \mathbf{x_{min}}) \tag{22}$$

R_i is a random number between [0, 1], and V_{max} is equal to a random number in the minimum and maximum range of the initial parameters of the problem.

If the value of v_i exceeds V_{max} at any moment, we set its value equal to V_{max}, and if the value of v_i becomes less than V_{min} at any moment, we set its value equal to V_{min}.

c_1 and c_2 parameters are for the influence of p_{best} or g_{best} results on the problem data. If the value of $c_1 \gg c_2$ means that the algorithm is more inclined towards local results, the value of $c_2 \gg c_1$ means that the algorithm is more inclined towards general results. To balance the local and general results, we set the parameters c_1 and c_2 in the range (Taghipourian et al., 2021; Kuranga et al., 2023) so that c_1 and c_2 have equal values. The best choice is $c_1 + c_2 = 4$, where c_1 and c_2 have almost equal values.

If we consider the values of c_1 and c_2 equal to 1, the particle's movement in the local responses will be somewhat slowed. As a result, it will hurt the general results (the particle has no noticeable movement in an interval.) But if the value of c_1 is close to the number 2 and the value of c_2 close to 1, the algorithm has many jumps in the local response space, but there are no noticeable changes in the general response space. Conversely, if we set c_2 equal to 2 and c_1 close to 1, the algorithm has weak changes in the local response space. But there are many changes in the space of general answers. These two modes prevent convergence in the algorithm. For this reason, it is better to give a balance between parameters c_1 and c_2 in the movement of particles.

4. Result analysis and validation

This section presents the prediction results of the university enrollment data, the TAIFEX index, and the TSEC weighted index dataset. We determine their accuracy by comparing the proposed models' forecasts with existing models based on the AFER index (see Table 12).

4.1. Prediction of university enrollment data set

To validate the proposed hybrid models (NTS-GA, NTS-PSO, and NTS-QOA), we compare them with well-known time series forecasting standard hybrid models such as the FTS-GA model (Chen and Chung, 2006), FTS-PSO model_1 (Kuo et al., 2009), FTS-PSO_2 model (Aladag et al., 2012), and FTS SA model (Radmehr and Gharneh, 2012). The predicted values of enrollment data obtained through the proposed models and the considered combined criteria models are shown in Table 13 regarding AFER.

Table 13 shows that the proposed combined models have an AFER of 0.166%, 0.167%, and 0.164%, respectively. In comparison, existing hybrid models like FTS GA model (Chen and Chung, 2006), FTS-PSO model_1 (Kuo et al., 2009), FTS PSO model_2 (Aladag et al., 2012), and FTS SA model (Radmehr and Gharneh, 2012) have AFERs of 2.19%, 3.11%, 1.58%, and 0.68% respectively. The proposed models' prediction accuracy is significantly better than the existing hybrid models. Additionally, a convergence rate analysis was carried out to compare the performance of the proposed models. As depicted in Fig. 1, the proposed model converges towards the optimal result much faster than the existing hybrid models.

Figs. 2–4 compare the actual and predicted enrollment based on the proposed models. Accordingly, it is apparent that the estimated enrollment closely matches the actual enrollment.

4.2. Prediction of TAIFEX index data set

The proposed NTS-GA, NTS-PSO, and NTS-QOA models were used to forecast the TAIFEX index data set from 3/8/1998–9/30/1998. The predicted results are shown in Table 14. In this table, the predicted results of the proposed models are compared with the existing ones. In comparison, it was observed that the proposed models performed better than the previous models.

Table 15 compares the results of the proposed NTS-QOA, NTS-QOA, and NTS-GA models and the prediction of the TAIFEX index dataset for the period between 1/19/2006 and 12/21/2006. The proposed models exhibit a significantly smaller AFER than the existing models.

4.3. Prediction of TSEC weighted index data sets

The proposed models have been verified by forecasting the TSEC weighted index from 2003 to 2004. The predicted results of the proposed models from 2003 to 2004 are compared with the previous works. Table 16 shows the comparison of other models for forecasting the TSEC weighted index from the period of 2003–2004, considering the average AFER. The proposed models have a much smaller AFER than the existing models.

Fig. 5 shows that the predicted TSEC-weighted index values closely align with the actual TSEC-weighted index values.

4.4. Statistical analysis of the results

The efficiency of the proposed NTS-QOA has been evaluated using statistical parameters that include observed and predicted mean deviation and standard deviation, correlation coefficient, and Theil's U statistic based on Eqns. 23–26.

$$\overline{\mathbf{M}} = \frac{\sum_{i=1}^{n} \mathrm{Act_i}}{\mathbf{n}} \tag{23}$$

$$\mathbf{S_d} = \sqrt{\frac{\mathbf{1}}{\mathbf{n}} \sum_{\mathbf{i=1}}^{\mathbf{n}} (\mathrm{Act_i} - \overline{\mathbf{M}})^2} \tag{24}$$

$$\mathrm{CC} = \frac{\boldsymbol{n} \sum \mathrm{Act_i} \boldsymbol{Fore_i} - (\sum \mathrm{Act_i})(\sum \mathrm{Fore_i})}{\sqrt{\boldsymbol{n} \sum \mathrm{Act_i^2} - (\sum \mathrm{Act_i})^2} \sqrt{\boldsymbol{n} \sum \mathrm{Fore_i^2} - (\sum \mathrm{Fore_i})^2}} \tag{25}$$

$$\mathbf{U} = \frac{\sqrt{\sum_{i=1}^{n} (\mathrm{Act_i} - \mathrm{Fore_i})^2}}{\sqrt{\sum_{i=1}^{n} \mathrm{Act_i^2}} + \sqrt{\sum_{i=1}^{n} \mathrm{Fore_i^2}}} \tag{26}$$

where $\mathrm{Fore_i}$ and $\mathrm{Act_i}$ denote the predicted and actual values for a particular day \ year i, respectively, and n is the total number of days \ years to be predicted. In Eqns. (23) and (24), ($\mathrm{Act_1}$, $\mathrm{Act_2}$, …, $\mathrm{Act_n}$) represent the real values of the time series, and $\overline{\mathrm{M}}$ is the average value of all these time series values. If the average and standard deviations of the actual time series values are close to the average and standard deviation of the predicted time series values, then the model's performance can be considered suitable for prediction accuracy.

In Eqn. (25), the correlation coefficient (CC) value must be between [−1, +1], that is, $1 \le \mathrm{CC} \le +1$. A model can be considered good if its CC value is greater than or equal to 0.8. In Eqn. (26), the value of U should be between [0, 1]. For good prediction accuracy, the value of U should be close to zero.

All the university enrollment prediction statistics, TAIFEX index, and TSEC weighted index datasets based on the proposed NTS-PSO, NTS-GA, and NTS-QOA models are shown in Tables 17–19. It can be seen from these tables that the averages of actual and predicted registrations are very close to each other. The minimum difference between the real and predicted averages indicates good forecast accuracy in all cases.

The difference between the actual and predicted standard deviation in the case of datasets is minimal, indicating that the proposed models'

Table 13
Comparing the proposed models with the existing ones (the university registration data set).

Year	Real enrollment	Model of (Lee and Chou, 2004)	Model of (Cheng et al., 2006)	Model of (Wong et al., 2010)	Model of (Qiu et al., 2011)	Model of (Gangwar and Kumar, 2012)	Model of (Liu, 2007)	Model of (Singh and Borah, 2013) (Singh and Dhiman, 2018)	Model of (Huarng, 2001b)	Model of (Aladag et al., 2012)	Model of (Kuo et al., 2009)	Model of (Chen and Chung, 2006)	Model of (Radmehr and Gharneh, 2012)	NTS-QOA	NTS-GA	NTS-PSO
1971	13,055	–	–	–	–	–	–	–	–	–	–	–	–	13373.41463	13365.83	13417.64
1972	13,563	14,000	14,025	15,430	14,195	13,563	13,500	13,563	13,563	13,480.38	14,000	–	13,706	13720.7093	13717.13	13741.46
1973	13,867	14,000	14,568	15,430	14,424	13,500	13,800	13,867	13,500	13,480.38	14,000	14,146	13,706	14182.15909	14175.16	14311.46
1974	14,696	14,000	14,568	15,430	14,593	14,500	14,700	14,696	14,500	14,780.95	14,000	14,878	14,749	14935.6087	14852.34	14961.59
1975	15,460	15,500	15,654	15,430	15,589	15,500	15,600	15,425	15,500	15,474.17	15,500	14,878	15,341	15427.45263	15460.00	15460.00
1976	15,311	16,000	15,654	15,430	15,645	15,500	15,400	15,420	15,466	15,523.89	16,000	15,609	15,346	15433.16223	15311.00	15311.00
1977	15,603	16,000	15,654	15,430	15,634	15,500	15,750	15,420	15,392	15,609.76	16,000	15,609	15,346	15570.15158	15763.86	15692.67
1978	15,861	16,000	15,654	15,430	16,100	15,500	15,400	15,923	15,549	15,562.98	16,000	16,214	15,923	16081.29167	16024.52	16281.56
1979	16,807	16,000	16,197	16,889	16,188	16,807	16,800	16,862	16,433	16,751.87	16,000	16,214	16,839	16849.875	16807.00	16899.35
1980	16,919	16,813	17,283	16,871	17,077	16,919	17,100	17,192	16,656	16,837.83	16,833	16,818	17,046	16962.16071	16919.00	16980.30
1981	16,388	16,813	17,283	16,871	17,105	16,500	17,100	17,192	16,624	16,837.83	16,833	16,818	16,400	16219.05155	16220.78	16023.82
1982	15,433	16,789	16,197	15,447	16,369	15,500	15,300	15,425	15,556	15,901.87	16,833	15,609	15,455	15400.50947	15433.00	15433.00
1983	15,497	16,000	15,654	15,430	15,643	15,500	15,750	15,420	15,524	15,524.96	16,000	15,609	15,346	15464.37474	15497.00	15497.00
1984	15,145	16,000	15,654	15,430	15,648	15,500	15,400	15,420	15,497	15,527.33	16,000	14,146	15,346	15265.83777	15251.28	15323.18
1985	15,163	16,000	15,654	15,430	15,622	15,500	15,300	15,627	15,305	15,492.78	16,000	14,146	15,341	15283.98138	15269.41	15341.39
1986	15,984	16,000	15,654	15,430	15,623	15,984	15,750	15,627	15,308	15,570.69	16,000	16,818	15,923	16,206	16148.78	16407.82
1987	16,859	16,000	16,197	16,889	16,231	16,859	16,800	16,862	16,402	16,906.56	16,000	16,818	16,839	16902.00765	16859.00	16920.08
1988	18,150	16,813	17,283	16,871	17,090	18,500	17,100	17,192	18,500	16,837.83	16,833	17,992	18,007	18,150	18150.00	18245.53
1989	18,970	19,000	18,369	19,333	18,325	18,500	18,900	18,923	18,534	19,144.40	19,000	19,126	19,059	18,970	18970.00	19068.80
1990	19,328	19,000	19,454	19,333	19,000	19,337	19,200	19,333	19,345	19,144.40	19,000	19,126	19,197	19,328	19328.00	19261.58
1991	19,337	19,000	19,454	19,333	19,000	19,500	19,050	19,136	19,423	19,144.40	19,000	19,126	19,197	19,337	19337.00	19270.55
1992	18,876	–	–	19,333	19,000	18,704	19,050	19,136	18,752	19,144.40	19,000	19,126	19,059	18,876	18876.00	18974.31
AFER	–	7.83	7.30	2.75	2.65	1.44	1.32	1.19	0.96	1.58	3.11	2.19	0.68	0.166	0.167	0.164

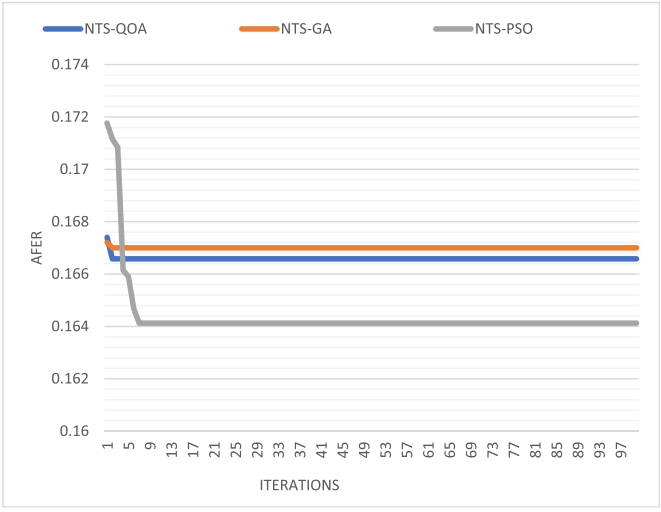

Fig. 1. AFER curve and convergence of the proposed models.

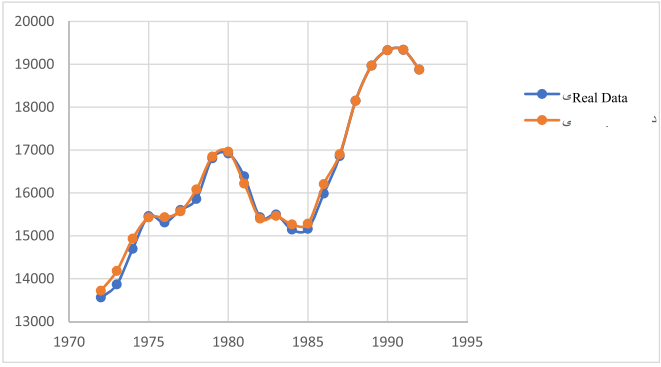

Fig. 2. Comparison of actual with predicted enrollment based on the proposed NTS-QOA model.

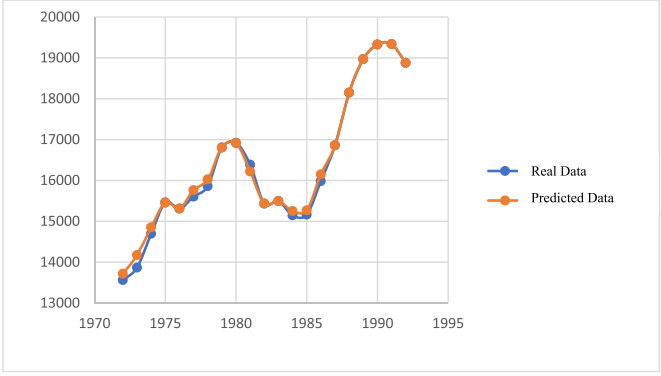

Fig. 3. Comparison of actual with predicted enrollment based on the proposed NTS-GA model.

prediction accuracy is good. The correlation coefficients between the real and predicted values in this data set also show the efficiency of the proposed models.

Theil's U statistic for all these data sets is close to zero, which shows that the proposed models are effective. Each statistic, therefore, supports the proposed models, and the TAIFEX index dataset and the TSEC weighted index dataset look very convincing, with excellent performance across all university enrollment cases.

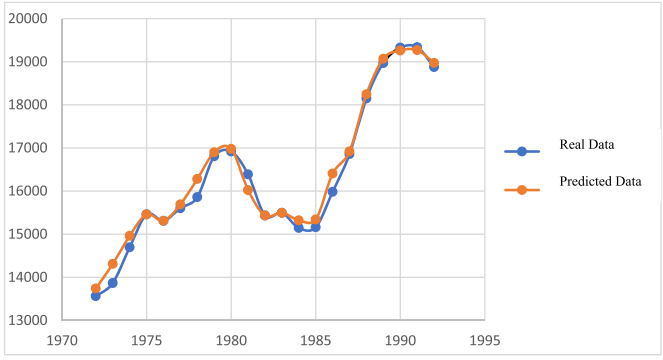

Fig. 4. Comparison of actual with predicted enrollment based on the proposed NTS-PSO model.

5. Conclusion

In this research, we sought a new hybrid model for predicting time series in the neutrosophic environment. Neutrosophic set theory was used to represent the time series data set. This form of representation is called NTS and is mainly used to model time series data sets. In the present study, with the help of the concept of entropy, NERs were created between NTS representations of time series. These NERs were used in the de-fuzzification operation to obtain the prediction results.

This study showed that the prediction accuracy of the NTS modeling method relies more on the optimal selection of the reference set and its corresponding distances. This research problem was solved using the integration of three optimization algorithms, GA, QOA, and PSO, with the NTS modeling approach. The combination of QOA, GA, and PSO in this NTS modeling method helped to search for the overall optimal solution for the reference set and its corresponding distances from the list of locally optimal solutions. Then, the proposed NTS QOA, NTS-GA, and NTS-PSO models were verified by effectively predicting three different data sets: the University of Alabama enrollment, the TAIFEX index, and the TSEC index.

Prediction results showed that the proposed models perform better than other existing models. Among the proposed models, the NTS-PSO model performs better than the proposed NTS-QOA and NTS-GA models regarding convergence speed and minimum AFER parameters.

The study's approach, combining neutrosophic logic and optimization algorithms, can significantly improve forecasting accuracy, practically impacting various industries, including finance, energy, and education, where accurate predictions are crucial for decision-making and planning. Using neutrosophic logic provides a more comprehensive way of understanding and handling uncertainty in time series data. This insight can lead to better decision support systems as decision-makers gain a deeper understanding of the complexities involved in the data.

The study's demonstration of the approach's versatility across different datasets suggests that it can be applied in various domains. This insight can inspire researchers and practitioners to explore the application of neutrosophic logic and optimization algorithms in fields beyond those presented in the study. The study's acknowledgment of limitations and recommendations for future research, such as deep learning and parallel processing, provide a roadmap for further exploration. This insight guides researchers toward areas where improvements can be made, particularly in handling large datasets. The inclusion of Smarandache's philosophical foundation for neutrosophic logic demonstrates the relevance of philosophical perspectives in addressing real-world problems. This insight may encourage researchers to explore interdisciplinary approaches to solving complex issues.

Several potential limitations of the study can be mentioned as follows.

Table 14
Comparison between the proposed and existing models (for the TAIFEX index data set).

Date	Real data	Model of (Lee et al., 2006)	Model of (Lee et al., 2006)	Model of (Lee et al., 2008)	Model of (Huarng et al., 2007)	Model of (Kuo et al., 2009)	Model of (Kuo et al., 2010)	NTS-QOA	NTS-GA	NTS-PSO
March 08, 1998	7552	–	–	–	–	–	–	7513.07	7513.47	7512.67
April 08, 1998	7560	7450	7450	–		–	–	7521.03	7521.43	7520.63
May 08, 1998	7487	7450	7450	–	–	–	–	7481.43	7487.00	7479.84
June 08, 1998	7462	7500	7450	7450	–	–	7452.54	7456.45	7462.00	7454.86
July 08, 1998	7515	7500	7500	7550	–	–	7518.11	7476.26	7476.66	7475.86
October 08, 1998	7365	7450	7450	7350	–	–	7359.49	7359.52	7365.00	7357.95
November 08, 1998	7360	7300	7350	7350	–	–	7359.49	7354.52	7360.00	7352.96
December 08, 1998	7330	7300	7300	7350	7329	7320.77	7331.62	7324.55	7330.00	7322.99
8/13/1998	7291	7300	7350	7250	7289.5	7289.56	7285.63	7285.58	7291.00	7284.02
8/14/1998	7320	7183.33	7100	7350	7329	7320.77	7331.62	7314.55	7320.00	7313.00
8/15/1998	7300	7300	7350	7350	7289.5	7289.56	7291.67	7294.57	7300.00	7293.01
8/17/1998	7219	7300	7300	7250	7215	7222.19	7217.15	7213.63	7219.00	7203.64
8/18/1998	7220	7183.33	7100	7250	7215	7222.19	7217.15	7214.63	7220.00	7204.64
8/19/1998	7285	7183.33	7300	7250	7289.5	7289.56	7285.63	7279.58	7285.00	7278.03
8/20/1998	7274	7183.33	7100	7250	7289.5	7289.56	7279.59	7268.59	7274.00	7267.04
8/21/1998	7225	7183.33	7100	7250	7215	7222.19	7217.15	7219.62	7225.00	7209.63
8/24/1998	6955	7183.33	7100	6950	6949.5	6952.76	6950.85	6951.11	6906.70	6938.97
8/25/1998	6949	6850	6850	6950	6949.5	6952.76	6941.88	6945.11	6900.74	6932.99
8/26/1998	6790	6850	6850	6750	6796	6800.07	6784.34	6786.20	6796.81	6802.30
8/27/1998	6835	6775	6650	6850	6848	6850.12	6843.35	6831.17	6841.85	6819.25
8/28/1998	6695	6850	6750	6650	6698.5	6713.46	6700.46	6705.28	6701.71	6707.13
8/29/1998	6728	6750	6750	6750	6726	6713.46	6721.85	6738.33	6734.74	6740.19
8/31/1998	6566	6775	6650	6550	6569.5	6568.29	6562.63	6551.73	6531.07	6529.92
September 1, 1998	6409	6450	6450	6450	6417	6416.74	6402.61	6395.07	6426.23	6426.80
September 2, 1998	6430	6450	6550	6450	6417	6416.74	6417.14	6416.02	6447.28	6447.86
September 3, 1998	6200	6450	6350	6250	6205	6195	6191.69	6268.89	6233.70	6234.83
September 4, 1998	6403.2	6450	6450	6450	6417	6416.74	6402.61	6389.28	6420.41	6420.99
September 5, 1998	6697.5	6150	6550	6650	6698.5	6713.46	6700.46	6707.79	6704.21	6709.63
September 7, 1998	6722.3	6750	6750	6750	6726	6713.46	6721.85	6732.63	6729.04	6734.48
September 8, 1998	6859.4	6775	6850	6850	6848	6850.12	6852.31	6855.56	6866.28	6843.59
September 9, 1998	6769.6	6850	6750	6750	6763	6767.08	6770.64	6765.81	6776.39	6781.86
September 10, 1998	6709.75	6775	6650	6750	6726	6713.46	6711.15	6720.06	6716.48	6721.91
September 11, 1998	6726.5	6775	6850	6750	6726	6713.46	6721.85	6736.83	6733.24	6738.69
9/14/1998	6774.55	6775	6850	6817	6763	6767.08	6784.34	6770.76	6781.34	6786.82
9/15/1998	6762	6775	6650	6817	6763	6767.08	6770.64	6758.21	6768.78	6774.25
9/16/1998	6952.75	6775	6850	6817	6949.5	6952.76	6950.85	6948.86	6904.47	6936.73
9/17/1998	6906	6850	6950	6950	6904.5	6905.44	6903.03	6902.13	6912.92	6890.09
9/18/1998	6842	6850	6850	6850	6848	6850.12	6843.35	6838.17	6848.86	6826.24
9/19/1998	7039	6850	6950	7050	7064	7033.83	7042.6	7001.95	6990.12	7024.02
9/21/1998	6861	6850	6850	6850	6848	6850.12	6852.31	6857.16	6867.88	6845.19
9/22/1998	6926	6850	6950	6950	6904.5	6905.44	6932.9	6922.12	6877.90	6910.04
9/23/1998	6852	6850	6850	6850	6848	6850.12	6852.31	6848.16	6858.87	6836.21
9/24/1998	6890	6850	6950	6850	6904.5	6905.44	6891.1	6886.14	6896.91	6874.12
9/25/1998	6871	6850	6850	6850	6848	6850.12	6861.26	6867.15	6877.89	6855.17
9/28/1998	6840	6850	6750	6850	6848	6850.12	6843.35	6836.17	6846.86	6824.24
9/29/1998	6806	6850	6750	6850	6796	6800.07	6798.04	6802.19	6812.82	6818.33
9/30/1998	6787	6850	6750	6750	6796	6800.07	6784.34	6783.20	6793.80	6799.30
AFER%	–	0.88	1/02	0.43	0.11	0.12	0.097	0.081	0.081	0.081

Table 15
Comparison between the proposed model and existing ones (for the TAIFEX index data set).

Date	Real data	Model of (Lee et al., 2006)	Model of (Lee et al., 2006)	Model of (Lee et al., 2008)	Model of (Huarng et al., 2007)	Model of (Kuo et al., 2009)	Model of (Kuo et al., 2010)	Proposed NTS-QOA	Proposed NTS-GA	Proposed NTS-PSO
1/19/ 2006	7649	7650	7650	7650	7650	7650	7650	7569.32	7609.57	7565.86
2/16/ 2006	7270	7279	7279	7279.5	7279.5	7271.5	7273.5	7231.33	7232.14	7229.61
3/16/ 2006	7002	7009	7009	7005	7005	7005	7005	6926.71	6957.78	6923.33
4/20/ 2006	6926	6910	6910	6916	6916	6916	6916	6850.72	6882.26	6886.65
5/18/ 2006	6750	6739	6739	6749	6741	6741	6741	6676.63	6678.19	6646.55
6/22/ 2006	6418	6428.5	6428.5	6428.5	6428.5	6418.5	6418.5	6514.15	6487.76	6518.67
7/20/ 2006	6413	6433.5	6423.5	6423.5	6421.5	6421.5	6421.5	6509.07	6482.71	6513.60
8/17/ 2006	6998	6988.1	6988.1	6988.1	6988.1	6988.1	6998.1	6922.75	6953.80	6919.37
9/21/ 2006	7064	7088	7088	7088	7088	7088	7088	6988.04	7019.39	6984.63
10/19/ 2006	6519	6538.5	6538.5	6538.5	6538.5	6538.5	6538.5	6555.22	6495.63	6556.90
11/16/ 2006	6656	6678	6678	6678	6668	6661	6661	6619.43	6632.14	6553.99
12/21/ 2006	6491	6471	6491	6491	6491	6491	6491	6527.06	6561.55	6592.82
AFER%	–	0.23	0.19	0.16	0.16	0.12	0.11	0.11	0.11	0.10

Table 16
Comparison between the proposed model and existing ones (for the TSEC weighted index data set).

Model	2003	2004	AFER
Huarng et al. (Singh et al., 2018)	5.41	6.87	6.14
Bivariate model (Qiao et al., 2022)	8.67	8.97	8.82
Univariate model (Qiao et al., 2022)	11.5	7.68	9.59
AR-1 (Duan et al., 2022)	6.09	5.69	5.89
AR-2 (Duan et al., 2022)	5.19	4.94	5.07
Jiang et al. (Jiang et al., 2017)	4.68	4.85	4.77
Gupta et al. (Gupta et al., 2018)	2.18	2.15	2.17
Singh et al. (Singh and Dhiman, 2018)	1.17	2.15	1.66
Proposed Model-1 (NTS-QOA)	0.126	0.055	0.09
Proposed Model-1 (NTS-GA)	0.126	0.054	0.09
Proposed Model-1 (NTS-PSO)	0.126	0.055	0.09

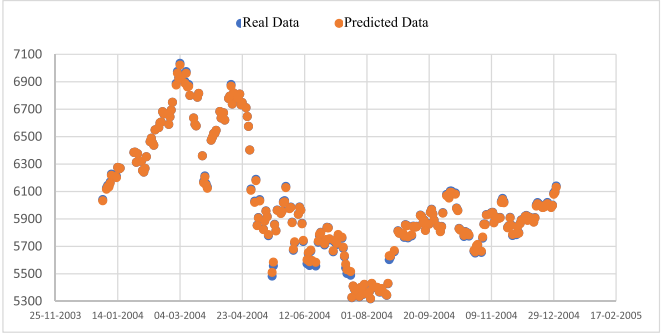

Fig. 5. The actual and predicted weighted index of TSEC for 2004 based on the proposed NTS-PSO model.

- Computational Intensity: The study acknowledges that when dealing with large datasets, the execution speed of the algorithm decreases significantly. The optimization algorithms' computational intensity, especially when combined, can be a limiting factor, particularly when dealing with extensive time series data. This limitation can make the approach less practical for very large datasets without efficient computational resources.
- Limited Generalization: While the study demonstrates the effectiveness of the proposed NTS models on the specific datasets used (university enrollment, financial indices, stock exchange data), its generalizability to other domains and datasets may be limited. It remains to be seen how well the model performs in diverse scenarios beyond the examples provided. The model may not perform equally well in all situations, and the choice of datasets can influence the perceived success of the approach.
- Hybrid Model Selection: The study introduces a hybrid model (NTS-QOA, -GA, and -PSO) but does not extensively explore the criteria for selecting the optimal combination of optimization algorithms. There may be an element of subjectivity in the selection process, which can affect the model's performance.
- Overfitting Concerns: The study does not explicitly address the potential for overfitting when employing multiple optimization algorithms. Overfitting could lead to models that perform exceptionally well on training data but poorly on new, unseen data.

Based on the identified limitations of the study, several potential future research directions can be proposed.

- Given the computational intensity of the optimization algorithms, future research could focus on developing efficient parallel processing techniques to make the proposed approach more scalable for handling large datasets. It could involve exploring distributed, cloud, or other parallel computing architectures.
- Future research could explore integrating deep learning techniques with Neutrosophic Logic and optimization algorithms to improve forecasting accuracy. It could involve developing neural network architectures specifically designed for time series forecasting in uncertain and complex environments. They may explore the potential of combining the proposed NTS-QOA model with other optimization techniques, such as simulated annealing, ant colony optimization, or reinforcement learning, to enhance forecasting accuracy.
- Future research could conduct extensive real-world case studies and applications across various industries to validate the proposed

Table 17
Statistical analysis of predicted results based on the proposed NTS-QOA model.

Data Set/Parameter	University enrollment (See Table 13)		TAIFEX (See Table 14)		TAIFEX (See Table 15)		TSEC (4-4) (See Table 16)	
	Real	Predicted	Real	Predicted	Real	Predicted	Real	Predicted
$\overline{M}$	16194.18	16.194	6964.25	6942	6846.3	6846.3	5597.84	5586.7
S_d	1816.49	1747/7	331.58	328.03	378.38	362.27	515.16	578
CC	0.9983		0.999		0.0997		0.999	
Theil's U	0.0063		0.0017		0.0072		0.0006	

Table 18
Statistical analysis of predicted results based on the proposed NTS-GA model.

Data Set/Parameter	University enrollment (See Table 13)		TAIFEX (See Table 14)		TAIFEX (See Table 15)		TSEC (4-4) (See Table 16)	
	Real	Predicted	Real	Predicted	Real	Predicted	Real	Predicted
$\overline{M}$	16194.18	16,194	6964.25	6964.2	6846.33	6846.3	5597.84	5597.85
S_d	1816.49	1774.7	331.58	328.03	378.38	362.27	515.16	514.2
CC	0.9987		0.9983		0.9935		0.999	
Theil's U	0.0056		0.0021		0.0053		0.0022	

Table 19
Statistical analysis of predicted results based on the proposed NTS-PSO model.

Data Set/Parameter	University enrollment (See Table 13)		TAIFEX (See Table 14)		TAIFEX (See Table 15)		TSEC (4-4) (See Table 16)	
	Real	Predicted	Real	Predicted	Real	Predicted	Real	Predicted
$\overline{M}$	16194.18	16,194	6964.25	6964.2	6846.33	6846.3	5597.84	5597.85
S_d	1816.49	1774.7	331.58	328.03	378.38	362.27	515.16	514.13
CC	0.9953		0.9992		0.9824		0.9999	
Theil's U	0.0094		0.0018		0.0085		0.0009	

approach's practical impact. It would provide valuable insights into the model's effectiveness in different domains. Future studies could perform benchmarking and comparative analyses by testing the proposed approach against a wider range of existing time series forecasting models, including both traditional statistical methods and other advanced techniques. It would help establish the approach's relative strengths and weaknesses.

- Future works may develop interactive decision support systems that utilize the proposed forecasting model to provide real-time decision support for users. It could involve creating user-friendly interfaces that enable domain experts to make informed decisions based on the model's predictions.
- Future studies may investigate the potential for transfer learning in time series forecasting. Can knowledge gained from one domain be effectively transferred to other disciplines, and how can transfer learning techniques be adapted for NTS modeling?

CRediT authorship contribution statement

Seyyed Ahmad Edalatpanah: Data curation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Final draft preparation. **Farnaz Sheikh Hassani:** Investigation, Methodology, Supervision, Writing – original draft, Writing – review & editing, Final draft preparation. **Florentin Smarandache:** Investigation, Methodology, Supervision, Writing – original draft, Writing – review & editing, Final draft preparation. **Ali Sorourkhah:** Investigation, Methodology, Supervision, Writing – original draft, Writing – review & editing, Final draft preparation. **Dragan Pamucar:** Supervision, Writing – original draft, Writing – review & editing, Final draft preparation. **Bing Cui:** Conceptualization, Data curation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Final draft preparation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

References

Abbasimehr, H., Khodizadeh Nahari, M., 2020. Improving demand forecasting with LSTM by taking into account the seasonality of data. Journal of Applied Research on Industrial Engineering 7 (2), 177–189. https://doi.org/10.22105/jarie.2020.232594.1168.

Abbaspour Ghadim Bonab, A., 2022. A comparative study of demand forecasting based on machine learning methods with time series approach. Journal of Applied Research on Industrial Engineering 9 (3), 331–353. https://doi.org/10.22105/jarie.2021.246283.1192.

Abdi, A., Abdi, A., Fathollahi-Fard, A.M., Hajiaghaei-Keshteli, M., 2021. A set of calibrated metaheuristics to address a closed-loop supply chain network design problem under uncertainty. Int. J. Syst. Sci.: Operations and Logistics 8 (1), 23–40. https://doi.org/10.1080/23302674.2019.1610197.

Abdolhosseini, M., 2023. Forecasting of COVID-19 sixth peak in Iran based on singular spectrum analysis. Journal of Decisions and Operations Research 8 (1), 123–132. https://doi.org/10.22105/dmor.2022.312966.1517.

Adebisi, S.A., Smarandache, F., 2022. On the Introduction to neutrosophic statistics and neutrosophic algebraic structures (involving the fuzziness, similarity and the symmetry properties on the neutrosophic interval probability) (1). Journal of Fuzzy Extension and Applications. https://doi.org/10.22105/jfea.2022.366160.1235.

Aladag, C.H., Yolcu, U., Egrioglu, E., Dalar, A.Z., 2012. A new time invariant fuzzy time series forecasting method based on particle swarm optimization. Appl. Soft Comput. 12 (10), 3291–3299. https://doi.org/10.1016/j.asoc.2012.05.002.

Atanassov, K.T., 1986. Intuitionistic fuzzy sets. Fuzzy Set Syst. 20 (1), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3.

Bas, E., Grosan, C., Egrioglu, E., Yolcu, U., 2018. High order fuzzy time series method based on pi-sigma neural network. Eng. Appl. Artif. Intell. 72, 350–356. https://doi.org/10.1016/j.engappai.2018.04.017.

Casal-Guisande, M., Bouza-Rodríguez, J.-B., Cerqueiro-Pequeño, J., Comesaña-Campos, A., 2023. Design and conceptual development of a novel hybrid intelligent

decision support system applied towards the prevention and early detection of forest fires. Forests 14 (2), 172. https://doi.org/10.3390/f14020172.
Chen, S.-M., 1996. Forecasting enrollments based on fuzzy time series. Fuzzy Set Syst. 81 (3), 311–319. https://doi.org/10.1016/0165-0114(95)00220-0. Aug.
Chen, M.-Y., Chen, B.-T., 2015. A hybrid fuzzy time series model based on granular computing for stock price forecasting. Inf. Sci. 294, 227–241. https://doi.org/10.1016/j.ins.2014.09.038.
Chen, S.-M., Chung, N.-Y., 2006. Forecasting enrollments using high-order fuzzy time series and genetic algorithms. Int. J. Intell. Syst. 21 (5), 485–501. https://doi.org/10.1002/int.20145.
Chen, S.-M., Zou, X.-Y., Gunawan, G.C., 2019. Fuzzy time series forecasting based on proportions of intervals and particle swarm optimization techniques. Inf. Sci. 500, 127–139. https://doi.org/10.1016/j.ins.2019.05.047.
Chen, L., Wu, T., Wang, Z., Lin, X., Cai, Y., 2023. A novel hybrid BPNN model based on adaptive evolutionary Artificial Bee Colony Algorithm for water quality index prediction. Ecol. Indicat. 146, 109882 https://doi.org/10.1016/j.ecolind.2023.109882.
Cheng, C.-H., Chang, J.-R., Yeh, C.-A., 2006. Entropy-based and trapezoid fuzzification-based fuzzy time series approaches for forecasting IT project cost. Technol. Forecast. Soc. Change 73 (5), 524–542. https://doi.org/10.1016/j.techfore.2005.07.004.
Cheng, C., Cheng, G., Wang, J., 2008. Multi-attribute fuzzy time series method based on fuzzy clustering. Expert Syst. Appl. 34 (2), 1235–1242. https://doi.org/10.1016/j.eswa.2006.12.013.
Daneshvar, A., Salahi, F., Ebrahimi, M., Nahavandi, B., 2023. Analyzing behavioral patterns of bus passengers using data mining methods (case study: rapid transportation systems). Journal of Applied Research on Industrial Engineering 10 (1), 11–24. https://doi.org/10.22105/jarie.2021.302560.1373.
Dong, Y., Xiao, L., Wang, J., Wang, J., 2023. A time series attention mechanism based model for tourism demand forecasting. Inf. Sci. 628, 269–290. https://doi.org/10.1016/j.ins.2023.01.095.
Duan, J., Wang, P., Ma, W., Fang, S., Hou, Z., 2022. A novel hybrid model based on nonlinear weighted combination for short-term wind power forecasting. Int. J. Electr. Power Energy Syst. 134, 107452 https://doi.org/10.1016/j.ijepes.2021.107452.
Erden, C., 2023. Genetic algorithm-based hyperparameter optimization of deep learning models for PM2.5 time-series prediction. Int. J. Environ. Sci. Technol. 20 (3), 2959–2982. https://doi.org/10.1007/s13762-023-04763-6.
Espinosa, R., Jiménez, F., Palma, J., 2023. Multi-surrogate assisted multi-objective evolutionary algorithms for feature selection in regression and classification problems with time series data. Inf. Sci. 622, 1064–1091. https://doi.org/10.1016/j.ins.2022.12.004.
Etebari, M., Feghhi Farahmand, N., Iranzadeh, S., 2021. Compilation a credit forecasting model for legal customers of private banks using meta-heuristic algorithms (Case study: pasargad Bank branches in the north, west of the country). Journal of Decisions and Operations Research 6 (3), 330–349. https://doi.org/10.22105/dmor.2021.247229.1220.
Etemadi, S., Khashei, M., 2023. Proposing a new class of statistical reliability-based methods for modeling and decision-making. Journal of Decisions and Operations Research 8 (1), 269–282. https://doi.org/10.22105/dmor.2022.311692.1514.
Eyo, I., Eyoh, J., Umoh, U., 2021. On the prediction of Covid-19 time series: an intuitionistic fuzzy logic approach. Journal of Fuzzy Extension and Applications 2 (2), 171–190. https://doi.org/10.22105/jfea.2021.263890.1070.
Fakhrehosseini, S.F., Kaviani, M., 2023. Exploring the interface: financial crisis-induced exchange rate fluctuations and implications for Iran's current account deficit (1989-2022). Journal of Operational and Strategic Analytics 1 (3), 115–123. https://doi.org/10.56578/josa010302.
Gangwar, S.S., Kumar, S., 2012. Partitions based computational method for high-order fuzzy time series forecasting. Expert Syst. Appl. 39 (15), 12158–12164. https://doi.org/10.1016/j.eswa.2012.04.039.
Gao, R., Duru, O., 2020. Parsimonious fuzzy time series modelling. Expert Syst. Appl. 156, 113447 https://doi.org/10.1016/j.eswa.2020.113447. Oct.
Goyal, G., Bisht, D.C.S., 2023. Adaptive hybrid fuzzy time series forecasting technique based on particle swarm optimization. Granular Computing 8 (2), 373–390. https://doi.org/10.1007/s41066-022-00331-4.
Guo, J., Lin, P., Zhang, L., Pan, Y., Xiao, Z., 2023. Dynamic adaptive encoder-decoder deep learning networks for multivariate time series forecasting of building energy consumption. Appl. Energy 350, 121803. https://doi.org/10.1016/j.apenergy.2023.121803.
Gupta, C., Jain, A., Tayal, D.K., Castillo, O., 2018. ClusFuDE: forecasting low dimensional numerical data using an improved method based on automatic clustering, fuzzy relationships and differential evolution. Eng. Appl. Artif. Intell. 71, 175–189. https://doi.org/10.1016/j.engappai.2018.02.015.
Hajirahimi, Z., Khashei, M., 2023. Hybridization of hybrid structures for time series forecasting: a review. Artif. Intell. Rev. 56 (2), 1201–1261. https://doi.org/10.1007/s10462-022-10199-0.
Hieu, N.D., Van Linh, M., Phong, P.D., 2023. A Co-optimization algorithm utilizing particle swarm optimization for linguistic time series. Mathematics 11 (7), 1597. https://doi.org/10.3390/math11071597.
Huarng, K., 2001a. Heuristic models of fuzzy time series for forecasting. Fuzzy Set Syst. 123 (3), 369–386. https://doi.org/10.1016/S0165-0114(00)00093-2.
Huarng, K., 2001b. Effective lengths of intervals to improve forecasting in fuzzy time series. Fuzzy Set Syst. 123 (3), 387–394. https://doi.org/10.1016/S0165-0114(00)00057-9.
Huarng, Kun-Huang, Hui, Tiffany, Kuang, Yu, Hsu, Yu Wei, 2007. A multivariate heuristic model for fuzzy time-series forecasting. IEEE Transactions on Systems, Man and Cybernetics, Part B (Cybernetics) 37 (4), 836–846. https://doi.org/10.1109/TSMCB.2006.890303.
Ihsan, M., Saeed, M., Rahman, A.U., 2023. Optimizing hard disk selection via a fuzzy parameterized single-valued neutrosophic soft set approach. Journal of Operational and Strategic Analytics 1 (2), 62–69. https://doi.org/10.56578/josa010203.
Jiang, P., Dong, Q., Li, P., Lian, L., 2017. A novel high-order weighted fuzzy time series model and its application in nonlinear time series prediction. Appl. Soft Comput. 55, 44–62. https://doi.org/10.1016/j.asoc.2017.01.043.
Jiang, F., Zhao, Z., Shao, X., 2023. Time series analysis of COVID-19 infection curve: a change-point perspective. J. Econom. 232 (1), 1–17. https://doi.org/10.1016/j.jeconom.2020.07.039.
Kuo, I.-H., Horng, S.-J., Kao, T.-W., Lin, T.-L., Lee, C.-L., Pan, Y., 2009. An improved method for forecasting enrollments based on fuzzy time series and particle swarm optimization. Expert Syst. Appl. 36 (3), 6108–6117. https://doi.org/10.1016/j.eswa.2008.07.043.
Kuo, I.-H., et al., 2010. Forecasting TAIFEX based on fuzzy time series and particle swarm optimization. Expert Syst. Appl. 37 (2), 1494–1502. https://doi.org/10.1016/j.eswa.2009.06.102.
Kuranga, C., Muwani, T.S., Ranganai, N., 2023. A multi-population particle swarm optimization-based time series predictive technique. Expert Syst. Appl. 233, 120935 https://doi.org/10.1016/j.eswa.2023.120935.
Lee, H.-S., Chou, M.-T., 2004. Fuzzy forecasting based on fuzzy time series. Int. J. Comput. Math. 81 (7), 781–789. https://doi.org/10.1080/00207160410001712288. Jul.
Lee, Li-Wei, Wang, Li-Hui, Chen, Shyi-Ming, Leu, Yung-Ho, 2006. Handling forecasting problems based on two-factors high-order fuzzy time series. IEEE Trans. Fuzzy Syst. 14 (3), 468–477. https://doi.org/10.1109/TFUZZ.2006.876367.
Lee, L.-W., Wang, L.-H., Chen, S.-M., 2008. Temperature prediction and TAIFEX forecasting based on high-order fuzzy logical relationships and genetic simulated annealing techniques. Expert Syst. Appl. 34 (1), 328–336. https://doi.org/10.1016/j.eswa.2006.09.007.
Li, P., Edalatpanah, S.A., Sorourkhah, A., Yaman, S., Kausar, N., 2023. An integrated fuzzy structured methodology for performance evaluation of high schools in a group decision-making problem. Systems 11, 3. https://doi.org/10.3390/systems11030159.
Liu, H.-T., 2007. An improved fuzzy time series forecasting method using trapezoidal fuzzy numbers. Fuzzy Optim. Decis. Making 6 (1), 63–80. https://doi.org/10.1007/s10700-006-0025-9. Mar.
Maghsoodi, A.I., 2023. Cryptocurrency portfolio allocation using a novel hybrid and predictive big data decision support system. Omega 115, 102787. https://doi.org/10.1016/j.omega.2022.102787.
Martin, N., Edalatpanah, S.A., 2023. Application of extended fuzzy ISOCOV methodology in nanomaterial selection based on performance measures. Journal of Operational and Strategic Analytics 1 (2), 55–61. https://doi.org/10.56578/josa010202.
Melkikh, A.V., 2015. Nonlinearity of quantum mechanics and solution of the problem of wave function collapse. Commun. Theor. Phys. 64 (1), 47–53. https://doi.org/10.1088/0253-6102/64/1/47. Jul.
Melkikh, A.V., 2017. Quantum paradoxes, entanglement and their explanation on the basis of quantization of fields. Mod. Phys. Lett. B 31 (2), 1750007. https://doi.org/10.1142/S0217984917500075.
Pramanik, S., Suman, D., Rakhal, D., Binod, C.T., 2023. Neutrosophic BWM-TOPSIS strategy under SVNS environme. Neutrosophic Sets and Systems 56 (1), 178–189.
Qiao, W., Li, Z., Liu, W., Liu, E., 2022. Fastest-growing source prediction of US electricity production based on a novel hybrid model using wavelet transform. Int. J. Energy Res. 46 (2), 1766–1788. https://doi.org/10.1002/er.7293.
Qiu, W., Liu, X., Li, H., 2011. A generalized method for forecasting based on fuzzy time series. Expert Syst. Appl. 38 (8), 10446–10453. https://doi.org/10.1016/j.eswa.2011.02.096.
Qiu, P., Sorourkhah, A., Kausar, N., Cagin, T., Edalatpanah, S.A., 2023. Simplifying the complexity in the problem of choosing the best private-sector partner. Systems 11, 2. https://doi.org/10.3390/systems11020080.
Radmehr, F., Gharneh, N.S., 2012. Forecasting method based on high order fuzzy time series and simulated annealing technique. S. Afr. J. Ind. Eng. 23 (2), 176–190.
Saberhoseini, S.F., Edalatpanah, S.A., Sorourkhah, A., 2022. Choosing the best private-sector partner according to the risk factors in neutrosophic environment. Big Data and Computing Visions 2 (2), 61–68. https://doi.org/10.22105/bdcv.2022.334005.1075.
Sharifi, A.M., Khalili Damghani, K., Abdi, F., Sardar, S., 2022. A hybrid model for predicting bitcoin price using machine learning and metaheuristic algorithms. Journal of Applied Research on Industrial Engineering 9 (1), 134–150. https://doi.org/10.22105/jarie.2021.291175.1343.
Shemshad, A., Karim, R.G., 2023. The effect of managerial ability on the timeliness of financial reporting: the role of audit firm and company size. Journal of Operational and Strategic Analytics 1 (1), 34–41. https://doi.org/10.56578/josa010105.
Shoaee, S., Gholi Keshmarzi, M.M., 2023. Generalization of stochastic mortality models to improve mortality prediction in life insurance and pension funds. Journal of Decisions and Operations Research 8 (2), 352–369. https://doi.org/10.22105/dmor.2022.312595.1515.
Singh, P., 2016. Applications of Soft Computing in Time Series Forecasting, vol. 330. Springer International Publishing, Cham. https://doi.org/10.1007/978-3-319-26293-2.
Singh, P., 2017. A brief review of modeling approaches based on fuzzy time series. International Journal of Machine Learning and Cybernetics 8 (2), 397–420. https://doi.org/10.1007/s13042-015-0332-y. Apr.

Singh, P., Borah, B., 2013. An efficient time series forecasting model based on fuzzy time series. Eng. Appl. Artif. Intell. 26 (10), 2443–2457. https://doi.org/10.1016/j.engappai.2013.07.012.

Singh, P., Dhiman, G., 2018. A hybrid fuzzy time series forecasting model based on granular computing and bio-inspired optimization approaches. J. Comput. Sci. 27, 370–385. https://doi.org/10.1016/j.jocs.2018.05.008. Jul.

Singh, P., Rabadiya, K., 2018. Information classification, visualization and decision-making: a neutrosophic set theory based approach. In: IEEE International Conference on Systems, Man, and Cybernetics (SMC). IEEE, pp. 409–414. https://doi.org/10.1109/SMC.2018.00079. Oct. 2018.

Singh, P., Dhiman, G., Kaur, A., 2018. A quantum approach for time series data based on graph and Schrödinger equations methods. Mod. Phys. Lett. A 33 (35), 1850208. https://doi.org/10.1142/S0217732318502085. Nov.

Smarandache, F., 1998. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis, no. October. In: Analytic Synthesis and Synthetic Analysis Series. American Research Press [Online]. Available: https://books.google.com/books?id=LgEZAQAAIAAJ.

Sohail, A., 2023. Genetic algorithms in the fields of artificial intelligence and data sciences. Annals of Data Science 10 (4), 1007–1018. https://doi.org/10.1007/s40745-021-00354-9. Aug.

Sohrabi, P., Jodeiri Shokri, B., Dehghani, H., 2023. Predicting coal price using time series methods and combination of radial basis function (RBF) neural network with time series. Mineral Economics 36 (2), 207–216. https://doi.org/10.1007/s13563-021-00286-z.

Song, Q., Chissom, B.S., 1993. Forecasting enrollments with fuzzy time series — Part I. Fuzzy Set Syst. 54 (1), 1–9. https://doi.org/10.1016/0165-0114(93)90355-L.

Song, Q., Chissom, B.S., 1994. Forecasting enrollments with fuzzy time series — part II. Fuzzy Set Syst. 62 (1), 1–8. https://doi.org/10.1016/0165-0114(94)90067-1.

Sun, B., Guo, H., Reza Karimi, H., Ge, Y., Xiong, S., 2015. Prediction of stock index futures prices based on fuzzy sets and multivariate fuzzy time series. Neurocomputing 151, 1528–1536. https://doi.org/10.1016/j.neucom.2014.09.018.

Taghipourian, M.J., Fazeli Veisari, E., Norashrafodin, S.M., Verij Kazemi, M., 2021. Usage the lazy learning meta-heuristic technique for predicting entrepreneurial marketing in the insurance industry. Journal of Applied Research on Industrial Engineering 8, 1–13. https://doi.org/10.22105/jarie.2021.277767.1277. Special Issue.

Udoka, Ch G., 2020. The impact of passenger's traffic on exchange rate and economic growth in Nigerian aviation industry. International Journal of Research in Industrial Engineering 9 (4), 364–378. https://doi.org/10.22105/riej.2020.257832.1149.

Veeramani, C., Venugopal, R., Edalatpanah, S.A., 2022. Neutrosophic DEMATEL approach for financial ratio performance evaluation of the NASDAQ Exchange. Neutrosophic Sets and Systems 51, 766–782.

Vovan, T., Lethithu, T., 2022. A fuzzy time series model based on improved fuzzy function and cluster analysis problem. Commun Math Stat 10 (1), 51–66. https://doi.org/10.1007/s40304-019-00203-5.

Wong, Wai-Keung, Bai, Enjian, Chu, A.W., 2010. Adaptive time-variant models for fuzzy-time-series forecasting. IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics) 40 (6), 1531–1542. https://doi.org/10.1109/TSMCB.2010.2042055.

Yang, H., Yang, X., Li, G., 2023. Forecasting carbon price in China using a novel hybrid model based on secondary decomposition, multi-complexity and error correction. J. Clean. Prod. 401, 136701 https://doi.org/10.1016/j.jclepro.2023.136701.

Zadeh, L.A., 1965. "Fuzzy sets," Information and Control 8 (3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

Zarghami, A., Doaei, M., Boostani, A., 2023. Predicting the delisted companies of Tehran Stock Exchange using machine learning based algorithms. Journal of Decisions and Operations Research. https://doi.org/10.22105/dmor.2023.340413.1604.

A Neutrosophic Approach to Study Agnotology: A Case Study on Climate Change Beliefs

Maikel Leyva Vázquez,[1] **Florentin Smarandache**[2]

[1] Universidad de Guayaquil, Guayas, Ecuador
[2] University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA

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Abstract

Misinformation and biased information significantly impact public perception and political decisions, especially on critical issues such as climate change and environmental conservation. This study aims to understand how indeterminacy and contradiction influence public perception and policy formulation by applying neutrosophic theory to model the complexity and multi-dimensionality of ignorance. Using neutrosophic Likert scales, we capture a nuanced spectrum of opinions on the scientific certainty of human impact on climate change. The results are analyzed through a k-means clustering algorithm to identify patterns and segment participants into groups based on their levels of truth, indeterminacy, and falsehood. This approach reveals deeper insights into public perceptions and aids in evaluating their implications for effective communication and policy-making.

Keywords: Neutrosophy, Agnotology, Climate Change Perception, Neutrosophic Likert Scales.

1 | Introduction

Misinformation and biased information have a significant impact on public perception and political decisions, especially on critical issues such as climate change and environmental conservation. This study aims to better understand how indeterminacy and contradiction influence public perception and policy formulation, applying neutrosophic theory to model the complexity and multi-dimensionality of ignorance. Agnotology [1] helps to understand how knowledge and ignorance are not simply opposites, but that ignorance can be manufactured and maintained through specific strategies, affecting public perception and political decisions.

The rise of generative artificial intelligence has revolutionized the creation and dissemination of disinformation, exacerbating phenomena already studied such as agnotology (the study of disinformation), agnoiology (the study of ignorance), aningmology (the study of creating doubt), and cognitronics (the study of perception distortion). The ability of these technologies to generate convincing content at scale has increased the efficiency and reach of disinformation, posing new challenges in the fight for truth and the integrity of knowledge in the digital age [2].

Neutrosophy, a philosophy developed by the Romanian scientist Florentin Smarandache, provides a theoretical framework that addresses the coexistence and interactions between elements of truth, falsehood, and neutrality [3]. In exploring agnotology through a neutrosophic lens, this study harnesses the mathematical formalisms associated with the social structures of truth (T), uncertainty (I), and falsehood (F) to examine the

deliberate production of ignorance. By adopting a neutrosophic approach, we move beyond the binary framework of true or false to incorporate the crucial aspect of uncertainty, providing a more comprehensive understanding of knowledge dynamics. This framework facilitates the interrogation of how different societal beliefs and perceptions contribute to the systematic obscuration or distortion of information.

Utilizing neutrosophic Likert scales [4] in our survey design allows for a nuanced capture and analysis of these T, I, and F structures, offering a robust methodological tool to assess their prevalence and influence on public discourse and policy. This approach not only aids in pinpointing areas of informational conflict and consensus but also highlights zones of indeterminacy that are critical for a deeper understanding of agnotological processes. In this paper, we explore public perceptions regarding the impact of human activities on Earth's climate, an area where there is great scientific consensus [5] but where ignorance persists or has been deliberately created [6].

2 | Preliminaries

2.1 | Neutrosophic Set

Neutrosophic sets [7] offer a method for analyzing questionnaire responses, surpassing the capabilities of traditional fuzzy sets or intuitionistic sets by incorporating a unique indeterminacy function [8]. This function captures not only the degrees of truth and falsehood but also the critical dimension of indeterminacy. This additional layer allows for a more comprehensive representation of human responses, particularly useful in scenarios where answers may contain elements of contradiction, ambiguity, or uncertainty. Neutrosophic sets enable respondents to communicate their true thoughts and emotions with greater precision, capturing a broader spectrum of human cognition and perception.

By employing neutrosophic sets in survey analysis [8], researchers can achieve a deeper and more accurate understanding of respondent attitudes and behaviors. The inclusion of indeterminacy helps reveal the complexities and nuances in responses that traditional methods might overlook. This approach enriches data interpretation and enhances decision-making processes by providing a clearer insight into the diverse perspectives and uncertainties that characterize human responses, making neutrosophic sets a superior tool for capturing the full range of possible answers in questionnaires [9]. Next, we present basic definitions and concepts concerning neutrosophic sets and single-valued neutrosophic sets.

Definition 1 [10]: Let U be a discourse universe. $N=\{(x, T(x), I(x), F(x)) : x \in U\}$ is a neutrosophic set, denoted by a truth-membership function, $TN : U \rightarrow]0-, 1+[$; an indeterminacy-membership function, $IN : U \rightarrow]0-, 1+[$; and a falsity-membership function, $FN : U \rightarrow]0-, 1+[$.

Single-valued neutrosophic sets provide a way to represent and analyze possible elements in the discourse universe U.

Definition 2 [11]: Let U be a discourse universe. A single-valued neutrosophic set is defined as $N=\{(x, T(x), I(x), F(x)) : x \in U\}$, which is identified by a truth-membership function, $TN : U \rightarrow [0, 1]$; indeterminacy-membership function, $IN : U \rightarrow [0, 1]$; and falsity-membership function, $FN : U \rightarrow [0, 1]$, with $0 \leq TN(x)+IN(x)+FN(x) \leq 3$.

Neutrosophic logic [12] is an emerging field in which each proposition is considered to have proportions of truth, indeterminacy, and falsehood. This ability to handle multiple and varied degrees of truth and falsehood is particularly useful in contexts where information is paradoxical, inconsistent, or incomplete.

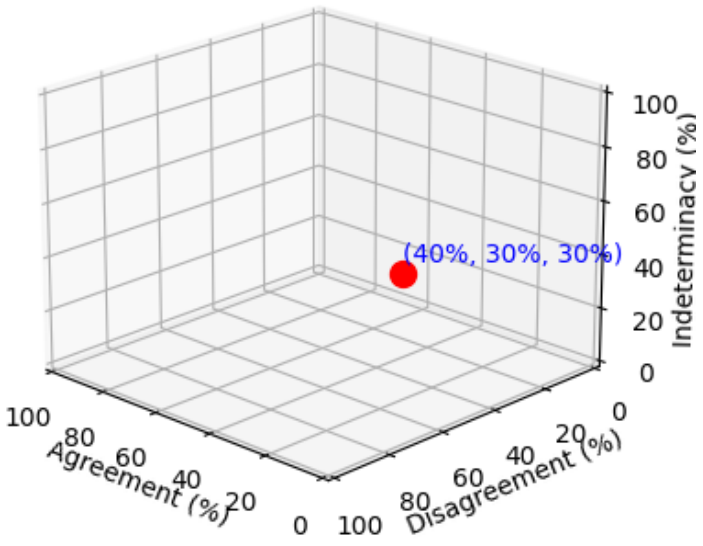

Figure 1. Neutrosophic representation space.

2.2 | Neutrosophic Likert Scales

In the study of Agnotology, social structures, identified through the categories of truth (T), uncertainty (I), and falsehood (F), can be captured and analyzed effectively through survey analysis [13eutrosophic sets, and consequently neutrosophic Likert scales, offer an expanded framework that includes indeterminacy as a fundamental element. In contrast, fuzzy sets facilitate the depiction and handling of data that lack precise specification. For survey responses in which participants' opinions not only vary across a spectrum (as accommodated by fuzzy sets) but also may include a degree of indecision or neutrality that is difficult to capture using traditional fuzzy logic or crisp Likert scales, neutrosophic Likert scales are, therefore, particularly well suited [14, 15].

The score function grounded in neutrosophic theory, effectively transforms responses from neutrosophic Likert scales into a consolidated single value.

Definition 3 [13]: Let A = (T, I, F) be a single-valued neutrosophic number. A score function S for this single-valued neutrosophic number, which is derived from the truth-membership degree, indeterminacy-membership degree, and falsity-membership degree, is defined as follows:

$$S(A) = \frac{1 + a - 2b - c}{2} \quad (1)$$

where S(A) ∈ [−1,1].

Surveys utilizing neutrosophic Likert scales effectively measure the diversity of opinions and their influence on public policy and social discourse, capturing areas of consensus, disagreement, and ambivalence. Neutrosophic logic, critical in Agnotology, analyzes the coexistence of different "truths" and the spectrum of "falsehoods," revealing how knowledge is shaped and sometimes obscured within cultural and social frameworks.

3 | Material and Methods

The survey was conducted with 25 university students at the University of Guayaquil in Ecuador. The primary question posed to participants was: "Based on your understanding, how scientifically certain is it that human activities are affecting Earth's climate?" The responses were structured on a neutrosophic Likert scale, allowing participants to express options that align with the neutrosophic framework.

- "Very sure (the scientific evidence is robust and widely accepted)", which corresponds to a high truth value (T).
- "Indeterminate (there are studies supporting it, but the consensus is unclear)", which captures the state of uncertainty (I).
- "Not sure (the scientific evidence is not convincing)", which is associated with falsehood (F).

Answers are classified according to the sum of the True, Indeterminate, and False components as follows:

- T+I+F<1: Incomplete
- T+I+F=1: Complete
- T+I+F>1: Contradictory

The k-means clustering analysis was performed on the dataset composed of the triplets (T, I, F) from each response. The objective was to identify patterns and segment the participants into homogeneous groups based on their levels of scientific certainty about human impact on climate change [16].

To perform the clustering, the following steps were followed:

i. Initialization: A number k of clusters was selected. In this case, $k = 3$ was chosen to identify three distinct groups of responses based on the Elbow Method [17] for the optimal value of k.

ii. Cluster Assignment: Each response (T, I, F) was assigned to the cluster whose centroid (mean of the points within the cluster) was nearest, by calculating the Euclidean distance. Given n responses (T_i, I_i, F_i) for i=1,2,…,n, i=1,2,…,n and a number of clusters k, the algorithm aims to minimize the sum of squared distances between each point and its corresponding centroid:

$$\min \sum_{j=1}^{k} \sum_{i \in C_j} \| (T_i, I_i, F_i) - \mu_j \|^2 \quad (2)$$

where C_j is the set of points assigned to cluster j and $\mu_j$is the centroid of cluster j.

iii. **Centroid Update:** The centroids of each cluster were recalculated based on the responses assigned to each one.

iv. **Iteration:** Steps 2 and 3 were repeated until the cluster assignments did not significantly change between iterations.

The scoring function (1) presented in this study effectively balances the influence of truth, indeterminacy, and falsehood, providing a nuanced view of the respondents' perceptions. To measure agreement with scientific certainty, the function assigns a greater negative weight to uncertainty. This approach permits a more complete and improved analysis of the data, capturing a detailed picture of respondents' levels of agreement.

4 | Results

Most of the responses (17 out of 24) fall into the "Contradictory" category as shown in Figure 2, suggesting that respondents perceive a higher level of complexity and inconsistency in the scientific certainty regarding human impact on climate change.

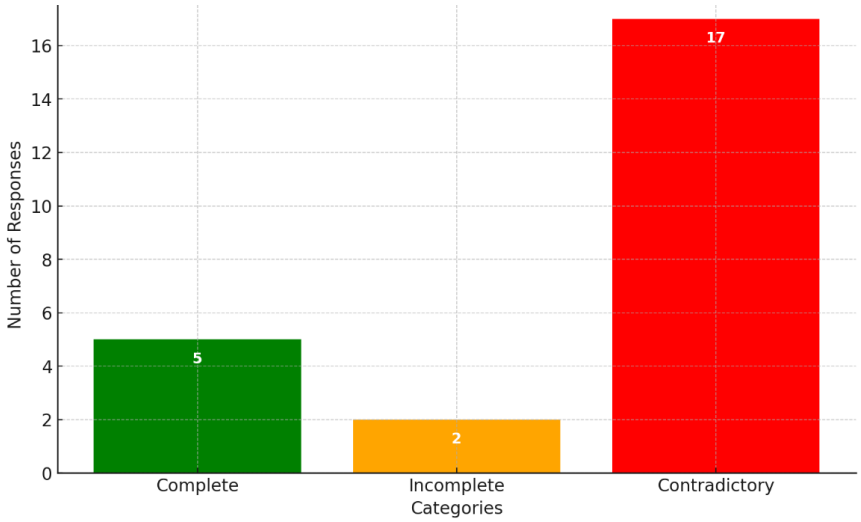

Figure 2. Neutrosophic classification of responses on scientific certainty about human impact on climate change.

This finding highlights the prevalence of conflicting beliefs and uncertainty among the public. The presence of more "Contradictory" responses suggests that people recognize the multifaceted nature of scientific evidence and the difficulty in reaching a definitive conclusion. This underscores the importance of addressing gaps in knowledge and understanding the reasons behind contradictory viewpoints to foster better communication and education on climate science.

The analysis utilized a k-means clustering algorithm to categorize the responses into three distinct clusters based on the variables Truth (T), Indeterminacy (I), and Falsehood (F) as presented in Figure 3. This approach aimed to uncover patterns in respondents' perceptions regarding the scientific certainty of human impact on climate change.

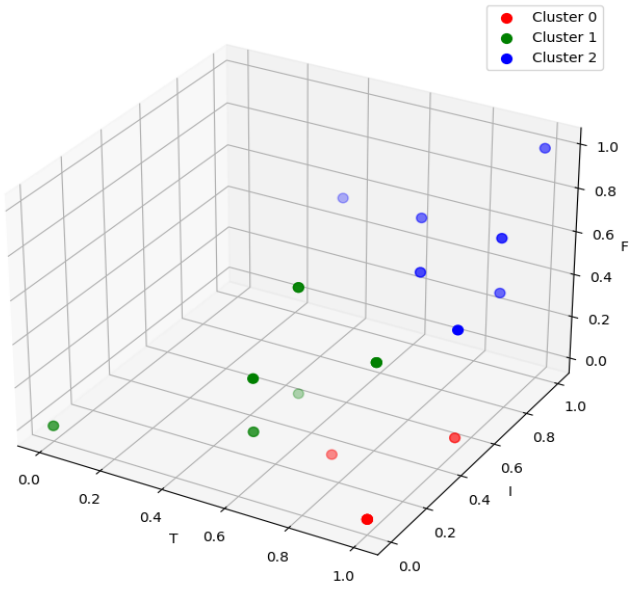

Figure 3. Distribution of clusters in three-dimensional space for variables T, I, and F.

In the context of the clusters derived from the data, the following interpretations can be made:

- Cluster 0 (Red): This cluster represents individuals who have higher values of certainty (T) but lower values in indeterminate (I) and false (F). This suggests that these people have a firmer and more certain understanding of human influence on climate change, with fewer doubts or contradictory beliefs.

- Cluster 1 (Green): This cluster is characterized by varied but generally lower values in certainty (T) and greater dispersion in indeterminate (I) and false (F). This might indicate individuals who have greater uncertainty and contradictory beliefs about human influence on climate change, reflecting a less firm understanding and more doubts.
- Cluster 2 (Blue): This cluster includes individuals with a wide variety of values, with some having high certainty (T) and others having higher values in indeterminate (I) and false (F). This may indicate a mix of undecided or skeptical people regarding the scientific certainty of human activities affecting the climate, showing the complexity and multifaceted nature of ignorance and uncertainty.

The graph of the survey responses, as shown in Figure 4, illustrates the distribution of scores. The scores range from −0.5 to 1.0 with an average score of approximately 0.245. This average indicates a slight overall agreement with the scientific certainty of human impact on climate change. Positive scores suggest a higher confidence in the scientific certainty, while negative scores highlight areas of doubt and disagreement.

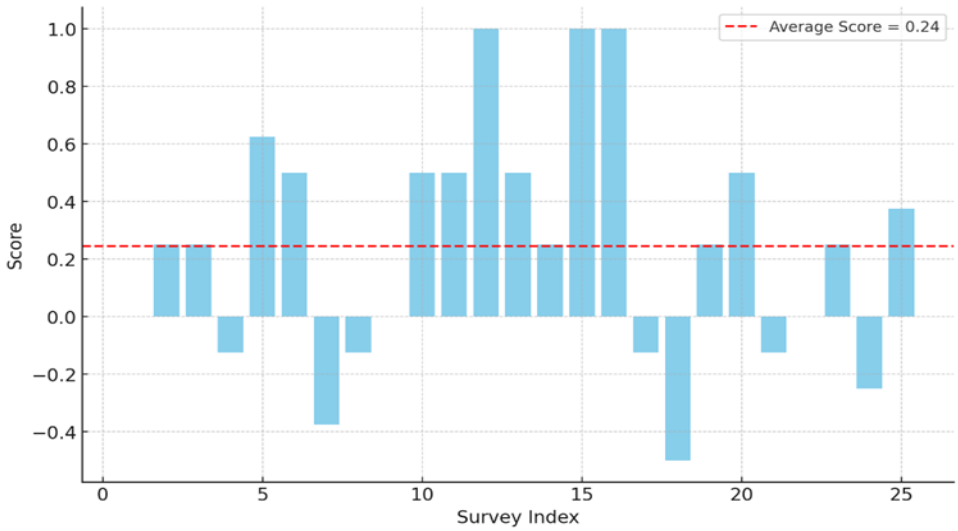

Figure 4. Score function results by survey.

The interpretation of the scores reveals significant insights into public perceptions:

Positive Scores: These scores, which range up to 1.0, indicate respondents who have a higher confidence in the scientific evidence of human impact on climate change. Scores close to 1.0 represent strong agreement, while those around 0.5 indicate moderate agreement.

Negative Scores: Scores down to −0.5 reveal respondents with higher levels of doubt and disagreement. These negative scores suggest that indeterminacy and falsehood values outweigh the truth values in these responses, reflecting significant uncertainty and conflicting beliefs.

From an agnostology perspective, the distribution of scores, particularly the presence of negative values, underscores the prevalent uncertainty and doubt among the public. This highlights the need for targeted communication and educational efforts to address these areas of ignorance and improve public understanding of climate science.

5 | Conclusion

The use of neutrosophy in this paper allows for a more detailed collection of public opinions, moving beyond simple binary responses to include degrees of uncertainty and ambivalence regarding the scientific consensus on climate change. Employing neutrosophic Likert scales provides deeper insights into the spectrum of perceptions and aids in evaluating their potential implications for effective communication and policy-making in addressing climate change challenges. Traditional statistical methods have been employed; however, the integration of neutrosophic statistics could offer more robust tools for analyzing complex data.

Future research should explore the application of neutrosophic approaches to other critical areas of public perception and policy, such as public health and socio-political issues, to capture a broader range of uncertainties and beliefs. By expanding the use of neutrosophic logic and statistics, researchers can better understand and address the intricacies of ignorance and misinformation, ultimately improving educational strategies and policy interventions. This could pave the way for more comprehensive and inclusive models of public understanding and engagement in various domains.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] Proctor, R. N., & Schiebinger, L. (Eds.). (2008). Agnotology: The making and unmaking of ignorance. Stanford University Press.
[2] Berghel, H. (2024). Generative Artificial Intelligence, Semantic Entropy, and the Big Sort. Computer, 57(01), 130-135.
[3] Smarandache, F. (1999). A unifying field in Logics: Neutrosophic Logic. In Philosophy (pp. 1-141). American Research Press.
[4] Kandasamy, I., Kandasamy, W. V., Obbineni, J. M., & Smarandache, F. (2020). Indeterminate Likert scale: feedback based on neutrosophy, its distance measures and clustering algorithm. Soft Computing, 24, 7459-7468.
[5] Cook, J., Oreskes, N., Doran, P. T., Anderegg, W. R., Verheggen, B., Maibach, E. W., ... & Rice, K. (2016). Consensus on consensus: a synthesis of consensus estimates on human-caused global warming. Environmental research letters, 11(4), 048002.
[6] Crockford, S. (2023). That Which They Will Not See: Climate Denial as a Vector of Epistemological Crisis in the Contemporary United States. Ethnos, 1-21.
[7] El-Hefenawy, N., Metwally, M. A., Ahmed, Z. M., & El-Henawy, I. M. (2016). A review on the applications of neutrosophic sets. Journal of Computational and Theoretical Nanoscience, 13(1), 936-944.
[8] Martinez, C. R., Hidalgo, G. A., Matos, M. A., & Sma-randache, F. (2020). Neutrosophy for Survey Analysis in Social Sciences. Neutrosophic Sets and Systems, (37), 403-411.
[9] Bodur, S., Topal, S., Gürkan, H., & Edalatpanah, S. A. (2024). A Novel Neutrosophic Likert Scale Analysis of Perceptions of Organizational Distributive Justice via a Score Function: A Complete Statistical Study and Symmetry Evidence Using Real-Life Survey Data. Symmetry, 16(5), 598.
[10] Smarandache, F. (1999). A unifying field in logics: Neutrosophic logic. En Philosophy (pp. 1-141). American Research Press.

[11] Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2012). Single valued neutrosophic sets. Rev. Air Force Acad.
[12] Smarandache, F. (2010). Neutrosophic logic-a generalization of the intuitionistic fuzzy logic. Multispace & multistructure. Neutrosophic transdisciplinarity (100 collected papers of science), 4, 396.
[13] Şahin, R. (2014). Multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment. arXiv preprint arXiv:1412.5202.
[14] Smarandache, F. (2015). (t, i, f)-Neutrosophic Structures & I-Neutrosophic Structures (Revisited). Neutrosophic Sets & Systems, 8.
[15] Gutiérrez-Gómez, E., Moscoso-Paucarchuco, K. M., Beraún-Espíritu, M. M., Moreno-Menéndez, F. M., Fernández-Jaime, R. J., Paco-Huamani, W. F., ... & Sandoval-Trigos, J. C. (2023). Study of the Pampay Mass (burial of offering) at the summit of the Andean snow-capped mountain as an example of (t, i, f)-Neutrosophic social structure. Neutrosophic Sets and Systems, 60(1), 3.
[16] Vázquez, M. Y. L., Ricardo, J. E., Hernández, N. B., Casanova, R. S., & Smarandache, F. (2024). Análisis neutrosófico de las actitudes hacia la máquina de experiencia de Nozick. Revista Investigación Operacional, 45(4)
[17] Syakur, M. A., Khotimah, B. K., Rochman, E. M. S., & Satoto, B. D. (2018, April). Integration k-means clustering method and elbow method for identification of the best customer profile cluster. In IOP conference series: materials science and engineering (Vol. 336, p. 012017). IOP Publishing.

On Neutrosophic Implicative Filters of BL-Algebra

A. Ibrahim,[1] **S. Karunya Helen Gunaseeli,**[1] **Florentin Smarandache**[2]

[1]*P.G. and Research Department of Mathematics, H.H. The Rajah's College, Pudukkottai (affiliated to Bharathidasan University), Trichirappalli, Tamilnadu, India*

[2]*Department of Mathematics, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA*

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Abstract. We put forward the ideas of the neutrosophic implicative and n-fold implicative filters of BL-algebras. Additionally, we demonstrate that every implicative filter, including the n-fold implicative filter, is a neutrosophic filter. Moreover, we obtain an extension property for the neutrosophic implication. We then look at some comparable circumstances for neutrosophic implicative filters.

Keywords. BL-algebra, Filter, Neutrosophic filter, Neutrosophic implicative filter, Neutrosophic n-fold implicative filter

Mathematics Subject Classification (2020). 03G25,03E72,03F55,06F35

1. Introduction

Ward and Dilworth [11] developed the concept of residuated lattices as a generalisation of the form of a ring's set of ideals. BL-algebras are the most well-known example of residuated lattices in logic. Hájek [2] created Basic Logic algebra (BL-algebra), a type of logical algebra, to offer an algebraic demonstration of completeness of '*Basic Logic*'. Xu and Qin [12] first proffered the conception of filter and implication filter in lattice implication algebras.

Filter theory is a crucial component of the study of innumerable logical algebras (Park and Ahn [7], and Zhang *et al*. [14]). They play a key role in the case made for the completeness of certain logical algebras. Researchers from several academic fields have looked into the conception of filters. Neutrosophy is acknowledged as a scientific study, investigates the origin, nature, and scope of neutralities (Salama and Alagamy [8], and Smarandache [9]). Fuzzy [13],

intuitionistic fuzzy sets and logic are generalised as neutrosophic sets and neutrosophic logic (Atanassov [1]). Recently, the authors examined some of the features of the neutrosophic filter, neutrosophic fantastic filter of BL-algebras (Ibrahim and Gunaseeli [4,5]).

In Section 2, the basic notions and outcomes are recalled. In Section 3, we explore the concept of neutrosophic implicative filter. In Section 4, we exhibit the conception of neutrosophic n-fold implicative filter.

2. Preliminaries

In this part, few of the definitions and findings from the literature are referred to progress the major conclusions.

Definition 2.1 ([2, 3])**.** A BL-algebra $(\mathcal{G}, \vee, \wedge, \circ, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ such that the subsequent requirements are persuaded for all $g_1, h_1, i_1 \in \mathcal{G}$,

(i) $(\mathcal{G}, \vee, \wedge, 0, 1)$ is a bounded lattice,

(ii) $(\mathcal{G}, \circ, 1)$ is a commutative monoid,

(iii) '$\circ$' and '$\rightarrow$' form an adjoint pair, that is, $i_1 \leq g_1 \rightarrow h_1$ if and only if $g_1 \circ i_1 \leq h_1$, for all $g_1, h_1, i_1 \in \mathcal{G}$,

(iv) $g_1 \wedge h_1 = g_1 \circ (g_1 \rightarrow h_1)$,

(v) $(g_1 \rightarrow h_1) \vee (h_1 \rightarrow g_1) = 1$.

Proposition 2.2 ([6, 10])**.** *The succeeding requirements are persuaded in a BL-algebra $\mathcal{G}$ for all $g_1, h_1, i_1 \in \mathcal{G}$,*

(i) $h_1 \rightarrow (g_1 \rightarrow i_1) = g_1 \rightarrow (h_1 \rightarrow i_1) = (g_1 \circ h_1) \rightarrow i_1$,

(ii) $1 \rightarrow g_1 = g_1$,

(iii) $g_1 \leq h_1$ *if and only if* $g_1 \rightarrow h_1 = 1$,

(iv) $g_1 \vee h_1 = ((g_1 \rightarrow h_1) \rightarrow h_1) \wedge ((h_1 \rightarrow g_1) \rightarrow g_1)$,

(v) $g_1 \leq h_1$ *implies* $h_1 \rightarrow i_1 \leq g_1 \rightarrow i_1$,

(vi) $g_1 \leq h_1$ *implies* $i_1 \rightarrow g_1 \leq i_1 \rightarrow h_1$,

(vii) $g_1 \rightarrow h_1 \leq (i_1 \rightarrow g_1) \rightarrow (i_1 \rightarrow h_1)$,

(viii) $g_1 \rightarrow h_1 \leq (h_1 \rightarrow i_1) \rightarrow (g_1 \rightarrow i_1)$,

(ix) $g_1 \leq (g_1 \rightarrow h_1) \rightarrow h_1$,

(x) $g_1 \circ (g_1 \rightarrow h_1) = g_1 \wedge h_1$,

(xi) $g_1 \circ h_1 \leq g_1 \wedge h_1$,

(xii) $g_1 \rightarrow h_1 \leq (g_1 \circ i_1) \rightarrow (h_1 \circ i_1)$,

(xiii) $g_1 \circ (h_1 \rightarrow i_1) \leq h_1 \rightarrow (g_1 \circ i_1)$,

(xiv) $(g_1 \rightarrow h_1) \circ (h_1 \rightarrow i_1) \leq g_1 \rightarrow i_1$,

(xv) $(g_1 \circ g_1^*) = 0$.

Definition 2.3 ([9]). A neutrosophic subset C of the universe U is a triple (T_C, I_C, F_C) where $T_C : U \to [0,1]$, $I_C : U \to [0,1]$ and $F_C : U \to [0,1]$ represents truth membership, indeterminacy and false membership functions respectively where $0 \le T_C(g_1) + I_C(g_1) + F_C(g_1) \le 3$, for all $g_1 \in U$.

Definition 2.4 ([5]). A neutrosophic set C of an algebra $\mathcal{G}$ is called a neutrosophic filter, if it persuades the following:

(i) $T_C(g_1) \le T_C(1)$, $I_C(g_1) \ge I_C(1)$ and $F_C(g_1) \ge F_C(1)$,

(ii) $\min\{T_C(g_1 \to h_1), T_C(g_1)\} \le T_C(h_1)$,
$\min\{I_C(g_1 \to h_1), I_C(g_1)\} \ge I_C(h_1)$, and
$\min\{F_C(g_1 \to h_1), F_C(g_1)\} \ge F_C(h_1)\}$, for all $g_1, h_1 \in \mathcal{G}$.

Proposition 2.5 ([5]). *Let C be a neutrosophic filter of $\mathcal{G}$ if and only if*

(i) *If $g_1 \le h_1$ then $T_C(g_1) \le T_C(h_1)$, $I_C(g_1) \ge I_C(h_1)$ and $F_C(g_1) \ge F_C(h_1)$,*

(ii) *$T_C(g_1 \circ h_1) \ge \min\{T_C(g_1), T_C(h_1)\}$, $I_C(g_1 \circ h_1) \le \min\{I_C(g_1), I_C(h_1)\}$ and $F_C(g_1 \circ h_1) \le \min\{F_C(g_1), F_C(h_1)\}$, for all $g_1, h_1 \in \mathcal{G}$.*

Proposition 2.6 ([4,5]). *Let C be a neutrosophic filter of $\mathcal{G}$, for all $g_1, h_1, i_1 \in \mathcal{G}$ then the following hold:*

(i) *$T_C(g_1 \to h_1) = T_C(1)$, then $T_C(g_1) \le T_C(h_1)$,*
$I_C(g_1 \to h_1) = I_C(1)$, then $I_C(g_1) \ge I_C(h_1)$,
$F_C(g_1 \to h_1) = F_C(1)$, then $F_C(g_1) \ge F_C(h_1)$,

(ii) *$T_C(g_1 \wedge h_1) = \min\{T_C(g_1), T_C(h_1)\}$,*
$I_C(g_1 \wedge h_1) = \min\{I_C(g_1), I_C(h_1)\}$,
$F_C(g_1 \wedge h_1) = \min\{F_C(g_1), F_C(h_1)\}$,

(iii) *$T_C(g_1 \circ h_1) = \min\{T_C(g_1), T_C(h_1)\}$,*
$I_C(g_1 \circ h_1) = \min\{I_C(g_1), I_C(h_1)\}$,
$F_C(g_1 \circ h_1) = \min\{F_C(g_1), F_C(h_1)\}$,

(iv) *$T_C(0) = \min\{T_C(g_1), T_C(g_1^*)\}$,*
$I_C(0) = \min\{I_C(g_1), I_C(g_1^*)\}$,
$F_C(0) = \min\{F_C(g_1), F_C(g_1^*)\}$.

3. Neutrosophic Implicative Filter

Here we put forward the conception of a neutrosophic implicative filter and confer its features with illustrations.

Definition 3.1. Let C be a neutrosophic filter of a BL-algebra $\mathcal{G}$. C is called a neutrosophic implicative filter if it persuades the following:

(i) $T_C(g_1) \le T_C(1)$, $I_C(g_1) \ge I_C(1)$ and $F_C(g_1) \ge F_C(1)$,

(ii) $\min\{T_C(g_1 \to (h_1 \to i_1)), T_C(g_1 \to h_1)\} \le T_C(g_1 \to i_1)$,
$\min\{_C(g_1 \to (h_1 \to i_1)), I_C(g_1 \to h_1)\} \ge I_C(g_1 \to i_1)$,
$\min\{F_C(g_1 \to (h_1 \to i_1)), F_C(g_1 \to h_1)\} \ge F_C(g_1 \to i_1)$, for all $g_1, h_1, i_1 \in \mathcal{G}$.

Example 3.2. Let $C = \{0, g_1, h_1, i_1, 1\}$. The bi-fold operations are specified by Tables 1 and 2.

Table 1. '$\circ$' operation

$\circ$	0	g_1	h_1	i_1	1
0	1	g_1	0	0	0
g_1	g_1	1	g_1	g_1	g_1
h_1	0	g_1	1	h_1	h_1
i_1	0	g_1	h_1	1	i_1
1	0	g_1	h_1	i_1	1

Table 2. '$\rightarrow$' operation

$\rightarrow$	0	g_1	h_1	i_1	1
0	1	1	1	1	1
g_1	g_1	1	1	1	1
h_1	0	g_1	1	1	1
i_1	0	g_1	h_1	1	1
1	0	g_1	h_1	i_1	1

Consider
$C = \{(0,[0.5,0.4,0.4]),(g_1,[0.5,0.4,0.4]),(h_1,[0.5,0.4,0.4]),(i_1,[0.5,0.4,0.4]),(1,[0.6,0.3,0.3])\}$.
It is evident that C assures Definition 3.1. Hence, C is a neutrosophic implicative filter of $\mathcal{G}$.

Proposition 3.3. *Every neutrosophic implicative filter of $\mathcal{G}$ is a neutrosophic filter. But, the converse is not true.*

Proof. Let C be a neutrosophic implicative filter of $\mathcal{G}$.

To prove: C is a neutrosophic filter of $\mathcal{G}$.
Taking $g_1 = 1$ in Definition 3.1, we get

$$\min\{T_C(1 \rightarrow (h_1 \rightarrow i_1)), T_C(1 \rightarrow h_1)\} \leq T_C(1 \rightarrow i_1), \quad \text{for all } g_1, h_1, i_1 \in \mathcal{G},$$

which implies

$$T_C(i_1) \geq \min\{T_C(h_1 \rightarrow i_1), T_C(h_1)\}.$$

Similarly,

$$I_C(i_1) \leq \min\{I_C(h_1 \rightarrow i_1), I_C(h_1)\}, F_C(i_1) \leq \min\{F_C(h_1 \rightarrow i_1), F_C(h_1)\}.$$

Thus, from Definition 2.4, C is a neutrosophic filter of $\mathcal{G}$. □

The converse part may not be true. This can be proved by an illustration.

Example 3.4. Let $C = \{0, g_1, h_1, 1\}$. The bi-fold operations are specified by Tables 3 and 4.

Table 3. '$\circ$' operation

$\circ$	0	g_1	h_1	1
0	0	0	0	0
g_1	0	0	g_1	h_1
h_1	0	g_1	h_1	h_1
1	0	g_1	h_1	1

Table 4. '$\rightarrow$' operation

$\rightarrow$	0	g_1	h_1	1
0	1	1	1	1
g_1	g_1	1	1	1
h_1	0	g_1	1	1
1	0	g_1	h_1	1

Consider $C = \{(0,[0.9,0.2,0.1]),(g_1,[0.5,0.3,0]),(h_1,[0.5,0.3,0]),(1,[0.9,0.2,0.1])\}$.
Here, C is not a neutrosophic implicative filter.
Since, $T_C(h_1 \rightarrow 1) = T_C(h_1) = 0.5 \not\geq 0.9 = T_C(0)$.

Proposition 3.5. *Let C be a neutrosophic filter of a BL-algebra $\mathcal{G}$. The following are equivalent for all $g_1, h_1, i_1 \in \mathcal{G}$.*

(i) *C is a neutrosophic implicative filter.*

(ii) $T_C(g_1 \to h_1) \geq T_C(g_1 \to (g_1 \to h_1))$,
$I_C(g_1 \to h_1) \leq I_C(g_1 \to (g_1 \to h_1))$,
$F_C(g_1 \to h_1) \leq F_C(g_1 \to (g_1 \to h_1))$,

(iii) $T_C(g_1 \to h_1) = T_C(g_1 \to (g_1 \to h_1))$,
$I_C(g_1 \to h_1) = I_C(g_1 \to (g_1 \to h_1))$,
$F_C(g_1 \to h_1) = F_C(g_1 \to (g_1 \to h_1))$.

Proof. (i)⇒(ii): Assume that C is a neutrosophic implicative filter of $\mathcal{G}$.
Put $i_1 = h_1$, $h_1 = g_1$ in Definition 3.1, we get

$$\begin{aligned} T_C(g_1 \to h_1) &\geq \min\{T_C(g_1 \to (g_1 \to h_1)), T_C(g_1 \to g_1)\} \\ &\geq \min\{T_C(g_1 \to (g_1 \to h_1)), T_C(1)\} \\ &= T_C(g_1 \to (g_1 \to h_1)). \end{aligned}$$

Therefore,

$$T_C(g_1 \to h_1) \geq T_C(g_1 \to (g_1 \to h_1)).$$

Similarly, we can prove for I_C, F_C.
Hence (ii) holds.

(ii)⇒(iii): Let $T_C(g_1 \to h_1) \geq T_C(g_1 \to (g_1 \to h_1))$.
Since $g_1 \to h_1 \leq g_1 \to (g_1 \to h_1)$ and from Proposition 2.6, we have

$$T_C(g_1 \to h_1) \leq T_C(g_1 \to (g_1 \to h_1)), \quad \text{for all } g_1, h_1 \in \mathcal{G}$$

and from (ii) we get

$$T_C(g_1 \to h_1) = T_C(g_1 \to (g_1 \to h_1)).$$

Similarly, we can prove for I_C, F_C. Hence (iii) holds.

(iii)⇒(i): Let $T_C(g_1 \to h_1) = T_C(g_1 \to (g_1 \to h_1))$.
If C is a neutrosophic filter of $\mathcal{G}$, then from Proposition 2.6,

$$\min\{T_C(g_1 \to (h_1 \to i_1)), T_C(g_1 \to h_1)\} \leq T_C(g_1 \to i_1), \quad \text{for all } g_1, h_1, i_1 \in \mathcal{G}.$$

Similarly, we can prove for I_C, F_C.
Hence, C is a neutrosophic implicative filter of $\mathcal{G}$. □

Proposition 3.6. *Let C and D be two neutrosophic filters of $\mathcal{G}$. Let $C \subseteq D$, $T_C(1) = T_D(1)$, $I_C(1) = I_D(1)$, $F_C(1) = F_D(1)$. If C is a neutrosophic implicative filter, then so is D.*

Proof. Let C and D be two neutrosophic filters of $\mathcal{G}$.
From Proposition 3.5, we only prove that

$$T_C(g_1 \to h_1) \geq T_C(g_1 \to (g_1 \to h_1)),$$
$$I_C(g_1 \to h_1) \leq I_C(g_1 \to (g_1 \to h_1)),$$
$$F_C(g_1 \to h_1) \leq F_C(g_1 \to (h_1 \to i_1)).$$

Let $x_1 = g_1 \to (g_1 \to h_1)$.
Then, $g_1 \to (g_1 \to (x_1 \to h_1)) = x_1 \to (g_1 \to (g_1 \to h_1)) = 1$.
Suppose, C is a neutrosophic implicative filter of $\mathcal{G}$, then from (iii) of Proposition 3.5 and since $C \subseteq D$, $T_C(1) = T_D(1)$,

$$\begin{aligned} T_D(x_1 \to (g_1 \to h_1)) &= T_D(g_1 \to (x_1 \to h_1)) \\ &\geq T_C(g_1 \to (x_1 \to h_1)) \\ &= T_C(g_1 \to (g_1 \to (x_1 \to h_1)) \\ &= T_C(x_1 \to (g_1 \to (g_1 \to h_1)) \\ &= T_C(1) \\ &= T_D(1). \end{aligned}$$

Thus,

$$T_D(x_1 \to (g_1 \to h_1)) \geq T_D(1).$$

This together with (i) of Definition 3.1,

$$T_D(x_1 \to (g_1 \to h_1)) \leq T_D(1)$$

imply that

$$T_D(x_1 \to (g_1 \to h_1)) = T_D(1).$$

Since, D is a neutrosophic filter then by Definition 2.4, we have

$$\begin{aligned} T_D(g_1 \to h_1) &\geq \min\{T_D(x_1 \to (g_1 \to h_1)), T_D(x_1)\} \\ &= \min\{T_D(1), T_D(x_1)\} \\ &= T_D(x_1) \\ &= T_D(g_1 \to (g_1 \to h_1)). \end{aligned}$$

Hence,

$$T_D(g_1 \to h_1) \geq T_D(g_1 \to (g_1 \to h_1)).$$

Similarly, we can prove for I_D, F_D.
Therefore, from (ii) of Proposition 3.5, D is a neutrosophic implicative filter. □

4. Neutrosophic n-fold Implicative Filter

Here, we put forward the conception of the neutrosophic n-fold implicative filter and confer its features with illustrations.

For any element g_1 and h_1 of a BL-algebra $\mathcal{G}$ and a positive integer n, let $g_1^n \to h_1$ signify $g_1 \to (g_1 \to \ldots (g_1 \to h_1))$ where g_1 happens n-times and $g_1^0 \to h_1 = h_1$.

Definition 4.1. Let C be a neutrosophic filter of a BL-algebra $\mathcal{G}$. C is called a neutrosophic n-fold implicative filter if it persuades,

(i) $T_C(1) \geq T_C(g_1)$, $I_C(1) \leq I_C(g_1)$ and $F_C(1) \leq F_C(g_1)$,

(ii) $T_C(g_1^n \to i_1) \geq \min\{T_C(g_1^n \to (h_1 \to i_1)), T_C(g_1^n \to h_1)\}$,
$I_C(g_1^n \to i_1) \leq \min\{I_C(g_1^n \to (h_1 \to i_1)), I_C(g_1^n \to h_1)\}$,
$F_C(g_1^n \to i_1) \leq \min\{F_C(g_1^n \to (h_1 \to i_1)), F_C(g_1^n \to h_1)\}$, for all $g_1, h_1, i_1 \in \mathcal{G}$.

Note. The neutrosophic 1-fold implicative filter is the same as neutrosophic implicative filter.

Example 4.2. Let $C = \{0, g_1, h_1, i_1, j_1, 1\}$. The bi-fold operations are specified by Tables 5 and 6. Consider $C = \{(0,[0.6,0.4,0.4]),(g_1,[0.6,0.4,0.4]),(h_1,[0.8,0.3,0.3]),(i_1,[0.8,0.3,0.3]),(j_1,[0.6,0.4,0.4]),(1,[0.8,0.3,0.3])\}$.

Table 5. '$\circ$' operation

$\circ$	0	g_1	h_1	i_1	j_1	1
1	1	1	1	1	1	1
g_1	i_1	1	h_1	i_1	h_1	1
h_1	j_1	g_1	1	h_1	g_1	1
i_1	g_1	g_1	1	1	g_1	1
j_1	h_1	1	1	h_1	1	1
1	0	g_1	h_1	i_1	j_1	1

Table 6. '$\rightarrow$' operation

$\rightarrow$	0	g_1	h_1	i_1	j_1	1
0	1	1	1	1	1	1
g_1	1	1	h_1	i_1	h_1	i_1
h_1	1	g_1	1	h_1	g_1	j_1
i_1	1	g_1	1	1	g_1	g_1
j_1	1	1	1	h_1	1	h_1
1	1	g_1	h_1	i_1	j_1	0

It is evident that C assures Definition 3.1. Hence, C is a neutrosophic n-fold implicative filter of $\mathcal{G}$.

Proposition 4.3. *Every neutrosophic n-fold implicative filter of a BL-algebra $\mathcal{G}$ is a neutrosophic filter but the adverse is not true.*

Proof. Let C be a neutrosophic n-fold implicative filter of $\mathcal{G}$.
Taking $g_1 = 1$ in (ii) of Definition 4.1 and from (ii) of Proposition 2.2, we get

$$T_C(i_1) \geq \min\{T_C(h_1 \rightarrow i_1), T_C(h_1)\},$$
$$I_C(i_1) \leq \min\{I_C(h_1 \rightarrow i_1), I_C(h_1)\},$$
$$F_C(i_1) \leq \min\{F_C(h_1 \rightarrow i_1), F_C(h_1)\}, \quad \text{for all } h_1, i_1 \in \mathcal{G}.$$

Thus, (ii) of Definition 2.4 holds.
Hence, C is a neutrosophic filter of $\mathcal{G}$. □

The adverse of the proposition may not be true. It can be verified by an illustration.

Example 4.4. Let $D = \{0, g_1, h_1, i, j_1, 1\}$. The bi-fold operations are specified by Tables 5 and 6. Consider $D = \{(0,[0.6,0.4,0.4]),(g_1,[0.6,0.4,0.4]),(h_1,[0.6,0.4,0.4]),(i_1,[0.6,0.4,0.4]),(j_1,[0.6,0.4,0.4]),(1,[0.8,0.3,0.3])\}$.
Here, D is not a neutrosophic n-fold implicative filter of $\mathcal{G}$.
Since, $T_D(j_1 \rightarrow i_1) = T_D(h_1) = 0.6 \not\geq 0.8 = T_D(1)$.

Proposition 4.5. *Let C be a neutrosophic filter of a BL-algebra $\mathcal{G}$.Then the succeeding requirements are equivalent.*

(i) *C is a neutrosophic n-fold implicative filter of $\mathcal{G}$.*

(ii) $T_C(g_1^n \rightarrow h_1) \geq T_C(g_1^{n+1} \rightarrow h_1), I_C(g_1^n \rightarrow h_1) \leq I_C(g_1^{n+1} \rightarrow h_1)$
$F_C(g_1^n \rightarrow h_1) \leq F_C(g_1^{n+1} \rightarrow h_1)$, *for all* $g_1, h_1 \in \mathcal{G}$.

(iii) $T_C((g_1^n \to h_1) \to (g_1^n \to i_1)) \geq T_C(g_1^n \to (h_1 \to i_1))$,
$I_C((g_1^n \to h_1) \to (g_1^n \to i_1)) \leq I_C(g_1^n \to (h_1 \to i_1))$,
$F_C((g_1^n \to h_1) \to (g_1^n \to i_1)) \leq F_C(g_1^n \to (h_1 \to i_1))$, *for all* $g_1, h_1, i_1 \in \mathcal{G}$.

Proof. (i)⇒(ii): Let C be a neutrosophic n-fold implicative filter of $\mathcal{G}$.
Putting $i_1 = h_1$, $h_1 = g_1$ in (ii) of Definition 4.1,

$$\begin{aligned} T_C(g_1^n \to h_1) &\geq \min\{T_C(g_1^n \to (g_1 \to h_1)), T_C(g_1^n \to g_1)\} \\ &= \min\{T_C(g_1^{n+1} \to h_1), T_C(1)\} \\ &= T_C(g_1^{n+1} \to h_1). \end{aligned}$$

Hence,

$$T_C(g_1^n \to h_1) \geq T_C(g_1^{n+1} \to h_1), \quad \text{for all } g_1, h_1 \in \mathcal{G}.$$

Similarly, we can prove for I_C, F_C.

(ii)⇒(iii): Let (ii) holds.
Since,

$$g_1^n \to (h_1 \to i_1) \leq g_1^n \to (g_1^n \to h_1) \to (g_1^n \to i_1)),$$

we have

$$T_C(g_1^n \to ((g_1^n \to h_1) \to (g_1^n \to i_1))) \geq T_C(g_1^n \to (h_1 \to i_1)) \quad \text{(from Definition 3.1).}$$

Since,

$$\begin{aligned} g_1^{n+1} \to ((g_1^{n-1} \to ((g_1^n \to h_1) \to i_1)) &= g_1^n \to ((g_1^n \to ((g_1^n \to h_1) \to i_1)) \\ &= g_1^n \to ((g_1^n \to h_1) \to (g_1^n \to i_1)) \end{aligned}$$

and using (ii), we have

$$\begin{aligned} T_C(g_1^{n+1} \to ((g_1^{n-2} \to ((g_1^n \to h_1) \to i_1))) &= T_C(g_1^n \to ((g_1^{n-1} \to ((g_1^n \to h_1) \to i_1)))) \\ &\geq T_C(g_1^{n+1} \to ((g_1^{n-1} \to ((g_1^n \to h_1) \to i_1))) \\ &= T_C(g_1^n \to ((g_1^n \to h_1) \to (g_1^n \to i_1))) \\ &\geq T_C(g_1^n \to (h_1 \to i_1)). \end{aligned}$$

Repeating the process, we conclude that

$$\begin{aligned} T_C((g_1^n \to h_1) \to (g_1^n \to i_1)) &= T_C(g_1^n \to ((g_1^n \to h_1) \to i_1)) \\ &\geq T_C(g_1^n \to (h_1 \to i_1)). \end{aligned}$$

Similarly, we can prove for I_C, F_C.
Therefore, (iii) holds.

(iii)⇒(i): Let (iii) holds.
By (iii) and (ii) of Definition 3.1,

$$\begin{aligned} T_C(g_1^n \to i_1) &\geq \min\{T_C((g_1^n \to h_1) \to (g_1^n \to i_1)), T_C(g_1^n \to h_1)\} \\ &\leq \min\{T_C(g_1^n \to (h_1 \to i_1)), T_C(g_1^n \to h_1)\}, \quad \text{for all } g_1, h_1, i_1 \in \mathcal{G}. \end{aligned}$$

Similarly, we can prove for I_C, F_C. Thus, C is a neutrosophic n-fold implicative filter.

5. Conclusion

In BL-algebras, we have put forth the conception of a neutrosophic implication in filters. We have also demonstrated the neutrosophic nature of every implicative and n-fold implicative filter. Moreover, other analogous circumstances for neutrosophic implicative filters are conferred. Further, research on the structure of BL-algebras and the above study will give us a wide range of applications in medical, industrial, and other fields.

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986), 87 – 86, DOI: 10.1016/S0165-0114(86)80034-3.

[2] P. Hájek, *Metamathematics of Fuzzy Logic*, 1st edition, Trends in Logic series (TREN, Volume 4), Springer, Dordrecht, viii + 299 pages (1998), DOI: 10.1007/978-94-011-5300-3.

[3] M. Haveshki, A. B. Saeid and E. Eslami, Some types of filters in BL algebras, *Soft Computing* **10** (2006), 657 – 664, DOI: 10.1007/s00500-005-0534-4.

[4] A. Ibrahim and S. K. H. Gunaseeli, On neutrosophic filter and fantastic filter of BL-algebras, *International Journal of Neutrosophic Science* **21**(2) (2023), 59 – 67, DOI: 10.54216/IJNS.210205.

[5] A. Ibrahim and S. K. H. Gunaseeli, On neutrosophic filter of BL-algebras, *Ratio Mathematica* **47** (2023), 141 – 150, DOI: 10.23755/rm.v39i0.816.

[6] L. Liu and K. Li, Fuzzy filters of BL-algebras, *Information Sciences* **173**(1-3) (2005), 141 – 154, DOI: 10.1016/j.ins.2004.07.009.

[7] S.-T. Park and S.-S. Ahn, On n-fold implicative vague filters in BE-algebras, *The Pure and Applied Mathematics* **19**(2) (2012), 127 – 136, DOI: 10.7468/jksmeb.2012.19.2.127.

[8] A. A. Salama and H. Alagamy, Neutrosophic filters, *International Journal of Computer Science Engineering and Information Technology Research* **3**(1) (2013), 307 – 312, DOI: 10.5281/zenodo.23184.

[9] F. Smarandache, *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic*, American Research Press, Rehoboth (1998).

[10] E. Turunen, Boolean deductive systems of BL-algebras, *Archive for Mathematical Logic* **40** (2001), 467 – 473, DOI: 10.1007/s001530100088.

[11] M. Ward and R. P. Dilworth, Residuated lattices, *Transactions of the American Mathematical Society* **45** (1939), 335 – 354, DOI: 10.1090/S0002-9947-1939-1501995-3.

[12] Y. Xu and K. Y. Qin, On filters of lattice implication algebra, *The Journal of Fuzzy Mathematics* **2** (1993), 251 – 260.

[13] L. A. Zadeh, Fuzzy sets, *Information and Control* **8**(3) (1965), 338 – 353, DOI: 10.1016/S0019-9958(65)90241-X.

[14] X. Zhang, X. Mao, Y. Wu and X. Zhai, Neutrosophic filters in pseudo-BCI algebras, *International Journal for Uncertainty Quantification* **8**(6) (2018), 511 – 526, DOI: 10.1615/Int.J.UncertaintyQuantification.2018022057.

Generation of Creative Game Scene Patterns by the Neutrosophic Genetic CoCoSo Method

Aurimas PETROVAS[1], Romualdas BAUSYS[1], Edmundas Kazimieras ZAVADSKAS[2], Florentin SMARANDACHE[3]

[1] Vilnius Gediminas Technical University, Department of Graphical Systems, 11 Saulėtekio Avenue, Vilnius, 10223, Lithuania
aurimas.petrovas@vilniustech.lt, romualdas.bausys@vilniustech.lt

[2] Vilnius Gediminas Technical University, Institute of Sustainable Construction, 11 Saulėtekio Avenue, Vilnius, 10223, Lithuania
edmundas.zavadskas@vilniustech.lt (**Corresponding author*)

[3] University of New Mexico, Department of Mathematics, 705 Gurley Avenue, Gallup, NM 87301, USA
fsmarandache@gmail.com

Abstract: Creative procedural generation requires, by its nature, a nondeterministic result. Each set of outputs has to satisfy the required criteria and, at the same time, has to show unique results to be considered creative. One of the keys to increasing nondeterminism is to use fuzzy neutrosophic sets. A combination of a CoCoSo multi-criteria decision algorithm with a genetic algorithm to generate game scenes was proposed. The algorithm satisfies the required game rules and improves the aesthetics value of the visual game scene. The proposed approach seeks to produce a creative solution which satisfies the replayability principle, when the solution is not optimal in the optimization procedure sense. In this paper, improvements of the CoCoSo method were proposed in order to work with iterative genetic algorithm.

Keywords: MCDM, CoCoSo, Genetic algorithm, Procedural generation, Games, Neutrosophic sets.

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1. Introduction

Automation improves the efficiency and quality of people's lives (Filip, 2021). And one of the tasks that is usually executed by humans is the creative/ intelligent problem solving. During the last decade, researchers paid a lot of attention to model creativity problems using strict mathematical approaches. The popularity of algorithmic solutions for creative tasks is increasing, and the traits that define creativity usually involve created value and novelty (Pichot et al., 2022). Creativity is a difficult task to achieve algorithmically, as there are some difficulties in constructing a mathematical model of this problem. It creates difficulties because traditional algorithms are not usually suitable for creative tasks. With the rise of computing power and new approaches meant to tackle the problem of creativity, many new methods are being developed to solve creativity tasks; however, it is still difficult to compete with human creativity in most cases (Franceschelli & Musolesi, 2021). Procedural Content Generation (PCG) is an algorithmic generation method that generates assets or asset composition. The results are usually represented by a computer-generated content that can be exploited at different levels of the game development process. This approach is commonly used to generate creative results and can modify a variety of different elements in video games. One of the common elements used in procedural game content generation is the layout of game objects. The main motivations for PCG are personalization, replayability, and costs (Togelius et al., 2011). In the research proposed in this paper, the focus is on the replayability side, by creating unique and varied game scenes with the same algorithm. Replayability represents a novelty in the context of creativity. This means that new results differ enough to be considered interesting after more than one instance of the same game level. Replayable content is important for video games if the intention is to use the same PCG algorithm for video game level generation.

There are numerous ways to procedurally generate content, and for the expansion of the variety of levels, the genetic algorithms (Herrmann, 1999) with an abstract fitness function can be used to increase the uniqueness aspect of the generated game level scene as shown in the related work for puzzle (Pereira et al., 2016; Thakkar et al., 2019) or arcade genre games (Safak et al., 2016). Unique content can be considered replayable because each instance of the generated level varies enough for it to still be interesting. One of the key aspects of the genetically created levels is the fitness function. The fitness functions determine the direction of evolution of the proposed task. The problem considered in this article is an optimization one, in which one tries to reach nondeterministic results, while keeping it functional. It needs to be vague to reach a sufficiently large numbers of satisfying results. At the same time, a criterion needs to be defined for the generated level. Usually, this criterion is divided into two categories: functional and aesthetic. Functional

criteria define the rules and limitations of game design. Game design rules can be optimized, and limitations always have to be true. Aesthetic criteria define how appealing the generated levels or objects look (Statham et al., 2022). With this mathematical model, all the criteria have to be combined into a single fitness number and, for reaching this goal, multi-criteria decision-making algorithms can be used (MCDM) (Zavadskas et al., 2014). Nowadays, many modern MCDM algorithms use fuzzy sets, enabling nondeterministic results and increasing replayability of the generated game level. There are many examples of applications of fuzzy MCDM methods employed in the research industry, but they are mostly used with a static environment and a single iteration. Some examples of these method applications are solving communication (Peng & Li, 2021; Zhao et al., 2021), networking (Peng & Garg, 2021), risk and failure analysis (Yazdani et al., 2021; Yousefi et al., 2021), logistics (Svadlenka et al., 2020; Kieu et al., 2021), and pathfinding (Semenas et al., 2021) problems. By its nature, the genetic algorithm requires more than one subsequent iteration of the algorithm to converge properly. This research proposes a new strategy that enables MCDM for a set of subsequent iterations, explores the fuzzy limitations of MCDM, and updates the general algorithms to be suitable for the creative PCG task. Nondeterministic creativity problems can be modelled by different versions of modern fuzzy sets.

Fuzzy set theory was established almost 60 years ago (Zadeh, 1965), but successful applications of it started to emerge in the 1980s. Fuzzy set theory is evolving, and new ways to redefine and update fuzzy set theory are constantly evolving (Kahraman et al., 2016, Wu et al., 2021). Some examples of applications include software selection problems, intuitionistic linguistic aggregation, human resources management, smartphone selection, and others (Haque et al., 2020). There are different types and applications of fuzzy sets, the neutrosophic set being the one which can increase nondeterminism. They consist of three numbers which define three neutrosophic set values (truth, indeterminacy, falsity) (Broumi et al., 2018; Smarandache, 1999). All three of these members should be independent. The neutrosophy, by its nature, deals with neutralities and their interaction with the ideational spectra (Zavadskas et al., 2020). In this research, the neutrosophic set environment of the CoCoSo method (Yazdani et al., 2018) was applied to calculate the final fitness score for each chromosome in the genetic algorithm.

This paper proposes a new extension of the neutrosophic CoCoSo method that is suitable for the genetic scene layout generator in which the fitness function is calculated via this method. To solve this content generation problem, adaptation aspects of the CoCoSo method were investigated. Normalization is usually calculated in the local Min-Max range, which includes a single iteration of the algorithm, so, it needs to be updated with global Min-Max values, including all possible value ranges (Choi & Moon, 2003). For this reason, the range can be normalized in each criterion to fit into the normalized value range. Another aspect taken into consideration is the speed of the algorithm. The genetic algorithm calculates the fitness function by using the CoCoSo method, in each generation, for each chromosome, so it can be quite taxing to perform repeated calculations. Some values that are in a short range and have little impact on the result can be changed to constant values. A conversion of linear crisp number to neutrosophic set will also be needed, as the slightest nonproportional change can skew the results in unsuitable ways and be higher than mutation differences, which can numb the evolution (Herrmann, 1999).

This paper proposes methods that can solve the problems discussed. The structure of this paper is established as follows. Section 2 will explain the fitness calculation by using the CoCoSo method, while section 3 will explain the CoCoSo modifications and implementation in the genetic game-level scene generator. Section 4 will present a study case and the results obtained for the replayable PCG case. The last section provides the conclusion of the proposed research.

2. Classical CoCoSo Method

The following steps explain how the general CoCoSo method works. The alternatives correspond to the population of the genetic algorithm, and the individual chromosome criterion corresponds to the criterion of the CoCoSo method.

The initial decision-making matrix is determined as follows (1), where $i = 1,2 \ldots, m; j = 1,2, \ldots n.$

$$X_{ij} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \quad (1)$$

The normalization of the values for the benefit criterion (2) and for the cost criterion (3) are determined as follows:

$$r_{ij} = \frac{x_{ij} - \min\limits_{i} x_{ij}}{\max\limits_{i} x_{ij} - \min\limits_{i} x_{ij}} \quad (2)$$

$$r_{ij} = \frac{\max\limits_{i} x_{ij} - x_{ij}}{\max\limits_{i} x_{ij} - \min\limits_{i} x_{ij}} \quad (3)$$

An alternative sum of the weighted comparability (4) and the amount of the power weight of the comparability sequences (5) are calculated next.

$$R_i = \sum_{j=1}^{n} (w_j r_{ij}) \quad (4)$$

$$P_i = \sum_{j=1}^{n} (r_{ij})^{w_j} \quad (5)$$

Three appraisal score strategies are used to generate relative weights. k_{ia} expresses the arithmetic mean of the sums of the WSM and WPM scores (6). k_{ib} expresses the sum of the relative scores of WSM and WPM compared to the best (7), and k_{ic} releases the balanced compromise of the WSM and WPM model scores (8). λ is a manually selected constant which should be between 0 and 1, and defines the weight between R and P. λ is defined as 0.5.

$$k_{ia} = \frac{R_i + P_i}{\sum_{i=1}^{m} (R_i + P_i)} \quad (6)$$

$$K_{ib} = \frac{R_i}{\min\limits_{i} R_i} + \frac{P_i}{\min\limits_{i} P_i} \quad (7)$$

$$K_{ic} = \frac{\lambda(R_i) + (1-\lambda)(P_i)}{\left(\lambda \max\limits_{i} R_i + (1-\lambda) \max\limits_{i} P_i \right)} \quad (8)$$

Finally, the ranking of alternatives is determined (9):

$$k_i = (k_{ia} k_{ib} k_{ic})^{\frac{1}{3}} + \frac{1}{3}(k_{ia} + k_{ib} + k_{ic}) \quad (9)$$

3. Neutrosophic Genetic CoCoSo Procedural Game Scene Generator

CoCoSo is used to calculate the final fitness score for each chromosome in a genetic algorithm generation. For the decision matrix, the criterion is used of each chromosome.

```
------------------------------------------------------------
InitializeRandomPopulation:
DoFullEvolution:
  for amountOfEvolutionCycles
    CalculateAllCriteria
    FindUnderperformersAndPerformers
      for populationSize
        calculateFitness
          CoCoSo method
EvolveUnderperformersWithGeneticAlgorithm
DrawGrid(best fitness):
------------------------------------------------------------
```

The core of the procedural generator is a genetic algorithm. The original MCDM alternative (i) and the criteria matrix (j) are transformed into a three-dimensional grid. The alternatives in the MCDM algorithm correspond to the population and chromosomes (i) in the genetic algorithm and the third dimension adds generations (j) in the time layer (see Figure 1).

After the fixed number of generations, the algorithm constructs the layouts of the game levels. It initializes random-level layouts and then develops them. Criteria scores are calculated and normalized to fit in the range of 0.1 to 0.9.

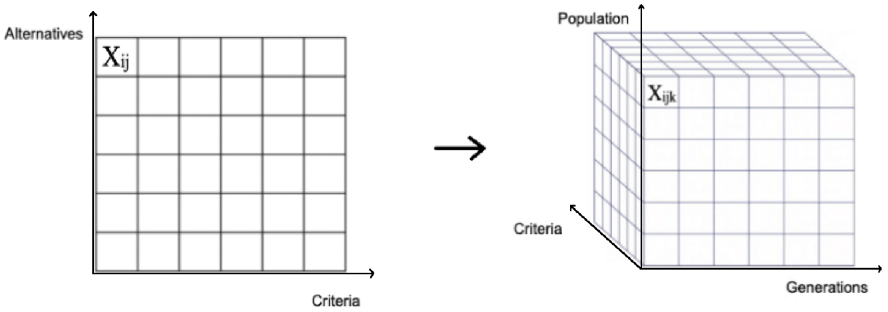

Figure 1. Grid transformation

In order to obtain a more reliable result for the MCDM algorithm, one avoids getting close to 0 and 1 values. The proposed set of criteria consists of 7 criteria. The first part of them are validation criteria, which check for the existence of a player, of an exit and of a path between them. The second part consists of optimizable criteria, which are symmetry, empty space balance, player exit distance, and safe zone calculation. After that, a final fitness score is calculated using the CoCoSo method and genetic operators are applied to the population. For each generation, the fitness calculation is repeated.

The linear conversion of crisp numbers to neutrosophic numbers is used, so the nonlinear differences would not compound over separate generations (10). It can skew the result as the modification can be higher than the evolved difference in a single generation.

$$N(t,i,f) = \begin{cases} C \\ 1-C \\ 1-C \end{cases} \tag{10}$$

Normalization is updated to use the smallest possible criterion values across all generations instead of the single ones generated across a single generation.

Beforehand, every criterion is manually normalised to fit into the 0-1 range, in order to achieve this goal, as the future generation values cannot be predicted and they have to be globally normalised for each generation (11).

$$0 < X_{ik} < 1 \tag{11}$$

The general normalization can also be skipped, as each criterion is normalized separately. This update also allows skipping the separate storage of data sets from neutrosophic sets in memory and incrementally adding and multiplying them to generate R_i (12) and P_i (13) values, without storing each of the single elements, where l is the number of generations for the genetic algorithm and n is the population size. Values r_{ij} are replaced by the values N_{ij} of the neutrosophic sets.

$$R_i = \sum_{j=1}^{nl} (w_j(N_{ij}) \tag{12}$$

$$P_i = \sum_{j=1}^{nl} (N_{ij})^{w_j} \tag{13}$$

It also allows the creation of an approximate constant A (14) for the divisor (15) of the *kia* (16) formula, rather than calculating a sum for each chromosome of the generation. It dramatically improves the performance of the iterative algorithm without visible loss of quality.

$$A = \frac{\sum_{i=1}^{nl} x_i}{nl} \tag{14}$$

$$d = nA \tag{15}$$

$$k_{ia} = \frac{S(R_{ik}) + S(P_{ik})}{d} \tag{16}$$

It uses global min (r_1, r_2) and max (p_1, p_2) values of Ri and Pi, which are calculated before revealing future values (17, 18, 19, 20) based on the normalization of the criteria formula, so it would be consistent in interactions between generations (21, 22).

$$r_1 = \min_{i \in nl}(R_i) \tag{17}$$

$$p_1 = \max_{i \in nl}(R_i) \tag{18}$$

$$r_2 = \min_{i \in nl}(P_i) \tag{19}$$

$$p_2 = \max_{i \in nl}(P_i) \tag{20}$$

$$k_{ib} = \frac{S(R_{ik})}{r_1} + \frac{S(P_{ik})}{r_2} \tag{21}$$

$$k_{ic} = \frac{\lambda S(R_{ik}) + (1-\lambda) S(P_{ik})}{\lambda p_1 + (1-\lambda) p_2} \tag{22}$$

Finally, the original k_i formula is used to calculate the final fitness function for a single chromosome.

4. Results

The base used for the algorithm was taken from the original WASPAS procedural scene generator (Petrovas & Bausys, 2022). The CPU used for the calculations has 8-core processor with 2.4 GHz speed. Convergent fitness scores begin to stabilize after about 80–120 generations, and, compared to the WASPAS implementation, the convergence occurs at an early stage. Note that the fitness scores cannot be directly compared, because the CoCoSo method produces fitness values outside the 0-1 range. In the proposed demo, it usually ranges between 4 and 8, and it has a tighter relative value range. The comparison between the fitness curves of CoCoSo and WASPAS methods, in terms of calculation time, can be seen in Figure 2.

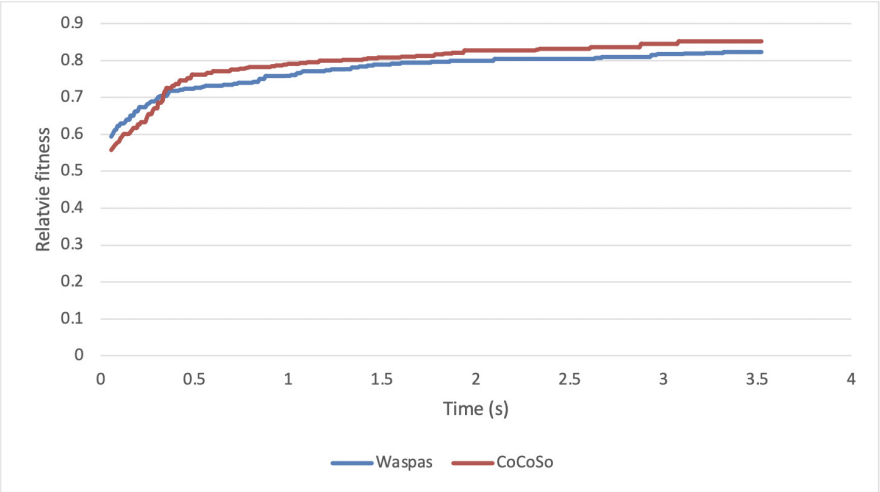

Figure 2. Fitness curve

It takes around 23 seconds for 2000 generations. But around 350 generations are usually enough to have satisfying results with the CoCoSo method, so the increased calculation time can be saved with a reduced number of generations. The visual results generated by the algorithm can be observed in Figures 3-6.

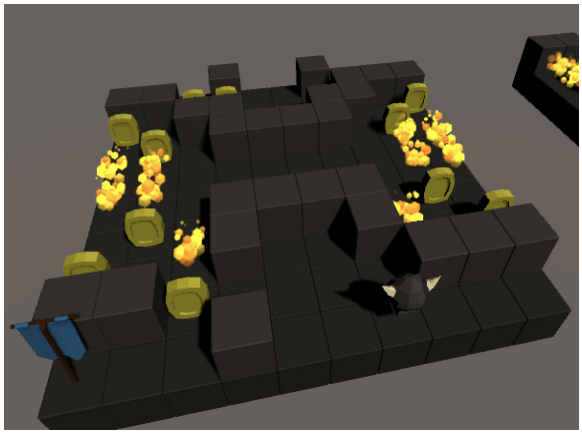

Figure 3. CoCoSo generated example 1

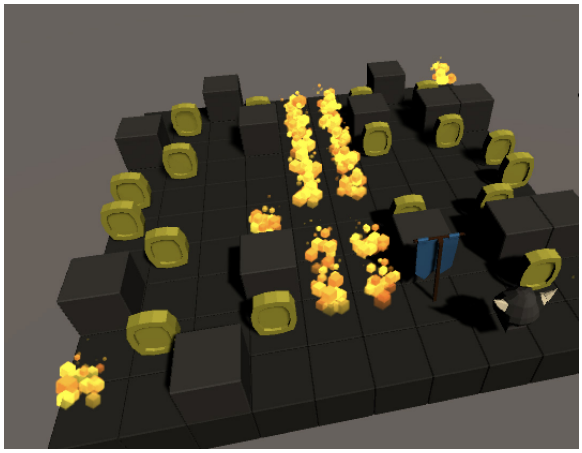

Figure 4. CoCoSo generated example 2

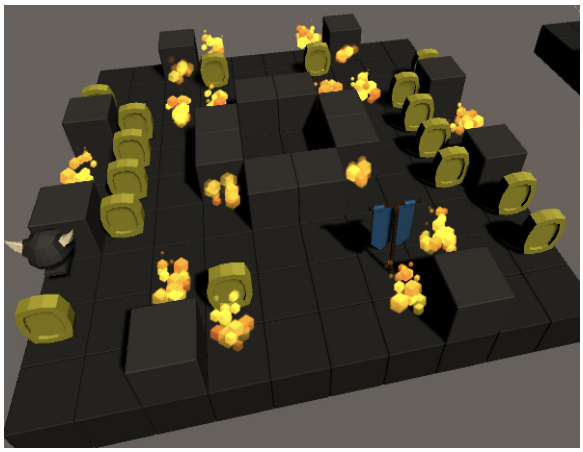

Figure 5. CoCoSo generated example 3

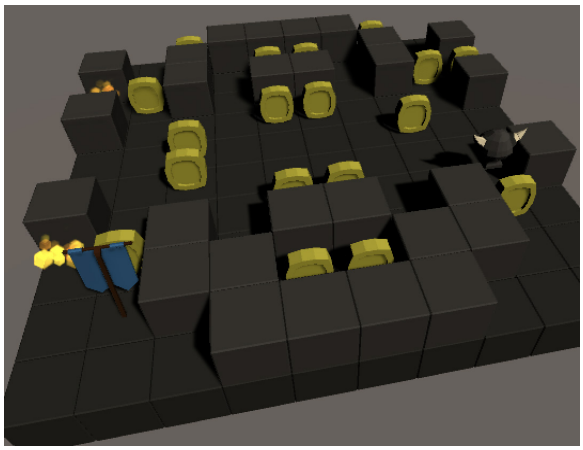

Figure 6. CoCoSo generated example 4

It can be seen that the generated game scene-level layouts satisfy both the aesthetics and the functional criteria and that the patterns between these two simultaneous results are different. It can also produce interesting patterns faster than the original algorithm, but calculation time

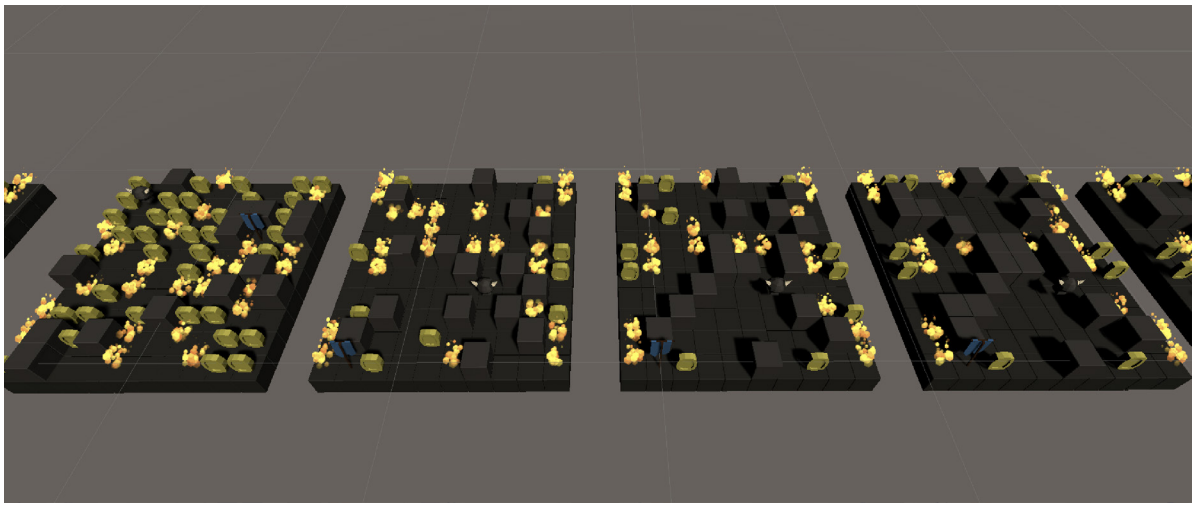

Figure 7. CoCoSo pattern progression over generations

is increased by 10% for the same number of generations and for the same population size. It can be noticed that, even in the initial steps of the algorithm and with a lower number of generations, the desired patterns begin to form (see Figure 7).

5. Conclusion

Applying the proposed approach, the genetic procedural scene layout generator was extended with the CoCoSo method. The CoCoSo method generates satisfying results with a smaller number of generations and takes into account more various aspects of the problem compared to the WASPAS method. Neutrosophic sets are a great tool to work with uncertain information due to modelling of truth, indeterminacy, and falsity membership functions. The implementation of neutrosophic set environment in the CoCoSo approach allows improving the variety of the generated levels. Within the framework of the genetic algorithm, the proposed neutrosophic CoCoSo method is improved compared to the original CoCoSo approach. This improvement can be stated as: if one has very close values of the divisor in equations (21) and (22), one sets them to constant values. The conversions from the crisp numbers to the neutrosophic ones were updated to suit the iterative evolution. The normalization procedures were updated to work with global min and max values, by adapting them to the set of generations, in which the values depend on the previous generations. The generated patterns satisfy the replayability principle for both the aesthetics and the functional criteria.

REFERENCES

Broumi, S., Bakali, A., Talea, M., Smarandache, F., Uluçay, V., Sahin, M, Dey, A., Dhar, M, Tan, R. P., Bahnasse, A. & Pramanik, S. (2018). Neutrosophic Sets: An Overview. In Smarandache, F. & Pramani, S. (Eds.), *New Trends in Neutrosophic Theory and Applications, vol. 2*, 403-434. Brussels: Pons Edition.

Choi, S. S. & Moon, B. R. (2003). Normalization in genetic algorithms. In *Genetic and Evolutionary Computation Conference* (pp. 862-873).

Filip, F. G. (2021). Automation and Computers and Their Contribution to Human Well-being and Resilience, *Studies in Informatics and Control*, *30*(4), 5-18. DOI: 10.24846/v30i4y202101

Franceschelli, G. & Musolesi, M. (2021). Creativity and machine learning: A survey, *eprint arXiv:2104.02726*. DOI: 10.48550/arXiv.2104.02726

Haque, T. S., Chakraborty, A., Mondal, S. P. & Alam, S. (2020). Approach to solve multi-criteria group decision-making problems using exponential operational law in a generalised spherical fuzzy environment, *CAAI Transactions on Intelligence Technology*, *5*(2), 106-114.

Herrmann, J. W. (1999). A genetic algorithm for minimax optimization problems. In *Proceedings of the 1999 Congress on Evolutionary Computation-CEC99, Vol. 2* (pp. 1099-1103).

Kahraman, C., Öztayşi, B. & Çevik Onar, S. (2016). A comprehensive literature review of 50 years of fuzzy set theory, *International Journal of Computational Intelligence Systems*, *9*(1), 3-24.

Kieu, P. T., Nguyen, V. T., Nguyen, V. T. & Ho, T. P. (2021). A Spherical Fuzzy Analytic Hierarchy Process (SF-AHP) and Combined Compromise Solution (CoCoSo) Algorithm in Distribution Center Location Selection: A Case Study in Agricultural Supply Chain, *Axioms*, *10*(2), article id: 53.

Peng, X. & Garg, H. (2021). Intuitionistic fuzzy soft decision making method based on CoCoSo and CRITIC for CCN cache placement strategy selection, *Artificial Intelligence Review*, *55*, 1567–1604.

Peng, X. & Li, W. (2021). Spherical fuzzy decision making method based on combined compromise solution for IIoT industry evaluation, *Artificial Intelligence Review*, *55*, 1857-1886.

Pereira, L. T., Toledo, C., Ferreira, L. N. & Lelis, L. H. (2016). Learning to speed up evolutionary content generation in physics-based puzzle games. In *2016 IEEE 28th International Conference on Tools with Artificial Intelligence (ICTAI)*, (pp. 901-907).

Petrovas, A. & Bausys, R. (2022). Procedural Video Game Scene Generation by Genetic and Neutrosophic WASPAS Algorithms, *Applied Sciences*, *12*(2), 772.

Pichot, N., Bonetto, E., Pavani, J. B., Arciszewski, T., Bonnardel, N. & Weisberg, R. W. (2022). The construct validity of creativity: empirical arguments in favor of novelty as the basis for creativity, *Creativity Research Journal*, *34*(1), 2-13.

Safak, A. B., Bostanci, E. & Soylucicek, A. E. (2016). Automated Maze Generation for Ms. Pac-Man Using Genetic Algorithms, *International Journal of Machine Learning and Computing*, *6*(4), 226-240.

Semenas, R., Bausys, R. & Zavadskas, E. K. (2021). A Novel Environment Exploration Strategy by m-generalised q-neutrosophic WASPAS, *Studies in Informatics and Control*, *30*(3), 19-28. DOI: 10.24846/v30i3y202102

Smarandache, F. A. (1999). *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set, and Logic*. Rehoboth: American Research Press.

Statham, N., Jacob, J. & Fridenfalk, M. (2022). Game environment art with modular architecture, *Entertainment Computing*, *41*, article id: 100476.

Svadlenka, L., Simic, V., Dobrodolac, M., Lazarevic, D. & Todorovic, G. (2020). Picture Fuzzy Decision-Making Approach for Sustainable Last-Mile Delivery, *IEEE Access*, *8*, 209393-209414.

Thakkar, S., Cao, C., Wang, L., Choi, T. J. & Togelius, J. (2019). Autoencoder and evolutionary algorithm for level generation in lode runner. In *2019 IEEE Conference on Games (CoG)*, (pp. 1-4).

Togelius, J., Kastbjerg, E., Schedl, D. & Yannakakis, G. N. (2011). What is procedural content generation? Mario on the borderline. In *Proceedings of the 2nd International Workshop on Procedural Content Generation in Games* (pp. 1-6).

Wu, Z., Liao, H., Lu, K. & Zavadskas, E. K. (2021). Soft Computing Techniques and Their Applications in Intelligent Industrial Control Systems: A Survey, *International Journal of Computers, Communications & Control*, *16*(1), 1-28.

Yazdani, M., Mohammed, A., Bai, C. & Labib, A. (2021). A novel hesitant-fuzzy-based group decision approach for outsourcing risk, *Expert Systems with Applications*, *184*, article id: 115517.

Yazdani, M., Zarate, P., Zavadskas, E. K. & Turskis, Z. (2018). A Combined Compromise Solution (CoCoSo) method for multi-criteria decision-making problems, *Management Decision*, *57*(9), 2501-2519.

Yousefi, S., Valipour, M. & Gul, M. (2021). Systems failure analysis using Z-number theory-based combined compromise solution and full consistency method, *Applied Soft Computing*, *113*(Part A), article id: 107902.

Zadeh, L. A. (1965). Fuzzy sets, *Information and Control*, *8*(3), 338-353.

Zavadskas, E. K., Bausys, R., Lesciauskiene, I., & Omran, J. (2020). M-generalised q-neutrosophic MULTIMOORA for Decision Making, *Studies in Informatics and Control*, *29*(4), 389-398. DOI: 10.24846/v29i4y202001

Zavadskas, E. K., Turskis, Z. & Kildienė, S. (2014). State of art surveys of overviews on MCDM/MADM methods, *Technological and Economic Development of Economy*, *20*(1), 165-179.

Zhao, H., Zhang, R., Zhang, A. & Zhu, X. (2021). Multi-attribute Group Decision Making Method with Unknown Attribute Weights Based on the Q-rung Orthopair Uncertain Linguistic Power Muirhead Mean Operators, *International Journal of Computers, Communications & Control*, *16*(3), 1-20.

Neutrosophic Genetic Algorithm for solving the Vehicle Routing Problem with uncertain travel times

Rafael Rojas-Gualdron,[1] **Florentin Smarandache**[2]

[1] Department of Industrial Engineering, Universidad de Investigación y Desarrollo, Barrancabermeja, Colombia;
E-mail: rrojas7@udi.edu.co
[2] Mathematics Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA
E-mail: smarand@unm.edu

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Abstract: The Vehicle Routing Problem (VRP) has been extensively studied by different researchers from all over the world in recent years. Multiple solutions have been proposed for different variations of the problem, such as Capacitive Vehicle Routing Problem (CVRP), Vehicle Routing Problem with Time Windows (VRP-TW), Vehicle Routing Problem with Pickup and Delivery (VRPPD), among others, all of them with deterministic times. In the last years, researchers have been interested in including in their different models the variations that travel times may experience when exposed to all kind of phenomena, mainly vehicle traffic. This article addresses the VRP from this perspective, proposing the design and implementation of a genetic algorithm based on neutrosophic theory for calculating the fitness function of each route, considering the variability and uncertainty present in travel times. A deterministic genetic algorithm is also implemented with the average travel times to compare it with the neutrosophic algorithm using simulation. As conclusion, a deterministic algorithm does not necessarily generate the best solution in the real world, full of uncertainty. Also, the quantification of uncertainty using neutrosophic theory can be used in route planning, opening a broad and interesting field of research for future investigations.

Keywords: Vehicle Routing Problem (VRP); Neutrosophic Theory; Uncertain Travel Times; Stochastic Vehicle Routing Problem (SVRP); Genetic Algorithm

1. Introduction

One of the characteristics that most affect the proper functioning of supply chains today is the variability and uncertainty present in their transportation systems [1], mainly due to three causes: firstly, the high complexity of its large-scale processes, secondly, to the occurrence of complex and random traffic phenomena, and, thirdly, to the susceptibility of being affected by external and unpredictable factors.

It is a common practice in companies to face the challenge of planning routes with many customers and demands using the empirical knowledge of their most experienced workers [2]. However, they face exceptionally tough working conditions due to the increasing number of routes and diversity of demands. For this reason, it is necessary to develop decision support systems that facilitate this task and allow planning efficient routes in short times applicable to the daily operation of companies.

It should also be considered that the flow of traffic on urban roads in developing countries is characterized by being heterogeneous, due to the large number of buses on its main roads [3]. This condition makes it especially difficult to model traffic behavior in its most populated cities and, particularly complex, to estimate vehicle travel times through its streets.

Among the external factors that affect the planning of the routes and cause the calculations to move away from reality, generating an increase in costs, we can consider the unpredictable behavior of some users, the various weather conditions that may occur, some of them extreme, in addition to the errors in the measurement or estimation of the travel times of each one of the routes [4].

Stochastic vehicle routing problems (SVRP) arise whenever some elements of the problem are random [5]. The most common cases found in the literature can be classified into three large groups: vehicle routing problems

with stochastic demand, vehicle routing problems with stochastic travel times, or problems in which both conditions are present.

SVRPs differ from their deterministic counterpart in several fundamental aspects, mainly in a different solution concept, which results in much more complex solution methodologies, which are often considered computationally intractable, since only relatively small instances can be solved optimally and the difficulty programmers face when designing and evaluating good heuristics for this kind of problem.

For the scope of this article, the authors focus on vehicle routing problems with stochastic travel times, mainly affected by unpredictable traffic behavior over time. This uncertainty is mainly due to two possible causes: the first, the probability of occurrence of traffic accidents and, the second, the variability of the demand of the transport networks, which generates phenomena such as peak hours or hours of high flow of vehicles and off-peak hours or hours with low vehicle traffic. These factors are the reason why the travel time between any two nodes in the transport network is considered stochastic [6].

A widely used tool to treat this stochasticity is the information available through historical data that is collected and analyzed to be converted into models or probability distributions of the random variables available, for our case study, of the travel times of each one of the arcs of the network [7]. In a pure SVRP, these distributions are available, and the route optimization process is static and is performed only once, without considering the changes that occur during the route.

Among the different solutions found in the literature, we can cite branch and price algorithms [8], [9], simulation-based heuristics [10], simheuristics [11], [12], adaptive local search algorithms, ALNS, [13], scatter search [14], tabu search [15], iterative local search, ILS [16], particle swarm optimization, PSO [17], genetic algorithms [18], non-dominated sorting genetic algorithms, NSGA [19] and memetic algorithms [20], [21].

For the scope of this article, the authors focus on the difficulty of modeling traffic, due to its complexity and randomness, especially on last-mile route planning. Authors based on the open-source data published by Uber Movement of the historical travel times to get the minimum, maximum and average values for travel times between two nodes. Also, a function is proposed to convert this data into a triangular neutrosophic number [22] to make use of the score function to compare two neutrosophic numbers. This score function is used to calculate the fitness function of the proposed genetic algorithm. Finally, the results are compared and a good behavior of the neutrosophic algorithm is observed in stochastic scenarios.

2. The proposed algorithm

The methodology proposed for the development of this research consists of the following phases: the definition of the structure of the data available to carry out the study, the design of a function to convert the available data into neutrosophic triangular numbers, the definition of a fitness function based on a neutrosophic score function to compare two triangular neutrosophic numbers, the definition of the parameters and the design of the genetic algorithm, the implementation of a deterministic genetic algorithm to solve the VRP problem using the mean values of the travel times to compare it with the neutrosophic algorithm and finally the simulation through the generation of scenarios to test the performance of both algorithms.

2.1. Definition of the structure of the data

This research is based on the open-source data provided by Uber Movement [23] on the history of travel times between two nodes, in our case, in the city of Bogotá, Colombia. The data structure that we are going to use includes the minimum, maximum and mean value of the data history of each route for the period between March 1 and March 31, 2020.

2.2. Design of a function to convert the data to neutrosophic triangular number

Once the travel times between a pair of nodes are obtained as a triplet [a, b, c] where a is the minimum value of the interval, b is the mean value and c the maximum value of the interval, we must proceed to convert this triplet into a neutrosophic triangular number $\langle [a, b, c], (T, I, F) \rangle$. For this purpose, the authors propose the use of the following function:

$$TNS\ number = \langle [a, b, c], (T, I, F) \rangle$$

where:

$$\Delta_T = \min\begin{pmatrix} b-a \\ c-b \end{pmatrix}$$

$$\Delta_F = \max\begin{pmatrix} b-a \\ c-b \end{pmatrix}$$

$$T = \begin{bmatrix} 1-\frac{\Delta_T}{b}, & \Delta_T < b \\ 0, & otherwise \end{bmatrix}$$

$$I = \begin{bmatrix} \frac{c-a}{b}, & c-a < b \\ 1, & otherwise \end{bmatrix}$$

$$F = \begin{bmatrix} \frac{\Delta_F}{b}, & \Delta_F < b \\ 1, & otherwise \end{bmatrix}$$

2.3. Definition of the fitness function

To calculate the fitness function, we rely on the score function used to compare two neutrosophic triangular numbers.

Definition 1 [24]**:** (Comparison of any two random TNS numbers). Let $\hat{r}^{NS} = \langle [\hat{r}_T, \hat{r}_I, \hat{r}_P], (T_{\hat{r}}, I_{\hat{r}}, F_{\hat{r}}) \rangle$ be a TNS number, and then the score function is defined as follows:

$$s(r^{NS}) = \frac{1}{12} \cdot [\hat{r}_T + 2 \cdot \hat{r}_I + \hat{r}_P] \cdot [2 + T_{\hat{r}} - I_{\hat{r}} - F_{\hat{r}}]$$

Let $\hat{r}^{NS} = \langle [\hat{r}_T, \hat{r}_I, \hat{r}_P], (T_{\hat{r}}, I_{\hat{r}}, F_{\hat{r}}) \rangle$ and $\hat{s}^{NS} = \langle [\hat{s}_T, \hat{s}_I, \hat{s}_P], (T_{\hat{s}}, I_{\hat{s}}, F_{\hat{s}}) \rangle$ be two arbitrary TNSNs, the ranking of $\hat{r}^{NS}$ and $\hat{s}^{NS}$ by score function is defined as follows:

if $s(\hat{r}^{NS}) \succ s(\hat{s}^{NS})$ then $\hat{r}^{NS} \succ \hat{s}^{NS}$
if $s(\hat{r}^{NS}) \prec s(\hat{s}^{NS})$ then $\hat{r}^{NS} \prec \hat{s}^{NS}$
if $s(\hat{r}^{NS}) \approx s(\hat{s}^{NS})$ then $\hat{r}^{NS} \approx \hat{s}^{NS}$

The value of the fitness function of each individual is equal to the sum of the score function of the neutrosophic triangular number that represents the distance between each pair of nodes of the route.

2.4. Definition of parameters and design of the genetic algorithm

2.4.1. Individual

Each individual is represented by a chromosome of n+2 positions, where n is the number of customers that are visited on the route. For example, for 10 clients, the chromosome would have 12 positions, one for each of the clients represented by 1, 2, 3..., n plus a zero at the beginning and at the end of the chromosome that represent the deposit from which the route starts and where the route finishes as seen on figure 1.

0	3	1	5	6	9	2	4	8	10	7	0

Figure 1. Chromosome for an individual with 10 clients.

2.4.2. Parent selection

The selection is carried out by a selection function, proportional to the fitness function, in which each individual has a probability of being selected as a parent that is proportional to the value of its fitness function. Because for the VRP the best individual is the one with the lowest value of the fitness function, the probability of being selected will be inversely proportional to their fitness.

2.4.3. Crossover function

The crossover function generates two random positions from the first parent to cut a chromosome fragment between the two positions. That fragment is then inserted at a random location within the second parent. Finally, the values that are repeated within the chromosome are eliminated from this resulting chromosome to generate the child. Figure 2 shows an example of the crossover function.

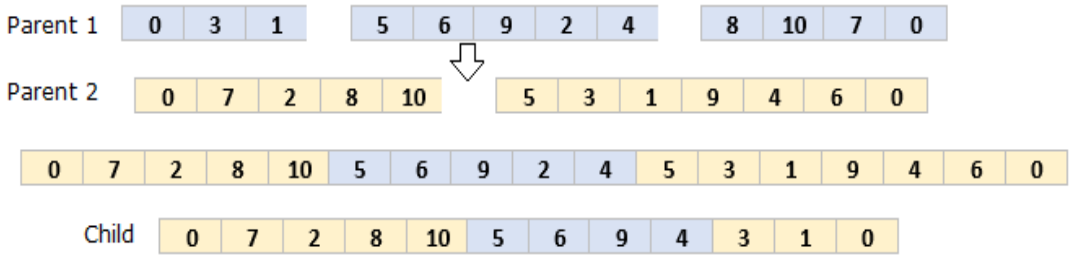

Figure 2. Example of the crossover function.

2.4.4. Mutation function

The mutation probability was set at 5 percent to diversify the individuals and avoid falling into a local optimum. If an individual is chosen for the mutation, two chromosome positions are randomly selected, and their values are swapped. Figure 3 shows an example of a mutation.

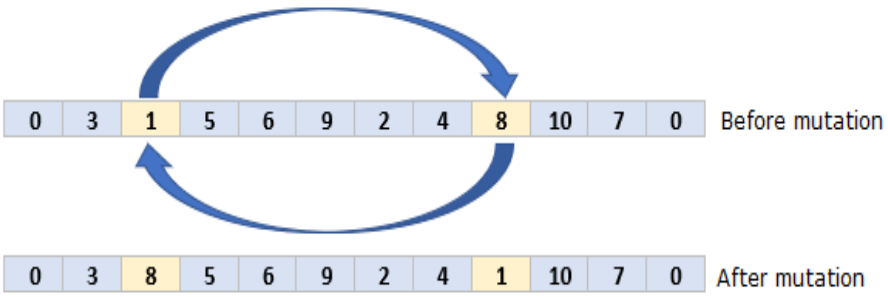

Figure 3. Example of the mutation function.

2.4.5. Pseudocode

Step 1. Generate random population
Step 2. Calculate fitness
Step 3. Update the incumbent
Step 4. For the number of generations:
    Step 4.1. Parent selection
    Step 4.2. Crossover function
    Step 4.3. Mutation function
    Step 4.4. Calculate fitness
    Step 4.5. Update the incumbent

2.5. Design of the deterministic genetic algorithm

To test the results of the proposed neutrosophic algorithm, we are going to implement a deterministic algorithm, which will have the same design as the neutrosophic algorithm with the difference that its fitness function will be equal to the sum of the mean travel times of each pair of nodes.

2.6. Simulation

In order to simulate the behavior of the algorithm in the real world we are going to randomly generate one hundred scenarios based on the data we have. To randomly generate the travel times between any pair of nodes, we need to fit the data we have to a probability distribution. For our case, the probability distribution that is closest to the behavior of the travel times and to the data we have is the triangular distribution, for which we need the values of the maximum, minimum, and mode. The maximum and minimum values are available in the data; however, the mode is not. To calculate the mode, we rely on the following equation applicable to a triangular distribution:

$$Mean = \frac{Maximum + Mode + Minimum}{3}$$

Bearing in mind that we have the value of the mean in the data, we only have to get the mode from the equation to be able to calculate it for the travel times between each pair of nodes.

$$Mode = 3 \cdot Mean - Maximum - Minimum$$

Having the values of maximum, minimum and mode we proceed to randomly generate the travel times between each pair of nodes using the triangular distribution. One hundred scenarios are generated, and the best solution obtained by the neutrosophic algorithm is compared against the best solution obtained by the deterministic algorithm to evaluate the results and draw conclusions.

3. Results

To test the performance of the algorithm, eleven locations will be selected from the Uber Movement database. Ten will be the customers and will be numbered from 1 to 10 and one will be the supplier and will have the number zero. The values in seconds for the maximum value of the range of travel times between each location can be seen in table 1.

Table 1. Maximum value for the range of travel times between each pair of nodes.

CUSTOMER	0	1	2	3	4	5	6	7	8	9	10
0	0	2217	1076	1965	2780	2518	3402	1344	2071	1778	2242
1	1945	0	1732	1505	2586	2926	3635	2283	1339	1037	2000
2	1490	1425	0	719	2095	1719	3692	1371	1231	855	1187
3	2125	1367	1141	0	2323	2758	2376	950	889	984	737
4	2486	2668	2200	1945	0	784	1015	2968	1798	1419	2511
5	1984	1986	1715	1542	773	0	1690	2544	1341	907	2113
6	3086	3558	3491	2517	1273	1583	0	3743	2754	2299	2901
7	1236	2503	1219	1278	3055	2723	3103	0	1762	1823	1058
8	2051	1194	1873	927	1954	2240	2963	1770	0	1009	1552
9	1016	998	809	819	2000	1964	2870	1597	542	0	1362
10	2064	2647	1894	1229	3900	3676	4547	1318	2301	2507	0

The values in seconds for the mean value of the range of travel times between each location can be seen in table 2.

Table 2. Mean value for the range of travel times between each pair of nodes.

CUSTOMER	0	1	2	3	4	5	6	7	8	9	10
0	0	1604	739	1333	1938	1768	2448	922	1491	1228	1550
1	1392	0	1194	1046	1705	1930	2432	1587	942	685	1354
2	896	971	0	481	1379	1147	2351	837	824	546	734
3	1370	989	699	0	1471	1857	1582	578	574	600	405
4	1742	1821	1475	1341	0	342	633	1979	1203	888	1706
5	1459	1393	1199	1089	327	0	988	1726	928	602	1450
6	2234	2576	2376	1865	798	991	0	2487	1889	1533	2164
7	805	1743	711	775	2098	1902	2134	0	1191	1208	585
8	1419	824	1198	600	1215	1435	1856	1171	0	615	976
9	726	688	507	550	1129	1108	1792	1016	364	0	877
10	1375	1905	1194	669	2658	2529	3008	724	1412	1684	0

The values in seconds for the mean value of the range of travel times between each location can be seen in table 3.

Table 3. Minimum value for the range of travel times between each pair of nodes.

CUSTOMER	0	1	2	3	4	5	6	7	8	9	10
0	0	1160	507	904	1350	1241	1761	632	1073	847	1071
1	995	0	823	726	1124	1272	1626	1102	662	452	916
2	538	661	0	321	907	765	1496	510	551	348	453
3	882	715	428	0	931	1250	1052	351	370	365	222
4	1220	1242	988	924	0	149	394	1319	804	555	1158
5	1072	976	837	768	138	0	577	1170	642	399	994
6	1617	1864	1616	1381	500	620	0	1652	1295	1021	1613
7	524	1213	414	469	1490	1328	1467	0	804	800	323
8	981	568	766	387	755	919	1162	774	0	374	613
9	518	473	317	369	637	624	1118	646	244	0	564
10	915	1370	752	364	1811	1739	1989	397	866	1131	0

Using the function defined in section 2.2, we proceed to calculate the degrees of truth, indeterminacy, and falsity of each triplet. The degrees of truth of each of the travel times between each pair of nodes can be seen in table 4.

Table 4. Degree of truth for the values of travel times between each pair of nodes.

CUSTOMER	0	1	2	3	4	5	6	7	8	9	10
0		0,723	0,686	0,678	0,697	0,702	0,719	0,685	0,720	0,690	0,691
1	0,715		0,689	0,694	0,659	0,659	0,669	0,694	0,703	0,660	0,677
2	0,600	0,681		0,667	0,658	0,667	0,636	0,609	0,669	0,637	0,617
3	0,644	0,723	0,612		0,633	0,673	0,665	0,607	0,645	0,608	0,548
4	0,700	0,682	0,670	0,689		0,436	0,622	0,666	0,668	0,625	0,679
5	0,735	0,701	0,698	0,705	0,422		0,584	0,678	0,692	0,663	0,686
6	0,724	0,724	0,680	0,740	0,627	0,626		0,664	0,686	0,666	0,745

CUSTOMER	0	1	2	3	4	5	6	7	8	9	10
7	0,651	0,696	0,582	0,605	0,710	0,698	0,687		0,675	0,662	0,552
8	0,691	0,689	0,639	0,645	0,621	0,640	0,626	0,661		0,608	0,628
9	0,713	0,688	0,625	0,671	0,564	0,563	0,624	0,636	0,670		0,643
10	0,665	0,719	0,630	0,544	0,681	0,688	0,661	0,548	0,613	0,672	

The degrees of indeterminacy of each of the travel times between each pair of nodes can be seen in table 5.

Table 5. Degree of indeterminacy for the values of travel times between each pair of nodes.

CUSTOMER	0	1	2	3	4	5	6	7	8	9	10
0		0,659	0,770	0,796	0,738	0,722	0,670	0,772	0,669	0,758	0,755
1	0,682		0,761	0,745	0,857	0,857	0,826	0,744	0,719	0,854	0,801
2	1,000	0,787		0,827	0,861	0,832	0,934	1,000	0,825	0,929	1,000
3	0,907	0,659	1,000		0,946	0,812	0,837	1,000	0,904	1,000	1,000
4	0,727	0,783	0,822	0,761		1,000	0,981	0,833	0,826	0,973	0,793
5	0,625	0,725	0,732	0,711	1,000		1,000	0,796	0,753	0,844	0,772
6	0,658	0,658	0,789	0,609	0,969	0,972		0,841	0,772	0,834	0,595
7	0,884	0,740	1,000	1,000	0,746	0,733	0,767		0,804	0,847	1,000
8	0,754	0,760	0,924	0,900	0,987	0,921	0,970	0,851		1,000	0,962
9	0,686	0,763	0,970	0,818	1,000	1,000	0,978	0,936	0,819		0,910
10	0,836	0,670	0,956	1,000	0,786	0,766	0,850	1,000	1,000	0,817	

The degrees of falsity for each of the travel times between each pair of nodes can be seen in table 6.

Table 6. Degree of falsity for the values of travel times between each pair of nodes.

CUSTOMER	0	1	2	3	4	5	6	7	8	9	10
0		0,382	0,456	0,474	0,434	0,424	0,390	0,458	0,389	0,448	0,446
1	0,397		0,451	0,439	0,517	0,516	0,495	0,439	0,421	0,514	0,477
2	0,663	0,468		0,495	0,519	0,499	0,570	0,638	0,494	0,566	0,617
3	0,551	0,382	0,632		0,579	0,485	0,502	0,644	0,549	0,640	0,820
4	0,427	0,465	0,492	0,450		1,000	0,603	0,500	0,495	0,598	0,472
5	0,360	0,426	0,430	0,416	1,000		0,711	0,474	0,445	0,507	0,457
6	0,381	0,381	0,469	0,350	0,595	0,597		0,505	0,458	0,500	0,341
7	0,535	0,436	0,714	0,649	0,456	0,432	0,454		0,479	0,509	0,809
8	0,445	0,449	0,563	0,545	0,608	0,561	0,596	0,512		0,641	0,590
9	0,399	0,451	0,596	0,489	0,771	0,773	0,602	0,572	0,489		0,553
10	0,501	0,390	0,586	0,837	0,467	0,454	0,512	0,820	0,630	0,489	

Now with the data of the travel times converted to neutrosophic triangular numbers we proceed to execute the genetic algorithm for the proposed example. The best solution obtained by the neutrosophic genetic algorithm with a fitness of 3456.4 can be seen in figure 4.

0	2	1	9	4	6	5	8	3	10	7	0

Figure 4. Best individual for the neutrosophic genetic algorithm.

The next step is to execute the deterministic genetic algorithm using the average values of the travel times between each pair of nodes. The best individual, with a fitness of 8508, for the deterministic genetic algorithm, can be seen in figure 5.

0	2	4	6	5	9	1	8	3	10	7	0

Figure 5. Best individual for the deterministic genetic algorithm.

Afterwards, the travel times between each pair of nodes are simulated for one hundred scenarios. Then the travel time of each of the best routes obtained by both the neutrosophic algorithm and the deterministic algorithm is calculated. The results can be seen in figure 6.

The neutrosophic algorithm obtained a shorter total travel time than the deterministic algorithm in 52 of the 100 simulated scenarios, while the deterministic algorithm obtained a shorter total travel time in 48 of the 100 simulated scenarios. The graph of the percentage of success of each of the algorithms can be seen in figure 7.

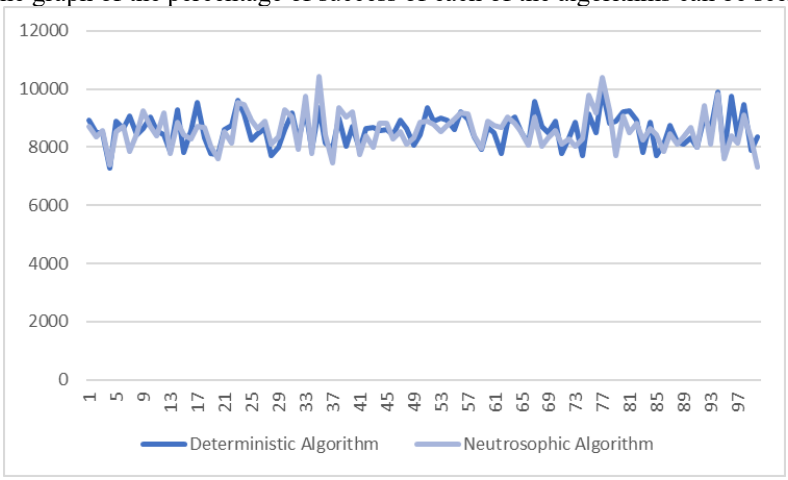

Figure 6. Travel times for both, the neutrosophic algorithm and the deterministic algorithm in the simulation.

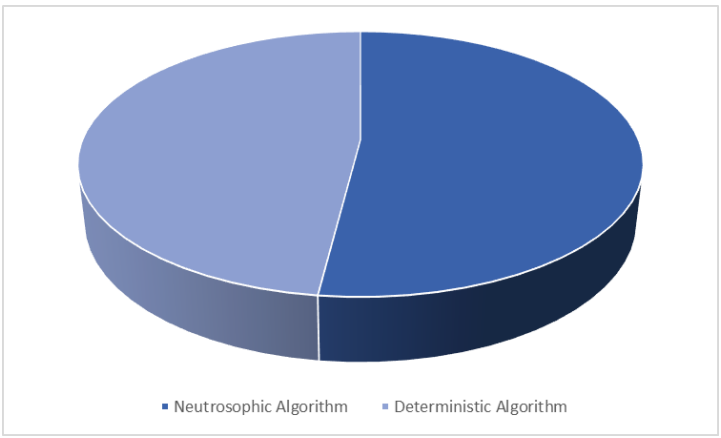

Figure 7. Percentage of successes of each algorithm.

4. Conclusions

Throughout this investigation we were able to verify the importance of including the uncertainty present in the data in the models we use for route planning. Working with mathematical models that only consider deterministic values can lead to generating solutions that in real life will not have the same behavior that was observed when solving the model.

The good performance of the neutrosophic theory was once again verified to work with values subject to different types of uncertainty, since it allows quantifying this uncertainty in order to take it into account in mathematical models and, as in this case, in the algorithms used to solve complex problems.

Even though the simulation was based on fitting the data to a triangular distribution, which greatly reduces the uncertainty present in them, the algorithm presented good results for the problem we were studying. This research is expected to open a new branch in neutrosophic research by combining metaheuristics with neutrosophic theory to solve complex problems that are also subject to high levels of uncertainty.

References

1. D. Kubek, P. Więcek, and K. Chwastek, "The Impact of Short Term Traffic Forecasting on the Effectiveness of Vehicles Routes Planning in Urban Areas," Transp. Res. Procedia, vol. 18, no. June, pp. 172–179, 2016, doi: 10.1016/j.trpro.2016.12.024.
2. X. Yu and K. Liu, "A spatial decision support system for large scale vehicle routing," 2009 Int. Conf. Meas. Technol. Mechatronics Autom. ICMTMA 2009, vol. 3, pp. 444–449, 2009, doi: 10.1109/ICMTMA.2009.270.
3. Z. Yao, B. Zhao, L. Qin, Y. Jiang, B. Ran, and B. Peng, "An efficient heterogeneous platoon dispersion model for real-time traffic signal control," Phys. A Stat. Mech. its Appl., vol. 539, p. 122982, 2020, doi: 10.1016/j.physa.2019.122982.
4. D. Kubek, "Zastosowanie optymalizacji odpornej w problematyce wyznaczania tras pojazdom: charakterystyka oraz kierunki bada," Logistyka, no. 3, pp. 2654–2663, 2015.
5. M. Gendreau, G. Laporte, and R. Séguin, "Stochastic vehicle routing," Eur. J. Oper. Res., vol. 88, no. 1, pp. 3–12, 1996, doi: 10.1016/0377-2217(95)00050-X.
6. J. Gao, "Model and algorithm of vehicle routing problem with time windows in stochastic traffic network," 2010 Int. Conf. Logist. Syst. Intell. Manag. ICLSIM 2010, vol. 2, pp. 848–851, 2010, doi: 10.1109/ICLSIM.2010.5461065.
7. U. Ritzinger, J. Puchinger, and R. F. Hartl, "A survey on dynamic and stochastic vehicle routing problems," Int. J. Prod. Res., vol. 54, no. 1, pp. 215–231, 2016, doi: 10.1080/00207543.2015.1043403.
8. G. Laporte, F. Louveaux, and H. Mercure, "Vehicle routing problem with stochastic travel times," Transp. Sci., vol. 26, no. 3, pp. 161–170, 1992, doi: 10.1287/trsc.26.3.161.
9. B. Rostami, G. Desaulniers, F. Errico, and A. Lodi, "Branch-price-and-cut algorithms for the vehicle routing problem with stochastic and correlated travel times," Oper. Res., vol. 69, no. 2, pp. 436–455, 2021.
10. Z. Wang and L. Lin, "A Simulation-based algorithm for the capacitated vehicle routing problem with stochastic travel times," J. Appl. Math., vol. 2013, 2013, doi: 10.1155/2013/127156.
11. D. Guimarans, O. Dominguez, J. Panadero, and A. A. Juan, "A simheuristic approach for the two-dimensional vehicle routing problem with stochastic travel times," Simul. Model. Pract. Theory, vol. 89, pp. 1–14, 2018, doi: 10.1016/j.simpat.2018.09.004.
12. J. Panadero, A. A. Juan, A. Freixes, C. Serrat, M. Grifoll, and M. Dehghanimohamamdabadi, "A SIMHEURISTIC FOR THE UNMANNED AERIAL VEHICLE SURVEILLANCE-ROUTING PROBLEM WITH STOCHASTIC TRAVEL TIMES AND RELIABILITY CONSIDERATIONS," in 2019 WINTER SIMULATION CONFERENCE (WSC), 2019, pp. 1883–1893.
13. D. Taş, N. Dellaert, T. van Woensel, and T. de Kok, "The time-dependent vehicle routing problem with soft time windows and stochastic travel times," Transp. Res. Part C Emerg. Technol., vol. 48, pp. 66–83, 2014, doi: 10.1016/j.trc.2014.08.007.
14. T. Zhang, W. A. Chaovalitwongse, and Y. Zhang, "Scatter search for the stochastic travel-time vehicle routing problem with simultaneous pick-ups and deliveries," Comput. Oper. Res., vol. 39, no. 10, pp. 2277–2290, 2012, doi: 10.1016/j.cor.2011.11.021.
15. X. Li, P. Tian, and S. C. H. Leung, "Vehicle routing problems with time windows and stochastic travel and service times: Models and algorithm," Int. J. Prod. Econ., vol. 125, no. 1, pp. 137–145, 2010, doi: 10.1016/j.ijpe.2010.01.013.
16. E. Messaoud, "A chance constrained programming model and an improved large neighborhood search algorithm for the electric vehicle routing problem with stochastic travel times," Evol. Intell., no. 0123456789, 2021, doi: 10.1007/s12065-021-00648-0.
17. Z. Shao, S. Gao, and S. Wang, "A Hybrid Particle Swarm Optimization Algorithm for Vehicle Routing Problem with Stochastic Travel Time," in FUZZY INFORMATION AND ENGINEERING, VOL 1, 2009, vol. 54, pp. 566–574.
18. B. Xie and S. An, "Applying genetic algorithm to vehicle routing problem with stochastic travel times," Dyn. Contin. Discret. Impuls. Syst. A-MATHEMATICAL Anal., vol. 13, no. 2, S, pp. 693–697, 2006.

19. J. Oyola, “The capacitated vehicle routing problem with soft time windows and stochastic travel times,” Rev. Fac. Ing. Univ. Pedagog. Y Tecnol. Colomb., vol. 28, no. 50, pp. 19–32, 2019, doi: 10.19053/01211129.v28.n50.2019.8782.
20. A. Gutierrez, L. Dieulle, N. Labadie, and N. Velasco, “A multi population memetic algorithm for the vehicle routing problem with time windows and stochastic travel and service times,” IFAC-PapersOnLine, vol. 49, no. 12, pp. 1204–1209, 2016, doi: 10.1016/j.ifacol.2016.07.673.
21. D. M. Miranda, J. Branke, and S. V. Conceição, “Algorithms for the multi-objective vehicle routing problem with hard time windows and stochastic travel time and service time,” Appl. Soft Comput. J., vol. 70, pp. 66–79, 2018, doi: 10.1016/j.asoc.2018.05.026.
22. S. I. A. Aal, M. M. A. Abd Ellatif, and M. M. Hassan, “Two ranking methods of Single Valued Triangular Neutrosophic Numbers to rank and evaluate Information Systems Quality,” Neutrosophic Sets Syst., vol. 19, pp. 133–143, 2018.
23. Uber, “Uber Movement,” 2022. https://movement.uber.com/.
24. R. Kumar, S. A. Edaltpanah, S. Jha, S. Broumi, and A. Dey, “Neutrosophic shortest path problem,” Neutrosophic Sets Syst., vol. 23, pp. 5–15, 2018.

An Efficient Optimal Solution Method for Neutrosophic Transport Models: Analysis, Improvements, and Examples

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract: Transport issues aim to determine the number of units that will be transferred from the production centers to consumption areas so that the cost of transportation is as low as possible, taking into account the conditions of supply and demand. Due to the great importance of these issues and to obtain more accurate results that take into account all circumstances, we conducted two research studies. In the first research, we presented a formulation of neutrosophic transport issues, and in the second research, we presented some ways to find a preliminary solution to these issues, but we do not know whether the preliminary solution is optimal or not, so we will present in this research a study whose purpose is to shed light on some important methods used to improve the optimal solution to transportation issues and then reformulating them using the concepts of neutrosophic science, a science that leaves nothing to chance or circumstances but rather provides solutions with neutrosophic values. Unspecified values take into account the best and worst conditions.

Keywords: Neutrosophic Science; Optimal Solution; Transportation; Stepping-Stone Method; Modified Distribution Method.

1. Introduction

The linear programming method is one of the most important operations research methods that companies and institutions have benefited from in their workflows by building linear models for which science has provided ways to find the optimal solution. Given this importance, we have reformulated these models in two previous studies and found one of the most important ways to solve them using the concepts of neutrosophic science [1, 2]. The issue of transportation is one of the most important issues that have been dealt with using the linear programming method because transportation problems appear frequently in practical life. We need to transfer materials from production centers to consumption centers to secure the regions; we need the lowest possible cost, To solve these recurring and daily problems, we use operations research methods, specifically the linear programming method, where we transform the issue data into a classical linear mathematical model when the data are classical and neutrosophic model when the data are neutrosophic. And in the research [3], there is a full explanation of the neutrosophic transport issues. Since these models are linear neutrosophic, we can get an optimal solution by using the direct simplex neutrosophic method described in the research [1].

But we know that these models have special characteristics, in terms of restrictions and objective function, which enabled scientists and researchers to find special methods, the methods in which we get preliminary solutions, and we explained how to obtain a preliminary solution for the neutrosophic transport in the research [4], and we recall that the neutrosophic transport issues are

transport issues in which the required quantities and the available quantities are neutrosophic values of the form $Na_i \in a_i + \varepsilon_i$, where ε_i the indeterminacy of the produced quantities can take the forms $\varepsilon_i \in [\lambda_{i1}, \lambda_{i2}]$. In addition, the required quantities of neutrosophic values of the form $Nb_j \in b_j + \delta_j$, where δ_j is the indeterminacy of the quantities produced we take it as one of the forms $\delta_j \in [\mu_{j1}, \mu_{j2}]$. and the cost of transportation is neutrosophic values form $Nc_{ij} \in c_{ij} + \alpha_{ij}$, where α_{ij} is the indeterminacy of the cost of transportation, we take it as one of the form $c_{2j} \in \{\alpha_{1_{2j}} \alpha_{2_{2j}}\}$.In order to review the basic concepts of neutrosophic science and its stages of development and the most important topics of operations research that have been reformulated using the concepts of this science, the following research can be found [5-16].

2. Methods

The purpose of this research, as we mentioned in the abstract, is to highlight some of the methods used according to classical logic to improve the preliminary solution to transportation issues, where we will present:

i. The Stepping-Stone Method
ii. Modified Distribution Method

As stated in some references [17-20], with a focus on the scientific basis on which these methods were based, and then we will reformulate them using the concepts of neutrosophic science and use them to improve the solution of neutrosophic models after finding a preliminary solution for them using one of the methods mentioned in the reference [4].

2.1 The Stepping-Stone Method

To reach the optimal solution in this way, we follow the following steps:

i. We find the preliminary solution by one of the three aforementioned methods, then we calculate the total cost according to the preliminary solution.
ii. We identify the basic variables from the non-basic variables from the preliminary solution table.
iii. We determine the indirect cost by finding closed paths, as each closed path has its beginning and end as a non-basic variable and consists of horizontal and vertical lines whose pillars are basic variables, as it happens that there are two basic variables in the way of the path, so we deviate from the basic non-basic variable and in general, the closed path represents in the following Figure 1:

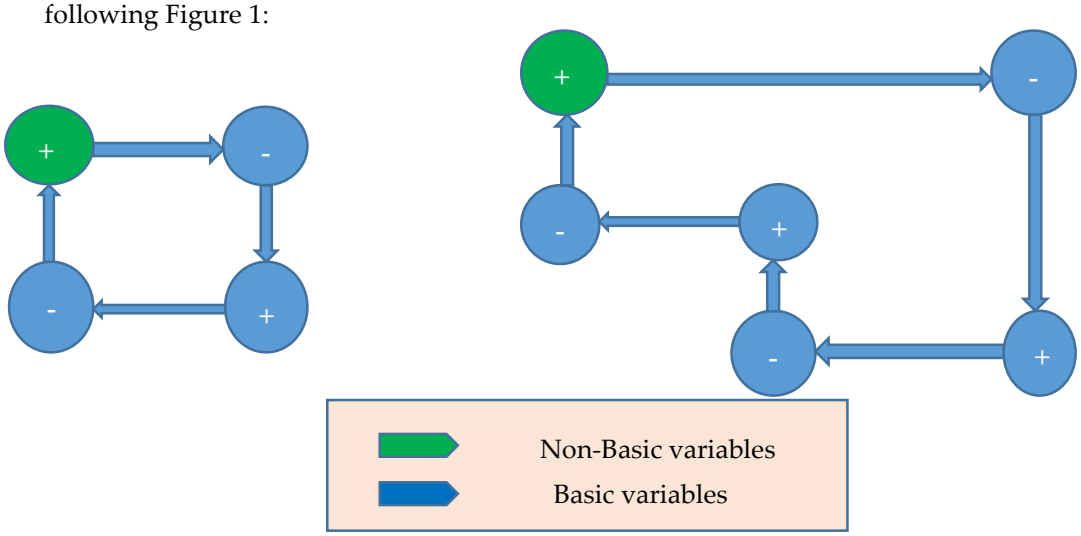

Figure 1. A representation of closed pathways.

We calculate the indirect cost of each non-basic variable by giving the cost of the non-basic variable a positive sign, and the cost of the basic variables we give it alternating negative and then positive signs, and so on. If the basic variables are positive or zero, this means that the solution that

we got is optimal and we stop. But if at least one of the indirect costs is negative, then we must develop the solution by choosing one of the non-basic variables to become basic and the exit of one of the basic variables.

Note:

To determine the basic internal variable, we take the non-basic variable that achieved the most negative in the indirect cost, and to make the solution the best possible, we try to pass in it the largest possible amount, we explain the above through the following example 1:

Example 1:

The following Table 1 represents the cost of transporting goods from sources $A_i;\ i = 1,2,3,4$ to distribution centers $B_j;\ j = 1,2,3,4$ it is required to use the mobile quarantine method to improve the solution and obtain the ideal solution**:**

Table 1. Issue data.

Consumption center / Production centers	B_1	B_2	B_3	Available Quantities
A_1	2	4	1	051
A_2	$\{3, 4.5\}$	$\{1, 2\}$	$\{5, 8\}$	211
A_3	6	2	4	525
A_4	0	7	9	25
Required quantities	081	521	211	711 / 711

In this example, the cost of transportation of the product at the production center A_2 is neutrosophic values we take in forms $c_{2j} \in \{\alpha_{1_{2j}}\alpha_{2_{2j}}\}$.

The solution:

We find the initial solution using the least cost method; we get the following preliminary solution Table 2.

Table 2. Preliminary solution.

Consumption center / Production centers	B_1	B_2	B_3	Available Quantities
A_1	2	4	1 150	**150**
A_2	$\{3,4.5\}$	$\{1,2\}$ 200	$\{5,8\}$	**022**
A_3	6 155	2 120	4 50	**325** **170** **50**
A_4	0 25	7	9	**02**
Required quantities	180 155	320 120	200 50	711 / 711

We note that the number of occupied squares is equal to $m + n - 1 = 6$

The total transportation cost according to the preliminary solution is:

$$Z_1 = 0 \times 150 + \{1,2\} \times 200 + 6 \times 155 + 2 \times 120 + 4 \times 50 + 1 \times 25$$

For $c_{22} = 1 \Rightarrow Z_1 = 1595$

For $c_{22} = 2 \Rightarrow Z_1 = 1795$

That is, against this preliminary solution, we have a neutrosophic value for the total transportation cost:

$$Z_1 \in \{1595,1795\}$$

Now we see whether this solution is an optimal solution or not?

For this we define basic variables and non-basic variables, it is clear that

The basic variables are:

$$x_{13}, x_{22}, x_{31}, x_{32}, x_{33}, x_{41}$$

The non-basic variables are:

$$x_{11}, x_{12}, x_{21}, x_{23}, x_{42}, x_{43}$$

We have six basic variables and six non-basic variables, so we get six closed paths are formed it is in Figure 2:

***Note**: The non-basic variables are green.*

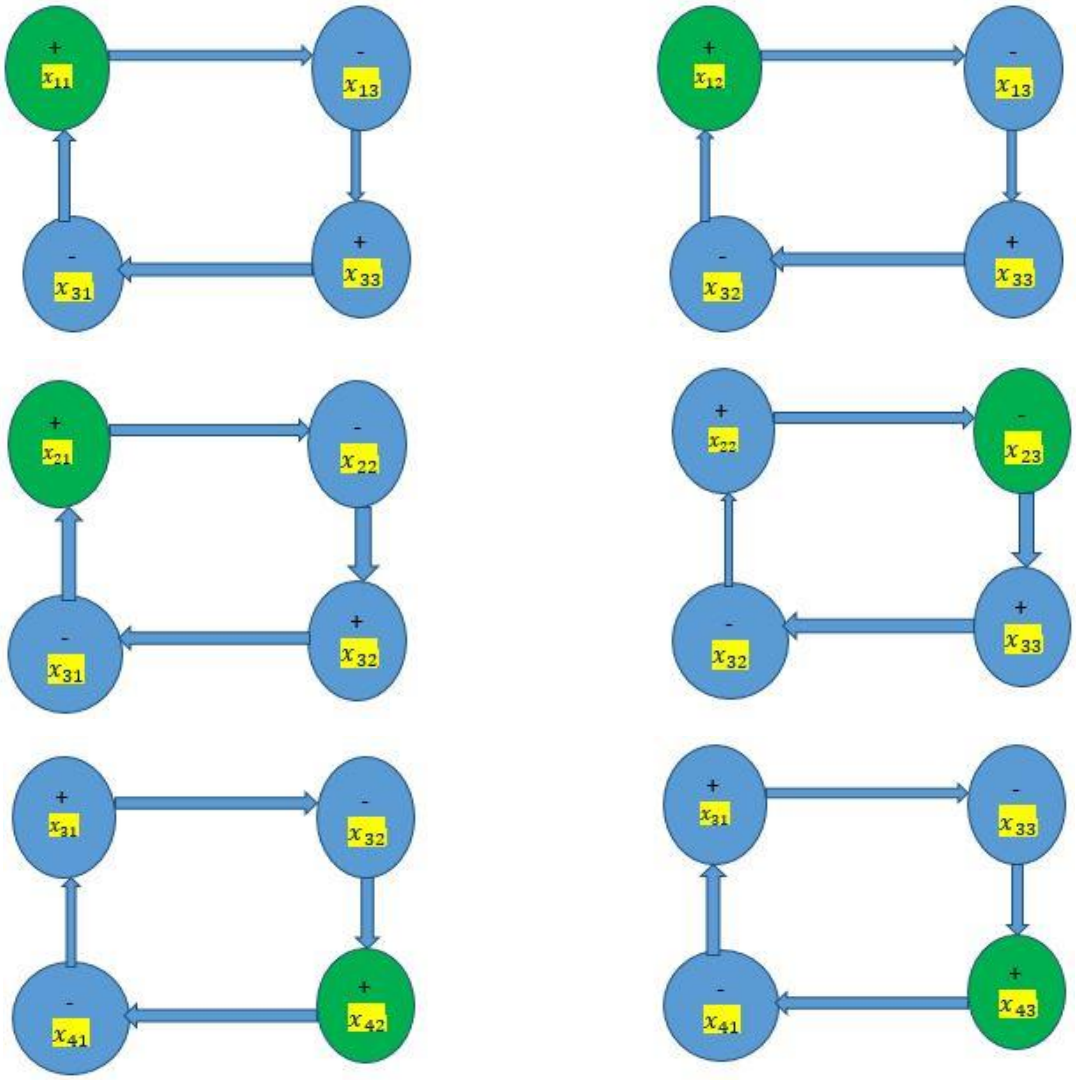

Figure 2. Possible closed paths after finding the initial solution.

The following Table 3, shows how the path is formed:

Table 3. Closed path identification.

Consumption center / Production centers	B_1	B_2	B_3	Available Quantities
A_1	2	4	1 051	051
A_2	{3,4.5}	{1,2} 211	{5,8}	211
A_3	6 055	2 021	4 51	525
A_4	0 25	7	9	25
Required quantities	081	521	211	711 / 711

We calculate the indirect cost, we find:

From the previous table and according to the drawn path, we explain how to calculate the indirect cost:

We start from the room (A_1B_1) we have the cost $c_{11} = 2$,_we take it with a positive sign because it is the room of the non-basic variable, then to the room (A_1B_3) we have the cost $c_{13} = 0$ We take here the minus sign and the variable in this room is a basic variable then to the room (A_3B_3)we have the cost $c_{33} = 4$, we take here the sign is positive, and the variable in this room is a basic variable then to the room (A_3B_1) we have the cost $c_{31} = 6$ We take here the minus sign, and the variable in this room is a basic variable, So the indirect cost of the non-essential variable is x_{11} is :

$$x_{11}: 2 - 0 + 4 - 6 = 0$$

In the same way, we calculate the cost for all non-essential variables we find:

$$x_{12} : 4 - 0 + 4 - 2 = 6$$

$$x_{21} : \{3,4.5\} - \{1,2\} + 2 - 6 = \{-2, -3, -0.5, -1,5\}$$

$$x_{23} : \{5,8\} - 4 + 2 - \{1,2\} = \{2,1,5,4\}$$

$$x_{42} : 7 - 1 + 6 - 2 = 10$$

$$x_{43} : 9 - 1 + 6 - 4 = 10$$

We note that the indirect cost corresponding to the basic variable x_{21} is a negative amount and it is the only one, so we enter this variable and it becomes one of the basic variables and we exit instead of x_{31}

We notice that we can pass the quantity $x_{21} = 155$, then it becomes:

$$x_{31} = 0, \quad x_{32} = 275 \ , \ x_{22} = 45$$

We get the following Table 4:

Table 4. The first improvement.

Consumption center / Production centers	B_1	B_2	B_3	Available Quantities
A_1	2	4	0 150	051
A_2	$\{3, 4.5\}$ 155	$\{1, 2\}$ 45	$\{5, 8\}$	211
A_3	6	2 275	4 50	525
A_4	0 25	7	9	25
Required quantities	081	521	211	711 711

We note that the transportation cost is according to the previous solution:

$$Z_2 \in (0 \times 150 + \{3{,}4.5\} \times 155 + \{1{,}2\} \times 45 + 2 \times 275 + 4 \times 50 + 1 \times 25)$$

For $c_{21} = 3 \; and \; c_{22} = 1 \Rightarrow Z_2 = 1285$

For $c_{21} = 3 \; and \; c_{22} = 2 \Rightarrow Z_2 = 1330$

For $c_{21} = 4.5 \; and \; c_{22} = 1 \Rightarrow Z_2 = 1517.5$

For $c_{21} = 4.5 \; and \; c_{22} = 2 \Rightarrow Z_2 = 1562.5$

Therefore:

$$Z_2 \in \{1285 \, , 1330 \, , 1517.5 \, , 1562.5 \, \}$$

$$\forall \, Z_2 \in \{1285 \, , 1330 \, , 1517.5 \, , 1562.5 \, \} \Longrightarrow Z_2 < Z_1 \in \{1595{,}1795\}$$

That is, this solution is better than the previous one, the question now is whether this solution is the optimal solution, for this we define the basic variables and the non-basic variables we find:

The basic variables are:

$$x_{41} \, , x_{33} \, , x_{32} \, , x_{22} \, , x_{21} \, , x_{13}$$

The non-basic variables are:

$$x_{43} \, , x_{42} \, , x_{31} \, , x_{23} \, , x_{12} \, , x_{11}$$

We have six basic variables and six non-basic variables, so we get six closed paths; we form the closed paths for the six non-basic variables in Figure 3:

Note: *The non-basic variables are green.*

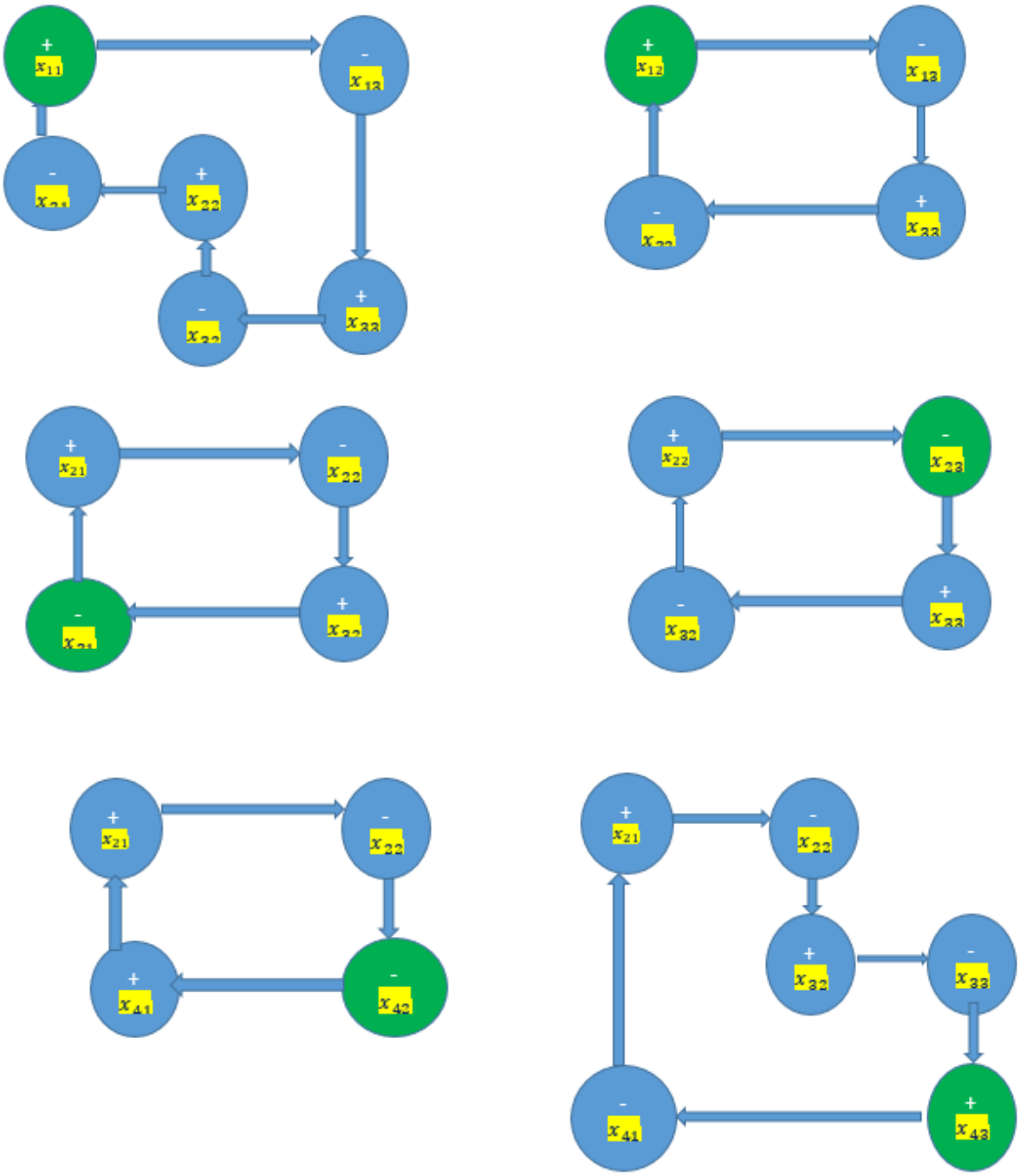

Figure 3. Possible closed paths after the first optimization.

We calculate the indirect cost:

$$x_{11} : 2 - 0 + 4 - \{1,2\} + 1 - 3 = \{3,2\}$$

$$x_{12} : 4 - 0 + 4 - 2 = 8$$

$$x_{32} : 6 - 2 + \{1,2\} - \{3,4.5\} = \{2,0.5,3,1.5\}$$

$$x_{42} : 7 - \{1,2\} + \{3,4.5\} - 1 = \{8,9.5,7,8.5\}$$

$$x_{43} : 9 - 4 + 2 - \{1,2\} + \{3,4.5\} - 1 = \{8,9.5,7,8.5\}$$

We note that the indirect cost for each non-basic variable is positive, and therefore we cannot introduce any non-basic variable to the basic rule. Therefore, the solution that we obtained in the first

improvement is an optimal vinegar, and the minimum transportation cost is the one that we obtained previously:
Therefore the optimal solution is:

$x_{13} = 150$, $x_{21} = 155$, $x_{22} = 45$, $x_{32} = 275$, $x_{33} = 50$, $x_{41} = 25$

The minimum cost of transportation is:

$$Z_2 \in \{1285 , 1330 , 1517.5 , 1562.5 \}$$

2.2 *Modified Distribution Method*

This method is another method of finding the optimal solution for transportation issues, and it is also similar to the previous method (the mobile stone method) conjunction.
To find the optimal solution to the transportation issue according to this method, we follow the following steps:

i. We find the initial solution in one of the previously mentioned ways
ii. We define the essential variables and non-basic variables for the solution
iii. We associate with each line $\boldsymbol{i}$ multiplied by $\boldsymbol{u_i}$, and with each column j multiplied, we call it v_j, so it is:

For each basic variable x_{ij} we have:

$$u_i + v_j = c_{ij} \qquad (*)$$

Where $\boldsymbol{c_{ij}}$ is the cost from A_i to B_j:
Since the number of basic variables is $m + n - 1$, we get the $m + n - 1$ equation from the previous Figure $(*)$ and by solving these equations we must find the values of u_i , v_j which have $m + n$ so we must give one of these multipliers an optional value, then we solve these equations according to this value.
After we found the values u_i , v_j , for each non-basic variable x_{ij} we calculate the quantities $c_q = c_{ij} - u_i - v_j$
In a similar way to the moving stone method, but if one of these quantities is negative, then we must introduce a non-basic variable to the set of basic variables and output instead of a basic variable, and the primary variable entered is chosen in the same previous way:

Example 2:

Let's take the previous example, where we found the preliminary solution according to the cost method:

Table 5. The preliminary solution.

Consumption center / Production centers		$\boldsymbol{v_1}$	$\boldsymbol{v_2}$	$\boldsymbol{v_3}$	Available Quantities
		B_1	B_2	B_3	
u_1	A_1	2	4	1 051	051
u_2	A_2	{3,4.5}	{1,2} 211	{5,8}	211
u_3	A_3	6 055	2 021	4 50	525
u_4	A_4	0 25	7	9	25
Required quantities		081	521	211	700 / 700

Transportation cost is: $Z_1 = 1595$

Basic variables:

$$x_{13}, x_{22}, x_{31}, x_{32}, x_{33}, x_{41}$$

Non- basic variables:

$$x_{11}, x_{12}, x_{21}, x_{23}, x_{42}, x_{43}$$

Multipliers is: $\boldsymbol{u_i}$; i = 1,2,3,4 and v_j ; $j = 1,2,3$.

For basic variables, we have:

For x_{13}, we have $u_1 + v_3 = 0$

For x_{22}, we have $u_2 + v_2 = \{1,2\}$

For x_{31}, we have $u_3 + v_1 = 6$

For x_{32}, we have $u_3 + v_2 = 2$

For x_{33}, we have $u_3 + v_3 = 4$

For x_{41}, we have $u_4 + v_1 = 1$

It is six equations with seven unknowns. To solve them, we impose $u_1 = 0$, so we find the rest of the variables:

$$u_1 = 0\,, u_2 = 3\,, u_3 = 4\,, \qquad u_4 = -1$$

$$v_1 = 2\,, v_2 = -2\,, v_3 = 0$$

For non-basic variables, we have:

$$x_{11}, x_{12}, x_{21}, x_{23}, x_{42}, x_{43}$$

For x_{11} it is: $\bar{c}_{11} = c_{11} - u_1 - v_1 = 2 - 0 - 2 = 0$

For x_{12} it is: $\bar{c}_{12} = c_{12} - u_1 - v_2 = 4 - 0 + 2 = 6$

For x_{21} it is: $\bar{c}_{21} = c_{21} - u_2 - v_1 = \{3,4.5\} - 3 - 2 = \{-2\,, -3.5\}$

For x_{23} it is: $\bar{c}_{23} = c_{23} - u_2 - v_3 = \{5,8\} - 3 - 0 = \{2\,, 5\}$

For x_{42} it is: $\bar{c}_{42} = c_{42} - u_4 - v_2 = 7 + 1 + 2 = 10$

For x_{43} it is: $\bar{c}_{43} = c_{43} - u_4 - v_3 = 9 + 1 - 0 = 10$

We note that the quantity $\bar{c}_{21} = -2$ is a negative value, and therefore the initial solution that we got is not optimal, we must develop this solution, and for that, we form the closed path for the non-basic variable x_{21}, so we find it from the Figure 4:

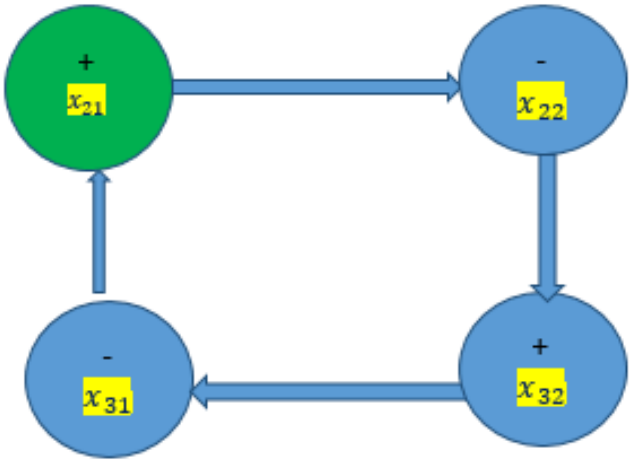

Figure 4. Possible closed pathways for the non-basic variable x_{21}.

We enter x_{21} into the set of basic variables, by giving it the value $x_{21} = 155$, and we remove the variable x_{31} so it becomes a non-basic variable, and then it becomes $x_{22} = 45$ and $x_{32} = 275$, so we get the following Table 6:

Table 6. The first improvement.

Consumption center / Production centers		v_1	v_2	v_3	Available quantities
		B_1	B_2	B_3	
u_1	A_1	2	4	1 051	051
u_2	A_2	{3,4.5} 155	{1,2} 45	{5,8}	211
u_3	A_3	6	2 275	4 50	525
u_4	A_4	0 25	7	9	25
Required quantities		081	521	211	700 / 700

We enter x_{21} into the set of basic variables, by giving it the value $x_{21} = 155$, and we remove the variable x_{31} so it becomes a non-basic variable, and then it becomes $x_{22} = 45$ and$x_{32} = 275$, so we get the following Table 6:

The new transfer cost is:

$$Z_2 \in \{1285\ , 1330\ , 1517.5\ , 1562.5\ \}$$

$$\forall\ Z_2 \in \{1285\ , 1330\ , 1517.5\ , 1562.5\ \} \Longrightarrow Z_2 < Z_1 \in \{1595, 1795\}$$

This solution is better than the previous solution, but is it the optimal solution?

For basic variables, we have:

For x_{13} we have $u_1 + v_3 = 0$

For x_{21} we have $u_2 + v_1 = 3$

For x_{22} we have $u_2 + v_2 = 1$

For x_{32} we have $u_3 + v_2 = 2$

For x_{33} we have $u_3 + v_3 = 4$

For x_{41} we have $u_4 + v_1 = 1$

We assume $u_1 = 0$ and solve the system of equations we find:

$$u_1 = 0,\ u_2 = 3,\ u_3 = 4,\ u_4 = 1$$

$$v_3 = 0,\ v_2 = -2,\ \ ,\ v_1 = 0,$$

For non-basic variables:

For x_{11} it is: $\bar{c}_{11} = c_{11} - u_1 - v_1 = 2 - 0 - 0 = 2$

For x_{12} it is: $\bar{c}_{12} = c_{12} - u_1 - v_2 = 4 - 0 + 2 = 6$

For x_{21} it is: $\bar{c}_{21} = c_{21} - u_2 - v_1 = \{3,4.5\} - 3 + 0 = \{0,1.5\}$

For x_{23} it is: $\bar{c}_{23} = c_{23} - u_2 - v_3 = \{5,8\} - 3 + 0 = \{2,5\}$

For x_{42} then: $\bar{c}_{42} = c_{42} - u_4 - v_2 = 7 - 1 + 2 = 8$

For x_{43} it is: $\bar{c}_{43} = c_{43} - u_4 - v_3 = 9 - 1 - 0 = 8$

We note that all quantities $\bar{c}_{ij}$ are positive quantities, so the solution that we got is optimal, and the minimum cost of transportation is:

$$Z_2 \in \{1285\ , 1330\ , 1517.5\ , 1562.5\ \}$$

$$\forall\ Z_2 \in \{1285\ , 1330\ , 1517.5\ , 1562.5\ \} \Longrightarrow Z_2 < Z_1 \in \{1595, 1795\}$$

Therefor the optimal solution is:

$x_{13} = 150\ ,\ x_{21} = 155\ , x_{22} = 45\ , x_{32} = 275\ ,\ x_{33} = 50\ , x_{41} = 25$

The minimum cost of transportation is:

$$Z_2 \in \{1285\ , 1330\ , 1517.5\ , 1562.5\ \}$$

3. Conclusion

In many practical issues, we encounter cases in which we are unable to provide confirmed data on the reality of the state of the system under study; the data is affected by circumstances surrounding the working environment, and this matter affects the future and may cause large losses. In the example that was presented in this research, we noticed the study gave us a neutrosophic transport cost suitable for all conditions because it was obtained through neutrosophic data.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Maissam Jdid, AA Salama, Huda E Khalid ,Neutrosophic Handling of the Simplex Direct Algorithm to Define the Optimal Solution in Linear Programming ,International Journal of Neutrosophic Science, Vol.18,No. 1, 2022
2. Maissam Jdid, Huda E Khalid, Mysterious Neutrosophic Linear Models, International Journal of Neutrosophic Science, Vol.18, No. 2, 2022
3. Maissam Jdid, Huda E. Khalid, Neutrosophic Mathematical formulas of Transportation Problems, Neutrosophic sets and Systems, NSS, Vol .51,2022
4. Maissam Jdid, Huda E Khalid, An Investigation in the Initial Solution for Neutrosophic Transportation Problems (NTP), Neutrosophic sets and Systems NSS, Vol.50,2022
5. Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Applied Mathematics and Data Analysis, Vo .2, No .1, 2023
6. Maissam Jdid, Neutrosophic Nonlinear Models, Journal Prospects for Applied Mathematics and Data Analysis, Vo .2, No .1, 2023
7. Florentin Smarandache, Victor Christianto, A few little steps beyond Knuth's Boolean Logic Table with Neutrosophic Logic: A Paradigm Shift in Uncertain Computation, Vol 2, No 2, 2023
8. Aslam, M. (2019). Control Chart for Variance using Repetitive Sampling under Neutrosophic Statistical Interval System, IEEE Access, 7 (1), 25253-25262.
9. Maissam Jdid, Neutrosophic Mathematical Model of Product Mixture Problem Using Binary Integer Mutant, Journal of Neutrosophic and Fuzzy Systems (JNFS), Vo .6, No .2, 2023
10. Maissam Jdid, Florentin Smarandache, Said Broumi, Inspection Assignment Form for Product Quality Control, Journal of Neutrosophic Systems with Applications, Vol. 1, 2023

11. Maissam Jdid, Said Broumi, Neutrosophic Rejection and Acceptance Method for the Generation of Random Variables, Neutrosophic Sets and Systems, NSS, Vol.56,2023
12. Maissam Jdid, Florentin Smarandache, The Use of Neutrosophic Methods of Operation Research in the Management of Corporate Work, Journal of Neutrosophic Systems with Applications, Vol. 3, 2023
13. Maissam Jdid, Florentin Smarandache, Lagrange Multipliers and Neutrosophic Nonlinear Programming Problems Constrained by Equality Constraints, Journal of Neutrosophic Systems with Applications, Vol. 6, 2023
14. Maissam Jdid, Florentin Smarandache, Neutrosophic Treatment of Duality Linear Models and the Binary Simplex Algorithm, Applied Mathematics and Data Analysis, Vo .2, No .2, 2023
15. Maissam Jdid, Florentin Smarandache, Neutrosophic Treatment of the Modified Simplex Algorithm to find the Optimal Solution for Linear Models, International Journal of Neutrosophic Science (IJNS), Vol. 23, No.1,2023
16. Maissam Jdid, NEUTROSOPHIC TRANSPORT AND ASSIGNMENT ISSUES, Publisher: Global Knowledge's: 978_0 _59975_771_6, (Arabic version).
17. Al Hamid. Mohammed Dabbas, Mathematical programming, Aleppo University, Syria, 2010. (Arabic version).
18. Bukajah J.S., Mualla, W... and others - Operations Research Book translated into Arabic - The Arab Center for Arabization, Translation, Authoring and Publishing -Damascus -1998. (Arabic version).
19. Alali. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004. (Arabic version)
20. Maissam Jdid, Operations Research, Published in ASPU, faculty of Information Technology, 2021.

An Innovative Approach on Yao's Three-Way Decision Model Using Intuitionistic Fuzzy Sets for Medical Diagnosis

Wajid Ali,[1] **Tanzeela Shaheen,**[1] **Iftikhar Ul Haq,**[1] **Florentin Smarandache,**[2]
Hamza Ghazanfar Toor,[3] **Faiza Asif**[4]

[1] Department of Mathematics, Air University, E-9, Islamabad 44000, Pakistan. Emails: wajidali00258@gmail.com, tanzeela.shaheen@mail.au.edu.pk, chibmathteacher@gmail.com.
[2] University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu.
[3] Biomedical Engineering Department, Riphah International University, Islamabad 44000, Pakistan; hamza.ghazanfar@riphah.edu.pk.
[4] Management and Finance Department, Leeds Beckett University Leeds United Kingdom; f.asif2456@student.leedsbeckett.ac.uk.

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Abstract: In the realm of medical diagnosis, intuitionistic fuzzy data serves as a valuable tool for representing information that is uncertain and imprecise. Nevertheless, decision-making based on this kind of knowledge can be quite challenging due to the inherent vagueness of the data. To address this issue, we employ power aggregation operators, which prove effective in combining several sources of data, such as expert thoughts and patient information. This allows for a more correct diagnosis; a particularly crucial aspect of medical practice where precise and timely diagnoses can significantly impact medication policy and patient results. In our research, we introduce a novel methodology to the three-way decision idea. Initially, we revamp the three-way decision model using rough set theory and incorporate interval-valued classes to handle intuitionistic fuzzy data. Secondly, we explore the use of intuitionistic fuzzy power weighted and intuitionistic fuzzy power weighted geometric aggregation operators to consolidate attribute values within the data system. Furthermore, we present a case study in the medical field to exhibit the validity and efficiency of our offered technique. This innovative method enables us to classify participants into three distinct zones based on their symptoms. The manuscript concludes with a summary of key points provided by the authors.

Keywords: Intuitionistic Fuzzy Sets; Aggregation Operators; Information System; Three-Way Decision; Medical Diagnosis; Decision Making; Optimization.

1. Introduction

Medical conditions can manifest with a variety of symptoms, which can complicate the process of accurate diagnosis. Some typical symptoms include elevated body temperature, fatigue, coughing, nausea, diarrhea, and skin eruptions. The identification of medical illnesses typically necessitates the involvement of healthcare professionals, who rely on a combination of patient history, physical assessments, laboratory tests, medical imaging, and other diagnostic measures to establish an analysis. This decision-making [1] process in medical diagnosis is influenced by numerous features, involving the patient's age, medical history, and genetic susceptibility. Physicians may also utilize diagnostic algorithms or decision trees to aid in the assessment of intricate medical conditions. In recent times, advancements in medical technology have considerably enhanced the precision and speed of medical diagnoses. This includes the integration of new imaging methods, genetic testing, and decision-making tools based on artificial intelligence [2-5].

The rough set model is a mathematical framework designed to systematically address situations involving incomplete and uncertain information. Zdzislaw Pawlak [6] first introduced this theory in

the early 1980s to handle vague and doubtful information. In the context of medical diagnosis, a rough set model proves valuable in determining the presence or absence of specific diseases or conditions, even when dealing with incomplete or uncertain data sources [7], such as symptoms, medical history, test results, and other related information [8]. The core principle following the rough set model revolves around the segmentation of information into subsets, primarily driven by their attributes, such as symptoms or test outcomes. This partitioning procedure aids in the discernment of the critical features or factors most closely linked to a specific ailment or medical condition. Once the data is organized into these distinct subsets, rough set theory can be leveraged to unveil rules or shapes that facilitate making predictions about whether a given patient is afflicted by an actual disease or condition. A multitude of scientists have contributed to the advancement of innovative algorithms for disease diagnosis using this methodology. For instance, El-Bably and colleagues [8-10] started the concept of soft, rough approximation and implemented it in the realm of medical difficulty diagnosis. Hosny et al. [11] expanded the application of rough sets by introducing the maximal right neighborhood system and exploring its uses in the field of medicine. Additionally, Al-Shami et al. [12] defined maximal rough neighborhoods and employed this approach for the diagnosis of medical conditions.

Atanassov [13] established the theory of an intuitionistic fuzzy set (IFS), which represents an expansion of the conventional fuzzy set (FS). Within the realm of IFS, one can express both the degree of membership and non-membership of an element within a universal set. IFSs hold significant importance in the field of medicine, particularly in the context of disease identification and problem-solving. Researchers have extensively investigated the utilization of IFS in medical diagnosis, especially in scenarios characterized by substantial hesitation and flexibility in symptoms and test outcomes. The intuitionistic fuzzy set offers a valuable tool for capturing and conveying this uncertainty, thus enhancing the precision of diagnostic knowledge. For instance, in the context of diagnosing complex conditions such as cancer, Intuitionistic Fuzzy Sets (IFS) can portray the degree of confidence or vagueness related to the diagnosis, considering a wide range of diagnostic criteria like blood test results, imaging studies, and biopsy conclusions. This approach ultimately leads to more precise and dependable diagnoses while also facilitating the creation of personalized treatment plans. Jiang et al. [14] employed IFS in medical image fusion, utilizing entropy measures, whereas Mehmood et al. [15, 16] extended the concept of intuitionistic fuzzy sets and applied these principles to the field of medical diagnosis. De et al. [17] also delved into the application of IFS in medical diagnosis, much like the work of Davvaz et al. [18]. Szmidt et al. [19] investigated the utilization of IFS in intelligent data analysis for medical diagnosis. In the decision-making process involving IFS, aggregation operators play a crucial role in computing attribute values. As a result, several experts have proposed a variety of aggregation operators for this purpose. For instance, Xu et al. [20] developed and implemented power aggregation operators for IFS in Multi-Attribute Decision Making (MADM). In 2006, Xu et al. [21] introduced geometric aggregation operators tailored for IFS. Mehmood et al. [22] presented similarity measures and power aggregation operators based on Intuitionistic Hesitant Fuzzy Sets (IHFS). More recently, Senapati et al. [23] and Garg et al. [24] have explored novel operators in this context.

The three-way decision (TWD) concept represents a notable extension of the RS theory, initially introduced by Yao [25, 26]. In the realm of medical diagnosis, a three-way decision entails the assessment of three potential outcomes: positive, negative, or inconclusive. In the case of a Positive Outcome, when a medical diagnosis yields a positive result, it confirms that the patient indeed has the specific condition or disease under examination. This necessitates treatment for the diagnosed ailment, with healthcare professionals closely monitoring the patient's progress. Conversely, a Negative Outcome in a medical diagnosis indicates that the patient does not have the particular condition or disease under investigation. In such instances, the patient may not need any medication, and healthcare providers may need to explore other potential causes for the patient's symptoms. An

Inconclusive Outcome arises when the test results do not provide sufficient clarity to decide whether the patient possesses the condition or disease under scrutiny. In such situations, additional tests or evaluations may be necessary to succeed at a more definitive diagnosis. Lately, Li et al. [27, 28] applied TWD procedures to improve decision-making in medical diagnosis. Hu et al. [29, 30] introduced the notion of a lattice model for medical diagnosis, incorporating TWD. Jia and Fan [31] devised TWD models for multi-criteria situations, while Ye et al. [32] integrated the TWD concept into the emerging field of fuzzy information systems. In a similar vein, numerous scholars have explored this field, proposing innovative approaches across various extensions of fuzzy sets [33-35].

In our exploration of the literature, we discovered that TWD models prove highly beneficial for medical problem diagnosis. The fusion of IFS and TWD, as described in reference [36], yields a robust framework for addressing situations characterized by vagueness and ambiguity. It should be highlighted that aggregating the outcomes of numerous participants through TWD poses a significant challenge. Researchers have traditionally employed conventional methods to compute alternatives for TWD, as evidenced by references [37-40]. In the existing TWD model, as outlined in references [25, 37], an external concept becomes necessary for determining equivalence classes. Additionally, a threshold is employed to categorize the alternatives into three distinct regions.

The primary objective behind creating this piece of work is to create an innovative algorithm for the TWD by utilizing aggregation operators and enhancing the TWD decision process through interval-valued equivalence classes for Interval-Valued Fuzzy Sets (IVFS). This approach aims to address the existing deficiencies and challenges in TWD computation. The following is a depiction of the key contribution made by this analysis.

i. Establish the notion of intervals to represent the degrees of membership in Interval Fuzzy Sets (IFS) by utilizing the step size function.
ii. Create equivalence classes by leveraging intervals and refer to them as interval-valued classes.
iii. To address concerns related to computational efficiency and timesaving, we introduce the IFPWA and IFPWG aggregation operators specifically tailored for the TWD theory.
iv. Present an algorithm designed for the classification of diverse patients and the diagnosis of diseases using multiple symptom criteria.

The rest of the article follows this structure: Section 2 presents an overview of essential concepts, including IFS, power aggregation operators, and Three-way Decisions in Section 3, we define membership grade intervals using the step size function and create equivalence classes. We then proceed to adapt the TWD for IFS based on these intervals. Section 4 encompasses the development of a well-defined algorithm, including a flow chart, and an in-depth, step-by-step explanation of the approach. Section 5 delves into a case study where we apply the offered methodology to diagnose a medical issue and classify alternatives using power aggregation operators for IFS. We also extensively discuss the benefits and benefits of the suggested models. Finally, Section 6 encapsulates the authors' conclusion and their plan.

2. Preliminaries

In this section, we take a closer look at several fundamental concepts within intuitionistic fuzzy sets (IFS) and explore some notions about power aggregation operators.

2.1 Intuitionistic Fuzzy Sets and Aggregation Operators

Atanassov [13] proposed the theory of IFS as an expansion of FS. While FS offers the membership grade (MG) of an element within a specific set [0, 1], IFS simultaneously provides both MG and non-membership grade (NMG).

Definition 1: [13] An IFS T on set E is represented using the two mappings $l(e)$ and $m(e)$. Mathematically, this representation is expressed through the following structure:

$$T = \langle e, l_T(e), m_T(e)) | e \in E \rangle,$$

Where, $l_T(e): E \to [0,1]$ and $m_T(e): E \to [0,1]$ signifies the MG and NMG including the condition $0 \leq l(e) + m(e) \leq 1$, for each $e \in E$.

Definition 2: let $T = (l_T, m_T)$ be an IFN, then the score function and accuracy function are stated and represented as:

$$S(T) = l_T - m_T, \qquad S(T) \in [-1,1];$$
$$H(T) = l_T + m_T, \qquad H(T) \in [0,1].$$

Definition 3: Suppose $T_1 = (l_1,\ m_1),\ T_2 = (l_2,\ m_2)$ be intuitionistic fuzzy sets (IFSs), then some basic operations are described as below:

i. $T_1 \oplus T_2 = (\{l_1 + l_2 - l_1 l_2\}, \{m_1 m_2\});$
ii. $T_1 \otimes T_2 = (\{l_1 l_2\}, \{m_1 + m_2 - m_1 m_2\});$
iii. $\lambda T_1 = \left(1 - (1-l)^{\lambda}, m^{\lambda}\right), \lambda > 0;$
iv. $T_1^{\lambda} = \left((l)^{\lambda}, 1 - (1-m)^{\lambda}\right), \lambda > 0;$
v. $T_1^{c} = (m_1, l_1).$

Definition 4: [21] Suppose that $T_j = \left(l_j, m_j\right)$ is a collection of IFS and the $k_j = (k_1, k_2, \dots, k_m)^T$ is weight vector for T_j, and $\sum_{j=1}^{m} k_j = 1$. Then $IFPWP_k$ an operator is a mapping $IFPWP_k: T^m \to T$ where

$$IFPWP_k(T_1, T_2, \dots, T_m) = \frac{\oplus_{j=1}^{m}(k_j(1 + J(T_j)T_j)}{\sum_{j=1}^{m} k_j(1 + J(T_j))} =$$
$$\left(1 - \prod_{j=1}^{m}(1 - (l_j)^{\frac{k_j((1+J(T_j))}{\sum_{j=1}^{m} k_j(1+J(T_j))}}, \prod_{j=1}^{m}(m_j)^{\frac{k_j(1+J(T_j))}{\sum_{j=1}^{m} k_j(1+TJT_j))}}\right),$$

Where,

$$J(T_j) = \sum_{\substack{i=1 \\ i \neq j}}^{m} k_j Sup(T_j, T_i),$$
$$Sup(T_j, T_i) = 1 - d(T_j, T_i),$$
$$d(T_j, T_i) = \frac{1}{l}\sum_{\substack{i=1 \\ i \neq j}}^{l}\left(|l_i - l_j| + |m_i - m_j|\right).$$

Definition 5: For IFSs $T_i = (l_i, m_i)$ with k_j such that $k_j > 0$ and $\sum_{j=1}^{m} k_j = 1$. A mapping $IFPOWP_k$: $T^m \to T$, is stated as:

$$IFPOWP_k(T_1, T_2, \dots, T_m) = \frac{\oplus_{j=1}^{m}(k_j(1 + J(T_{\sigma(j)})T_{\sigma(j)})}{\sum_{j=1}^{m} k_j(1 + J(T_{\sigma(j)}))}$$
$$= \left(1 - \prod_{j=1}^{m}(1 - (l_{\sigma(j)})^{\frac{(k_j(1+J(T_{\sigma(j)}))}{\sum_{j=1}^{m} k_j(1+J(T_{\sigma(j)}))}}, \prod_{j=1}^{m}(m_{\sigma(j)})^{\frac{(k_j(1+J(T_{\sigma(j)}))}{\sum_{j=1}^{m} k_j(1+J(T_{\sigma(j)}))}}\right).$$

Definition 6: For a set of IFS $T_j = \left(l_j, m_j\right)$ and the weights k_j for T_j, and $\sum_{j=1}^{m} k_j = 1$. Then $IFPWG_k$ an operator is a mapping $IFPWG_k: T^m \to T$.

$$IFPWG_k(T_1, T_2, \dots, T_m) = \frac{\otimes_{j=1}^{m}(k_j(1 + J(T_j)T_j)}{\sum_{j=1}^{m} k_j(1 + J(T_j))} =$$
$$\left(\prod_{j=1}^{m}(l_j)^{\frac{k_j\left(1+J(T_j)\right)}{\sum_{j=1}^{m} k_j(1+J(T_j))}}, 1 - \prod_{j=1}^{m}(1 - (m_j)^{\frac{k_j((1+J(T_j))}{\sum_{j=1}^{m} k_j(1+J(T_j))}}\right).$$

Where, $J(T_j) = \sum_{\substack{i=1 \\ i \neq j}}^{m} k_j Sup(T_j, T_i).$

Definition 7: For IFSs $T_i = (l_i, m_i)$ with their weights k_j such that $k_j > 0$ and $\sum_{j=1}^{m} k_j = 1$. A mapping $IFPOWG_k$: $T^m \to T$, is expressed as:

$$IFPOWG_k(T_1, T_2, \dots, T_m) = \frac{\otimes_{j=1}^{m}(k_j(1 + J(T_{\sigma(j)})T_{\sigma(j)})}{\sum_{j=1}^{m} k_j(1 + J(T_{\sigma(j)}))}$$
$$= \left(\prod_{j=1}^{m}(l_{\sigma(j)})^{\frac{(k_j(1+J(T_{\sigma(j)}))}{\sum_{j=1}^{m} k_j(1+J(T_{\sigma(j)}))}}, 1 - \prod_{j=1}^{m}(1 - (m_{\sigma(j)})^{\frac{(k_j(1+J(T_{\sigma(j)}))}{\sum_{j=1}^{m} k_j(1+J(T_{\sigma(j)}))}}\right).$$

2.2 Three-Way Decision Model Based on Rough Sets

Rough sets [6] constitute a mathematical theory that was formulated by Zdzislaw Pawlak, a Polish computer scientist, during the early 1980s. This notion deals with a structured methodology for addressing uncertainty and handling incomplete information within the realm of data analysis. The theory of three-way decision (TWD) [25] is an expansion of a rough set model, designed to accommodate the notion of "don't know" or "undetermined" elements in decision-making processes. In the traditional two-way decision-making framework, data is typically categorized into two sets: one that fulfills specific conditions and another that does not. The primary constraints associated with this approach are detailed below.

Definition 8: [38] Let $\mathcal{S} = (E, At, V_a, f)$ be an information system (IS), where $E = \{P_1, P_2, \dots P_m\}$ is the universe of discourse. $At = \{c_1, c_2, \dots, c_l\}$ is the set of the attributes, $V_a = \bigcup_{c \in Pt} \mathbb{V}_c$ is the range of values, $\mathbb{V}_c$ represents the value under attribute c, and $f = E \times At \to V_a$ is an information mapping function.

Furthermore, within these IS, two different categories of attributes exist condition attributes (C) and decision attributes (D), which collectively form the set of attributes denoted as $At = C \cup D$. These information systems are occasionally referred to as decision IS. To collect the parts in E based on the features in Pt, equivalence classes are constructed in rough sets (RSs). In this context, equivalence classes of the relations $BND(C)$ and $BND(D)$ will be called condition and decision classes, respectively.

Definition 9: Let $\mathcal{S} = (E, At, V_a, f)$ be an IS, and B is, $B \subseteq At$, an equivalence relation R is defined as:

$$R_B = \left\{ \left(P_{c_i}, P_{c_j} \right) \in E \times E | \, for \; all \; c \in B \; \left(P_{c_i} = P_{c_j} \right) \right\}.$$

With this relationship, the equivalence class of an element P_B in set E is established as follows,

$$\left[P_{c_i} \right]_B = \left\{ P_{c_j} \in E \mid \left(P_{c_i}, P_{c_i} \right) \in R_B \right\}$$

The main objective of an equivalence relation is to show the inability to distinguish objects. Using the equivalence relation denoted as R_B, the IS can be separated into three different segments by approximation classes.

Definition 10: Pawlak [6] started the concept of approximation classes for the approximation space $Appr(E, R)$ of E, defined for all $U \subseteq E$ as follows:

$$\underline{Appr}(U) = \{P \in E \mid [P]_B \subseteq U\},$$
$$\overline{Appr}(U) = \{P \in E \mid [P]_B \cap U \neq \emptyset\},$$

These categories are referred to as the lower approximation class denoted as $\underline{Appr}(U)$ and the upper approximation class indicated as $\overline{Appr}(U)$ with $[P]_B$ representing the equivalence class of P.

Definition 11: Using the classification of approximations, three distinct regions are defined in the following manner.

$$POS(U) = \underline{Appr}(U),$$
$$NEG(U) = E - \overline{Appr}(U),$$
$$BND(U) = \overline{Appr}(U) - \underline{Appr}(U).$$

Definition 12: Let $\mathcal{S} = (E, At, V_a, f)$ be an IS and a subset of attributes $B \subseteq At$, then the decision rules of $U \subseteq E$ and $z \in U$ are designed as:

$(\mathcal{A})$ If $q \vDash Des([P]_B) \; for \; [P]_B \in POS(U)$, then accept q,

$(\mathcal{R})$ If $q \vDash Des([P]_B) \; for \; [P]_B \in NEG(U)$, then reject q,

$(\mathcal{N})$ If $q \vDash Des([P]_B) \; for \; [P]_B \in BND(U)$, then neither accept nor reject q.

3. A Novel Three-Way Decision Model Based on Interval-Valued Classes

In this portion, we introduce an innovative approach to model TWD by creating intervals. These intervals lead to the creation of unique sets of interval-valued equivalence classes, which, in turn, are

used to categorize participants into three different regions: POS (positive), NEG (negative), and BND (boundary), facilitating their classification.

To transform the information system into a discrete form, we replace traditional equivalence classes with interval-valued equivalence classes, guided by the step size function. This function assists in dividing the alternatives into intervals, and its definition is as follows.

Definition 13: For the collection of IFNs (T_i), the intervals $(\mathcal{I}_m)$ for approximation classes established on MGs are defined and denoted as:

$$\mathcal{I}_m = [Mim(l_i), Mim(l_i) + h]$$

When the step size function (h) is established for the MGs of IFNs, it is defined by:

$$h = \frac{Max(l_i) - Mim(l_i)}{m}$$

Where m is the number of intervals $\mathcal{I}_m$ which we required.

According to the parental concept of TWD by Yao [25], by using the equivalence classes, we can provide the approximation classes. Continually, by the definition of intervals $\mathcal{I}_n$ in definition 13, nth interval-valued equivalence classes $[P]_B$ for the participants are developed as below:

Definition 14: The design of interval-valued equivalence classes $[P]_B$ for the alternatives P_i is structured such that:

$$[P]_B = \{P: P_i \in \mathcal{I}_m\}$$

Definition 15: The approximation classes within the approximation space $Appr\ (E, R)$ for all $U \subseteq E$, as defined by:

$$\underline{Appr}(U) = \{P \in E | [P]_B \subseteq U\},$$
$$\overline{Appr}(U) = \{P \in E | [P]_B \cap\ U \neq \emptyset\}.$$

Definition 16: Using the approximation classes outlined in definition 15, we can introduce three distinct regions as below.

$$POS(U) = \underline{Appr}(U),$$
$$NEG(U) = E - \overline{Appr}(U),$$
$$BND(U) = \overline{Appr}(U) - \underline{Appr}(U).$$

Definition 17: The three kinds of decision rules $(\mathcal{A}2 - \mathcal{N}2)$ of $U \subseteq E$ for an IS $\mathcal{S} = (E, At, V_a, f)$ are described as:

$(\mathcal{A}2)$ If $q \vDash Des([P]_B) for\ [P]_B \in POS(U)$, then accept q,

$(\mathcal{R}2)$ If $q \vDash Des([P]_B) for\ [P]_B \in NEG(U)$, then reject q,

$(\mathcal{N}2)$ If $q \vDash Des([P]_B) for\ [P]_B \in BND(U)$, then neither accept nor reject q.

4. An Algorithm for the Proposed Model

This portion delves into the detailed application of $IFPWP_k$ and $IFPWG_k$ aggregation operators under IF information for three-way decision-making. We plan five stages for choosing the TWD rules for distinct partakers. Let $\mathrm{E} = \{P_1, P_2 \ldots P_n\}$ represent the set of participants and consider $U = \{Yes, No\}$ denote the set of states indicating the decisions of participants, where $U \subseteq E$. The flowchart of the TWD model is displayed in Figure 1.

Step 1. Assess the data system with conditional and decision attributes using an intuitionistic fuzzy approach.

Step 2. For participants $P_i (i = 1,2, \ldots, m)$ aggregate all the IF attributes $P_{ij} (j = 1,2, \ldots, l)$ into a general result P_i utilizing $IFPWA_{\mathrm{k}}$ and $IFPWG_k$operators as below:

$$IFPWA_k(T_1, T_2, \ldots, T_m) = \frac{\oplus_{j=1}^{m}(k_j(1 + J(T_j)T_j)}{\sum_{j=1}^{m} k_j(1 + J(T_j))} =$$
$$\left(1 - \prod_{j=1}^{m}(1 - (l_j)^{\frac{k_j((1+J(T_j))}{\sum_{j=1}^{m} k_j(1+J(T_j))}}, \prod_{j=1}^{m}(m_j)^{\frac{k_j(1+J(T_j))}{\sum_{j=1}^{m} k_j(1+J(T_j))}}\right),$$

and

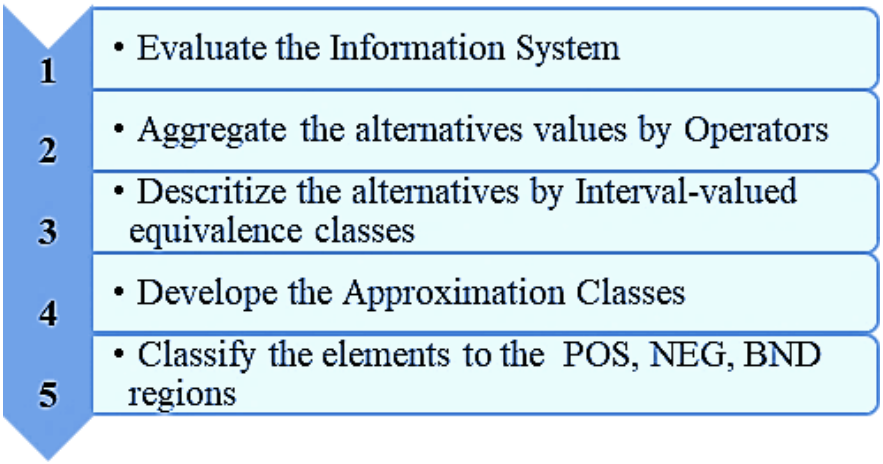
1
• Evaluate the Information System
2
• Aggregate the alternatives values by Operators
3
• Descritize the alternatives by Interval-valued equivalence classes
4
• Develope the Approximation Classes
5
• Classify the elements to the POS, NEG, BND regions

Figure 1. Five steps flow chart of interval-valued TWD model.

$$IFPWG_k(T_1, T_2, \dots, T_m) = \frac{\otimes_{j=1}^{m} (k_j(1 + J(T_j)T_j)}{\sum_{j=1}^{m} k_j(1 + J(T_j))} =$$

$$\left(\prod_{j=1}^{m} (l_j)^{\frac{k_j(1+J(T_j))}{\sum_{j=1}^{m} k_j(1+J(T_j))}}, 1 - \prod_{j=1}^{m} (1 - (m_j)^{\frac{k_j((1+J(T_j))}{\sum_{j=1}^{m} k_j(1+J(T_j))}} \right),$$

Step 3. Determine the interval-valued equivalence classes based on the interval specified in Definition 13.

Step 4. Discretize the upper approximation class and lower approximation class defined in Definition 15.

Step 5. Categorize the options into the POS, NEG, and BND regions based on their approximate classes.

5. Mathematical Model

Now, we offer an illustrative case that serves as a practical example for making decisions regarding the investigation of medical issues, with a focus on confirming or ruling out diseases in patients.

5.1 Explanation of the Problem

Medical diagnosis involves identifying the specific illness or condition that matches a person's symptoms. Healthcare professionals strive to make precise determinations by evaluating a patient's symptoms. It's a process where doctors select a particular disease based on the symptoms exhibited by an individual. The use of IFRS aids healthcare experts in handling complex linguistic concepts and minimizes inaccuracies. The effectiveness of IFRS in medical diagnosis is demonstrated in references [17, 28]. Figure 2 provides a visual representation of the medical diagnosis procedure.

Suppose that a collection of alternatives denoted as P_i (where $i = 1,2, \dots,15$), participates in the investigation to diagnose the disease "Coronavirus."

Additionally, let I be the group of conditional attributes, specifically $B = \{B_1\ (Chest\ paim), B_2\ (Fever), B_3(Fatigue), B_4(Cough),\}$. Furthermore, the set U, represented as $U = \{P_1, P_2, P_4, P_{15}, P_{11}\}$, indicates the decision attributes that provide the concept of "Yes" for the disease. Let the experts diagnose the disease for all participants and classify their decisions using the weighty vector $k = \{0.2, 0.3, 0.4, 0.1\}$. We will now employ a step-by-step algorithm to provide a detailed explanation of this medical condition.

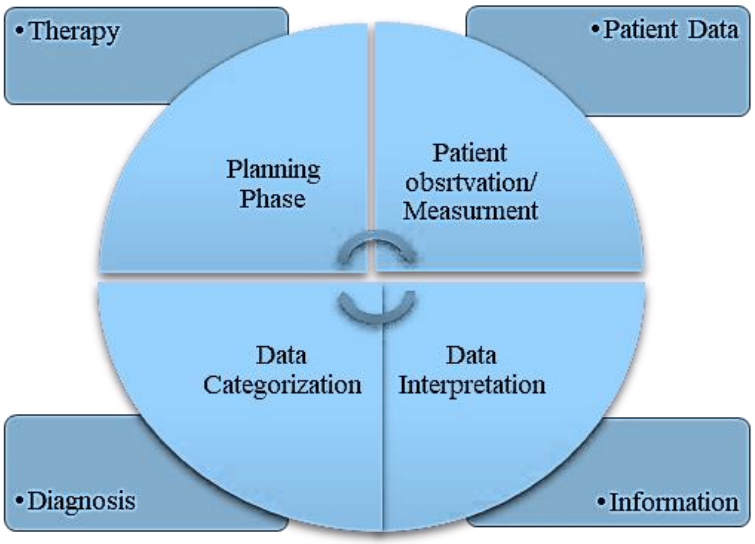

Figure 2. Medical diagnosis diagram.

Step 1: Table 1 represents the IF data of patients.

Table 1. An information Table of alternative conditions.

At	B_1	B_2	B_3	B_4	D
P_1	(0.1,0.3)	(0.4,0.5)	(0.1,0.5)	(0.1,0.5)	Yes
P_2	(0.4,0.5)	(0.5,0.4)	(0.5,0.3)	(0.2,0.6)	Yes
P_3	(0.2,0.3)	(0.2,0.4)	(0.6,0.2)	(0.4,0.5)	No
P_4	(0.4,0.2)	(0.1,0.2)	(0.7,0.4)	(0.3,0.1)	Yes
P_5	(0.5,0.3)	(0.5,0.2)	(0.3,0.2)	(0.4,0.2)	No
P_6	(0.6,0.2)	(0.7,0.1)	(0.4,0.1)	(0.4,0.4)	No
P_7	(0.7,0.1)	(0.2,0.2)	(0.5,0.2)	(0.5,0.2)	No
P_8	(0.3,0.4)	(0.3,0.3)	(0.6,0.2)	(0.2,0.3)	No
P_9	(0.4,0.2)	(0.5,0.2)	(0.7,0.2)	(0.3,0.5)	No
P_{10}	(0.5,0.2)	(0.8,0.1)	(0.2,0.3)	(0.4,0.3)	No
P_{11}	(0.6,0.2)	(0.9,0.1)	(0.5,0.3)	(0.5,0.4)	Yes
P_{12}	(0.8,0.1)	(0.0,0.9)	(0.6,0.4)	(0.2,0.2)	No
P_{13}	(0.9,0.1)	(0.3,0.2)	(0.4,0.3)	(0.4,0.3)	No
P_{14}	(0.1,0.2)	(0.2,0.2)	(0.6,0.3)	(0.3,0.4)	No
P_{15}	(0.8,0.1)	(0.1,0.3)	(0.3,0.4)	(0.4,0.2)	Yes

Step 2. For participants P_i calculate all the conditional attributes information applying $IFPWA_{\mathrm{k}}$ and $IFPWG_{\mathrm{k}}$ operators in the following:

$$IFPWA_k(T_1, T_2, \dots, T_m) = \frac{\oplus_{i=1}^{m}(k_j(1+J(T_i)T_i)}{\sum_{i=1}^{m} k_j(1+J(T_i))} =$$

$$\left(1 - \prod_{i=1}^{m}(1-(l_i)^{\frac{k_j((1+J(T_i))}{\sum_{i=1}^{m} k_j(1+J(T_i))}}, \prod_{i=1}^{m}(m_i)^{\frac{k_j(1+J(T_i))}{\sum_{i=1}^{m} k_j(1+J(T_i))}}\right),$$

and

$$IFPWG_k(T_1, T_2, \dots, T_m) = \frac{\otimes_{i=1}^{m}(k_j(1+J(T_i)T_i)}{\sum_{i=1}^{m} k_j(1+J(T_i))} =$$

$$\left(\prod_{i=1}^{m}(l_i)^{\frac{k_j(1+J(T_i))}{\sum_{j=1}^{m}k_j(1+J(T_i))}}, 1-\prod_{i=1}^{m}(1-(m_i)^{\frac{k_j((1+J(T_i))}{\sum_{i=1}^{m}k_j(1+J(T_i))}}\right).$$

The results are presented in Table 2.

Table 2. Aggregated results of all alternatives.

At	$\boldsymbol{IFPWA_k}$	$\boldsymbol{IFPWG_k}$
$\boldsymbol{P_1}$	$(0.203, 0.458)$	$(0.151, 0.470)$
$\boldsymbol{P_2}$	$(0.466, 0.374)$	$(0.450, 0.393)$
$\boldsymbol{P_3}$	$(0.429, 0.281)$	$(0.347, 0.306)$
$\boldsymbol{P_4}$	$(0.500, 0.261)$	$(0.333, 0.292)$
$\boldsymbol{P_5}$	$(0.409, 0.214)$	$(0.389, 0.217)$
$\boldsymbol{P_6}$	$(0.545, 0.124)$	$(0.507, 0.142)$
$\boldsymbol{P_7}$	$(0.471, 0.177)$	$(0.401, 0.183)$
$\boldsymbol{P_8}$	$(0.452, 0.261)$	$(0.400, 0.275)$
$\boldsymbol{P_9}$	$(0.581, 0.213)$	$(0.541, 0.226)$
$\boldsymbol{P_{10}}$	$(0.523, 0.201)$	$(0.372, 0.227)$
$\boldsymbol{P_{11}}$	$(0.703, 0.205)$	$(0.615, 0.236)$
$\boldsymbol{P_{12}}$	$(0.507, 0.384)$	$(0.00, 0.618)$
$\boldsymbol{P_{13}}$	$(0.535, 0.220)$	$(0.420, 0.239)$
$\boldsymbol{P_{14}}$	$(0.411, 0.252)$	$(0.302, 0.262)$
$\boldsymbol{P_{15}}$	$(0.395, 0.276)$	$(0.259, 0.312)$

Step 3. Determine the interval-based equivalence classes applying the prescribed method with a step size of $n = 5$ as illustrated in Table 3.

Table 3. Interval-valued equivalence classes.

$\boldsymbol{IFPWA_k}$	$[\boldsymbol{P_1}] = \{\boldsymbol{P_1}\}$ $[\boldsymbol{P_2}] = \{\boldsymbol{P_2}, \boldsymbol{P_3}, \boldsymbol{P_4}, \boldsymbol{P_5}, \boldsymbol{P_7}, \boldsymbol{P_8}, \boldsymbol{P_{14}}\}$ $[\boldsymbol{P_6}] = \{\boldsymbol{P_6}, \boldsymbol{P_9}, \boldsymbol{P_{10}}, \boldsymbol{P_{12}}, \boldsymbol{P_{13}}\}$ $[\boldsymbol{P_{11}}] = \{\boldsymbol{P_{11}}\}$ $[\boldsymbol{P_{15}}] = \{\boldsymbol{P_{15}}\}$
$\boldsymbol{IFPWG_k}$	$[P_1] = \{P_1\}$ $[P_2] = \{P_2, P_5, P_7, P_8, P_{10}, P_{13}\}$ $[P_3] = \{P_3, P_4, P_{14}, P_{15}\}$ $[P_6] = \{P_6, P_9, P_{11}\}$ $[P_{12}] = \{P_{12}\}$

Step 4. Determine the lower approximation and upper approximation by Definition 15 for the given decision attributes $U = \{P_1, P_2, P_4, P_{15}, P_{11}\}$.

Table 4. Approximation classes.

IFPA	$\underline{\boldsymbol{Appr}}(\boldsymbol{U}) = \{\boldsymbol{P_1}, \boldsymbol{P_{15}}, \boldsymbol{P_{11}}\}$ $\overline{\boldsymbol{Appr}}(\boldsymbol{U}) = \{\boldsymbol{P_1}, \boldsymbol{P_{15}}, \boldsymbol{P_3}, \boldsymbol{P_5}, \boldsymbol{P_7}, \boldsymbol{P_8}, \boldsymbol{P_4}, \boldsymbol{P_{14}}, \boldsymbol{P_2}\}$
IFPG	$\underline{Appr}(U) = \{P_1\}$, $\overline{Appr}(U) =$ $\{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{13}, P_{14}\}$

Step 5. Finally, the classification of the elements for POS, NEG, and BND regions respectively represented in Table 5.

Table 5. Classification of alternatives accordingly.

IFPA	IFPG
$\mathbf{POS(U)} = \{\mathbf{P_1}, \mathbf{P_{15}}, \mathbf{P_{11}}\}$ $\mathbf{NEG(U)} = \{\mathbf{P_{12}}, \mathbf{P_6}, \mathbf{P_{13}}, \mathbf{P_9}, \mathbf{P_{10}}\}$ $\mathbf{BND(U)} = \{\mathbf{P_2}, \mathbf{P_3}, \mathbf{P_4}, \mathbf{P_8}, \mathbf{P_7}, \mathbf{P_{10}}\}$	$\mathrm{POS(U)} = \{\mathrm{P_1}\}$ $\mathrm{NEG(U)} = \{\mathrm{P_{12}}\}$ $\mathrm{BND(U)} = \{\mathrm{P_2}, \mathrm{P_3}, \mathrm{P_4}, \mathrm{P_5}, \mathrm{P_6}, \mathrm{P_7}, \mathrm{P_8}, \mathrm{P_9}, \mathrm{P_{10}}, \mathrm{P_{13}}, \mathrm{P_{14}}\}$

Thus, the findings indicate that individuals in the POS zone have tested positive for Coronavirus disease, those in the NEG regions are free from the virus, and individuals in the BND regions have not received confirmation. Additionally, we can categorize newcomers' choices based on the descriptions of previously tested elements.

5.2 Advantages Offered by the Proposed Model

The proposed approach offers several advantages as outlined below:

i. One of the key benefits of this approach is its broader applicability. It serves as a more generalized version of intuitionistic fuzzy sets. When the non-membership grades are set to zero, intuitionistic fuzzy sets transform into FSs.

ii. Power aggregation operators prove to be highly effective and straightforward tools for addressing decision-making problems in fuzzy environments. These operators facilitate the determination of attribute values for elements and can account for their significance when aggregating data.

iii. Many of the existing methods in the literature for TWD primarily adhere to conventional theories such as Yao's [37]. In contrast, our approach presents novel procedures for TWD, which encompass the development of power aggregation operators. Furthermore, we introduce interval-valued categories to categorize participants.

iv. In the context of medical diagnosis, especially in complex cases, such as the one presented here, accurate disease diagnosis is a critical concern for both experts and patients. To tackle this challenge, we have established an idea that accounts for distinct patient profiles and their disease attributes. Ultimately, decisions are made based on input from experts.

6. Conclusion and Future Work

In conclusion, it is imperative to categorize potential solutions and opt for the most practical choices. Decision-making can be quite challenging as it varies depending on the context. Therefore, it's crucial to weigh both the pros and cons of each option. Furthermore, effective decision-making is beneficial for your overall well-being and enhances the chances of identifying the most suitable choice. It's vital to determine the exact amount of essential information that decision-makers need. In the decision-making model, the most efficient strategy involves closely focusing on your objectives.

In the article, we initially explored the fundamental concept of three-way decisions introduced by Yao [25] and the utilization of power aggregation operators. We devised an innovative approach for discretizing the information table. For classifying participants, we employed interval-valued classes, creating three zones based on these classes. The use of aggregation operators is highly advantageous for consolidating results and combining attribute values into single values. Given the significance of these operators, we employed power aggregation operators. Furthermore, we developed an algorithm for disease identification utilizing the suggested method.

Moving forward, the outcomes of this study will be extended to encompass fuzzy and rough data [41- 44], and we will devise novel aggregation operators to address real-life issues.

References

1. Sox, H. C. (2007). Medical decision making. ACP Press.
2. Chen, E. L., Chung, P. C., Chen, C. L., Tsai, H. M., & Chang, C. I. (1998). An automatic diagnostic system for CT liver image classification. IEEE transactions on biomedical engineering, 45(6), 783-794.
3. Szolovits, P., Patil, R. S., & Schwartz, W. B. (1988). Artificial intelligence in medical diagnosis. Annals of internal medi-cine, 108(1), 80-87.
4. Ledley, R. S., & Lusted, L. B. (1959). Reasoning foundations of medical diagnosis: symbolic logic, probability, and value theory aid our understanding of how physicians' reason. Science, 130(3366), 9-21.
5. Awotunde, J. B., Matiluko, O. E., & Fatai, O. W. (2014). Medical diagnosis system using fuzzy logic. African Journal of Computing & ICT, 7(2), 99-106.
6. Pawlak, Z., Grzymala-Busse, J., Slowinski, R., & Ziarko, W. (1995). Rough sets. Communications of the ACM, 38(11), 88-95.
7. Salama, A. S. (2010). Bitopological rough approximations with medical applications. Journal of King Saud Universi-ty-Science, 22(3), 177-183.
8. El-Bably, M. K., Abu-Gdairi, R., & El-Gayar, M. A. (2023). Medical diagnosis for the problem of Chikungunya disease using soft, rough sets. AIMS Mathematics, 8(4), 9082-9105.
9. Bai, J., Sun, B., Chu, X., Wang, T., Li, H., & Huang, Q. (2022). Neighborhood rough set-based multi-attribute prediction approach and its application of gout patients. Applied Soft Computing, 114, 108127.
10. Ejegwa, P. A., Onyeke, I. C., & Adah, V. (2020). An algorithm for an improved intuitionistic fuzzy correlation measure with medical diagnostic application. Annals of Optimization Theory and Practice, 3(3), 51-66.
11. Hosny, M. (2022). Generalization of rough sets using maximal right neighborhood systems and ideals with medical applications. AIMS Mathematics, 7(7), 13104-13138.
12. Al-Shami, T. M. (2022). Maximal rough neighborhoods with a medical application. Journal of Ambient Intelligence and Humanized Computing, 1-12.
13. Atanassov, K. T. (2012). On intuitionistic fuzzy sets theory (Vol. 283). Springer.
14. Jiang, Q., Huang, J., Jin, X., Wang, P., Zhou, W., & Yao, S. (2023). Medical Image Fusion Using a New Entropy Measure between Intuitionistic Fuzzy Sets Joint Gaussian Curvature Filter. IEEE Transactions on Radiation and Plasma Medical Sciences.
15. Mahmood, T., Ahmmad, J., Ali, Z., & Yang, M. S. (2023). Confidence Level Aggregation Operators Based on Intuition-istic Fuzzy Rough Sets with Application in Medical Diagnosis. IEEE Access, 11, 8674-8688.
16. Mahmood, T., Ali, W., Ali, Z., & Chinram, R. (2021). Power aggregation operators and similarity measures based on improved intuitionistic hesitant fuzzy sets and their applications to multiple attribute decision making. Computer Modeling in Engineering & Sciences, 126(3), 1165-1187.
17. De, S. K., Biswas, R., & Roy, A. R. (2001). An application of intuitionistic fuzzy sets in medical diagnosis. Fuzzy sets and Systems, 117(2), 209-213.
18. Davvaz, B., & Hassani Sadrabadi, E. (2016). An application of intuitionistic fuzzy sets in medicine. International journal of biomathematics, 9(03), 1650037.

19. Szmidt, E., & Kacprzyk, J. (2001). Intuitionistic fuzzy sets in intelligent data analysis for medical diagnosis. In Compu-tational Science-ICCS 2001: International Conference San Francisco, CA, USA, May 28—30, 2001 Proceedings, Part II 1 (pp. 263-271). Springer Berlin Heidelberg.
20. Xu, Z. (2011). Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators. Knowledge-Based Systems, 24(6), 749-760.
21. Xu, Z., & Yager, R. R. (2006). Some geometric aggregation operators are based on intuitionistic fuzzy sets. International journal of general systems, 35(4), 417-433.
22. Mahmood, T., Ali, W., Ali, Z., & Chinram, R. (2021). Power aggregation operators and similarity measures based on improved intuitionistic hesitant fuzzy sets and their applications to multiple attribute decision making. Computer Modeling in Engineering & Sciences, 126(3), 1165-1187.
23. Senapati, T., Chen, G., Mesiar, R., & Yager, R. R. (2023). Intuitionistic fuzzy geometric aggregation operators in the framework of Aczel-Alsina triangular norms and their application to multiple attribute decision making. Expert Sys-tems with Applications, 212, 118832.
24. Garg, H., Ali, Z., Mahmood, T., Ali, M. R., & Alburaikan, A. (2023). Schweizer-Sklar prioritized aggregation operators for intuitionistic fuzzy information and their application in multi-attribute decision-making. Alexandria Engineering Journal, 67, 229-240.
25. Yao, Y. (2012, August). An outline of a theory of three-way decisions. In International conference on rough sets and current trends in computing (pp. 1-17). Berlin, Heidelberg: Springer Berlin Heidelberg.
26. Yao, Y. (2021). The geometry of three-way decision. Applied Intelligence, 51(9), 6298-6325.
27. Li, Z., Zhang, P., Xie, N., Zhang, G., & Wen, C. F. (2020). A novel three-way decision method in a hybrid information system with images and its application in medical diagnosis. Engineering Applications of Artificial Intelligence, 92, 103651.
28. Li, Z., Xie, N., Huang, D., & Zhang, G. (2020). A three-way decision method in a hybrid decision information system and its application in medical diagnosis. Artificial Intelligence Review, 53, 4707-4736.
29. Hu, J., Chen, D., & Liang, P. (2019). A novel interval three-way concept lattice model with its application in medical diagnosis. Mathematics, 7(1), 103.
30. Hu, B. Q. (2014). Three-way decisions space and three-way decisions. Information sciences, 281, 21-52.
31. Jia, F., & Liu, P. (2019). A novel three-way decision model under multiple-criteria environment. Information Sciences, 471, 29-51.
32. Ye, J., Zhan, J., & Xu, Z. (2020). A novel decision-making approach based on three-way decisions in fuzzy information systems. Information Sciences, 541, 362-390.
33. Ali, W., Shaheen, T., Haq, I. U., Toor, H., Akram, F., Garg, H., ... & Hassan, M. M. (2023). Aczel-Alsina-based aggregation operators for intuitionistic hesitant fuzzy set environment and their application to multiple attribute decision-making process. *AIMS Math*, *8*(8), 18021-18039.
34. Zhang, Q., Xie, Q., & Wang, G. (2018). A novel three-way decision model with decision-theoretic rough sets using utility theory. Knowledge-Based Systems, 159, 321-335.
35. Liu, D., Liang, D., & Wang, C. (2016). A novel three-way decision model based on incomplete information system. Knowledge-Based Systems, 91, 32-45.
36. Yang, J., & Yao, Y. (2021). A three-way decision-based construction of shadowed sets from Atanassov intuitionistic fuzzy sets. Information Sciences, 577, 1-21.
37. Yao, Y., & Deng, X. (2011, August). Sequential three-way decisions with probabilistic rough sets. In IEEE 10th International Conference on Cognitive Informatics and Cognitive Computing (ICCI-CC'11) (pp. 120-125). IEEE.
38. Zhang, Q., Pang, G., & Wang, G. (2020). A novel sequential three-way decisions model based on penalty function. Knowledge-Based Systems, 192, 105350.
39. Dai, J., Chen, T., & Zhang, K. (2023). The intuitionistic fuzzy concept-oriented three-way decision model. Information Sciences, 619, 52-83.
40. Gora, P. (2023). Intuitionistic Fuzzy Modulus Similarity Measure. International Journal of Decision Support System Technology (IJDSST), 15(1), 1-22.

41. Ali, W., Shaheen, T., Haq, I. U., Alballa, T., Alburaikan, A., & El-Wahed Khalifa, H. A. (2023). A Novel Generalization of Q-Rung Orthopair Fuzzy Aczel Alsina Aggregation Operators and Their Application in Wireless Sensor Networks. *Sensors*, *23*(19), 8105.
42. Radenovic, S., Ali, W., Shaheen, T., Haq, I. U., Akram, F., & Toor, H. (2023). Multiple attribute decision-making based on bonferroni mean operators under square root fuzzy set environment. *Journal of Computational and Cognitive Engineering*, *2*(3), 236-248.
43. Haq, I. U., Shaheen, T., Ali, W., Toor, H., Senapati, T., Pilla, F., & Moslem, S. (2023). Novel Fermatean Fuzzy Aczel–Alsina Model for Investment Strategy Selection. *Mathematics*, *11*(14), 3211.
44. Ali, W., Shaheen, T., Toor, H. G., Alballa, T., Alburaikan, A., & Khalifa, H. A. E. W. (2023). An Improved Intuitionistic Fuzzy Decision-Theoretic Rough Set Model and Its Application. *Axioms*, *12*(11), 1003

The Use of Neutrosophic Methods of Operation Research in the Management of Corporate Work

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract: The science of operations research is one of the modern sciences that have made a great revolution in all areas of life through the methods provided by it, suitable and appropriate to solve most of the problems that were facing researchers, scholars and those interested in the development of societies, and the most beneficiaries of this science were companies and institutions that are looking for scientific methods that help them manage their work so that they achieve the greatest profit and the lowest cost, and one of the important methods that have been used in the management of companies we offer in this research two methods, Dynamic programming method. This method has been used in many practical matters and helped decision-makers in companies to achieve a maximum profit and less cost by formulating the reality of the state of the company and the data provided by decision-makers with a dynamic mathematical model that is solved using methods of solving dynamic models and we will provide in this research an example of this through the issue of choosing the optimal investment for the budget of a company so that it achieves a maximum profit, and the method of programming with integers: the method that provided these companies with solutions with integer values suitable for the nature of its work, through the use of the binary integer in the formulation of the appropriate mathematical model on the one hand, and on the other hand, the use of the binary integer variable helped to convert some nonlinear models that lead to some practical problems into linear models, and it should be noted here that in the previous two methods there is something indeterminable because we must make a decision in choosing or not choosing something, but the optimal solution that we will get remains A specific value because we are building the mathematical model for any realistic issue through the data provided by those responsible for the work and these data are calculated quantities and therefore they are uncertain values because their validity depends on the circumstances surrounding the work environment, they may be exposed to increase or decrease, and therefore the optimal solution on which the company will base its decision is suitable for specific values and any change in them can cause the company an uncalculated loss, so in this research we will use the concepts of neutrosophic science, the branch of science founded by the American scientist Florentin Smarandache in 1995 based on his belief that there is no absolute truth, a science that is interested in the study of ideas and concepts that are neither true nor false, but just in-between, and we will take the data (calculated quantities) neutrosophic values that are specified or unspecified values are any set close to the calculated quantities, then the resulting mathematical model is a neutrosophic model and the optimal solution has neutrosophic values and thanks to the indefinite uncertainty that these values have, companies from the development of appropriate plans for all circumstances and thus achieve the greatest profit and the lowest cost, and we will clarify the above through two issues, the issue of optimal designation of a warehouse site, which we will formulate the mathematical model of using the neutrosophic integer programming method - and the issue of capital budget, which we will present in two different forms, we use in the first form the neutrosophic integer programming method and in the second the neutrosophic dynamic programming method.

Keywords: Operations Research; Neutrosophic Integer Programming; Binary Integer Variable; Neutrosophic Dynamic Programming; Neutrosophic Science; Neutrosophic Linear Mathematical Model; Neutrosophic Nonlinear Mathematical Model; The Problem of Locating a Repository; The Problem of Capital Budget.

1. Introduction

The splendor of any science is completed through its scientific methods that help solve practical issues facing us in our daily lives and work on the development of societies, and one of the important methods presented by the science of operations research is the method of programming with integers, which depends on building mathematical models using the binary correct variable, which enabled researchers to convert some nonlinear mathematical models into linear mathematical models, in addition to that when solving these models, beam compounds were the optimal solution correct values that meet the nature of the issue under study and do not need to rotate fractional values for the optimal solution to obtain correct values as we used to do previously, the greatness of the science of operations research is when it meets with the concepts of neutrosophic science, the latest and most important thing that science has offered in our time, the science that brought values out of determination to indeterminacy, this margin of freedom can be greatly benefited from when the methods of operations research are reformulated using its concepts, a complement to what we have presented from research in which we formulated some methods of process research according to the concepts of neutrosophic [1−21]. Due to the importance of the dynamic programming method, we also presented a paper entitled Neutrosophic Dynamic Programming, through which we explained how to formulate dynamic models using the concepts of neutrosophic science [22]. In another paper, we used the neutrosophic integer programming method entitled Neutrosophic Mathematical Model of the product mixture problem using the binary integer variable [23]. In this research, we will use the method of dynamic neutrosophic programming and the method of neutrosophic integer programming to help companies manage their work in a scientific manner that enables them to exploit the resources available to them in an ideal way that achieves the greatest profit and the least loss by formulating two issues: the question of optimal designation of a warehouse site and the question of capital budget using the concepts of neutrosophic science It should be noted that these two issues were studied using classical values, see reference [24, 25].

2. Discussion

Since the aim of this research is to help decision-makers in companies and institutions to make the optimal decision to ensure the greatest profit and the lowest cost, we must study the issue at hand a good study through which we can determine the data that are affected by the surrounding conditions, and then we take these neutrosophic data values any unspecified values take the form

$Na_i = a_i + \varepsilon_i$ where ε_i is the indeterminacy on the data can take one of the forms $[\lambda_{i1}, \lambda_{i2}]$ أو $\{\lambda_{i1}, \lambda_{i2}\}$.. Or----- otherwise, which is any neighborhood containing the value a_i.

We clarify the above through the following problems:

2.1 The first issue

An executive in one of the companies asked an expert in the science of operations research to help him obtain an optimal solution through which to achieve the lowest cost of transportation and operation of warehouses he wants to establish in order to expand the company's work and provided him with information through which the expert formulated the following issue: The text of the problem according to the concepts of neutrosophic science: A retail company plans to expand its activities in a specific area by establishing two new warehouses, the following Table 1 shows the potential locations, the number of customers and the possibility of meeting the demand for the sites where $(*)$ has been placed in the event that the site can meet the customer's request and put $(\times)$ the opposite and code $\boldsymbol{Nc_{ij}}$ For the cost of transferring one unit from site $\boldsymbol{i}$ to customer $\boldsymbol{j}$ he got the following Table:

Table 1. Transportation cost in case of location selection.

Customer / site	B_1	B_2	B_3	B_4
A_1	* Nc_{11}	* Nc_{12}	×	* Nc_{14}
A_2	* Nc_{21}	* Nc_{22}	* Nc_{23}	* Nc_{24}
A_3	×	* Nc_{32}	* Nc_{33}	* Nc_{34}
Customer orders	D_1	D_2	D_3	D_4

Table 2 shows the following information available for each of the candidate locations for warehouses.

Table 2. Operation information.

Information / site	Operating cost per unit (monetary unit)	Initial Invested Capital (Monetary Unit)	Site Capacity
first	Np_1	k_1	A_1
second	Np_2	k_2	A_2
third	Np_3	k_3	A_3

It is required to choose suitable locations for warehouses that make the total costs of investment, operation and transportation as small as possible.

Building the mathematical model: Each site has a fixed capital cost independent of the quantity stored in the warehouse referred to that site and also has a variable cost proportional to the quantity transported, and therefore the total cost of establishing and operating the warehouse is a non-linear function of the stored quantity and using binary integer variables can be formulated the issue of determining the location of the warehouse in a program with integers where we assume that the binary integer variable δ_i Symbolizes the decision to choose the site or not to choose it in other words

$$\delta_i = \begin{cases} 1 & \text{if we chose the site } i \\ 0 & \text{otherwise} \end{cases}$$

Suppose that x_{ij} is the quantity transferred from site i to customer j, so the constraint expressing the ability of the first site to meet the requests is as follows:

$$x_{11} + x_{12} + x_{14} \leq A_1\delta_1$$

When $\delta_1 = 1$, the first location with capacity A_1 is chosen. The quantity transported from the first site cannot exceed the capacity of that site A_1 When $\delta_1 = 0$ the non-negative variables $x_{11}, x_{12}, x_{14} = 0$ directly, indicating that it is not possible to ship from the first location

In a similar way, we obtain the following two constraints for the second and third signatories.

$$x_{21} + x_{22} + x_{23} + x_{24} \leq A_2\delta_2$$

$$x_{31} + x_{33} + x_{34} \leq A_3\delta_3$$

To choose exactly two locations, we need the following restriction:

$$\boldsymbol{\delta_1 + \delta_2 + \delta_3 = 2}$$

As $\boldsymbol{\delta_1}$ can take one of the values of 0 or 1 only, the new constraint will force two variables from among the three variables, $\boldsymbol{\delta_i}$ to be equal to one

The restrictions for customer requests can be written as follows:

first customer $\boldsymbol{x_{11} + x_{21} = D_1}$

Second customer $\boldsymbol{x_{12} + x_{22} + x_{32} = D_2}$

Third customer $\boldsymbol{x_{23} + x_{33} = D_3}$

Forth customer $\boldsymbol{x_{14} + x_{24} + x_{34} = D_4}$

To write the objective function, we note that the total cost of investment, operation and transportation for the first site is as follows:

$$\boldsymbol{k_1\delta_1 + Np_1(x_{11} + x_{12} + x_{14}) + Nc_{11}x_{11} + Nc_{12}x_{12} + Nc_{14}x_{14}}$$

When we do not choose the first site, variable $\boldsymbol{\delta_1 = 0}$ And that forces the variables

$\boldsymbol{x_{11}, x_{12}, x_{14} = 0}$

In a similar way, the cost functions of the second and third sites can be written, and thus the full formulation of the issue of assigning the location of the warehouse is reduced to the following correct mixed program: Z is meant to be made minimal

$$\boldsymbol{Z = k_1\delta_1 + Np_1(x_{11} + x_{12} + x_{14}) + Nc_{11}x_{11} + Nc_{12}x_{12} + Nc_{14}x_{14} + k_2\delta_2 + Np_2(x_{21} + x_{22} + x_{23} + x_{24}) + Nc_{21}x_{21} + Nc_{22}x_{22} + Nc_{23}x_{23} + Nc_{24}x_{24} + k_3\delta_3 + Np_3(x_{32} + x_{33} + x_{34}) + Nc_{32}x_{32} + Nc_{33}x_{33} + Nc_{34}x_{34}}$$

considering the following restrictions:

$$\boldsymbol{x_{11} + x_{12} + x_{14} \le A_1\delta_1}$$

$$\boldsymbol{x_{21} + x_{22} + x_{23} + x_{24} \le A_2\delta_2}$$

$$\boldsymbol{x_{31} + x_{33} + x_{34} \le A_3\delta_3}$$

$$\boldsymbol{\delta_1 + \delta_2 + \delta_3 = 2}$$

$$\boldsymbol{x_{11} + x_{21} = D_1}$$

$$\boldsymbol{x_{12} + x_{22} + x_{32} = D_2}$$

$$\boldsymbol{x_{23} + x_{33} = D_3}$$

$$\boldsymbol{\delta_i \text{ true variable for } i = 1,2,3}$$

$$\boldsymbol{x_{ij} \ge 0 \ ; \ i = 1,2,3 \ and \ j = 1,2,3,4}$$

2.2 The second issue

The second request addressed by the executive was how I can choose the appropriate projects to operate a limited capital available in the company through a number of projects presented, through the information provided by the official in charge of managing the company, the expert formulated the following issue:

The issue of the capital budget: A company plans to disburse its capital during the $\boldsymbol{T_j}$ periods. where $\boldsymbol{j = 1,2,---,n}$, and there is$\boldsymbol{A_i}$A proposed project where $\boldsymbol{i = 1,2,---,m}$ versus a limited capital $\boldsymbol{B_j}$ Available for investment in period $\boldsymbol{j}$ and when choosing any project $\boldsymbol{i}$ becomes in need of a certain capital in each period $\boldsymbol{j}$ we denote it$\boldsymbol{Na_{ij}}$. It is a neutrosophic value, the value of each project is measured in terms of the liquidity flow corresponding to the project in each period minus the value

of inflation, and this is called net present value (NPV), we denote it $\boldsymbol{Nv_i}$ Accordingly, the following Table 3 can be organized:

Table 3. Return on Investment during periods.

period / project	T_1	T_2	----------	T_n
A_1	Na_{11}	Na_{12}	----------	Na_{11}
A_2	Na_{21}	Na_{22}		Na_{2n}
--------	----------	----------	---------	----------
A_m	Na_{m1}	Na_{m2}	----------	Na_{mn}
Limited capital	B_1	B_2	----------	B_n

What is required in this problem is to select the right projects that maximize the total value (NPV) of all selected projects. Formulation of the mathematical model:

Here we assume a binary integer variable $\boldsymbol{x_j}$It takes the value one if the project $\boldsymbol{j}$ is selected and takes the value zero if the project $\boldsymbol{j}$ is not selected

$$x_i = \begin{cases} 1 & \text{if we chose project } i \\ 0 & \text{otherwise} \end{cases}$$

Then the objective function is given by the following relationship:

$$\boldsymbol{Z = \sum_{i=1}^{m} Nv_i\, x_i}$$

Then the objective function is given by the following relationship:

$$\boldsymbol{\sum_{i=1}^{m} Na_{ij}\, x_i \le B_j \quad ; j = 1, - - -, n}$$

Accordingly, we get the following mathematical model:

Find the maximum value of the function:

$$\boldsymbol{Z = \sum_{i=1}^{m} Nv_i\, x_i}$$

Considering the following restrictions:

$$\boldsymbol{\sum_{i=1}^{m} Na_{ij}\, x_i \le B_j \quad ; j = 1, - - -, n}$$

$\boldsymbol{x_i}$ A binary variable takes one of the values 0 or 1 for all values of $\boldsymbol{i = 1, - - -, m}$ in the previous two issues, we got models with integers that have special methods of solution. This research cannot be presented and we will present them in later research using the concepts of neutrosophic science.

2.3 The third issue

in the reference [25] an example of a dynamic programming problem, which is the capital budget problem using classical values, is presented. In this paper, we will reformulate the general model of this problem using neutrosophic values, and we will apply this to the example given in the reference so that we can compare the use of classical values and neutrosophic values using the optimal solution that we will get.

Another presentation of the problem of the neutrosophic capital budget in reference [25] the following question was raised: the budget of a company (5 million monetary units) wants to be distributed to four different investment projects, if you know that each investment has a certain return corresponding to each amount of one million and is shown in the attached table, it is required to find the optimal distribution of the budget if (5 million and 4 million) between the four projects in a way that obtains the greatest profit

Table 5. Periodic amounts classic values.

Invesment projects / **Invested amounts**	$F_1(x_1)$	$F_2(x_2)$	$F_3(x_3)$	$F_4(x_4)$
0	0	0	0	0
1	0.56	0.5	0.3	0.4
2	0.9	0.82	0.5	0.66
3	1.3	1.1	0.8	0.84
4	1.56	1.3	1.00	0.96
5	1.8	1.5	1.24	1.06

This issue is a dynamic programming issue that was addressed using the tables method and the summary table for the final solution was as follows:

Table 6. Final table of solution and classic values.

Z_4	x_4	$F_4(x_4)$	Z_3	$G_3^*(Z_3)$	$G_4(Z_4)$
5	0	0	5	2.12	2.12
	1	0.40	4	1.80	2.20
	2	0.66	3	1.40	2.06
	3	0.84	2	1.06	1.90
	4	0.96	1	0.56	1.52
	5	1.06	0	0	1.06
4	0	0	4	1.80	1.80
	1	0.40	3	1.40	1.80
	2	0.66	2	1.06	1.72
	3	0.84	1	0.56	1.40
	4	0.96	0	0	0.96

From the table, we find that the maximum return if (5) million is invested equals to $G_4^*(Z_4) = \mathbf{2.20}$

This value corresponds to $x_4 = \mathbf{1}$, $x_3 = \mathbf{0}$, $Z_3 = \mathbf{4}$ and therefore from the relations$Z_i = Z_{i-1} + x_i$

We find the following distribution:

$$G_4^*(Z_4) = 2.20 \Longrightarrow Z_4 = 5 \Longrightarrow \begin{cases} x_4 = 1 \\ x_3 = 0 \\ x_2 = 1 \\ x_1 = 3 \end{cases}$$

$x_4 = 1$ أو $x_4 = 0$:

From the table, we find that the maximum return if (4) million is invested equals $G_4^*(Z_4) = 1.80$

This value corresponds to , $x_4 = 1$ **or** $x_4 = 0$ and the distribution is as follows:

$$G_4^*(Z_4) = 1.80 \Rightarrow Z_4 = 4 \Rightarrow \begin{cases} x_4 = 0 \\ or \\ x_4 = 1 \end{cases} \Rightarrow \begin{cases} \begin{cases} x_4 = 0 \\ x_3 = 0 \\ x_2 = 1 \\ x_1 = 3 \end{cases} \\ 0r \\ \begin{cases} x_4 = 1 \\ x_3 = 0 \\ x_2 = 1 \\ x_1 = 2 \end{cases} \end{cases}$$

To clarify the extent to which neutrosophic values affect the final result of the solution, that is, on the maximum profit of the company In the beginning, we formulate the general model for this problem, neutrosophic science concepts, and accordingly we determine the data that are affected by the surrounding conditions of the work environment and here we take the values that express the profit from investing an amount of amounts in one of the projects neutrosophic values because they are the values that are affected by the surrounding conditions

We get the following problem:

The problem of the distribution of the neutrosophic capital budget: the budget of the company is M monetary unit that you want to distribute to n of different investment projects, $B_1, B_2, - - -, B_n$ if you know that for every amount invested in a project, there is a profit (i.e. the profit is related to the amount invested into the project), $NF_1(x_1), NF_2(x_2), - - -, NF_n(x_n)$، shown in the attached table, it is required to find the optimal distribution of the budget between n projects so that the company gets the greatest profit

Table 4. Return on Investment by amount used.

Ivestment projects / Invested amounts	B_1	B_2	---------	B_n
1	Nc_{11}	Nc_{12}	---------	Nc_{1n}
2	Nc_{21}	Nc_{22}	---------	Nc_{2n}
----------	------	--------	--------	--------
m	Nc_{m1}	Nc_{m2}	--------	Nc_{mn}

After studying the data of this issue, we note that it can be formulated in the form of a dynamic programming problem corresponding to the case of one dimension and we mean the criterion that affects the decision-making process at a certain stage of obtaining the optimal solution and one

influencer and we mean by it the goal follower and here we note that the goal is the only one which is to achieve the greatest profit so we can represent it with the following relationships:

$$\boldsymbol{Z_i = NF_i(x_i)} \quad ; \boldsymbol{i = 1, 2, - - -, n}$$

$\boldsymbol{Z_i}$ is the function that expresses the state reached by the problem at a certain partial point $\boldsymbol{x_i}$ is secision variable in stage $\boldsymbol{i}$

The relationship between the state in a given phase i−1 and the next phase i is written as follows:

$$\boldsymbol{Z_i = g_i(x_i , Z_{i-1})}$$

These relationships represent constraints

The goal function is written according to the partial goal dependencies in the previous stages with the following relationship:

$$\boldsymbol{NG_i(Z_i) = NF_i(x_i , Z_{i-1} , Z_i)} \quad ; \boldsymbol{i = 1, 2, - - -, n}$$

This function takes an optimal value at each stage and this value is given by the following relationship: $\boldsymbol{NG_n^*(Z_n) = \underbrace{opt}_{x_1 \to x_n} [NF_1(x_i , Z_{i-1} , Z_i) + - - - + NF_n(x_n , Z_{n-1} , Z_n)]}$

Which can be expressed as follows:

$$\boldsymbol{NG_n^*(Z_n) = \underbrace{opt}_{x_n} [NF_n(x_n , Z_{n-1} , Z_n) + NG_{n-1}^*(Z_{n-1})]}$$

whereas

$\boldsymbol{NG_n^*(Z_n)}$ – **optimal goal function at stage** $\boldsymbol{n}$

$\boldsymbol{NG_{n-1}^*(Z_{n-1})}$ – **optimal goal function at stage** $\boldsymbol{n-1}$

they be solved as follows:

$$\boldsymbol{NG_i^*(Z_i) = \underbrace{0pt}_{x_i} NF_i[(x_i , Z_{i-1} , Z_i)]} \quad ; \boldsymbol{i = 1, 2, - - -, n} \quad \boldsymbol{(1)}$$

Restrictions:

$$\boldsymbol{Z_i = g_i(x_i , Z_{i-1})} \quad ; \boldsymbol{i = 1, 2, - - -, n} \quad \boldsymbol{(2)}$$

The relationships,1. and restrictions,2. It includes all one-dimensional, one-effect neutrosophic dynamic programming problems and is solved in the form of tables that we will illustrate by solving the following example: Example: (We will take some neutrosophic values) The budget of the company M=$ 5 million wants to spread it among four projects $\boldsymbol{B_1 , B_2 , B_3 , B_4}$ of the various investment projects, if you know that for every million investment in a project, there is a profit (that is, the profit is related to the amount that is invested into the project), shown in the attached table, it is required to find the optimal distribution of the budget between the four projects so that the company achieves the greatest profit

Table 7. Return on Investment with neutrosophic values.

Investment projects / **Invested amounts**	B_1	B_2	B_3	B_4
0	0	0	0	0
1	[0.52 , 0.57]	0.5	0.3	0.4
2	0.9	[0.8 , 0.85]	0.5	0.66
3	1.3	1.1	[0.79 , 0.82]	0.84
4	1.56	1.3	1.00	[0.9 , 1.3]
5	1.8	1.5	1.24	[1.05 , 1.2]

This problem is formulated with a dynamic programming program and the optimal solution is found using the following relationships: We denote $\boldsymbol{NF_i(x_i)}$. -function at each stage (the value of the profit returned from the use of the amount $\boldsymbol{x_i}$ in project , $\boldsymbol{x_1, x_2, x_3, x_4}$ is the amount of money distributed over the four projects

$$\boldsymbol{Z_i = Z_{i-1} + x_i}$$

Solution equations:

$$\boldsymbol{NG_1^*(Z_1) = NF_1(x_1)}$$

$$\boldsymbol{NG_2^*(Z_2) = \underbrace{Max}_{x_2}[NF_2(x_2) + NG_1^*(Z_1)]}$$

$$\boldsymbol{NG_3^*(Z_3) = \underbrace{Max}_{x_3}[NF_3(x_3) + NG_2^*(Z_2)]}$$

$$\boldsymbol{NG_4^*(Z_4) = \underbrace{Max}_{x_4}[NF_4(x_4) + NG_3^*(Z_3)]}$$

1- Solution using tables

2- Table $\boldsymbol{Z_1}$

Table 8. Z_1 table.

Z_1	x_1	$G_1^*(Z_1)$	
0	0	0	
1	1	[0.52 , 0.57]	
2	2	0.9	$Z_0 = 0$
3	3	1.3	$Z_1 = Z_0 + x_1$
4	4	1.56	
5	5	1.8	

1- Table Z_2 where $Z_2 = Z_1 + x_2$

Table 9. Z_2 table.

Z_2	x_2	$NF_2(x_2)$	Z_1	$NG_1^*(Z_1)$	$NG_2(Z_2)$
0	0	0	0	0	0
1	0	0	1	[0.52 , 0.57]	[0.52 , 0.57]
	1	0.5	0	0	0.5
2	0	0	2	0.9	0.9
	1	0.5	1	[0.52 , 0.57]	[1.02 , 1.07]
	2	[0.8 , 0.85]	0	0	[0.8 , 0.85]
3	0	0	3	1.3	1.3
	1	0.5	2	0.9	1.4
	2	[0.8 , 0.85]	1	[0.52 , 0.57]	[1.32 , 1.42]
	3	1.1	0	0	1.1
4	0	0	4	1.56	1.56
	1	0.5	3	1.3	1.8
	2	[0.8 , 0.85]	2	0.9	[1.7 , 1.75]
	3	1.1	1	[0.52 , 0.57]	[1.62 , 1.67]
	4	1.3	0	0	1.3
5	0	0	5	1.8	1.8
	1	0.5	4	1.56	2.06
	2	[0.8 , 0.85]	3	1.3	[2.1 , 2.15]
	3	1.1	2	0.9	2.0
	4	1.3	1	[0.52 , 0.57]	[1.82 , 1.87]
	5	1.5	0	0	1.5

The values colored yellow represent the maximum values of profit returned for each amount in which the second project is supplied, so we write a table, $\boldsymbol{Z_2}$ the abbreviation that shows the optimal values.

Table 10. Optimal values of Z_2.

Z_2	x_2	$NG_2^*(Z_2)$	
0	0	0	$Z_2 = Z_1 + x_2$
1	0	[0.52 , 0.57]	
2	1	[1.02 , 1.07]	
3	2	[1.4 , 1.42]	
4	2	[1.7 , 1.75]	
5	2	[2.1 , 2.15]	

2- Z_3 table where $Z_3 = Z_2 + x_3$

Table 11. Z_3 table.

Z_3	x_3	$NF_3(x_3)$	Z_2	$NG_2^*(Z_2)$	$NG_3(Z_3)$
0	0	0	0	0	0
1	0	0	1	[0.52 , 0.57]	[0.52 , 0.57]
	1	0.3	0	0	0.3
2	0	0	2	[1.02 , 1.07]	[1.02 , 1.07]
	1	0.3	1	[0.52 , 0.57]	[0.82 , 0.87]
	2	0.5	0	0	0.5
3	0	0	3	[1.4 , 1.42]	[1.4 , 1.42]
	1	0.3	2	[1.02 , 1.07]	[1.32 , 1.37]
	2	0.5	1	[0.52 , 0.57]	[1.02 , 1.07]
	3	[0.79 , 0.82]	0	0	[0.79 , 0.82]
4	0	0	4	[1.7 , 1.75]	[1.7 , 1.75]
	1	0.3	3	[1.4 , 1.42]	[1.7 , 1.72]
	2	0.5	2	[1.02 , 1.07]	[1.52 , 1.57]
	3	[0.79 , 0.82]	1	[0.52 , 0.57]	[1.31 , 1.39]
	4	1.0	0	0	1.0
5	0	0	5	[2.1 , 2.15]	[2.1 , 2.15]
	1	0.3	4	[1.7 , 1.75]	[2 , 2.05]
	2	0.5	3	[1.4 , 1.42]	[1.9 , 1.92]
	3	[0.79 , 0.82]	2	[1.02 , 1.07]	[1.81 , 1.89]
	4	1.0	1	[0.52 , 0.57]	[1.52 , 1.57]
	5	1.24	0	0	1.24

The values colored yellow represent the maximum values of profit payable for each amount in which the second project is supplied, so we write a table $\boldsymbol{Z_3}$ The abbreviation that shows the optimal values.

Table 12. Optimal values.

Z_3	x_3	$NG_3^*(Z_3)$	
0	0	0	$Z_3 = Z_2 + x_3$
1	0	[0.52 , 0.57]	
2	0	[1.02 , 1.07]	
3	0	[1.4 , 1.42]	
4	0	[1.7 , 1.75]	
5	0	[2.1 , 2.15]	

3- Z_3 table where $Z_4 = Z_3 + x_4$

Table 13. Z_4 table.

Z_4	x_4	$NF_4(x_4)$	Z_3	$NG_3^*(Z_3)$	$NG_4(Z_4)$
0	0	0	0	0	0
1	0	0	1	[0.52 , 0.57]	[0.52 , 0.57]
	1	0.4	0	0	0.4
2	0	0	2	[1.02 , 1.07]	[1.02 , 1.07]
	1	0.4	1	[0.52 , 0.57]	[0.92 , 0.97]
	2	0.66	0	0	0.66
3	0	0	3	[1.4 , 1.42]	[1.4 , 1.42]
	1	0.4	2	[1.02 , 1.07]	[1.42 , 1.47]
	2	0.66	1	[0.52 , 0.57]	[1.18 , 1.23]
	3	0.84	0	0	0.84
4	0	0	4	[1.7 , 1.75]	[1.7 , 1.75]
	1	0.4	3	[1.4 , 1.42]	[1.8 , 1.82]
	2	0.66	2	[1.02 , 1.07]	[1.68 , 1.73]
	3	0.84	1	[0.52 , 0.57]	[1.36 , 1.41]
	4	[0.9 , 1.3]	0	0	[0.9 , 1.3]
5	0	0	5	[2.1 , 2.15]	[2.1 , 2.15]
	1	0.4	4	[1.7 , 1.75]	[2.1 , 2.15]
	2	0.66	3	[1.4 , 1.42]	[2.06 , 2.08]
	3	0.84	2	[1.02 , 1.07]	[1.86 , 1.91]
	4	[0.9 , 1.3]	1	[0.52 , 0.57]	[1.42 , 1.87]
	5	[1.05 , 1.2]	0	0	[1.05 , 1.2]

The values colored yellow represent the maximum values of profit returned for each amount in which the second project is supplied, so we write a table,Z-4. The abbreviation that shows the optimal values

Table 14. the optimal values of Z_4.

Z_4	x_4	$NG_4^*(Z_4)$	
0	0	0	$Z_4 = Z_3 + x_4$
1	0	[0.52 , 0.57]	
2	0	[1.02 , 1.07]	
3	1	[1.42 , 1.47]	
4	1	[1.8 , 1.82]	
5	0 *or* 1	[2.1 , 2.15]	

Conclusion of the optimal solution: Note that the maximum profit value when investing the amount (5) available in the company is achieved at two values of $\boldsymbol{x_4}$ Therefore, the optimal distribution is as follows:

$$NG_4^*(Z_4) = [2.1\,,2.15] \Rightarrow Z_4 = 5 \Rightarrow \begin{cases} x_4 = 0 \\ or \\ x_4 = 1 \end{cases} \Rightarrow \begin{cases} \begin{cases} x_4 = 0 \\ x_3 = 0 \\ x_2 = 2 \\ x_1 = 3 \end{cases} \\ 0r \\ \begin{cases} x_4 = 1 \\ x_3 = 0 \\ x_2 = 2 \\ x_1 = 2 \end{cases} \end{cases}$$

To obtain a maximum profit of a value belonging to the range, $[\mathbf{2.1}\,,\mathbf{2.15}]$ an amount of (5) million can be distributed in one of the two forms:

Figure 1: We provide the first project with (3) million dollars and the second project with (2) million dollars and cancel investment in the third and fourth projects Figure 2: We provide the first project (2) million dollars, the second project (2) million dollars, the fourth project (1) million dollars, and cancel the investment in the third project to obtain a maximum profit of a value belonging to the range$[\mathbf{1.8}\,,\mathbf{1.82}]$ The amount of (4) million can be distributed as follows: We provide the first project with (1) million dollars, the second project with (2) million dollars, and the fourth project with (1) million dollars, and cancel the investment in the third project

$$\boldsymbol{NG_4^*(Z_4) = [1.8\,,1.82] \Longrightarrow Z_4 = 4 \Longrightarrow \begin{cases} x_4 = 1 \\ x_3 = 0 \\ x_2 = 2 \\ x_1 = 1 \end{cases}}$$

Note: When calculating $\boldsymbol{Z_4}$ We can only calculate for (4) and (5) million, i.e. sufficiency in the following table:

Table 15. Table of optimal values for Z_4 with the short solution.

Z_4	x_4	$NF_4(x_4)$	Z_3	$NG_3^*(Z_3)$	$NG_4(Z_4)$
4	0	0	4	[1.7 , 1.75]	[1.7 , 1.75]
	1	0.4	3	[1.4 , 1.42]	[1.8 , 1.82]
	2	0.66	2	[1.02 , 1.07]	[1.68 , 1.73]
	3	0.84	1	[0.52 , 0.57]	[1.36 , 1.41]
	4	[0.9 , 1.3]	0	0	[0.9 , 1.3]
5	0	0	5	[2.1 , 2.15]	[2.1 , 2.15]
	1	0.4	4	[1.7 , 1.75]	[2.1 , 2.15]
	2	0.66	3	[1.4 , 1.42]	[2.06 2.08]
	3	0.84	2	[1.02 , 1.07]	[1.86 , 1.91]
	4	[0.9 , 1.3]	1	[0.52 , 0.57]	[1.42 , 1.87]
	5	[1.05 , 1.2]	0	0	[1.05 , 1.2]

Comparison for the investment of (5) million:

Table 16. comparison in case the amount is (5) millions.

Type of profit values / Projects	Optimal distribution of amount in case that some profit values are neutrosophic values		Optimal distribution of amount in case that some profit values are classical values
Forth project	$x_4 = 1 \quad 0r \quad x_4 = 0$		$x_4 = 1$
Third project	$x_3 = 0$	$x_3 = 0$	$x_3 = 0$
Second project	$x_2 = 2$	$x_2 = 2$	$x_2 = 1$
First project	$x_1 = 3$	$x_1 = 1$	$x_1 = 3$
Maximal general profit	$NG_4^*(Z_4) = [2.1\ , 2.15]$		$G_4^*(Z_4) = 2.20$

Comparison for the investment of (4) million:

Table 17. comparison in case the amount is (4) millions.

Type of profit values / Projects	Optimal distribution of amount in case that some profit values are neutrosophic values	Optimal distribution of amount in case that some profit values are classical values	
Forth project	$x_4 = 1$	$x_4 = 0 \quad or \quad x_4 = 1$	
Third project	$x_3 = 0$	$x_3 = 0$	$x_3 = 0$
Second project	$x_2 = 2$	$x_2 = 1$	$x_2 = 1$
First project	$x_1 = 1$	$x_1 = 3$	$x_1 = 2$
Maximal general profit	$NG_4^*(Z_4) = [1.8\ , 1.82]$	$G_4^*(Z_4) = 1.80$	

3. Conclusion and Results

In this research, we presented a study whose importance lies in the following points:

1. Obtaining a general formula for the issue of establishing new warehouses that help companies and institutions expand their work at the lowest cost by choosing the ideal locations for expansion points
2. Obtaining a general formula for the issue of distributing the companies' capital budget through the ideal investment for them in selected projects within a group of proposed projects so that the company achieves the maximum profit.
3. Clarifying the role of the binary integer in converting some nonlinear models into linear models and also obtain solutions with integers appropriate to the nature of the problems under study.
4. Focusing on the need to use the concepts of neutrosophic science to obtain optimal solutions suitable for all circumstances surrounding the work environment.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

References

1. Maissam jdid, Rafif Alhabib, and AA Salama ,The static model of inventory management without a deficit with Neutrosophic logic, International Journal of Neutrosophic Science, Vol. 16, 2021
2. Maissam Jdid, AA Salama, Huda E Khalid ,Neutrosophic Handling of the Simplex Direct Algorithm to Define the Optimal Solution in Linear Programming ,International Journal of Neutrosophic Science, Vol.11,No. 1, 2022
3. Maissam Ahmed Jdid _A. A. Salama _Rafif Alhabib _Huda E. Khalid_Fatima Suliman ,Neutrosophic Treatment of the Static Model of Inventory Management with Deficit, International Journal of Neutrosophic Science, Vol.11,No. 1, 2022
4. Maissam Jdid, Huda E Khalid ,Mysterious Neutrosophic Linear Models, International Journal of Neutrosophic Science, Vol.11,No. 2, 2022
5. Maissam Jdid ,Basel Shahin ,Fatima Suliman ,Important Neutrosophic Rules for Decision – Making in Case of Uncertain Data, International Journal of Neutrosophic Science, Vol .18,No.3, 2022
6. Maissam Jdid, Rafif Alhabib, and AA Salama, Fundamentals of Neutrosophical Simulation for Generating Random Numbers Associated with Uniform Probability Distribution, Neutrosophic sets and Systems , Vol 49,2022
7. Maissam Jdid , Rafif Alhabib , Ossama Bahbouh , A. A. Salama , Huda E. Khalid ,The Neutrosophic Treatment for Multiple Storage Problem of Finite Materials and Volumes, International Journal of Neutrosophic Science (IJNS) Vol. 18, No. 1, 2022
8. Maissam Jdid , Rafif Alhabib , Huda E. Khalid , A. A. Salama ,The Neutrosophic Treatment of the Static Model for the Inventory Management with Safety Reserve ,International Journal of Neutrosophic Science (IJNS) Vol. 18, No. 2, 2022
9. Maissam Jdid, Huda E Khalid, An Investigation in the Initial Solution for Neutrosophic Transportation Problems (NTP),Neutrosophic sets and Systems NSS , Vol.50,2022
10. Maissam Jdid , Huda E. Khalid ,Neutrosophic Mathematical formulas of Transportation Problems, Neutrosophic sets and Systems ,NSS , Vol .51,2022
11. Maissam jdid- Fatima AlSuliman ,A study in Integral of sine and Cosine Functions ,Journal of Mathematical Structures and Applications ,(GJMSA) , Vol .2 ,No. 1 ,2022
12. Maissam jdid ,Important Neutrosophic Economic Indicators of the Static Model of Inventory Management without Deficit, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .5,No .1, 2023
13. Maissam jdid- Hla Hasan, The state of Risk and Optimum Decision According to Neutrosophic Rules, International Journal of Neutrosophic Science (IJNS),Vol. 20, No.1,2023

14. Maissam Jdid , Rafif Alhabib ,A. A. Salam ,The Basics of Neutrosophic Simulation for Converting Random Numbers Associated with a Uniform Probability Distribution into Random Variables Follow an Exponential Distribution ,Journal Neutrosophic Sets and Systems ,NSS,Vol.53,2023
15. Mohammed Alshikho, Maissam Jdid, Said Broumi , Artificial Intelligence and Neutrosophic Machine learning in the Diagnosis and Detection of COVID 19 ,Journal Prospects for Applied Mathematics and Data Analysis ,Vol 01, No,02 USA,2023.
16. Mohammed Alshikho, Maissam Jdid, Said Broumi ,A Study of a Support Vector Machine Algorithm with an Orthogonal Legendre Kernel According to Neutrosophic logic and Inverse Lagrangian Interpolation, , Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .5,No .1, 2023
17. Maissam Jdid, Khalifa Alshaqsi, Optimal Value of the Service Rate in the Unlimited Model M\M\1, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .6,No .1, 2023
18. Maissam Jdid, A. A. Salam, Using the Inverse Transformation Method to Generate Random Variables that follow the Neutrosophic Uniform Probability Distribution. Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .6,No .2, 2023
19. Maissam Jdid ,Neutrosophic Nonlinear Models, Journal Prospects for Applied Mathematics and Data Analysis, Vo .2,No .1, 2023
20. Maissam Jdid, The Use of Neutrosophic linear Programming Method in the Field of Education, Handbook of Research on the Applications of Neutrosophic Sets Theory and Their Extensions in Education ,Chapter 15, IGI-Global,2023.
21. Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Applied Mathematics and Data Analysis, VO .2, No .1, 2023.
22. Maissam Jdid, Rafif Alhabib, ,Neutrosophical dynamic programming ,International Journal of Neutrosophic Science, Vol .18,No.3, 2022
23. Maissam Jdid, Neutrosophic Mathematical Model of Product Mixture Problem Using Binary Integer Mutant, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .6,No .2, 2023
24. Bukajah J.S. , Mualla, W... and others - Operations Research Book translated into Arabic - The Arab Center for Arabization, Translation, Authoring and Publishing -Damascus -1998. . (Arabic version).
25. Hamd. Ibtisam - Analysis of engineering systems. Tishreen University Publications.1998 (Arabic version).

On Refined Neutrosophic Finite *p*-Group

Sunday Adesina Adebisi,[1] Florentin Smarandache[2]

[1] Department of Mathematics, Faculty of Science, University of Lagos, Akoka, Yaba, Nigeria; adesinasunday71@gmail.com.
[2] Department of Mathematics, University of New Mexico, Gallup Campus, NM 87301, USA; smarand@unm.edu.

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Abstract

The neutrosophic automorphisms of a neutrosophic groups G (I) , denoted by Aut(G (I)) is a neutrosophic group under the usual mapping composition. It is a permutation of G (I) which is also a neutrosophic homomorphism. Moreover, suppose that $X_1 = X(G(I))$ is the neutrosophic group of inner neutrosophic auto-morphisms of a neutrosophic group G (I) and X_n the neutrosophic group of inner neutrosophic automorphisms of X_{n-1}. In this paper, we show that if any neutrosophic group of the sequence G (I), X_1, X_2, … is the identity, then G (I) is nilpotent.

Keywords: Neutrosophic automorphism, Commutator subgroup, Neutrosophic subgroup, Minimal condition, Mapping composition, Nilpotency.

1 | Introduction

The concepts of refined neutrosophic algebraic structures and studies of refined neutrosophic groups in particular were introduced by Agboola [1]. After the successful feat, many other neutrosophic researchers have as well tried to establish more further studies on the refined neutrosophic algebraic structures [2]. Further studies on refined neutrosophic rings and refined neutrosophic subrings, their presentations and fundamentals were also worked upon.

Also, Agboola [3] has examined and as well studied the refined neutrosophic quotient groups, where more properties of re ned neutrosophic groups were presented and it was shown that the classical isomorphism theorems of groups do not hold in the refined neu-trosophic groups. The existence of classical morphisms between refined neutrosophic groups G (I_1; I_2) and neutrosophic groups G (I) were established. The readers can as well consult [4–7] in order to have detailed knowledge concerning the refined neutrosophic logic, neutrosophic groups, refined neutrosophic groups and neutrosophy, in general. Please note the following: throughout this paper, our binary operation is strictly the usual ordinary addition (as the operation of multiplication may not be de ned due to the fact that I_1 does not exist).

Definition 1 ([3]). Suppose that $(X(I_1; I_2); +; .)$ is any re ned neutro-sophic algebraic structure. Here, + and . are ordinary addition and multiplication respectively. Then I_1 and I_2 are the split components of the indeterminacy factor I that is $I = \alpha_1 I_1 + \alpha_2 I_2$ with α_i in C (the set of complex numbers); $i = 1; 2$.

Definition 2 ([3]). Suppose that (G; *) is any group. Then, the couple $(G(I_1; I_2); *)$ can be referred to as the refined neutrosophic group. Furthermore, this group can be said to be generated by G, I_1 and I_2 and $(G(I_1; I_2); *)$ is said to be commutative if for all x; y in for all $G(I_1; I_2)$; we have $x*y = y*x$: otherwise, $(G(I_1; I_2); *)$ can be referred to as a non-commutative refined neutrosophic group.

Here, I has been refined as I_1 and I_2: note that it is possible to refine T and F as well as T_1, T_2 and F as F_1, F_2 (see [8] for some details on this. We hope to make substantial contributions and relevant considerations in this regards as future possible studies.)

Theorem 1 ([3]). 1) every refined neutrosophic group is a semigroup but not a group, and 2) every refined neutrosophic group contains a group.

Corollary 1 ([3]). Every refined neutrosophic group $(G(I_1; I_2); +)$ is a group.

Definition 3 ([3]). Let $(G(I_1; I_2);*)$ be a refined neutrosophic group and let $A(I_1; I_2)$ be a nonempty subset of $G(I_1; I_2)$: $A(I_1; I_2)$ is called a refined neutrosophic sub-group of $G(I_1; I_2)$ if $(A(I_1; I_2);*)$ is a refined neutrosophic group. It is essential that $A(I_1; I_2)$ contains a proper subset which is a group. Otherwise, $A(I_1; I_2)$ will be called a pseudo refined neutrosophic subgroup of $G(I_1; I_2)$.

Definition 4 ([3]). Let $H(I_1; I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1; I_2); .)$: define $x = (a; bI_1; cI_2)$ in $G(I_1; I_2)$.

Theorem 2 ([3]). Let $(G(I_1; I_2); +)$ be a refined neutrosophic group and let $(G(I); +)$ be a neutrosophic group such that where $I = xI_1 + yI_2$ with x; y in C. Let ϕ: $G(I_1; I_2) \longrightarrow G(I)$be a mapping defined by $((a; xI_1; yI_2)) = (a; (x + y)I)$ for all $(a; xI_1; yI_2)$ in $(G(I_1; I_2)$ with a; x; y in G: then ϕ is a group homomorphism.

An interesting type of neutrosophic isomorphism of a neutrosophic groups G (I) would occur when the image neutrosophic group G (I) coincides with G (I). The classical group concepts as regards to this has been discussed by [9]. A neutrosophic isomorphism α: $G(I) \longrightarrow G(I)$ of G (I) onto itself can be called a neutrosophic automorphism of G (I). In particular, permutes the elements of G(I). The collection of all neutrosophic automorphisms of G (I) forms a neutrosophic group under composition of maps.

If β: $G(I) \longrightarrow G(I)$ is another neutrosophic automorphism, we denote the product of α and β by $\alpha\beta$. The group of all neutrosophic automorphisms of G (I) denoted Aut (G(I)) can be called the neutrosophic automorphism group of G(I). The unit element of G(I) is the neutrosophic identity automorphism i. This which leaves every element of G (I) fixed i.e.,

$$ix = x, \left(\left(a;\ bI_1;\ cI_2\right) = x \in G(I)\right).$$

Definition 5. A neutrosophic group G (I) can be said to be nilpotent if it has a normal series of a finite length n. That is,

$$G(I) = G0(I) \geq G1(I) \geq G2(I) \geq \ldots \geq Gn(I) = \{e\},$$

where

$$G_i(I) / G_{i+1}(I) \leq Z(G(I) / G_{i+1}(I)).$$

By this notion, every finite neutrosophic p-group G(I) is nilpotent. The nilpotence property is an hereditary one. Thus

I. Any finite product of nilpotent neutrosophic group is nilpotent.
II. If G (I) is nilpotent of a class c, then, every neutrosophic subgroup as well as the neutrosophic quotient group of G (I) is nilpotent and of class ≤ c.

Definition 6. Suppose that (W (I); #) and (V (I); ⊕) are two neutrosophic groups. Define a neutrosophic homomorphism from α: W (I) to V (I) to be a mapping: W (I) ⊕ V (I) such that α(x#y) = α (x) α (y) where x = (a_1; b_1I_1; c_1I_2), and y = (a_2; b_2I_1; c_2I_2). A neutrosophic homomorphism α which maps a neutrosophic group W (I) on itself is called a neutrosophic endomorphism. A bijective neutrosophic endomorphism is known as a neutrosophic automorphism.

Now, let t = (a; bI_1; cI_2) be a fixed element of a group W (I). The mapping β_t: W (I)⟶W (I) which could be defined by β_t(x) = txt-1 for all (x_1; x_2I_1; x_3I_2) = x in W (I) is known as an inner neutrosophic automorphism of the group W (I).

Every other neutrosophic automorphism of W (I) is called outer neutrosophic automor-phism. (The classical group concepts on this was also discussed in [10] and [11].)

Theorem 3. A neutrosophic abelian group G (I) of order $p_1^{\alpha 1}p_2^{\alpha 2}\dots p_n^{\alpha n}$, where $p_1\, p_2 \dots p_n$ are distinct primes, is the direct product of groups G_{p1}(I), G_{p2}(I), G_{p3}(I), … G_{pn}(I) of respective orders $p_1^{\alpha 1}$, $p_2^{\alpha 2}$, … $p_n^{\alpha n}$.

The subgroup G_p(I) is formed of all the operations of G (I) whose orders are powers of p with the identical operation (see also [12] for the classical group concepts.)

2 | Statement of Proof of the Main Results

We are now about to prove the main results. Already, an inner neutrosophic automorphism of a neutrosophic group has been de ned. Now, given that X_1 = X(G(I)) is the neutrosophic group of inner neutrosophic automorphisms of a group W (I). Also X_n is the neutrosophic group of the inner neutrosophic automorphisms of X_{n-1}, n, an integer.

Definition 7. Suppose there exists the lower central series of a group G (I) given by:

G(I)=$G_{(0)}$(I) ⊇ $G_{(1)}$(I) ⊇ $G_{(2)}$(I) ⊇ …. Here, $G_{(i)}$(I)=[$G_{(i-1)}$(I), G(I)], i>0. i.e., $G_{(1)}$(I)=[$G_{(0)}$(I), G (I)]=[G (I), G (I)]=G"(I), the commutator subgroup of G (I) such that the lower central series terminates at {ε} after a finite number of steps (i.e. $G_{(n)}$(I) = {ε}, for some integer n). Then G (I) is said to be nilpotent.

Define $u^{-1}v^{-1}uv$=[u; v], the commutator of u and v, in a group G (I).

And $u^v = v^{-1}uv$. Here, u = (u_1; u_2I_1; u_3I_2), and $v = \left(v_1;\ v_2I_1;\ v_3I_2\right)u^{-1}v^{-1}uv$

$$= \left(u_1^{-1};\ u_2^{-1}I_1;\ u_3^{-1}I_2\right)\left(v_1^{-1};\ v_2^{-1}I_1;\ v_3^{-1}I_2\right)\left(u_1;\ u_2I_1;\ u_3I_2\right)\left(v_1;\ v_2I_1;\ v_3I_2\right)$$
$$= \left(u_1^{-1}v_1^{-1};\ u_2^{-1}v_2^{-1}I_1;\ u_3^{-1}v_3^{-1}I_2\right)\left(u_1v_1;\ u_2v_2I_1;\ u_3v_3I_2\right)$$
$$= \left(u_1^{-1}v_1^{-1}u_1v_1;\ u_2^{-1}v_2^{-1}u_2v_2I_1;\ u_3^{-1}v_3^{-1}u_3v_3I_2\right) = \left[u;\ v\right].$$

By the definition of inner neutrosophic automorphism,using induction on G (I),

$$X(G(I)) = X_1 = \left\{x_1^{-1}gx_1 j(a_1;\ a_2I_1;\ a_3I_2) = g \text{ in } G(I)\right\}$$
$$= \{g^{x1} \mid g \in G(I)\} = \{\, gg^{-1}x_1^{-1}gx_1 j(a_1;\ a_2I_1;\ a_3I_2) = g \in G \text{ and } x_1;\ a_1;\ a_2;\ a_3; \in G\}$$
$$= \{\, g[g;\ x_1]j(a_1;\ a_2I_1;\ a_3I_2) = g \in G(I)\}.$$

If n=2. Then X_2 is the neutrosophic group of the inner neutrosophic automorphisms of $X_1 = \{g[g; x_1] / (a_1; a_2I_1; a_3I_2) = gG(I)\}$.

Hence, there exists $x_2 \in G(I)$ such that

$$X_2 = \{x_2 -^1 (g^{x1})x_2 \mid g^{x1} \in X_1\} = \{(g^{x1})^{x2} \mid g^{x1} \in X_1\}.$$

Following similar trends, there exists $x_3 \in G(I)$ such that

$$X_3 = \{x_3 -^1 (g^{x_1x_2})x_3 | g^{x_1x_2} \in X_2\} = \{g^{x_1x_2x_3} \mid g^{x_1x_2} \in x_2\}.$$

Also, there exists $x_4 \in G(I)$, such that

$$X_4 = \{x_4^{-1}(g^{x_1x_2x_3})x_4 \mid g^{x_1x_2x_3} X_3\} = fg^{x_1x_2x_3x_4} \mid g^{x_1x_2x_3} \in X_3\}.$$

And for $n = k$, there exists $x_k \in G(I)$, $k \in N$, such that

$$X_k = \{X_k^{-1}(g^{x_1x_2x_{k-1}})X_{k+1} \mid g^{x_1x_2x_k} \in X_{k+1}.$$

If the truth of the last statement is assumed, there exists $x_{k+1} \in G(I)$, $k \in N$ such that

$$X_{k+1} = \{x_{k+1}^{-1}(g^{x1x2\,xk})x_{k+1} \mid g^{x1x2\,xk} X_{k+1}\}.$$

We have that

$$G(I) = X_0 \supseteq X_1 \supseteq X_2 \supseteq X_k \supseteq X_{k+1} \supseteq \ldots \supseteq X_n \supseteq \ldots.$$

Definition 8. A neutrosophic group A (I) is said to satisfy the Descending Chain Condition (DCC) for any neutrosophic subgroups if every descending chain, $A_1(I) \supseteq A_2(I) \supseteq$ i.e, of neutrosophic subgroups terminates, i.e., there exists t in N (the set of natural numbers) such that for all $n \geq t$, $A_n(I) = A_t(I)$. Hence, every non-empty subset of the neutrosophic subgroups of A (I) has a minimal element. By the original hypothesis, let X_{n+1} be the identity $\{\varepsilon\}$ of the sequence G (I), X_1, X_2, … .

Then, the minimal condition implies that

$$G(I) = X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots \supseteq X_n \supseteq X_{n+1} = \{\varepsilon\} \text{ and } X_i = X_{i-1}(G(I)).$$

This actually shows the nilpotence of G (I) (for more and extensive discussion regarding to the classical group concepts, please see [10] and [13].)

3 | Applications

This findings can be fully applicable to every other nite group in general, most especially those nite groups that are nilpotent.

4 | Conclusion

Finally, the nilpotent characteristics of every nite p-group has been observed to be highly hereditary and so, any other neutrosophic product groups formed which have origin from nite p-group would de nitely display neutrosopiy.

References

[1] Agboola, A. A. A. (2015). On refined neutrosophic algebraic structures. *Neutrosophic sets and systems*, *10*, 99–101. DOI:10.5281/ZENODO.571570

[2] Adeleke, E. O., Agboola, A. A. A., & Smarandache, F. (2020). Refined neutrosophic rings I. *International journal of neutrosophic science*, *2*(2), 77–81. DOI:10.5281/ZENODO.3728222

[3] Agboola, A. A. A. (2020). On refined neutrosophic quotient qroups. *International journal of neutrosophic science*, *5*(2), 76–82. DOI:10.5281/zenodo.3828609

[4] Adesina A. A., Akinola, A. D., & Oyebola, O. Y. (2011). Neutrosophic rings I. *International journal of mathematical combinatorics*, *4*, 1–14.

[5] Agboola, A. A. A., Akwu, A. D. & Oyebo, Y. T. (2012). Neutrosophic groups and subgroups. *International journal of mathematical combinatorics*, *3*, 1–9.

[6] Bera, T., & Mahapatra, N. K. (2017). On neutrosophic normal soft groups. *International journal of applied and computational mathematics*, *3*(4), 3047–3066.

[7] Kandasamy, W. B. V., Kandasamy, I., & Smarandache, F. (2019). Neutrosophic triplets in neutrosophic rings. *Mathematics*, *7*(6), 563. DOI:10.3390/MATH7060563

[8] Smarandache, F. (2013). N-valued refined neutrosophic logic and its applications to physics. *Infinite study*, *4*, 143–146.

[9] Ledermann, W. (1973). *Introduction to group theory*. Oliver and Boyd.

[10] Kuku, A. O. (1980). *Abstract algebra*. Ibadan University Press.

[11] Kurosh, A. G. (1960). *Theory of groups*. Chelsea Publishing Company.

[12] Berkovich, Y. (2008). *Groups of prime power order*. De Gruyter.

[13] Hall, M. (1959). *The theory of groups*. Macmillan.

Neutrosophic Industry 5.0 Inventory Model with Technology Driven Demand and Costs Parameters

A. Theeba[1], Nivetha Martin[2], and Florentin Smarandache[3]

[1]Assistant Professor, Department of Mathematics, G.T.N.Arts College, Dindigul, Tamil Nadu, India. E-mail: theebajoe@gmail.com
[2]Assistant Professor, Department of Mathematics, Arul Anandar College, Karumathur, Madurai, Tamil Nadu, India. E-mail: nivetha.martin710@gmail.com
[3]Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA. E-mail: smarand@unm.edu

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Abstract: In this age of technology, the manufacturing sectors are embracing the elements of industry 5.0 to setup a robust kind of production process. This research work proposes a novel neutrosophic production inventory model encompassing the cost parameters of technology inaddition to the conventional inventory costs. In this model the demand is expressed as function of technology of the form $\alpha e^{-\beta t} + \gamma t$ with the coefficients α, β and γdealing with the initial demand, decrease in demand over time and increase in demand with the adoption of new technology. The neutrosophic model developed in this work addresses the decision circumstances of indeterminacy in addition to uncertainty. The primary objective of this paper is to introduce the notion of technology driven demand and new types of costs associated with technology in a neutrosophic modelling environment. The proposed neutrosophic model is simulated and sensitively analyzed to draw inferences of the parameters over the production quantity q(t). The efficiency of this neutrosophic model is determined on making comparative analysis with crisp data sets. The neutrosophic model possesses high degree of flexibility and applicability in technology dominant manufacturing firms facilitating the decision makers to design optimal solutions.

Keywords: Neutrosophic sets, Inventory, Technology driven demand, Optimization

1. Introduction

Industry 5.0[1] is an advancement of industry 4.0 and it primarily focusses on the integration of digital technologies such as IoT (Internet of Things), Artificial Intelligence and big data analytics to evolve intelligent factory frameworks. Industry 4.0 is much concerned on automation whereas Industry 5.0[2] is the next evolution emphasizing on collaborative endeavours between humans and machines. The future manufacturing systems will be certainly embedded into the framework of industry 5.0[3] with key factors such as human-machine collaboration, customization, personalization, sustainability, resilience,flexibility and ethical responsibility.The incorporation of digital oriented technology into production firm costs the manufacturing sectors and this will forman integral part of product production in near future. Inventory modelling is primarily concerned on stock management together with costs optimization. In general, the inventory total cost comprisescosts of ordering, production, holding, shortages, backlog, deterioration, rework, remanufacture and many other related to product production and distribution. However, the modifications in the production system are reflected in the total inventory costs. The developments

in the production technology and the augmentations in a production framework incur costs of various kinds and that have to be considered in optimizing the inventory costs.

The production system is expected to be customer centered and function with sustainable consciousness. However, as the demands of the customers are varying with respect to time and with reference to advancements in technology, the production system embeds the facets of technology into their production framework and so as the relevant technology costs with the usual inventory management costs. Inventory modeling with technology driven demand is the key factor of industry 5.0 embraced production firms. The cost parameters considered in modelling may not be of deterministic in nature at all instances as there may exist few indeterminacies in such considerations. To resolve such cases, the theory of neutrosophy is incorporated into inventory modeling to evolve a comprehensive inventory model with neutrosophic parametric representations. Neutrosophic theory developed by Smarandache deals with three- valued function with truth, indeterminacy and falsity membership representations. The objective of this research work is to formulate a neutrosophic inventory model with technology driven demand and technology associated cost parameters. This work also performs a comparative analysis between neutrosophic model and deterministic model to showcase the efficacy of neutrosophic representations of the parameters. The attempt of considering technology dependent demand is a novel aspect of this research work and this has certainly more scope in discussing the nature of the demand under different conditions.

The remaining contents are presented in the following sections. Section 2 consists of the detailed literature review of the works associated with neutrosophic inventory modelling. Section 3 consists of basic definitions of neutrosophy. Section 4 discusses the key aspects of Industry 5.0 and Inventory 5.0. Section 5 develops the inventory model. Section 6 validates the proposed model with numerical examples considering both crisp and neutrosophic representations of the parameters. Section 7 compares the illustrations and draws inferences. Section 8 describes the industrial implications and the last section concludes the work with the ideas and scope of further extension.

2.Literature Review

This section outlines the state of art of neutrosophic inventory models focusing the robustness of neutrosophic representations of inventory parameters. Kar et al[4] modelled inventory model with space constraints using neutrosophic geometric programming. The multi-objective inventory model putforth effectively handled the uncertainty and indeterminacies in inventory management. Mullai and Broumi[5] devised neutrosophic based inventory model without shortages to determine the demand and supply flow in an inventory system. Das and Islam[6] developed fuzzy integrated multi-item inventory model with neutrosophic hesitant fuzzy programming approach to address the challenges of deterioration, space constraints with demand dependent production costs. Mullai and Surya[7] developed neutrosophic inventory backorder problem using triangular neutrosophic numbers considering uncertainty in demand and lead times. Mullai et al[8] formulated a single valued neutrosophic based inventory model with neutrosophic random variable to make accurate estimations of inventory decisive parameters. Islam et al[9] applied neutrosophic hesitant fuzzy programming approach in inventory modeling considering deterioration and space constraints with and without shortages. Mondal et al[10] presented a neutrosophic optimization approach considering inventory policies for seasonal items with logistic-growth demand rates. Rajeswari et al[11] developed reusable based inventory model with octagonal fuzzy neutrosophic numbers. Jdid et al[12] framed a static inventory model with neutrosophic logic and also devised neutrosophic based inventory model with safety reserve.

Sugapriya et al[13] modelled an effective container inventory model with bipolar neutrosophic representations. Garg et al proposed a model for container inventory using trapezoidal bipolar neutrosophic number. Bhavani et al[14] developed a neutrosophic based inventory system with particle swarm optimization algorithm for handling discounts, deterioration items. Sen and Chakrabarthi[15] proposed an industrial based production inventory model with deterioration using neutrosophic fuzzy optimizing approaches. Sarkar and Srivastava[16] presented a multi-item multi-

objective neutrosophic model considering sustainability cost parameters. Das and Islam [17]sketched out a neutrosophic based programming approach for optimizing multi-objective shortage follow inventory (SFI) model with ramp demand. Surya and Mullai[18] presented Neutrosophic multi-item inventory control models. Jayanthi [19]developed a neutrosophic fuzzy geometric approach in overage management. Rajeswari [20]employed neutrosophic approaches for optimizing efficient inventory systems. Barman et al[21] modelled rework-based inventory model with neutrosophic representations. Jdid[22] devised neutrosophic static inventory model with economic indicators. Pattnaik et al[23] designed neutrosophic inventory model to handle overage items. Bhavani and Mahapatra[24] proposed an inventory model with generalized triangular neutrosophic cost parameters. Mohanta et al[25] developed neutrosophic inventory model with two-level trade credit policy to handle perishable products. Mohanta[26] framed neutrosophic integrated smart manufacturing-oriented inventory model. Kalaiarasi and Swathi[27] proposed neutrosophic inventory model with quick returns. Miriam et al[28] discussed a neutrosophic rework warehouse inventory model for product distribution considering quality aspects. Dubey et al[29] presented a survey on neutrosophic based inventory problems. Supakar et al [30]developed neutrosophic inventory model and integrated with artificial bee colony algorithm to discuss green production inventory system. Moorthy et al[31] applied neutrosophic logic in inventory management. Surya et al[32] developed neutrosophic inventory model to handle decay items and price dependent demand. Kar et al[33] formulated multi-objective perishable multi-item green neutrosophic inventory models. Das and Islam[34] developed a multi-item inventory model with quadratic demand patterns and with neutrosophic Pythagorean hesitant fuzzy programming. Dubey and Kumar[35] modelled cost effective neutrosophic inventory model.Martin et al[36] designed a neutrosophic based industry 4.0 inventory model integrating neutrosophic logic with inventory cost parameters. These inventory models based on neutrosophy are modelled using different neutrosophic representations. However, neutrosophic trapezoidal interval valued fuzzy numbers are used in inventory optimization. Also, industry 4.0 based neutrosophic inventory model is more industrial centered and this contribution has motivated the authors to explore industry 5.0. Also, the demand nature discussed in the aforementioned inventory models are conventional. This research work has attempted to represent the demand as technology dependent. This work proposes a neutrosophic inventory model integrating the aspects of both industry and inventory 5

3.Preliminaries

This section presents the basic definitions of neutrosophic sets.

3.1 Neutrosophic set[37]

A neutrosophic set is characterized independently by a truth-membership function (x) , an indeterminacy-membership function $\beta(x)$, and a falsity-membership function $\gamma(x)$ and each of the function is defined from $X \rightarrow [0,1]$

3.2 Single valued Neutrosophic set(SVNS)[38]

A SVNS is denoted and defined as $\widetilde{A_N} = \{x, T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x)/x\epsilon X\}$, where for each generic point x in X, $T_{\widetilde{A_N}}(x)$ called truth membership function, $I_{\widetilde{A_N}}(x)$ called indeterminacy membership function and $F_{\widetilde{A_N}}(x)$ called falsity membership function in [0,1] and $0 \leq T_{\widetilde{A_N}}(x) + I_{\widetilde{A_N}}(x) + F_{\widetilde{A_N}}(x) \leq 3$. For continuous SNVS$\widetilde{A_N} = \int_{A_N} \langle T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x)\rangle / x_i, x_i \epsilon X$.

3.3 Neutrosophic number[38]

Let x be a generic element of a non empty set x. A neutrosophic number $\widetilde{A_N}$ in X is defined as $\widetilde{A_N} = \{x, \langle T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x)\rangle / x\epsilon X\}, \forall T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x) and F_{\widetilde{A_N}}(x)$ belongs to $]0^-, 1^+[$ where $T_{\widetilde{A_N}}: X \rightarrow]0^-, 1^+[, I_{\widetilde{A_N}} : X \rightarrow]0^-, 1^+[and F_{\widetilde{A_N}}: X \rightarrow]0^-, 1^+[$ are functions of truth – membership, indeterminacy membership and falsity – membership in $\widetilde{A_N}$respectively with

$0^- \leq T_{\widetilde{A_N}}(x) + I_{\widetilde{A_N}}(x) + F_{\widetilde{A_N}}(x) \leq 3^+$.

3.4 Interval – valued neutrosophic set[38]

Let X be a nonempty set. Then an interval – valued neutrosophic set [IVNS] $\widetilde{A_N}^{IV}$ of X is defined as:

$\widetilde{A_N}^{IV} = \{\langle x; [T_{\widetilde{A_N}}^L, T_{\widetilde{A_N}}^U], [I_{\widetilde{A_N}}^L, I_{\widetilde{A_N}}^U], [F_{\widetilde{A_N}}^L, F_{\widetilde{A_N}}^U]\rangle : x \epsilon X\}$ where

$[T_{\widetilde{A_N}}^L, T_{\widetilde{A_N}}^U], [I_{\widetilde{A_N}}^L, I_{\widetilde{A_N}}^U] and [F_{\widetilde{A_N}}^L, F_{\widetilde{A_N}}^U] \subset [0,1]\ \forall\, x \epsilon X\ T_{\widetilde{A_N}}^L = \inf(T_{\widetilde{A_N}})$, $T_{\widetilde{A_N}}^U = \sup(T_{\widetilde{A_N}})$; $I_{\widetilde{A_N}}^L = \inf(I_{\widetilde{A_N}})$, $I_{\widetilde{A_N}}^U = \sup(I_{\widetilde{A_N}})$; and $F_{\widetilde{A_N}}^L = \inf(F_{\widetilde{A_N}})$, $F_{\widetilde{A_N}}^U = \sup(F_{\widetilde{A_N}})$

3.5 Neutrosophic Trapezoidal Interval Valued Number[37]

A single valued trapezoidal Neutrosophic number $\tilde{a} = \langle (a_1, b_1, c_1, d_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is a special neutrosophic set on the real number set R, whose truth – membership, indeterminacy – membership, and a falsity – membership are given as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{(x-a_1)w_{\tilde{a}}}{b_1-a_1}, & (a_1 \leq x < b_1) \\ w_{\tilde{a},} & (b_1 \leq x \leq c_1) \\ \frac{(d_1-x)w_{\tilde{a}}}{d_1-c_1}, & (c_1 < x \leq d_1) \\ 0, & otherwise \end{cases}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} (b_1 - x + u_{\tilde{a}}(x-a_1))/(b_1-a_1), & (a_1 \leq x < b_1) \\ u_{\tilde{a}} & (b_1 \leq x \leq c_1) \\ \frac{x-c_1+u_{\tilde{a}}(d_1-x)}{(d_1-c_{1)}}, & (c_1 < x \leq d_1) \\ 1, & otherwise \end{cases}$$

and

$$\lambda_{\tilde{a}}(x) = \begin{cases} (b_1 - x + y_{\tilde{a}}(x-a_1))/(b_1-a_1), & (a_1 \leq x < b_1) \\ y_{\tilde{a}} & (b_1 \leq x \leq c_1) \\ \frac{x-c_1+y_{\tilde{a}}(d_1-x)}{(d_1-c_{1)}}, & (c_1 < x \leq d_1) \\ 1, & otherwise \end{cases}$$

respectively.

3.6 Operational Laws on IVTrNeNs[38]

Let $\widetilde{a_{N_1}}^{IV} = \langle [\{(a_1, b_1, c_1, d_1); u_{\widetilde{a_{N_1}}^{IV}}\}, \{(e_1, f_1, g_1, h_1); v_{\widetilde{a_{N_1}}^{IV}}\}, \{(l_1, m_1, n_1, p_1); w_{\widetilde{a_{N_1}}^{IV}}\}] \rangle$ and $\widetilde{a_{N_2}}^{IV} = \langle [\{(a_2, b_2, c_2, d_2); u_{\widetilde{a_{N_2}}^{IV}}\}, \{(e_2, f_2, g_2, h_2); v_{\widetilde{a_{N_2}}^{IV}}\}, \{(l_2, m_2, n_2, p_2); w_{\widetilde{a_{N_2}}^{IV}}\}] \rangle$ be two IVTrNeNs with twelve components, where $u_{\widetilde{a_{N_1}}^{IV}} = \left[u_{\widetilde{a_{N_1}}^{IV}}^L, u_{\widetilde{a_{N_1}}^{IV}}^U\right]; u_{\widetilde{a_{N_2}}^{IV}} = \left[u_{\widetilde{a_{N_2}}^{IV}}^L, u_{\widetilde{a_{N_2}}^{IV}}^U\right]$;

$v_{\widetilde{a_{N_1}}^{IV}} = \left[v_{\widetilde{a_{N_1}}^{IV}}^L, v_{\widetilde{a_{N_1}}^{IV}}^U\right]$; $v_{\widetilde{a_{N_2}}^{IV}} = \left[v_{\widetilde{a_{N_2}}^{IV}}^L, v_{\widetilde{a_{N_2}}^{IV}}^U\right]$; $w_{\widetilde{a_{N_1}}^{IV}} = \left[w_{\widetilde{a_{N_1}}^{IV}}^L, w_{\widetilde{a_{N_1}}^{IV}}^U\right]$;

$w_{\widetilde{a_{N_2}}^{IV}} = \left[w_{\widetilde{a_{N_2}}^{IV}}^L, w_{\widetilde{a_{N_2}}^{IV}}^U\right]$, then the following operations hold:

Addition of IVTrNeNs:

$$\widetilde{a_{N_1}}^{IV} + \widetilde{a_{N_2}}^{IV} = \langle \begin{matrix} \{(a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2); u_{\widetilde{a_{N_1}}^{IV} \wedge u_{\widetilde{a_2}}^{IV}}\} \\ \{(e_1+e_2, f_1+f_2, g_1+g_2, h_1+h_2); v_{\widetilde{a_{N_1}}^{IV} \vee v_{\widetilde{a_{N_2}}^{IV}}}\} \\ \left\{(l_1+l_2, m_1+m_2, n_1+n_2, p_1+p_2); w_{\widetilde{a_{N_1}}^{IV} \vee w_{\widetilde{a_{N_2}}^{IV}}}\right\} \end{matrix} \rangle$$

Negative of IVTrNeNs:

$$-\widetilde{a_{N_2}}^{IV} = \langle\left[\{-d_2,-c_2,-b_2,-a_2), u_{\widetilde{aN_2}^{IV}}\right\}, \{-h_2,-g_2,-f_2,-e_2), v_{\widetilde{aN_2}^{IV}}\}, \{-p_2,-n_2,-m_2,-l_2), w_{\widetilde{aN_2}^{IV}}\}]\rangle$$

Subtraction of IVTrNeNs:

$$\widetilde{a_{N_1}}^{IV} - \widetilde{a_{N_2}}^{IV} = \langle \begin{matrix} \left\{(a_1-d_2, b_1-c_2, c_1-b_2, d_1-a_2); u_{\widetilde{aN_1}^{IV}\wedge u_{\widetilde{a_2}^{IV}}}\right\} \\ \{(e_1-h_2, f_1-g_2, g_1-f_2, h_1-e_2);\ v_{\widetilde{aN_1}^{IV}\vee v_{\widetilde{aN_2}^{IV}}} \\ \left[\{(l_1-p_2, m_1-n_2, n_1-m_2, p_1-l_2);\ w_{\widetilde{aN_1}^{IV}\vee w_{\widetilde{aN_2}^{IV}}}\right] \end{matrix} \rangle$$

Scalar multiplication of SVTrNeN:

$$\lambda\widetilde{a_{N_1}}^{IV} = \begin{cases} \langle\left\{(\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1); u_{\widetilde{aN_1}^{IV}}\right\}, \left\{(\lambda e_1, \lambda f_1, \lambda g_1, \lambda h_1); v_{\widetilde{aN_1}^{IV}}\right\}, \left\{(\lambda l_1, \lambda m_1, \lambda n_1, \lambda p_1); w_{\widetilde{aN_1}^{IV}}\right\}\rangle\ if\lambda > 0 \\ \langle\left\{(\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1); u_{\widetilde{aN_1}^{IV}}\right\}, \left\{(\lambda e_1, \lambda f_1, \lambda g_1, \lambda h_1); v_{\widetilde{aN_1}^{IV}}\right\}, \left\{(\lambda e_1, \lambda f_1, \lambda g_1, \lambda h_1); w_{\widetilde{aN_1}^{IV}}\right\}\rangle\ if\lambda < 0 \end{cases}$$

Multiplication of SVTrNeN:

$$\widetilde{a_{N_1}}^{IV}.\ \widetilde{a_{N_2}}^{IV} = \begin{cases} \langle(a_1.a_2, b_1.b_2, c_1.c_2, d_1.d_2); (e_1.e_2, f_1.f_2, g_1.g_2, h_1.h_2); (l_1.l_2, m_1.m_2, n_1.n_2, p_1.p_2) \\ \left[u_{\widetilde{aN_1}^{IV}\wedge u_{\widetilde{a_2}^{IV}}}\right],\ [v_{\widetilde{aN_1}^{IV}\vee v_{\widetilde{a_2}^{IV}}}], \left[u_{\widetilde{aN_1}^{IV}\vee u_{\widetilde{a_2}^{IV}}}\right] if d_1>0, d_2>0, h_1>0, h_2>0, p_1>0, p_2>0\rangle \\ \langle(a_1.d_2, b_1.c_2, c_1.b_2, d_1.a_2); (e_1.h_2, f_1.g_2, g_1.f_2, h_1.e_2); (l_1.p_2, m_1.n_2, n_1.m_2, p_1.l_2) \\ \left[u_{\widetilde{aN_1}^{IV}\wedge u_{\widetilde{a_2}^{IV}}}\right],\ [v_{\widetilde{aN_1}^{IV}\vee v_{\widetilde{a_2}^{IV}}}], \left[u_{\widetilde{aN_1}^{IV}\vee u_{\widetilde{a_2}^{IV}}}\right] if d_1<0, d_2>0, h_1<0, h_2>0, p_1<0, p_2>0\rangle \\ (d_1.d_2, c_1.c_2, b_1.b_2, a_1.a_2); (h_1.h, g_1.g_2, f_1.f_2, e_1.e_2); (p_1.p_2, n_1.n_2, m_1.m_2, l_1.l_2) \\ \langle\left[u_{\widetilde{aN_1}^{IV}\wedge u_{\widetilde{a_2}^{IV}}}\right],\ [v_{\widetilde{aN_1}^{IV}\vee v_{\widetilde{a_2}^{IV}}}], \left[u_{\widetilde{aN_1}^{IV}\vee u_{\widetilde{a_2}^{IV}}}\right] if d_1<0, d_2<0, h_1<0, h_2<0, p_1<0, p_2<0\rangle \end{cases}$$

Inverse of SVTrNeN:

$$\boldsymbol{s(\widetilde{a_{N_1}}^{IV})^{-1}} = \frac{\mathbf{1}}{\widetilde{a_{N_1}}^{IV}} \begin{cases} \langle\left(\frac{1}{d_1}, \frac{1}{c_1}, \frac{1}{b_1}, \frac{1}{a_1}\right); \left(\frac{1}{h_1}, \frac{1}{g_1}, \frac{1}{f_1}, \frac{1}{e_1}\right); \left(\frac{1}{p_1}, \frac{1}{n}, \frac{1}{m_1}, \frac{1}{l_1}\right); \boldsymbol{u_{\widetilde{aN_1}^{IV}}, v_{\widetilde{aN_1}^{IV}}, w_{\widetilde{aN_1}^{IV}}}\rangle, \\ if\ a_1>0, b_1>0, c_1>0, d_1>0, e_1>0, f_1>0, g_1>0, h_1>0, l_1>0, m_1>0, n_1>0, p_1>0 \\ \langle\left(\frac{1}{a_1}, \frac{1}{b_1}, \frac{1}{c_1}, \frac{1}{d_1}\right); \left(\frac{1}{e_1}, \frac{1}{f_1}, \frac{1}{g_1}, \frac{1}{h_1}\right); \left(\frac{1}{l_1}, \frac{1}{m_1}, \frac{1}{n_1}, \frac{1}{p_1}\right); \boldsymbol{u_{\widetilde{aN_1}^{IV}}, v_{\widetilde{aN_1}^{IV}}, w_{\widetilde{aN_1}^{IV}}}\rangle \\ if\ a_1<0, b_1<0, c_1<0, d_1<0, e_1<0, f_1<0, g_1<0, h_1<0, l_1<0, m_1<0, n_1<0, p_1<0 \end{cases}$$

Division of SVTrNeN:

$$\frac{\widetilde{a_{N_1}}^{IV}}{\widetilde{a_{N_2}}^{IV}} = \begin{cases} \langle\left(\frac{a_1}{d_1}, \frac{b_1}{c_1}, \frac{c_1}{b_1}, \frac{d_1}{a_1}\right), \left(\frac{e_1}{h_1}, \frac{f_1}{g_1}, \frac{g_1}{f_1}, \frac{h_1}{e_1}\right); \left(\frac{l_1}{p_1}, \frac{m}{n_1}, \frac{n_1}{m_1}, \frac{p_1}{l_1}\right); \left[u_{\widetilde{aN_1}^{IV}\wedge u_{\widetilde{a_2}^{IV}}}\right], \left[v_{\widetilde{aN_1}^{IV}\vee v_{\widetilde{aN_2}^{IV}}}\right], \left[w_{\widetilde{aN_1}^{IV}\vee w_{\widetilde{aN_2}^{IV}}}\right]\rangle, \\ if d_1>0, d_2>0, h_1>0, h_2>0, p_1>0, p_2>0, \\ \langle\left(\frac{d_2}{d_1}, \frac{c_2}{c_1}, \frac{b_2}{b_1}, \frac{a_2}{a_1}\right), \left(\frac{h_2}{h_1}, \frac{g_2}{g_1}, \frac{f_2}{f_1}, \frac{e_2}{e_1}\right); \left(\frac{p_2}{p_1}, \frac{n_2}{n_1}, \frac{m_2}{m_1}, \frac{l_2}{l_1}\right); \left[u_{\widetilde{aN_1}^{IV}\wedge u_{\widetilde{a_2}^{IV}}}\right], \left[v_{\widetilde{aN_1}^{IV}\vee v_{\widetilde{aN_2}^{IV}}}\right], \left[w_{\widetilde{aN_1}^{IV}\vee w_{\widetilde{aN_2}^{IV}}}\right]\rangle, if \\ d_1<0, d_2>0, h_1<0, h_2>0, p_1<0, p_2>0, \\ \langle\left(\frac{d_2}{a_1}, \frac{c_2}{b_1}, \frac{b_2}{c_1}, \frac{a_2}{d_1}\right), \left(\frac{h_2}{e_1}, \frac{g_2}{f_1}, \frac{f_2}{g_1}, \frac{e_2}{h_1}\right); \left(\frac{p_2}{l_1}, \frac{n_2}{m_1}, \frac{m_2}{n_1}, \frac{l_2}{p_1}\right); \left[u_{\widetilde{aN_1}^{IV}\wedge u_{\widetilde{a_2}^{IV}}}\right], \left[v_{\widetilde{aN_1}^{IV}\vee v_{\widetilde{aN_2}^{IV}}}\right], \left[w_{\widetilde{aN_1}^{IV}\vee w_{\widetilde{aN_2}^{IV}}}\right]\rangle, if \\ d_1<0, d_2<0, h_1<0, h_2<0, p_1<0, p_2<0. \end{cases}$$

Where $u_{\widetilde{aN_1}^{IV}\wedge u_{\widetilde{a_2}^{IV}}} = \left[\min\left(u_{\widetilde{aN_1}^{IV}}{}^{L},\ u_{\widetilde{aN_2}^{IV}}{}^{L}\right), \min\left(u_{\widetilde{aN_1}^{IV}}{}^{U},\ u_{\widetilde{aN_2}^{IV}}{}^{U}\right)\right]$,

$v_{\widetilde{a_{N_1}}^{IV}} \vee v_{\widetilde{a_{N_2}}^{IV}} = [max\left(v_{\widetilde{a_{N_1}}^{IV}}{}^{L},\ v_{\widetilde{a_{N_2}}^{IV}}{}^{L}\right), max(v_{\widetilde{a_{N_1}}^{IV}}{}^{U},\ v_{\widetilde{a_{N_2}}^{IV}}{}^{U})$ and

$w_{\widetilde{a_{N_1}}^{IV}} \vee w_{\widetilde{a_{N_2}}^{IV}} = [max\left(w_{\widetilde{a_{N_1}}^{IV}}{}^{L},\ w_{\widetilde{a_{N_1}}^{IV}}{}^{L}\right), max(w_{\widetilde{a_{N_1}}^{IV}}{}^{U},\ w_{\widetilde{a_{N_1}}^{IV}}{}^{U})]$.

3.7 Score and Accuracy functions of IVTrNeNs[38]

The score function concept is used to find comparison between two IVTrNeNs. Greater of score function value demonstrate the greater of IVTrNeN. According the base of the score and accuracy functions of an IVTrNeN$\widetilde{a_N}^{IV}$canbe defined asfollows:

$$S(\widetilde{a_{N_1}}^{IV}) = \frac{1}{12}\big((8 + (a_1 + b_1 + c_1 + d_1) - (e_1 + f_1 + g_1 + h_1) - (l_1 + m_1 + n_1 + p_1)\big) \times (2 + u_{\widetilde{a_{N_1}}^{IV}}{}^{L} + u_{\widetilde{a_{N_2}}^{IV}}{}^{U} - v_{\widetilde{a_{N_1}}^{IV}}{}^{L} - v_{\widetilde{a_{N_1}}^{IV}}{}^{U} - w_{\widetilde{a_{N_1}}^{IV}}{}^{L} - w_{\widetilde{a_{N_1}}^{IV}}{}^{U}$$

$S(\widetilde{a_{N_1}}^{IV}) \epsilon [0,1]$. The accuracy function $A(\widetilde{a_{N_1}}^{IV}) = \frac{1}{4}(a_1 + b_1 + c_1 + d_1 - l_1 - m_1 - n_1 - p_1) \times (2 + u_{\widetilde{a_{N_1}}^{IV}}{}^{L} + u_{\widetilde{a_{N_2}}^{IV}}{}^{U} - v_{\widetilde{a_{N_1}}^{IV}}{}^{L} - v_{\widetilde{a_{N_1}}^{IV}}{}^{U} - w_{\widetilde{a_{N_1}}^{IV}}{}^{L} - w_{\widetilde{a_{N_1}}^{IV}}{}^{U})$

4. Industry 5.0 and Inventory 5.0

Industry 5.0

Industry 5.0 is an enhanced version of industry 4.0 which focuses more on collaboration of both human and machine to evolve a more technology centered scenario. The integration of digital technologies with industrial settingis the prime highlight of this advanced version. The key characteristic features of industry 5.0 are presented pictorially in Fig. 1.

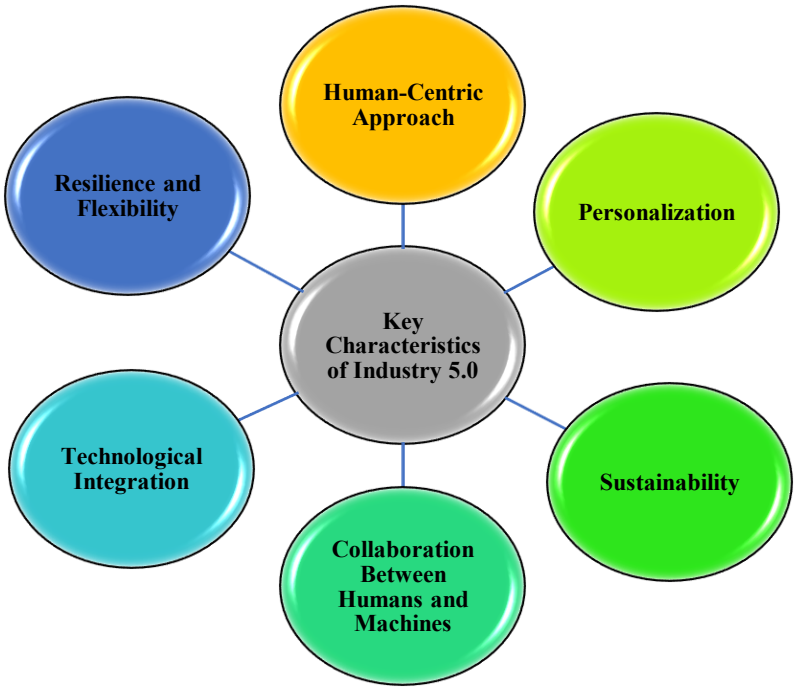

Fig.1. Key Characteristics of Industry 5.0

Human-Centric Approach: Industry 5.0 provides a setting to harmonize the associations between the human and the machines. The competency of the human beings is utilized in resolving the problems by making optimal decisions with the assistance of the machines in handling the tedious tasks.

Personalization: The advanced technical augmentation facilitates customization suiting to the needs of the customers. This is made possible with the collaborative roles of both human and machines enabling a flexible manufacturing process.

Sustainability: Industry 5.0 is more sustainable centric as it facilitates circular economy contributing to waste reduction and environmental sustainability. Resource optimization, remanufacturing and recycling are some of the pivotal roles of this enhanced version.

Collaboration Between Humans and Machines: Industry 5.0 envisages a synergetic relationship between human workers and machines. The working of human with robots in a parallel manner increases the productivity together with the leverage of manpower expertise in the required circumstances.

Technological Integration: Industry 5.0 is circumscribed with advanced technology comprising the approaches of AI, machine learning, data analytics to evolve a more intelligent production system. These well-developed systems are more adaptative in nature and capable of responding to the dynamic needs of the production.

Resilience and Flexibility: Industry 5.0 is potent in adapting to the disruptions occurring in the manufacturing systems thereby ensuring a more flexible and adaptable manufacturing system.

Inventory 5.0

Inventory 5.0 shall be characterized as the enhancement of inventory management in align wit the key components of industry 5.0. The integration of digital technology facets with inventory handling leverages the applications of real-time data analytics, AI and automation making the inventory system more flexible and sustainable.

The key components of Inventory 5.0 are presented in Fig.2 and it explicitly exhibits the attributes of this advanced inventory system.

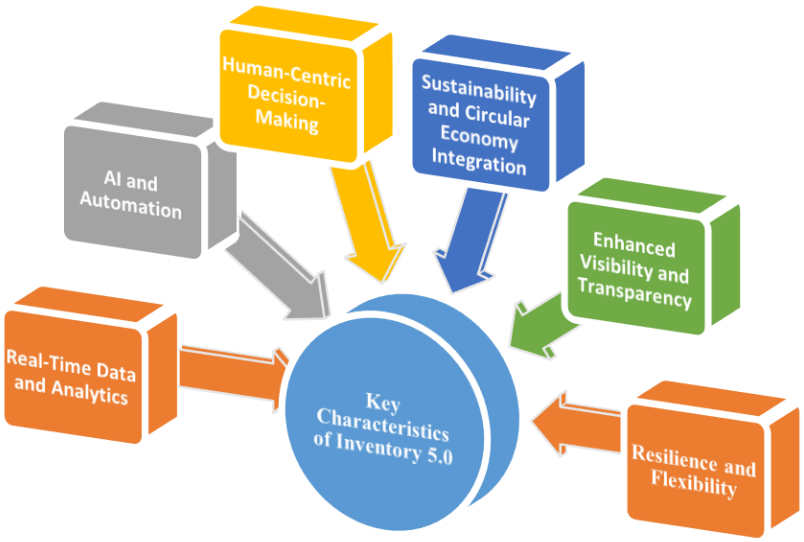

Fig.2. Key Characteristics of Inventory 5.0

On comparing the key components of industry 5.0 and inventory 5.0, it is highly evident that these two phenomena are closely related and there are functioning points of opportunities in align with one another. As both the advanced versions are relatively associated with digital technologies

attributing to resilience, flexibility and sustainability, the integration of inventory 5.0 with industry 5.0 may extremely benefit the manufacturing systems.

5. Model Development

This section presents the development of an inventory model characterizing the manufacturing system with technology dependence. Let us consider a technology-oriented inventory system which considers demand to be technology driven and incorporates technology associated cost parameters with other conventional cost parameters. The demand is expressed of the form $\alpha e^{-\beta t} + \gamma t$, where

$\alpha e^{-\beta t}$: Represent the decrease demand over time as initial technology adoption occurs. Here,α is the initial demand due to technologyand β is the rate at which the demand decreases.

γt: Represent the increase demand over time due to continuous market penetration and adoption of new technologies. Here, γ is the coefficient that indicates how demand increases with time.

The model is developed with the assumptions of P > D; no shortages are allowed; D is technology driven.

The notations used in this model are presented as follows,

$\alpha -$ Technology adoption rate coefficient

$\beta -$ Innovation level coefficient

$\gamma -$ Market penetration coefficient

$t -$ Time

P – Production

D – Demand

T_c– Technology Acquisition Costs

$M_c -$ Maintenance and Upgrade Costs

$T_{RC} -$ Training Costs

$I_c -$ Integration Costs

$C_c -$ Cyber security Costs

$D_c -$ Depreciation Costs

T – Total Production time

C_1 – Constant carrying cost per unit time

C_3 – Fixed Ordering cost

$C_2 -$ Purchase Costs

TC – Total Cost

TAC – Total average cost

Let us consider a system of differential equations with technology driven demand

For $0 \le t \le t_1$

$$\frac{dq(t)}{dt} = P - (\alpha e^{-\beta t} + \gamma t)$$

For $t_1 \le t \le T$

$$\frac{dq(t)}{dt} = (\alpha e^{-\beta t} + \gamma t)$$

$\alpha e^{-\beta t}$: Represent the decrease demand over time as initial technology adoption occurs. Here,α is the initial demand due to technology,and β is the rate at which the demand decreases.

γt: Represent the increase demand over time due to continuous market penetration and adoption of new technologies. Here, γ is the coefficient that indicates how demand increases with time.

For $0 \le t \le t_1$

The differential equation is

$\frac{dq(t)}{dt} = P - (\alpha e^{-\beta t} + \gamma t)$ (1)

This can be solved by integrating both sides with respect to t

$$\int_0^t dq(t) = \int_0^t (P - \alpha e^{-\beta t} - \gamma t) dt$$

$$q(t) - q(0) = Pt + \frac{\alpha e^{-\beta t}}{\beta} - \frac{\gamma t^2}{2} - \frac{\alpha}{\beta} + C_1$$

Given the initial condition $q(0) = 0$ (2)

That implies $C_1 = 0$

Therefore $q(t) = Pt + \frac{\alpha e^{-\beta t}}{\beta} - \frac{\gamma t^2}{2} - \frac{\alpha}{\beta}$ (3)

For $t_1 \le t \le T$

The differential equation is

$\frac{dq(t)}{dt} = (\alpha e^{-\beta t} + \gamma t)$ (4)

This can be solved by integrating both sides with respect to t

$$\int_t^T dq(t) = -\int_t^T (\alpha e^{-\beta t} + \gamma t)dt$$

We use the boundary condition $q(T) = 0$ (5)

That implies $C_2 = 0$

$q(t) = -\frac{\alpha e^{-\beta T}}{\beta} + \frac{\alpha e^{-\beta t}}{\beta} + \frac{\gamma T^2}{2} - \frac{\gamma t^2}{2}$ (6)

Let $q(t_1) = I_{max}$ (7)

Equation (3) and (6) we get

$I_{max} = Pt_1 + \frac{\alpha e^{-\beta t_1}}{\beta} - \frac{\gamma {t_1}^2}{2} - \frac{\alpha}{\beta}$

$t_1 = \frac{I_{max}}{P-\alpha}$ (8)

$$I_{max} = -\frac{\alpha e^{-\beta T}}{\beta} + \frac{\alpha e^{-\beta t_1}}{\beta} + \frac{\gamma T^2}{2} - \frac{\gamma {t_1}^2}{2}$$

$T - t_1 = \frac{I_{max}}{\alpha}$ (9)

$I_{max} = \frac{\alpha(P-\alpha)T}{P}$ (10)

Holding Cost = $C_1\left[\int_0^{t_1} q(t)dt + \int_{t_1}^{T} q(t)dt\right]$

$$= C_1\left(\int_0^{t_1} (P-\alpha)t_1 dt + \int_{t_1}^{T} \alpha(T-t_1)dt\right)$$

$$= C_1\left((P-\alpha)\frac{{t_1}^2}{2} + \alpha\frac{(T-t_1)^2}{2}\right)$$

Holding Cost $= \frac{C_1}{2}\left(1-\frac{\alpha}{P}\right)\alpha T^2$ [Using (8), (9) and (10)]

Total Cost = Purchase Costs +Fixed Ordering cost + Holding Cost + Technology Acquisition Costs + Maintenance and Upgrade Costs + Training Costs + Integration Costs + Cyber security Costs + Depreciation Costs

Total Average Cost (T) TC= $\frac{1}{T}\left(C_2 + C_3 + \frac{C_1}{2}\left(1-\frac{\alpha}{P}\right)\alpha T^2 + T_c + M_c + T_{RC} + I_c + C_c + D_c\right)$ ------- (11)

Differentiate with respect to T

$T = \sqrt{\frac{2(C_2+C_3+T_c+M_c+T_{RC}+I_c+C_c+D_c)}{C_1(1-\frac{\alpha}{P})\alpha}}$ ---------- (12)

6. Illustration

Let us consider an inventory system with the following parameters.

$$\alpha = 0.5, \beta = 0.1, \gamma = 0.05, P = 1000\ units, T_c = 2000, M_c = 1500, T_{Rc} = 1000, I_c = 500,$$
$$C_c = 300, D_c = 800, C_1 = 2, C_2 = 10000, C_3 = 500.$$

By using the above expressions of T, TC and t_1,the following values are obtained.

$$T = 182.25, t_1 = 0.091, TC = 33199.23$$

Neutrosophic Trapezoidal Fuzzy Numbers for Parameters:

$$\alpha = (0.45,0.5,0.55,0.6; [0.9,1], [0.1,0.15], [0,0.05])$$
$$\beta = (0.08,0.1,0.12,0.14; [0.85,0.95], [0.05,0.1], [0,0.02])$$
$$\gamma = (0.04,0.05,0.06,0.07; [0.8,0.9], [0.1,0.2], [0,0.05])$$
$$P = (950,1000,1050,1100; [0.95,1], [0.05,0.1], [0,0.05])$$
$$T_c = (1900,2000,2100,2200; [0.9,0.95], [0.05,0.1], [0,0.05])$$
$$M_c = (1425,1500,1575,1650; [0.9,0.95], [0.05,0.1], [0,0.05])$$
$$T_{RC} = (950,1000,1050,1100; [0.85,0.95], [0.05,0.1], [0,0.05])$$
$$I_c = (475,500,525,550; [0.85,0.95], [0.05,01], [0,0.05])$$
$$C_c = (285,300,315,330; [0.8,0.9], [0.1,0.15, [0,0.05])$$
$$D_c = (760,800,840,880; [0.85,0.9], [0.1,0.15], [0,0.05])$$
$$C_1 = (1.8,2.,2.2,2.4; [0.95,1], [0.05,0.1], [0,0.05])$$
$$C_2 = (9500,10000,10500,11000; [0.9,0.95], [0.05,0.1], [0,0.05])$$
$$C_3 = (475,500,525,550; [0.9,0.95], [0.05,0.1], [0,0.05])$$
$$I_{max} = (180,200,220,240; [0.9,0.95], [0.05,0.1], 0,0.05])$$
$$T = (148.04,165.49,186.76,212.38; [0.8,0.9], [0.05,0.95], [0,0.05])$$
$$t_1 = (0.164,0.191,0.22,0.25; [0.9,0.95], [0.05,0.1], 0,0.05])$$
$$TC = (24642.38,30319,38518.65,50715.82; [0.8,0.9], [0.05,0.95], [0,0.05])$$

Results and Discussion

The above formulated inventory model is validated with both crisp and interval valued trapezoidal neutrosophic numbers and the results obtained are presented in the Table 1.

Table 1. Result of both crisp and interval valued trapezoidal neutrosophic numbers

Results	Crisp data input	Interval valued trapezoidal neutrosophic data input
T	182.25	(148.04,165.49,186.76,212.38; [0.8,0.9], [0.05,0.95], [0,0.05])
t_1	0.091	(0.164,0.191,0.22,0.25; [0.9,0.95], [0.05,0.1], 0,0.05])
TC	33199.23	(24642.38,30319,38518.65,50715.82; [0.8,0.9], [0.05,0.95], [0,0.05])

8. Sensitivity Analysis

The parameter α are varied and the respective changes on T and TC are determined with both crisp and neutrosophic input.

Case (1) Variations of the parameter α with Crisp data input

The values obtained on varying α are presented in Table 2.

Table 2. Variation of α with Crisp data

α	T	Total Average Cost
0.1	407.451349	81.482121
0.2	288.126020	115.227358
0.3	235.265676	141.117058
0.4	203.756243	162.939793
0.5	182.254241	182.163114
0.6	166.382922	199.539710
0.7	154.048360	215.516737
0.8	144.106257	230.385555
0.9	135.871481	244.348554
1.0	128.905456	257.553101

The respective graphical representations are presented in Fig.3.

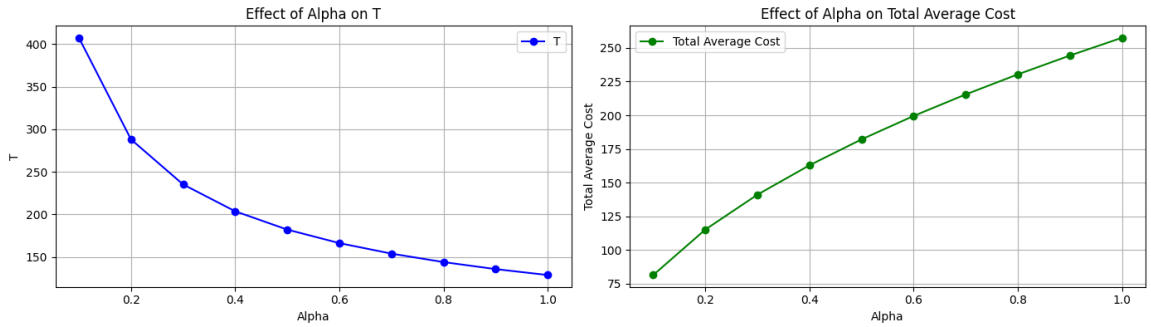

Fig.3. Graphical representations on Effect of α on T and Total Averaege cost using Crisp data

Case (2) Variations of the parameter α with Neutrosophic data input are presented in Table 3.

Table 3. Variation of α with Neutrosophic data

Parameter Value	α	T	Total Average Cost
0	0.400	194.754022	151.260034
1	0.425	188.941646	155.913218
2	0.450	183.620697	160.431261
3	0.475	178.725620	164.825278
4	0.500	174.202484	169.104937

The respective graphical representations are presented in Fig.4.

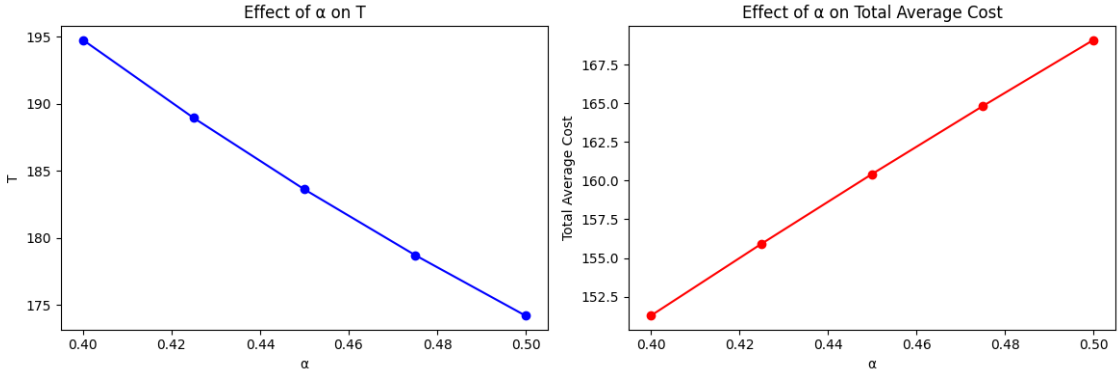

Fig.4. Graphical representations on Effect of α on T and Total Averaege cost using Neutrosophic data

In both the figures[3,4] the effects of α on T and TC are determined. However, the parameter α has positive correlation with TAC and negative correlation with T. The graphical representation of the effect of α is more vivid in terms of neutrosophic data input in comparison with the crisp data input. This shows the efficacy of neutrosophic data representations. In a similar fashion the other cost parametric effects shall be determined.

Case (3) Variations of the parameter α, β, γ on q(t):

The Sensitivity Analysis for α are presented in the below Table 4.

Table 4. Sensitivity Analysis for α

qα=0.45	qα=0.5	qα=0.55	qα=0.6
100.964620	100.959595	100.954570	100.949545
201.929185	201.919185	201.909185	201.899186
302.893689	302.878765	302.863841	302.848916
403.858129	403.838329	403.818530	403.798731
…	…	…	…
9590.881255	9590.572780	9590.264304	9589.955828
9691.825314	9691.514913	9691.204513	9690.894112
9792.769036	9792.456730	9792.144424	9791.832118
9893.712421	9893.398229	9893.084036	9892.769844
9994.655466	9994.339406	9994.023347	9993.707287

The graphical representation of Sensitivity Analysis for α is represented in fig.5.

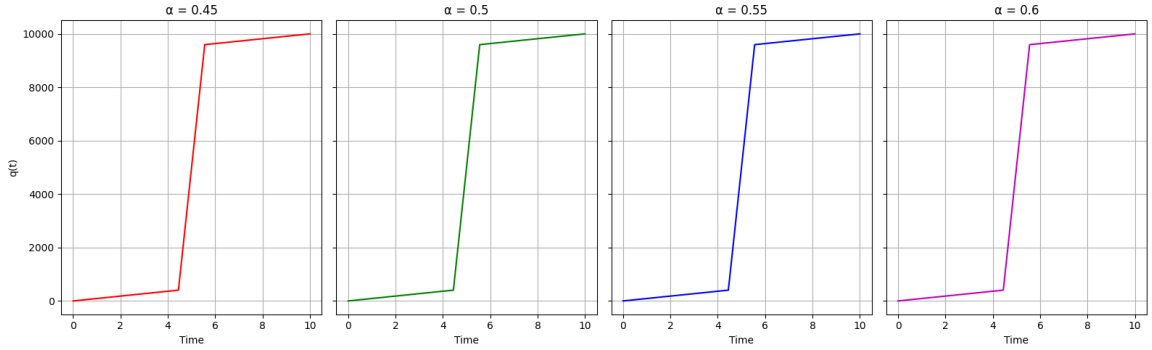

Fig.5. Graphical representation of Sensitivity Analysis for α

The Sensitivity Analysis for β are presented in the below Table 5.

Table 5. Sensitivity Analysis for β

qβ = 0.08	qβ =0.1	qβ = 0.12	qβ = 0.14
100.959544	100.959595	100.959646	100.959696
201.918983	201.919185	201.919386	201.919587
302.878314	302.878765	302.879214	302.879661
403.837533	403.838329	403.839122	403.839910
…	…	…	…
9590.308101	9590.572780	9590.808201	9591.018079
9691.246139	9691.514913	9691.753712	9691.966374
9792.183854	9792.456730	9792.698904	9792.914343
9893.121245	9893.398229	9893.643775	9893.861983

qβ = 0.08	qβ =0.1	qβ = 0.12	qβ = 0.14
9994.058312	9994.339406	9994.588323	9994.809291

The graphical representation of Sensitivity Analysis for β is represented in fig.6.

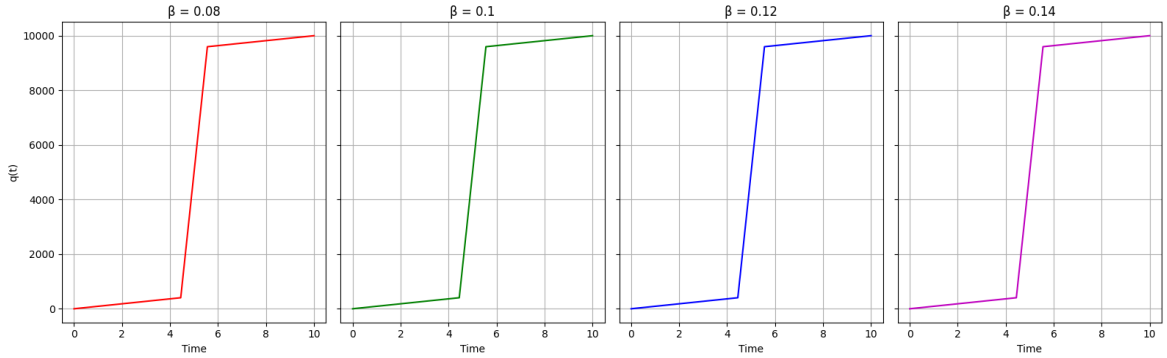

Fig.6. Graphical representation of Sensitivity Analysis for β

The Sensitivity Analysis for β are presented in the below Table 6.

Table 6. Sensitivity Analysis for γ

qγ = 0.04	qγ = 0.05	qγ = 0.06	qγ = 0.07
100.959646	100.959595	100.959544	100.959493
201.919389	201.919185	201.918981	201.918777
302.879224	302.878765	302.878306	302.877847
403.839146	403.838329	403.837513	403.836697
...	...	...	...
9591.033192	9590.572780	9590.112367	9589.651955
9691.985069	9691.514913	9691.044757	9690.574601
9792.936732	9792.456730	9791.976728	9791.496726
9893.888179	9893.398229	9892.908279	9892.418329
9994.839406	9994.339406	9993.839406	9993.339406

The graphical representation of Sensitivity Analysis for γ is represented in fig.7.

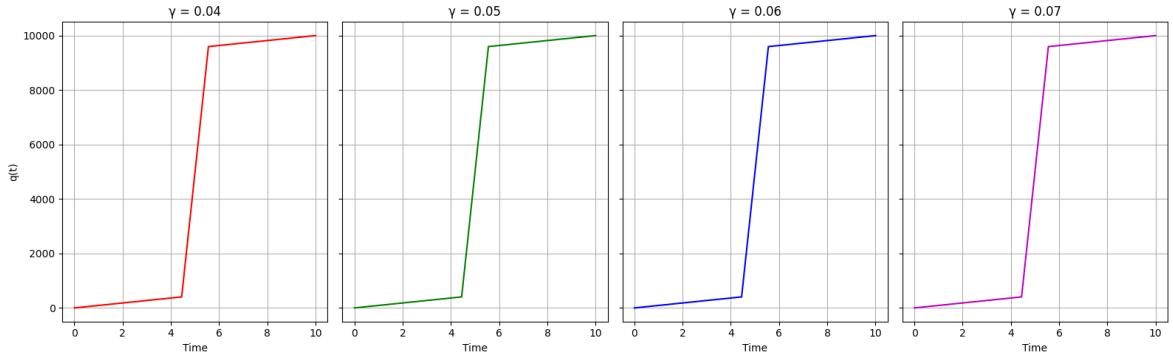

Fig.7. Graphical representation of Sensitivity Analysis for γ

From the above Table [4,5,6] values and respective graphical representations the following inferences on the trend patterns are obtained

The increase in α values causes slight decrease in q(t) over time and then increases rapidly.

The system behaviour is dampened by the impacts of variations in the values of α over a period of time.

The increase in β values causes marginalincrease in q(t) over time with very small variations initially.

The parameter β influences the long-term behaviour and resilience of the system.

The increase in γ values causes decrease in the values of q(t) over a period of time.

The system behaviour is dampened by the impacts of variations in the values of γover a period of time.

All the three parameters α, β and γ causes significant impacts on q(t) enhances the resilience of the system behaviour. The changes in the parameters enables to comprehend the behaviour of the system with respect to output maximization with assurance of the system stability.

9. Industrial Implications

The neutrosophic model with technology dependent demand pattern is more concerned with industry 5.0 and this has high association with industries especially embracing the advanced versions of human -machine collaboration. The discussion of inventory modelling with technology-oriented demand facilitates the decision makers to gain more insights on the costs associated with technology and their influences over the total inventory costs. The variation of the parameters brings the picturization of the demand impacts on inventory decisions. This model also supports manufacturing system with human-machine interferences to devise suitable strategies for cost minimization and profit maximizations. Also, the customer's demand depending on the technology advancements is well reflected in this modelling. The neutrosophic parameter representations model is studied by the influence of the parameters. This neutrosophic model proposed in this work shall be extended by studying the demand patterns influenced by the parameters of sustainability. This work provides room for developing more research ideas of industry and inventory 5.0provides a suitable framework for the managerial to optimize the production based on the demand pattern discussed in this model.

Conclusion

The novel inventory model proposed in this research work with neutrosophic representations is more feasible and comprehensive in nature. The illustration of the model with both crisp and neutrosophic data lays a clear picture of the efficiency of neutrosophic representations over crisp

forms. The sensitivity analysis also favours neutrosophic representations. The dynamic behaviour of the model is studied by the influence of the

parameters. This neutrosophic model proposed in this work shall be extended by studying the demand patterns influenced by the parameters of sustainability. This work provides room for developing more research ideas of industry and inventory 5.0

Reference

1. Hussein, G. S., Zaied, A. N. H., & Mohamed, M. (2023). ADM: Appraiser Decision Model for Empowering Industry 5.0-Driven Manufacturers toward Sustainability and Optimization: A Case Study. *Neutrosophic systems with applications*, *11*, 22-30.
2. Alojaiman, B. (2023). Technological modernizations in the industry 5.0 era: A descriptive analysis and future research directions. *Processes*, *11*(5), 1318.
3. Özdemir, V., & Hekim, N. (2018). Birth of industry 5.0: Making sense of big data with artificial intelligence,"the internet of things" and next-generation technology policy. *Omics: a journal of integrative biology*, *22*(1), 65-76.
4. Kar, C., Mondal, B., & Roy, T. K. (2018). An inventory model under space constraint in neutrosophic environment: a neutrosophic geometric programming approach. *Neutrosophic Sets and Systems: An International Book Series in Information Science and Engineering*, *21*(2018), 93-109.
5. Mullai, M., & Broumi, S. (2018). Neutrosophic inventory model without shortages. *Asian Journal of Mathematics and Computer Research*, *23*(4), 214-219.
6. Das, S. K., & Islam, S. (2019). *Fuzzy Multi Item Inventory Model with Deterioration and Demand Dependent Production Cost Under Space Constraint: Neutrosophic Hesitant Fuzzy Programming Approach*. Infinite Study
7. Mullai, M., & Surya, R. (2020). Neutrosophic inventory backorder problem using triangular neutrosophic numbers. *Neutrosophic Sets and Systems*, *31*(1), 148-155.
8. Mullai, M., Sangeetha, K., Surya, R., Kumar, G. M., Jeyabalan, R., & Broumi, S. (2020). *A single valued neutrosophic inventory model with neutrosophic random variable*. Infinite Study
9. Islam, S. (2021). Fuzzy Multi-Objective Inventory Model for Deteriorating Items, with Shortages Under Space Constraint: Neutrosophic Hesitant Fuzzy Programming Approach. *Neutrosophic Operational Research: Methods and Applications*, 197-219.
10. Mondal, B., Garai, A., Mukhopadhyay, A., & Majumder, S. K. (2021). Inventory policies for seasonal items with logistic-growth demand rate under fully permissible delay in payment: a neutrosophic optimization approach. *Soft Computing*, *25*(5), 3725-3750.
11. Rajeswari, S., Sugapriya, C., Nagarajan, D., Broumi, S., & Smarandache, F. (2021). Octagonal fuzzy neutrosophic number and its application to reusable container inventory model with container shrinkage. *Computational and Applied Mathematics*, *40*, 1-20.
12. Jdid, M., Alhabib, R., & Salama, A. A. (2021). The static model of inventory management without a deficit with Neutrosophic logic.
13. Sugapriya, C., Rajeswari, S., Nagarajan, D., & Al Amri, Z. S. M. (2022). An effective container inventory model under bipolar neutrosophic environment. *Neutrosophic Sets and Systems, vol. 50/2022: An International Journal in Information Science and Engineering*, 336.
14. Bhavani, G. D., Georgise, F. B., Mahapatra, G. S., & Maneckshaw, B. (2022). Neutrosophic cost pattern of inventory system with novel demand incorporating deterioration and discount on defective items using particle swarm algorithm. *Computational Intelligence and Neuroscience*, *2022*(1), 7683417.
15. Sen, J. D., & Chakrabarti, T. (2022). An industrial production inventory model with deterioration under neutrosophic fuzzy optimization. *Neuroquantology*, *20*(19), 4399.

16. Sarkar, D., & Srivastava, P. K. (2022). A dynamic optimization approach for multi-item multi-objective fixed-place neutrosophic fuzzy inventory and green management system. *Mathematics in Engineering, Science & Aerospace (MESA)*, *13*(2).
17. Das, K., & Islam, S. (2022). A multi-objective shortage follow inventory (SFI) model involving ramp-type demand, time varying holding cost and a marketing cost under neutrosophic programming approach. *Neutrosophic Sets and Systems*, *49*, 48-69.
18. Surya, R., & Mullai, M. (2022). Neutrosophic multi-item inventory control models with constraints. *Journal of Intelligent & Fuzzy Systems*, *43*(6), 8127-8136.
19. Jayanthi, K. (2022). A Fuzzy Differential Approach to An Overage Management with Customer Acquisition Inventory Problem Via Neutrosophic Fuzzy Geometric Programming Technique. *Journal of Algebraic Statistics*, *13*(3), 215-225
20. Rajeswari, S(2022). Optimization Techniques For Some Efficient Inventory Systems Using Fuzzy And Neutrosophic Models.
21. Barman, H., Roy, S. K., Sakalauskas, L., & Weber, G.W. (2023). Inventory model involving reworking of faulty products with three carbon policies under neutrosophic environment. *Advanced Engineering Informatics*, *57*, 102081.
22. Jdid, M. (2023). Important Neutrosophic Economic Indicators of the Static Model of Inventory Management without Deficit. *Journal of Neutrosophic and Fuzzy Systems (JNFS) Vol*, *5*(01), 08-14.
23. Pattnaik, S., Nayak, M. M., & Kumar, P. (2023). An Inventory Control Model in the Framework of COVID-19 Disruptions Considering Overage Items with Neutrosophic Fuzzy Uncertainty. *Neutrosophic Sets and Systems*, *56*(1), 9.
24. Bhavani, G. D., & Mahapatra, G. S. (2023). Inventory system with generalized triangular neutrosophic cost pattern incorporating maximum life-time-based deterioration and novel demand through PSO. *Soft Computing*, *27*(5), 2385-2402.
25. Mohanta, K., Jha, A. K., Dey, A., & Pal, A. (2023). An application of neutrosophic logic on an inventory model with two-level partial trade credit policy for time-dependent perishable products. *Soft Computing*, *27*(8), 4795-4822.
26. MOHANTA, K., Sarkar, S., Dey, A., De, D., & Pal, A. (2023). Cost Optimization of an Inventory Control Problem in Neutrosophic Environment for Smart Manufacturing
27. Kalaiarasi, K., & Swathi, S. (2023). Neutrosophic Inventory Model With Quick Return For Damaged Materials And Python-Analysis. *2023 Neutrosophic SuperHyperAlgebra And New Types of Topologies*, 219.
28. Renee Miriam, M., Martin, N., Aleeswari, A., & Broumi, S.(2023) Rework Warehouse Inventory Model for Product Distribution with Quality Conservation in Neutrosophic Environment. *Doi: https://doi. org/10.54216/IJNS*, *210215*
29. Dubey, A., Dey, A., Broumi, S., & Kumar, R. (2024). A Survey on Neutrosophic Principles for Inventory Management Problem. *Neutrosophic Sets and Systems*, *68*(1), 19.
30. Supakar, P., Manna, A. K., Mahato, S. K., & Bhunia, A. K. (2024). Application of artificial bee colony algorithm on a green production inventory problem with preservation for deteriorating items in neutrosophic fuzzy environment. *International Journal of System Assurance Engineering and Management*, *15*(2), 672-686.
31. Moorthy, C. B., Rajani, D., Pushpalatha, A. P., Ramya, S., Selvaraj, A., & Tiwari, M. (2024). Enhancing Inventory Management through Advanced Technologies and Mathematical Methods: Utilizing Neutrosophic Fuzzy Logic. *Full Length Article*, *24*(4), 50-0.
32. Surya, R., Mullai, M., & Vetrivel, G. (2024). Neutrosophic Inventory System for Decaying Items with Price Dependent Demand. *Neutrosophic Sets and Systems*, *66*, 55-75.
33. Kar, C., De, M., Maiti, M., & Das, P. (2024). Multi-objective perishable multi-item green inventory models with uncertain finite time horizons and constraints by neutrosophic optimisation approach. *International Journal of Mathematics in Operational Research*, *27*(2), 167-198.

34. Das, K., & Islam, S. (2024). A deterministic multi-item inventory model with quadratic demand under neutrosophic and pythagorean hesitant fuzzy programming approach. *Results in Control and Optimization*, *14*, 100367.
35. Dubey, A., & Kumar, R. (2024). Neutrosophic Inventory Management: A Cost-Effective Approach. *Economics*, *18*(1), 20220101
36. Martin, N., Mayan, M. K., & Smarandache, F. (2020). *Neutrosophic optimization of industry 4.0 production inventory model* (Vol. 38). Infinite Study.
37. Martin, N., Mayan, M. K., & Smarandache, F. (2020). *Neutrosophic optimization of industry 4.0 production inventory model* (Vol. 38). Infinite Study.
38. Saini, R. K., Sangal, A., & Ahirwar, A. (2022). A novel approach by using interval-valued trapezoidal neutrosophic numbers in transportation problem. *Neutrosophic Sets Syst*, *51*, 234-253.

Neutrosophic Logic to Navigate Uncertainty of Security Events in Mexico

Mohammad Saif Wajid[1], Hugo Terashima-Marin[2], Mohd Anas Wajid[3], Florentin Smarandache[4], Satya Bhushan Verma[5], and Mohammad Kaif Wajid[6]

[1]Department of Computer Science, School of Engineering and Sciences, Tecnológico de Monterrey, Monterrey, Mexico. E-mail: mohdsaif06@gmail.com
[2]Department of Computer Science, School of Engineering and Sciences, Tecnológico de Monterrey, Monterrey, Mexico. E-mail: terashima@tec.mx
[3]Institute For the Future of Education, Tecnologico de Monterrey, Monterrey, Mexico. E-mail: anaswajid.bbk@gmail.com
[4]University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA. E-mail: smarand@unm.edu
[5]Srmuniversity, Lucknow, India. E-mail: satyabverma1@gmail.com
[6]Institute of Management Sciences, Lucknow University, India.

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Abstract: A significant degree of uncertainty surrounds security events in Mexico because of the interaction of multiple socioeconomic and political elements. Conventional probabilistic and logical approaches frequently fall short of properly handling this uncertainty. To effectively navigate and manage security incidents in Mexico, this study suggests applying neutrosophic logic, an extension of classical logic that allows for indeterminacy and partial knowledge. Neutrosophic logic provides a more adaptable framework for making decisions in uncertain contexts by simultaneously handling true, false, and indeterminate values. In this study, we have created a neutrosophic-based model that includes metrics like social unrest, economic instability, and crime rates to provide a thorough evaluation of the security environment. When the model is put to the test using actual data from different parts of Mexico, it outperforms conventional techniques in terms of prediction and interpretation of security occurrences. The findings demonstrate how neutrosophic logic helps security analysts and policymakers make more thoughtful and nuanced judgments by offering insightful information about the ambiguous nature of security occurrences. The ability to foresee and address possible risks is much improved when neutrosophic logic is incorporated into security event analysis. This not only improves overall security management but also creates new opportunities for its application in other sectors where uncertainty is crucial.

Keywords: Neutrosophic Logic, Security Events, Uncertainty, Mexico, Decision-Making, Predictive Analysis, Crime Rates, Socio-Economic Factors, Indeterminacy

1. Introduction

Mexico's security landscape has long been fraught with complexities, shaped by a convergence of organized crime, drug trafficking, violence, political corruption, and social instability. In recent decades, the country has faced escalating security threats, particularly from powerful drug cartels, transnational criminal organizations, and increasing levels of violent crime. The unpredictability of these threats presents a daunting challenge to government agencies, security forces, and policymakers who must continuously adapt to an evolving and uncertain environment [1]. Traditional approaches to security management, which often rely on deterministic models and

binary logic, struggle to effectively handle the complexity and uncertainty intrinsic to these events. The crux of the problem lies in the uncertainty surrounding security events. Security analysts and decision-makers are frequently confronted with incomplete, contradictory, or imprecise information [2].

For example, intelligence reports may be based on conflicting sources, and the outcomes of potential interventions may be difficult to predict due to rapidly changing circumstances. In such scenarios, the ability to incorporate and reason with uncertainty is crucial for making informed decisions. Traditional binary logic—where a statement is either true or false—fails to account for the nuances of real-world data, where information may be partially true, indeterminate, or evolving [3].

To address this gap, neutrosophic logic offers a novel approach. Introduced by Florentin Smarandache in the late 1990s, neutrosophic logic extends classical and fuzzy logic by introducing three distinct components: truth (T)**,** indeterminacy (**I**), and falsity (F)**.** These parameters allow for a more flexible interpretation of reality, where propositions are not simply true or false, but can also include elements of uncertainty or contradiction. This is particularly relevant in security contexts, where data can often be incomplete, unreliable, or fluid. Neutrosophic logic provides the tools to model these ambiguities, allowing for better decision-making under uncertain conditions. In the context of Mexico's security challenges**,** neutrosophic logic can be a powerful tool for navigating the complex, uncertain, and rapidly changing landscape. By offering a more comprehensive framework for understanding and responding to security threats, neutrosophic logic can assist in areas such as crime prediction, risk assessment, and strategy formulation. This logic allows decision-makers to model not just the known elements of a security event, but also the unknowns and contradictions that often accompany them. This paper aims to explore the application of neutrosophic logic to security events in Mexico, focusing on how it can provide a structured approach to handling uncertainty. We will begin by examining Mexico's current security environment, highlighting the unpredictable nature of crime and violence in the country. Next, we will introduce the principles of neutrosophic logic and discuss its theoretical foundations. Finally, we will present practical examples of how this logic can be applied to real-world security scenarios in Mexico, offering insights into how it can improve decision-making and policy formulation in the face of uncertainty [4].

1.1. Security Events

The incidents or occurrences that pose a risk to the safety, stability, and well-being of a nation, its citizens, and its institutions are referred to as security events. These events can range from organized crime, terrorism, and political unrest to cyber-attacks, natural disasters, and civil disobedience. Security events may involve both internal and external threats, such as domestic violence, insurgencies, cross-border conflicts, or cyber warfare. In the context of Mexico, for instance, security events often include cartel violence, kidnappings, organized crime, extortion, and widespread corruption. However, security events aren't limited to criminal activity; they can also include mass protests, economic instability leading to civil unrest, and even natural disasters like earthquakes or hurricanes, which can destabilize regions and strain government resources [5].

The significance of security events lies in their profound impact on a nation's stability, governance, and quality of life. These events, such as violence, terrorism, organized crime, and political unrest, directly threaten public safety, erode trust in institutions, and destabilize economies. In countries like Mexico, the persistence of security events, particularly driven by drug cartels and organized crime, affects social cohesion, disrupts daily life, and hinders economic growth by deterring investment and tourism. Security issues also undermine the rule of law, leading to impunity and weakening democratic governance. Addressing these events is critical for protecting citizens, preserving social order, and ensuring long-term national development [6]. A secure environment enables not only the physical safety of individuals but also promotes political stability,

economic prosperity, and international standing, all essential for the betterment of a country. Determinate factors are the underlying causes or contributors that shape and influence the occurrence of security events. These factors can vary based on geographical, socio-economic, political, and historical contexts, but they generally include elements such as crime, corruption, poverty, governance failures, and social inequality. Understanding these determinate factors is crucial for identifying the root causes of security issues and developing effective strategies to prevent and mitigate violence. For the current study, they are considered as determinate factors throughout this paper. These are as follows[7], [8], [9]:

- Drug Trafficking and Organized Crime (Df1)
- Corruption and Weak Governance (Df2)
- Poverty and Social Inequality (Df3)
- Weak Rule of Law and Impunity (Df4)
- Availability of Firearms (Df5)
- Territorial Control and Fragmentation of Cartels (Df6)
- Migration and Human Trafficking (Df7)
- Political Instability and Social Unrest (Df8)

The rest of the paper is divided into 5 sections. Section 2 and Section 3 present extensive literature and methodology respectively. Section 4 is focused on results and discussion whereas Section 5 presents the conclusion of the present work.

2. Literature Survey

Many studies focus on analyzing the root causes of violence and insecurity in Mexico. For instance, Astorga (2020) explores the roles of political corruption, social inequality, and the weakness of the rule of law as key contributors to the rise of organized crime in the country. Similarly, Buscaglia (2018) provides a comprehensive analysis of how the lack of government control in certain regions has allowed criminal organizations to flourish, leading to greater uncertainty about security outcomes [10].

However, despite the extensive body of work on the causes of security issues in Mexico, there is relatively little research on how to manage the uncertainty that these security threats produce. This gap presents an opportunity to apply advanced logical frameworks, such as neutrosophic logic, to better understand and address the unpredictable nature of security events in the country.

Neutrosophic logic has gained traction in recent years for its applications in decision-making processes, particularly in fields that involve uncertainty, such as engineering, medicine, and economics. One notable example is the use of neutrosophic sets in multi-criteria decision-making (MCDM) frameworks. These frameworks are often employed in situations where decision-makers must choose between multiple alternatives based on incomplete or uncertain information. In a study by Ye, neutrosophic logic was applied to assess risks in industrial processes, demonstrating its effectiveness in environments characterized by uncertain and indeterminate conditions. Other research has focused on using neutrosophic logic to improve decision-making in medical diagnosis, where conflicting data can lead to ambiguous results. These applications suggest that neutrosophic logic could similarly be applied to the complex decision-making processes involved in security events in Mexico, where risks must be assessed under conditions of uncertainty and incomplete information [11]. The potential of neutrosophic logic for risk assessment has been demonstrated in several studies. For instance, Smarandache applied neutrosophic models to security risk analysis in infrastructure systems, proving its effectiveness in handling indeterminate situations where traditional logic failed to provide conclusive outcomes. Given the complexity of security threats in Mexico, this approach could be highly relevant, as it would allow for more comprehensive risk assessments that take into account not only known threats but also the uncertainties surrounding

those threats. Indeterminacy, or the presence of uncertain, ambiguous, or incomplete factors, plays a significant role in the perpetuation of violence, particularly in complex environments like Mexico's security landscape. Indeterminate factors make it difficult to predict or fully understand the dynamics behind violent events, complicating efforts to devise effective responses [12]. For instance, inconsistent or unreliable intelligence reports, fluctuating alliances between criminal organizations, and blurred lines between corrupt officials and law enforcement contribute to an indeterminate environment. Additionally, the role of external influences, such as illegal arms trafficking and international drug markets, adds another layer of unpredictability to the situation. These indeterminate factors exacerbate the uncertainty faced by policymakers and law enforcement, making it harder to implement sustainable security solutions. The unpredictability of violence in Mexico, driven by a combination of socio-economic, political, and international variables, underscores the need for flexible decision-making frameworks like neutrosophic logic that can accommodate and navigate such indeterminacies.

Many researchers have emphasized the assessments happening in institutions for the successful implementation of educational models. Black and Wiliam analyze various assessment techniques and their effects on learning [13].

They highlight that the effectiveness of assessments depends on their design, implementation, and students' perceptions. The impact of assessment techniques is indeterminate due to differences in assessment types, teacher practices, and student responses to assessments.

There are as follows as are termed as indeterminate or uncertain throughout the study [14], [15], [16], [17], [18], [19]:

- Exposure to Violence (Idf1)
- Family Dynamics (Idf2)
- Socioeconomic Conditions (Idf3)
- Brain Abnormalities (Idf4)
- Immediate Stressors (Idf5)
- Provocation (Idf6)
- Substance Abuse (Idf7)

3. Methodology

The proposed solution to indeterminacy uses the concept of Neutrosophic Cognitive Map (NCMs). It is a technique in Neutrosophy introduced by W. B. Vasantha Kandasamy [20]. The concept of Neutrosophic logic introduced by Florentine Smarandache [21], which is a merger of the fuzzy logic together with the inclusion of indeterminacy. When data under scrutiny contains concepts which are indeterminate, we are not able to formulate mathematical expression. Presentation of Neutrosophic logic by Florentine Smarandache [22] has put forward a panacea to this problem. It is the reason Neutrosophy has been introduced as an additional notion for evaluation educational models. Fuzzy theory evaluates the existence or non-existence of associateship but it has failed to attribute the indeterminate relations among concepts and most data collected in educational setup has much indeterminate and uncertain concepts. Therefore we have employed Neutrosophic Cognitive Maps (NCMs) in place of Fuzzy Cognitive Maps (FCMs) to show the significance of indeterminate factors of violence in Mexico.

3.1. Neutrosophic Concepts

3.1.1. Neutrosophic Sets:

Neutrosophic sets are an extension of classical sets and fuzzy sets that handle uncertainty, indeterminacy, and contradiction in data. A classical set only allows elements to be either fully part

of the set or not (0 or 1). Fuzzy sets extend this by allowing partial membership (values between 0 and 1), but neutrosophic sets go further by introducing three components:

1. Truth (T): The degree to which a statement is true.
2. Indeterminacy (I): The degree to which it is uncertain or indeterminate.
3. Falsity (F): The degree to which a statement is false.

In a neutrosophic set, each element has a degree of truth, indeterminacy, and falsity, all ranging from 0 to 1, and they don't have to sum to 1 (unlike fuzzy sets). This flexibility allows for more nuanced representation of uncertainty and contradictory information.

3.1.2. Neutrosophic Cognitive Maps (NCMs)

3.2. FCM Adjacency Matrix

An **FCM Adjacency Matrix** is a mathematical tool used to represent the relationships between concepts in a Fuzzy Cognitive Map (FCM). It is a square matrix where each row and column corresponds to a concept in the system, and the entries in the matrix reflect the strength and type of influence one concept has on another. The values in the matrix typically range from -1 to 1, where positive values indicate a positive influence, negative values signify a negative influence, and zero means there is no direct influence between the concepts. This matrix helps to model complex systems by showing how changes in one concept can affect others, enabling the analysis and simulation of dynamic behaviors. The FCM adjacency matrix is particularly useful for studying decision-making processes, forecasting, and understanding the interactions in systems with interdependent factors, such as social, economic, or environmental systems.

Let's consider a system with 3 concepts: C1C_1C1 (Economic Growth), C2C_2C2 (Employment Rate), and C3C_3C3 (Public Spending). The FCM adjacency matrix could look like this:

$$\begin{pmatrix} 0 & 0.6 & -0.2 \\ 0.7 & 0 & 0.3 \\ 0 & -0.4 & 0 \end{pmatrix}$$

- w_12=0.6 means that "Economic Growth" has a positive influence on "Employment Rate".
- w_13 means that "Economic Growth" negatively affects "Public Spending".
- w_21=0.7 means that "Employment Rate" positively affects "Economic Growth".
- w_23=0.3 means that "Employment Rate" positively influences "Public Spending".
- w_32=−0.4 means that "Public Spending" negatively influences "Employment Rate"
- Diagonal entries are typically 0 because a concept does not directly influence itself.

3.3. NCM Adjacency Matrix

The **NCM Adjacency Matrix** is an extension of the Fuzzy Cognitive Map (FCM) adjacency matrix, used to represent the relationships between concepts in a **Neutrosophic Cognitive Map (NCM)**. Unlike FCMs, where each connection between concepts is represented by a single value (indicating positive or negative influence), the NCM adjacency matrix incorporates three components to account for the uncertainty and indeterminacy in relationships. Each entry in the NCM adjacency matrix consists of three values: **Truth (T)**, **Indeterminacy (I)**, and **Falsity (F)**.

Example:

Consider a system with 3 concepts: C1C_1C1 (Economic Stability), C2C_2C2 (Inflation Rate), and C3C_3C3 (Employment Levels). The NCM adjacency matrix could look like this:

$$\begin{pmatrix} (0,0,0) & (0.6,0.2,0.1) & (0.4,0.3,0.1) \\ (0.5,0.3,0.2) & (0,0,0) & (0.7,0.1,0.1) \\ (0.3,0.4,0.2) & (0.6,0.2,0.2) & (0,0,0) \end{pmatrix}$$

- The relationship between "Economic Stability" and "Inflation Rate" is (0.6,0.2,0.1)(0.6, 0.2, 0.1)(0.6,0.2,0.1), meaning there is a 60% positive influence, 20% uncertainty, and 10% negative influence.
- The relationship between "Inflation Rate" and "Employment Levels" is (0.7,0.1,0.1)(0.7, 0.1, 0.1)(0.7,0.1,0.1), indicating a strong positive influence with little uncertainty and falsity.

Neutrosophic Cognitive Maps (NCMs) are an advanced extension of Fuzzy Cognitive Maps (FCMs), designed to model complex systems where uncertainty, indeterminacy, and contradiction play a significant role. Like FCMs, NCMs consist of concepts representing variables within a system, connected by edges that depict the influence between these concepts. However, in NCMs, each connection is characterized by three components: truth (T), indeterminacy (I), and falsity (F), allowing for a more nuanced understanding of relationships. This structure helps capture not just the positive or negative influence between concepts but also the uncertainty and incomplete information that might exist in real-world scenarios. NCMs are particularly useful for dynamic, evolving systems—such as social, economic, or environmental models—where information is often ambiguous or contradictory, providing a more flexible and realistic framework for analysis and decision-making.

In the following NCM diagram, each node represents a concept such as "Economic Stability" or "Inflation Rate," and each directed edge between nodes represents the influence one concept has on another. The relationships between nodes are described by three values: Truth (T),

Indeterminacy (I), and Falsity (F), are displayed as labels on the edges.

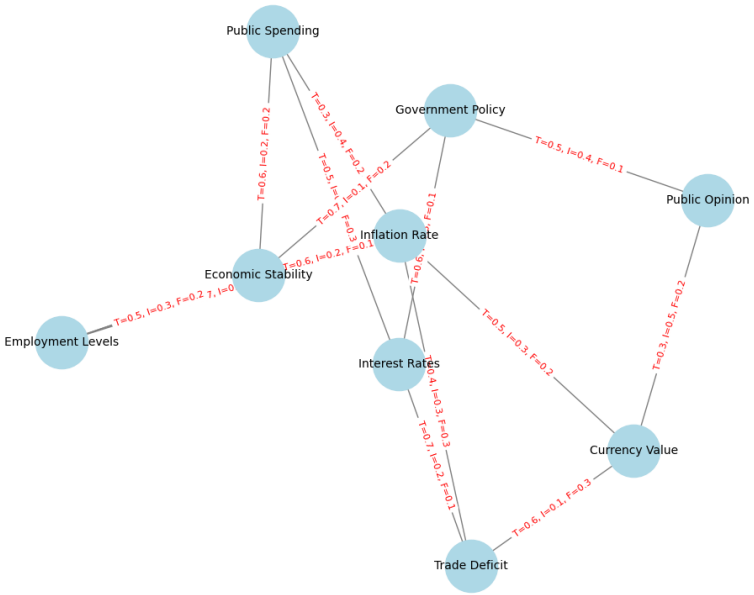

Figure 1: Neutrosophic Cognitive Maps for Economic Stability

4. Results

After taking all these factors & their relationship to violence, we must model the condition of violence based on the concept of Neutrosophic Cognitive Maps. This is the graph for mapping the condition of factors related to violence. Nodes denote factors and edges denote the relationship from node to node or from node to factors. Some nodes have representation like *Df*1, *Df*2, *Df*3, ……., and *D*f7 known as Determinate Factors, and other nodes have representation like *Idf*1, *Idf*2, *Idf*3, ……*I*df8 known as Indeterminate Factors. The edges have weight "1" represent known or determinate edges and the edges having representation "*I*" represent Indeterminate edges. The graph contains two types of edges, the first type of edge is the complete edge as we have seen in the normal graph, this edge shows determinacy and another type of edge is a dotted edge which represents indeterminacy. We took Blue color for showing Determinate Factors and Red color for showing Indeterminate Factors respectively as we have in Figure 2. The straight simple line shows the relationship between Determinate Factors and dotted lines show the relationship between Indeterminate Factors. The edges have

weight "*I*" representing known or determinate edges and the edges having representation "*I*" represent Indeterminate edges. We have formulated the adjacency matrix given in Table 1 for the graph(Figure 2).

4.1 Determinate and Indeterminate Factors | for Affecting Violence

The literature review identified numerous factors that contribute to shaping the conditions of violence. Table 1 presents the factors categorized as determinate, while Table 2 highlights those considered indeterminate.

Table 1. Determinate Factors

S. No.	Determinate Factors	Presentations
1	Drug Trafficking and Organized Crime	Df1
2	Corruption and Weak Governance	Df2
3	Poverty and Social Inequality	Df3
4	Weak Rule of Law and Impunity	Df4
5	Availability of Firearms	Df5
6	Territorial Control and Fragmentation of Cartels	Df6
7	Migration and Human Trafficking	Df7

Table 2. Indeterminate Factors

S. No.	Indeterminate Factors	Presentations
1	Exposure to Violence	Idf1
2	Family Dynamics	Idf2
3	Socioeconomic Conditions	Idf3

S. No.	Indeterminate Factors	Presentations
4	Brain Abnormalities	Idf4
5	Immediate Stressors	Idf5
6	Provocation	Idf6
7	Substance Abuse	Idf7
8	Political Instability and Social Unrest	Df8

4.2 NCM graph of factors

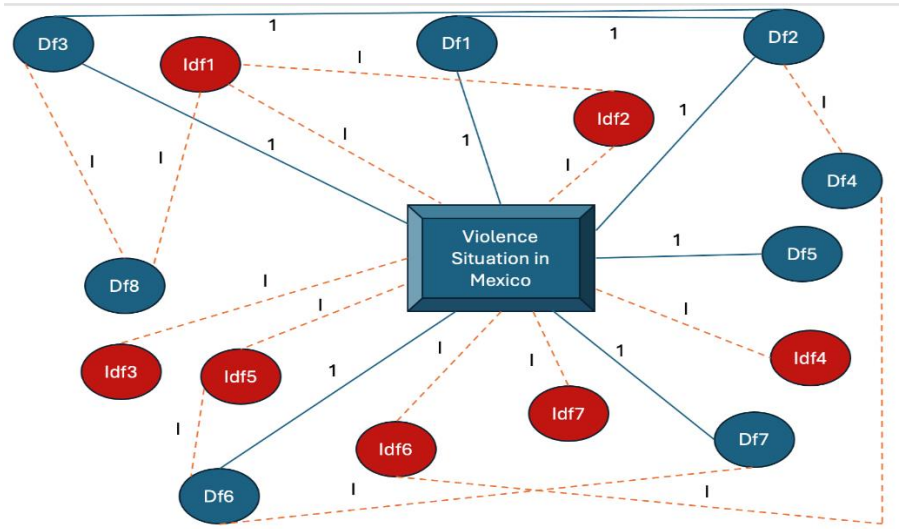

Figure 2: Neutrosophic Cognitive Map based on determinate and indeterminate factors affecting violence in Mexico

4.3 Adjacency matrix

Table 3. AdjecencyMarix

	VSI_M	Df1	Idf1	Df2	Idf2	Df3	Idf3	Df4	Idf4	Df5	Idf5	Df6	Idf6	Df7	Idf7	Df8
VSI_M	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
Df1	**1**	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
Idf1	**0**	0	0	0	I	0	0	0	0	0	0	0	0	0	0	I
Df2	**1**	0	0	0	0	0	0	I	0	0	0	0	0	0	0	0
Idf2	**0**	0	I	0	0	0	0	0	0	0	0	0	0	0	0	0
Df3	**1**	0	0	1	0	0	0	0	0	0	0	0	0	0	0	I
Idf3	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Df4	**1**	0	0	I	0	0	0	0	0	0	0	I	0	0	0	0
Idf4	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Df5	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Idf5	**0**	0	0	0	0	0	0	0	0	0	0	0	I	1	0	0

	VSI_M	Df1	Idf1	Df2	Idf2	Df3	Idf3	Df4	Idf4	Df5	Idf5	Df6	Idf6	Df7	Idf7	Df8
Df6	**1**	0	0	0	0	0	0	0	0	I	0	0	0	I	0	0
Idf6	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Df7	**1**	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
Idf7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Df8	**1**	0	I	0	0	I	0	0	0	0	0	0	0	0	0	0
Idf8	**0**	0	0	I	0	I	0	0	0	0	I	0	0	0	0	0

Now, this Neutrosophy Adjacency Matrix will be evaluated to show the significance of factors causing violence. We will take vector FA as an "ON" state i.e. the state vector is given as input. Based on the methodology mentioned above following iterations are carried out. These iterations are performed till we obtain a limit cycle.

This limit cycle is also referred to as a constant state vector. The Limit cycle shows hidden patterns that are used in mapping inferences. These inferences are used to show the joint effect of interacting knowledge. To get the current result we used NCM which shows that when a crime is in an ON state all the factors. All these factors show that they have a direct effect on crime.

4.4 Iterative Process

Now, this Neutrosophy Adjacency Matrix will be evaluated to show the significance of factors causing violence. We will take vector FA as an "ON" state i.e. the state vector is given as input. Violence in Mexico (VM) = (1000000000000000000) and the combined system is VM1*N(E) The symbol → denotes that the resultant vector is updated and threshold. Based on the methodology mentioned above following iterations are carried out. These iterations are performed till we obtain a limit cycle. This limit cycle is also referred to as a constant state vector. The first iteration starts by keeping the critical thinking state as ON rest all other factors are considered null at this time. So, the Initial State becomes,

$$S(0) = [1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$$

The iterations for both FCM and NCM are done separately. The further iterations are carried out using the formula mentioned below. Iterations for FCM:

Iteration 1: $S(1)=S(0)\times WS(1)=S(0)\times W$

Iteration 2: $S(2)=S(1)\times WS(2)=S(1)\times W$

Iteration 1: $S(1)=S(0)\times WFCMS(1)=S(0)\times WFCM$

Iteration 2: $S(2)=S(1)\times WFCMS(2)=S(1)\times WFCM$

Combining the intermediate results of iteration, we get the final state vector (1,1,0,1,0,0,0,1,0,1, I,1,0,1,0,0). This clearly shows the presence of I mentioning the importance of indeterminate factors in crime analysis which was absent in the final result obtained by the FCM.

Conclusion

The crime situation in Mexico has seen an unprecedented change due to prevailing uncertainty surrounding security events. This is due to the complex mix of social, economic, and political factors. Traditional methods struggle to handle this uncertainty effectively. Therefore, this study proposes the use of neutrosophic logic, which can deal with uncertainty by considering true, false, and indeterminate values present in any security event captioning. By creating a model based on neutrosophic logic, we can assess factors like social unrest, economic instability, and crime rates more accurately. Our results show that this approach predicts and interprets security events better than traditional methods, helping analysts and policymakers make better decisions in uncertain situations.

References

1. Deci, E. L., & Ryan, R. M. (2000). The" what" and" why" of goal pursuits: Human needs and the self-determination of behavior. *Psychological inquiry*, *11*(4), 227-268.
2. Hattie, J. (2008). *Visible learning: A synthesis of over 800 meta-analyses relating to achievement*. routledge.
3. Garrison, D. R., & Kanuka, H. (2004). Blended learning: Uncovering its transformative potential in higher education. *The internet and higher education*, *7*(2), 95-105.
4. Fraser, B. J. (1998). Classroom environment instruments: Development, validity and applications. *Learning environments research*, *1*, 7-34.
5. Black, P., & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education: principles, policy & practice*, *5*(1), 7-74.
6. Membrillo-Hern´andez J, Ramírez-Cadena MJ, Martínez-Acosta M, Cruz-G´omez E, Mu˜noz-Díaz E, Elizalde H. Challenge-based learning: the importance of world-leading companies as training partners. Int J Interact Design Manuf 2019;13(3):1103–13. https://doi.org/10.1007/s12008-019-00569-4.
7. Membrillo-Hern´andez J, Mu˜noz-Soto RB, Rodríguez-S´anchez AC, Díaz-Qui˜nonez JA, Villegas PV, Castillo-Reyna J, Ramírez-Medrano A. Student engagement outside the classroom: analysis of a challenge-based learning strategy in biotechnology engineering. In: *IEEE Global Engineering Education Conference, EDUCON, April-2019* 617-621; 2019. https://doi.org/10.1109/EDUCON.2019.8725246.
8. Zafar, A., & Wajid, M. A. (2019). *Neutrosophic cognitive maps for situation analysis*. Infinite Study.
9. Smith, C., Onofre-Martínez, K., Contrino, M. F., & Membrillo-Hernández, J. (2021). Course design process in a technology-enhanced learning environment. *Computers & Electrical Engineering*, *93*, 107263.
10. Belchior-Rocha, H., Casquilho-Martins, I., & Simões, E. (2022). Transversal competencies for employability: from higher education to the labour market. *Education Sciences*, *12*(4), 255.
11. Cushen, J., & Durkin, L. (2022). The Future of Transversal Competencies in Higher Education Assessment. In *Handbook of Research on Future of Work and Education: Implications for Curriculum Delivery and Work Design* (pp. 253-268). IGI Global.
12. García-Álvarez, J., Vázquez-Rodríguez, A., Quiroga-Carrillo, A., & Priegue Caamaño, D. (2022). Transversal competencies for employability in university graduates: A systematic review from the employers' perspective. *Education Sciences*, *12*(3), 204.
13. Hanesová, D., & Theodoulides, L. (2022). Mastering transversal competences in a higher education environment: through processes of critical thinking and reflection.

14. Wajid, M. S., & Wajid, M. A. (2021). The importance of indeterminate and unknown factors in nourishing crime: a case study of South Africa using neutrosophy. *Neutrosophic Sets and Systems*, *41*(2021), 15.
15. Abdel-Basset, M., Manogaran, G., Gamal, A., & Chang, V. (2019). A Novel Intelligent Medical Decision Support Model Based on Soft Computing and IoT. IEEE Internet of Things Journal.
16. Wajid, M. S., Terashima-Marin, H., Paul Rad, P. N., & Wajid, M. A. (2022). Violence detection approach based on cloud data and Neutrosophic cognitive maps. *Journal of Cloud Computing*, *11*(1), 85.
17. Abdel-Basset, M., Mohamed, R., Zaied, A.E.N.H. and Smarandache, F., 2019. A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. Symmetry, 11(7), p.903.
18. Abdel-Basset, M. and Mohamed, M., 2019. A novel and powerful framework based on neutrosophic sets to aid patients with cancer. Future Generation Computer Systems, 98, pp.144-153.a
19. Wajid, M. A., Zafar, A., Terashima-Marín, H., & Wajid, M. S. (2023). Neutrosophic-CNN-based image and text fusion for multimodal classification. *Journal of Intelligent & Fuzzy Systems*, *45*(1), 1039-1055.
20. Kandasamy Vasantha W. B. and Smarandache Florentin (2003). Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps. Xiquan.
21. Wajid, M. A., & Zafar, A. (2022). Neutrosophic Image Segmentation: An Approach for the Treatment of Uncertainty in Multimodal Information Systems. *International Journal of Neutrosophic Science (IJNS)*, *19*(1).
22. Smarandache, F. (2001). A Unifying Field in Logics: Neutrosophic Logic, Preface by Charles Le, American Research Press, Rehoboth, 1999, 2000. Second edition of the Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics. University of New Mexico, Gallup. http://www.gallup.unm.edu/~smarandache/eBook- Neutrosophics2.pdf

A Python Class for Neutrosophic Morphology

Lorenzo Affè[1,*], Giorgio Nordo[2] and Florentin Smarandache[3]

[1]MIFT Department – Mathematical and Computer Science, Physical Sciences and Earth Sciences University of Messina, 98166 Sant'Agata, Messina, Italy; lorenzo.affe1@gmail.com

[2]MIFT Department – Mathematical and Computer Science, Physical Sciences and Earth Sciences University of Messina, 98166 Sant'Agata, Messina, Italy; giorgio.nordo@unime.it

[3]Mathematics, Physics, and Natural Science Division University of New Mexico 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu

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Abstract. In this work, we introduce a Python class, named `NSmorph`, developed to facilitate image manipulation through neutrosophic morphological operations. This innovative approach extends traditional image processing methods by leveraging the flexibility of neutrosophic logic to handle uncertainty, indeterminacy, and noise in digital images. The class offers implementations of essential morphological operators, such as neutrosophic dilation, erosion, opening, and closing, providing a robust tool for applications where image clarity is often compromised, like medical imaging and surveillance. We detail the class structure and functions and provide multiple examples to demonstrate its practical applications and comparative advantages over classical morphological methods.

Keywords: neutrosophic set; neutrosophic morphology; morphological image processing; Python programming; uncertainty in image analysis.

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1. Introduction

In contemporary society, marked by increasing complexity and interconnected systems, classical mathematical theories face significant limitations when applied to real-world scenarios where vagueness, uncertainty, and indeterminacy are predominant. Such challenges are particularly pronounced in the field of image analysis, where accurate data interpretation plays a crucial role in diverse applications ranging from medical diagnostics to security surveillance. Traditional image processing techniques often struggle to handle these uncertainties effectively, necessitating alternative mathematical approaches capable of modeling and managing ambiguity [19].

Mathematical Morphology, originally developed by Matheron and Serra [13, 14], provides a powerful set of tools in nonlinear image analysis, primarily focusing on the geometric structure of images. Classical morphological operators, extensively discussed in Dougherty's works [2,3], have proven effective for tasks like noise filtering, shape extraction, and image segmentation. However, these operators are built on deterministic models of data, which can be limiting in applications where noise or uncertainty inherently affects image quality. Researchers have thus explored various generalizations, including Zadeh's fuzzy sets [20] and Atanassov's intuitionistic fuzzy sets [1], to accommodate situations where precise binary interpretations are insufficient. These theories laid the groundwork for the emergence of neutrosophic sets, introduced by Smarandache [17], which extend the representation of uncertainty by considering degrees of truth, indeterminacy, and falsehood independently, providing a nuanced framework suited to complex, real-world data.

Neutrosophic set theory has been applied successfully in various domains, including graph theory [4], decision-making [6], and soft topological spaces [7], where traditional mathematical approaches fall short. In image analysis, Neutrosophic Morphology applies Smarandache's principles to extend classical morphological operations, providing a framework that captures and preserves information about uncertainty and noise within images. This has proven especially beneficial in applications where data reliability varies significantly, allowing each pixel in an image to possess distinct degrees of membership, non-membership, and indeterminacy [18]. Studies have demonstrated that neutrosophic operators maintain structural transformations without losing important information about the underlying uncertainty, making them particularly effective for medical image analysis [12] and other fields requiring enhanced data fidelity [8, 16].

The importance of computational applications and frameworks developed in Python has significantly grown within neutrosophic studies. These frameworks facilitate practical implementations of neutrosophic concepts, making them more accessible to researchers and practitioners. Recent Python-based tools, such as those by El-Ghareeb [5] and Sleem et al. [16], have laid foundational work for performing neutrosophic operations efficiently. Nordo et al. [9] further extended this effort by developing a comprehensive Python framework specifically tailored for neutrosophic sets and mappings, highlighting the demand for adaptable and open-source tools in this field. Such frameworks not only simplify the computational aspects but also expand the applicability of neutrosophic methods across various disciplines, including image processing, where tools that address uncertainty and indeterminacy are crucial.

Despite these advancements, there remains a need for a comprehensive and user-friendly Python-based solution that can perform neutrosophic morphological operations in image analysis with flexibility and precision. Existing frameworks either focus narrowly on numerical

neutrosophic operations or lack the modular design needed for broader applicability [5, 16]. Recognizing this gap, we propose `NSmorph`, a Python class designed to implement fundamental neutrosophic morphological operators, including neutrosophic dilation, erosion, opening, and closing. These operators extend traditional morphological techniques into the neutrosophic domain, providing a versatile toolkit for image manipulation in environments with prevalent uncertainty, noise, and indeterminacy [11].

The structure of the paper is organized as follows: in Section 2, we delve into the data structure and the specific methods that constitute the `NSmorph` class, detailing each component's functionality. Section 3 presents practical examples that demonstrate the efficacy of the class in various scenarios, along with observations on the performance of neutrosophic operators relative to their classical counterparts. Finally, in Section 4, we provide our conclusions, underscoring the advantages of neutrosophic morphology in image analysis and the potential for future developments in this field [15].

2. The `NSmorph` class

The `NSmorph` class is designed for manipulating neutrosophic images using common morphological operators, including NS-dilation, NS-erosion, NS-opening, and NS-closing. The class constructor can initialize a neutrosophic image from various input types, such as a file path to a PNG or JPEG image, a NumPy array, or an instance of the `NSmorph` class itself. The following code shows how the constructor is implemented:

```
import os.path
import cv2 as cv
import numpy as np
from matplotlib import pyplot as plt

class NSmorph:
   def __init__(self, image, radius=0):
      if radius < 0:
      raise ValueError(f"The radius '{radius}' of the neighbourhood cannot be
    negative")
      if type(image) == np.ndarray:
         if image is None:
            raise ValueError("The image is not valid")
         (height, width)=image.shape
         self.__height = height
         self.__width = width
         self.__image_orig = image
         self.__radius = radius
         i_med = np.zeros((height, width), dtype = np.float32)
         for y in range(height):
            for x in range(width):
               md = 0
               n_pixel = 0
               for j in range(y - radius, y + radius + 1):
```

```
24                  for i in range(x - radius, x + radius +1):
25                     if (0<=i<width) and (0<=j<height) :
26                        md += image[j][i]
27                        n_pixel += 1
28               md /= n_pixel
29               i_med[y][x] = md
30         i_med_min = i_med.min()
31         i_med_max = i_med.max()
32         i_med_size = i_med_max - i_med_min
33         delta = np.zeros((height, width), dtype=np.float32)
34         for y in range(height):
35            for x in range(width):
36               delta[y][x] = abs(image[y][x] - i_med[y][x])
37         delta_min = delta.min()
38         delta_max = delta.max()
39         delta_size = delta_max - delta_min
40         ns_image = np.zeros((height, width, 3), dtype = np.float32)
41         for y in range(height):
42            for x in range(width):
43               ns_image[y][x][0] = (i_med[y][x] - i_med_min)/i_med_size if
   i_med_size !=0 else 0
44               ns_image[y][x][1] = (delta[y][x] - delta_min)/delta_size if
   delta_size !=0 else 0
45               ns_image[y][x][2] = 1 - ns_image[y][x][0]
46         self.__ns_image = ns_image
47      elif type(image) == str:
48         tmp_imgns = NSmorph(NSmorph.load(image), radius)
49         self.__ns_image = tmp_imgns.get()
50         self.__image_orig = tmp_imgns.getOrig()
51         self.__height = tmp_imgns.height()
52         self.__width = tmp_imgns.width()
53         self.__radius = radius
54      elif type(image) == NSmorph :
55         self.__ns_image = image.get()
56         self.__image_orig = image.getOrig()
57         self.__height = image.height()
58         self.__width = image.width()
59         self.__radius = image.radius()
60      else:
61         raise ValueError("The first parameter must be a matrix or a string")
```

Lines 1 to 4 import necessary libraries, including OpenCV and NumPy for image handling and `pyplot` for visualization. The constructor is defined in line 7, where the `radius` parameter is checked for validity in lines 8 and 9. The constructor can handle three types of input:

- NumPy array (lines 10–46): If `image` is a NumPy array, the image's height, width, and radius are stored. Then, from lines 18 to 32, the average intensity $\overline{I}$ for each pixel is computed by averaging the values within the kernel centered at each pixel. This forms the first channel of the neutrosophic image. In lines 33 to 39, the `delta` matrix, which measures intensity deviation from the local average (representing the homogeneity function), is computed and stored as the second channel. Finally, lines 40

to 46 construct the neutrosophic image by normalizing the intensity and delta values and calculating the non-membership degree as $1 - \texttt{membership}$.

- File path (string) (lines 47–53): If `image` is a file path, the constructor calls the `load` method to load the image as a grayscale NumPy array. The `NSmorph` instance for this image is then initialized, and the original image dimensions and radius are set.
- Existing `NSmorph` object (lines 54–59): If `image` is an existing `NSmorph` instance, the neutrosophic image and related attributes are directly copied.

Finally, in lines 60 and 61, if the input does not match any valid types, an error is raised.

The `NSmorph` class also includes several methods to retrieve the complete neutrosophic image or its individual components—membership, non-membership, and indeterminacy—or to convert it back to the original binary format. These functions provide flexible access and modification capabilities for each aspect of the neutrosophic image representation.

A particularly useful method is the static method `load(img_path)`, which loads an image from a specified file path `img_path`. This method reads the image in grayscale, thus simplifying conversion to the neutrosophic format. If the provided path is invalid, not found, or inaccessible, the method raises an error, leveraging `os.path.isfile()` to verify the file's existence. This ensures that only valid paths are processed, preventing runtime issues due to incorrect paths.

```
@staticmethod
def load(img_path):
    if img_path is None:
        raise ValueError("The path of the image is not valid")
    if not os.path.isfile(img_path):
        raise FileNotFoundError(f"The file '{img_path}' does not exist or it is not accessible.")
    image = cv.imread(img_path, cv.IMREAD_GRAYSCALE)
    if image is None:
        raise IOError(f"It has not been possible to read the image file '{img_path}'")
    return image
```

The `get()` method gives back the neutrosophic image.

```
def get(self):
    return self.__ns_image
```

Methods are also available for retrieving and assigning specific components of the neutrosophic image:

- Membership (`getM()` and `setM()`): `getM()` returns the membership degree for each pixel, while `setM(x, y, mu)` allows for setting the membership degree at a specific pixel location `(x, y)`.
- Non-membership (`getNM()` and `setNM()`): `getNM()` retrieves the non-membership degree matrix, and `setNM(x, y, omega)` sets the non-membership degree for the specified pixel.
- Indeterminacy (`getI()` and `setI()`): `getI()` returns the indeterminacy degree, while `setI(x, y, sigma)` assigns a new indeterminacy degree to a given pixel.

Each method supports fine-grained control over image processing, allowing users to either view or modify the neutrosophic image's distinct properties individually or as a composite.

```
def getM(self):
    return self.__ns_image[:,:,0]

def getI(self):
    return self.__ns.image[:,:,1]

def getNM(self):
    return self.__ns.image[:,:,2]


def setM(self, x, y, mu):
    self.__ns_image[y][x][0] = mu

def setI(self, x, y, sigma):
    self.__ns_image[y][x][1] = sigma

def setNM(self, x, y, omega):
    self.__ns_image[y][x][2] = omega
```

The `getOrig()` method returns the original grayscale image from which the neutrosophic image was derived. This can be useful for comparison purposes, allowing users to access the unprocessed version of the image alongside its neutrosophic representation. The method is implemented as follows:

```
def getOrig(self):
    return self.__image
```

The `width()` and `height()` methods return the width and height of the image, respectively. These methods are particularly useful when constructing other morphological operations, such as dilation and erosion, as they provide easy access to the dimensions of the image, which is essential for managing boundary conditions and kernel application across the entire image. The implementations are straightforward:

```
def width(self):
    return self.__width

def height(self):
    return self.__height
```

The `radius()` method returns the radius of the neighborhood associated with the uploaded image. This radius defines the size of the local region around each pixel that is used in various neutrosophic morphological operations, such as calculating local intensity averages or applying structuring elements in dilation and erosion. Accessing the radius through this method is particularly useful when the neighborhood size impacts processing or when adapting the morphological operators to different kernel sizes. The implementation is as follows:

```
def radius(self):
    return self.__radius
```

The `getBinary()` method performs thresholding on the current image, producing a binary image based on a threshold value specified by the user. The method returns a binary representation of the image as a NumPy matrix, where rows and columns are indexed from $(0, 0)$ at the top left corner. In this binary matrix, pixel values are set to 0 (black) for intensities below the threshold and 1 (white) for intensities equal to or above the threshold. The method is implemented as follows:

```
def getBinary(self, threshold):
    (ret, bin_image) = cv.threshold(self.__image_orig, threshold, 1, cv.
    THRESH_BINARY)
    return bin_image
```

The `getRepresentation()` method returns a grayscale image that represents an interpolation of the three neutrosophic levels: membership, indeterminacy, and non-membership. This interpolation combines the three levels based on customizable weights provided as optional parameters. By default, these weights prioritize the membership degree, reflecting its primary influence on the pixel intensity in the resulting grayscale image. The following code illustrates this implementation:

```
def getRepresentation(self, weightM=0.85, weightI=0.25, weightNM=-0.1, binary
    =False, limit_value=128):
    img_M = self.getM()
    img_I = self.getI()
    img_NM = self.getNM()
    mat_rap = np.zeros((self.__height, self.__width, 3), dtype = np.float32)
```

```
    for y in range(self.__height):
        for x in range(self.__width):
            mat_rap[y][x][0] = weightM*img_M[y][x] + weightI*img_I[y][x] +
     weightNM*img_NM[y][x]
    img_rap = cv.cvtColor(mat_rap, cv.COLOR_BGR2GRAY)
    if binary == True:
        (ret, img_rap) = cv.threshold(img_rap, limit_value, 1, cv.THRESH_BINARY)
    return img_rap
```

The method signature is as follows:

- `weightM`: Weight assigned to the membership degree. Default is 0.85.
- `weightI`: Weight assigned to the indeterminacy degree. Default is 0.25.
- `weightNM`: Weight assigned to the non-membership degree. Default is -0.1.
- `binary`: Boolean flag to enable binarization of the resulting image based on a threshold.
- `limit_value`: Threshold value used for binarization if `binary` is set to `True`. Default is 128.

The grayscale image is generated by combining the three components, emphasizing areas with higher membership and optionally applying thresholding if `binary` is `True`. If thresholding is active, the method uses `limit_value` to convert the grayscale image to a binary one, mapping values above the threshold to white and those below to black.

This function is particularly useful for visualizing the overall neutrosophic information in a single image, allowing for quick interpretation of complex neutrosophic properties in greyscale or binary format.

2.1. *Neutrosophic dilation*

The `dilation(self, kernel)` method performs the dilation operation on a neutrosophic image, using another neutrosophic image as the structuring element (kernel). The method returns the dilated neutrosophic image and it is implemented as follows:

```
def dilation(self, kernel):
    (height, width) = (self.__height, self.__width)
    (height_k, width_k) = (kernel.height(), kernel.width())

    img_M = self.getM()
    img_I = self.getI()
    img_NM = self.getNM()
    kernel_M = kernel.getM()
    kernel_I = kernel.getI()
    kernel_NM = kernel.getNM()

    mat_generating = np.zeros((height, width, 3), dtype=np.uint8)
    im_empty = cv.cvtColor(mat_generating, cv.COLOR_BGR2GRAY)
```

```
    im_dil = NSmorph(im_empty)

    for y in range(height):
        for x in range(width):

            width_sx = x - width_k//2
            if width_sx<0 :
                width_sx = 0
            width_dx = x + width_k//2 +1
            if width_dx > width - 1 :
                width_dx = width - 1
            height_up = y - height_k//2
            if height_up < 0 :
                height_up = 0
            height_down = y+ height_k//2 + 1
            if height_down > height - 1 :
                height_down = height - 1

            mat_M = img_M[height_up:height_down, width_sx:width_dx]
            mat_I = img_I[height_up:height_down, width_sx:width_dx]
            mat_NM = img_NM[height_up:height_down, width_sx:width_dx]

            (n_rows, n_columns) = mat_M.shape

            width_sx_kernel = width_k//2 - n_columns//2
            if width_sx_kernel > n_columns//2 - 1:
                width_sx_kernel = 0
            width_dx_kernel = width_k//2 + n_columns//2 + 1
            if width_dx_kernel > n_columns//2 + 1:
                width_dx_kernel = n_columns
            height_up_kernel = height_k//2 - n_rows//2
            if height_up_kernel > n_rows//2 - 1:
                height_up_kernel = 0
            height_down_kernel = height_k//2 + n_rows//2 + 1
            if height_down_kernel > n_rows//2 + 1 :
                height_down_kernel = n_rows

            mat_kernelM = kernel_M[height_up_kernel:height_down_kernel,
    width_sx_kernel:width_dx_kernel]
            mat_kernelI = kernel_I[height_up_kernel:height_down_kernel,
    width_sx_kernel:width_dx_kernel]
            mat_kernelNM = kernel_NM[height_up_kernel:height_down_kernel,
    width_sx_kernel:width_dx_kernel]

            minimum_M = np.minimum(mat_M, mat_kernelM)
            minimum_I = np.minimum(mat_I, mat_kernelI)
            maximum_NM = np.maximum(mat_NM, 1 - mat_kernelNM)

            mu = minimum_M.max()
            sigma = minimum_I.max()
            omega = maximum_NM.min()

            im_dil.setM(x, y, mu)
            im_dil.setI(x, y, sigma)
            im_dil.setNM(x, y, omega)
    return im_dil
```

In lines 2 and 3, the dimensions of both the input image and the kernel are stored in separate tuples. Then, in lines 5 to 10, the membership, indeterminacy, and non-membership matrices of the image and kernel are retrieved.

From lines 12 to 14, an empty matrix, `mat_generating`, is created using `np.zeros()` from the NumPy library. This matrix is converted into a grayscale image, `im_empty`, with OpenCV's `cv.cvtColor()` function and serves as the base for creating the dilated neutrosophic image `im_dil`.

Lines 16 and 17 use nested `for` loops to iterate through each pixel of the input image. Within each loop, lines 19 to 30 define the bounds for the image region centered on the current pixel (x, y), ensuring that the extracted region matches the size of the kernel. This is done by adjusting for any out-of-bounds areas along the edges of the image, where the kernel might partially exceed the image boundaries.

In lines 32 to 34, the membership, indeterminacy, and non-membership matrices are extracted for the region defined by these bounds, centered around (x, y). Line 36 then evaluates the actual dimensions of these matrices, accounting for cases where the region is truncated by image boundaries.

Lines 38 to 49 set up further bounds to ensure that the extracted kernel matrices align precisely with the region around (x, y) in the image. This step helps reduce the kernel dimensions to match those of the extracted region when scanning near boundaries.

The actual dilation operation occurs in lines 55 to 61, applying the following neutrosophic morphological formulas:

$$\begin{aligned}
\mu_{A\oplus B}(v) &= \sup_{u\in\mathbb{U}} \min\left\{\mu_A(v+u), \mu_B(u)\right\}, \\
\sigma_{A\oplus B}(v) &= \sup_{u\in\mathbb{U}} \min\left\{\sigma_A(v+u), \sigma_B(u)\right\}, \\
\omega_{A\oplus B}(v) &= \inf_{u\in\mathbb{U}} \max\left\{\omega_A(v+u), 1-\omega_B(u)\right\},
\end{aligned}$$

where $\mu_{A\oplus B}(v)$, $\sigma_{A\oplus B}(v)$, and $\omega_{A\oplus B}(v)$ represent the degrees of membership, indeterminacy, and non-membership, respectively, after dilation.

In lines 63 to 65, the calculated values for membership, indeterminacy, and non-membership are stored in the `im_dil` image at position (x, y). Finally, in line 66, the fully processed dilated neutrosophic image is returned.

2.2. *Neutrosophic erosion*

The `erosion(self, kernel)` method applies the neutrosophic erosion operation to the input image using a specified *kernel*, which is also a neutrosophic image. The method returns the eroded neutrosophic image and is implemented as follows:

```
def erosion(self, kernel):
   (height, width) = (self.__height, self.__width)
   (height_k, width_k) = (kernel.height(), kernel.width())

   img_M = self.getM()
   img_I = self.getI()
   img_NM = self.getNM()
   kernel_M = kernel.getM()
   kernel_I = kernel.getI()
   kernel_NM = kernel.getNM()

   mat_generating = np.zeros((height, width, 3), dtype=np.uint8)
   im_empty = cv.cvtColor(mat_generating, cv.COLOR_BGR2GRAY)
   im_er = NSmorph(im_empty)

   for y in range(height):
      for x in range(width):

         width_sx = x - width_k//2
         if width_sx<0 :
            width_sx = 0
         width_dx = x + width_k//2 +1
         if width_dx > width - 1 :
            width_dx = width - 1
         height_up = y - height_k//2
         if height_up < 0 :
            height_up = 0
         height_down = y+ height_k//2 + 1
         if height_down > height - 1 :
            height_down = height - 1

         mat_M = img_M[height_up:height_down, width_sx:width_dx]
         mat_I = img_I[height_up:height_down, width_sx:width_dx]
         mat_NM = img_NM[height_up:height_down, width_sx:width_dx]

         (n_rows, n_columns) = mat_M.shape

         width_sx_kernel = width_k//2 - n_columns//2
         if width_sx_kernel > n_columns//2 - 1:
            width_sx_kernel = 0
         width_dx_kernel = width_k//2 + n_columns//2 + 1
         if width_dx_kernel > n_columns//2 + 1:
            width_dx_kernel = n_columns
         height_up_kernel = height_k//2 - n_rows//2
         if height_up_kernel > n_rows//2 - 1:
            height_up_kernel = 0
         height_down_kernel = height_k//2 + n_rows//2 + 1
         if height_down_kernel > n_rows//2 + 1 :
            height_down_kernel = n_rows

         mat_kernelM = kernel_M[height_up_kernel:height_down_kernel,
   width_sx_kernel:width_dx_kernel]
         mat_kernelI = kernel_I[height_up_kernel:height_down_kernel,
   width_sx_kernel:width_dx_kernel]
         mat_kernelNM = kernel_NM[height_up_kernel:height_down_kernel,
   width_sx_kernel:width_dx_kernel]
```

```
54
55          maximum_M = np.maximum(mat_M, 1 - mat_kernelM)
56          maximum_I = np.maximum(mat_I, 1 - mat_kernelI)
57          minimum_NM = np.minimum(mat_NM, mat_kernelNM)
58
59          mu = maximum_M.min()
60          sigma = maximum_I.min()
61          omega = minimum_NM.max()
62
63          im_er.setM(x, y, mu)
64          im_er.setI(x, y, sigma)
65          im_er.setNM(x, y, omega)
66      return im_er
```

Similar to the neutrosophic dilation method, the dimensions of both the input image and kernel are stored in tuples (lines 2 and 3), and the membership, indeterminacy, and non-membership matrices of the input image and kernel are assigned to `img_M`, `img_I`, `img_NM` and `kernel_M`, `kernel_I`, `kernel_NM` respectively (lines 5 to 10).

Lines 12 to 14 create an empty matrix, `mat_generating`, which serves as the base for the output eroded image, `im_er`. The matrix is initialized as a grayscale image using `cv.cvtColor()` from the OpenCV library.

Nested `for` loops in lines 16 and 17 iterate over each pixel in the input image. Lines 19 to 30 define the region within the input image centered at each pixel (x, y), ensuring the extracted region matches the kernel size. These boundary checks ensure that when the kernel partially overlaps the image edges, the region adjusts accordingly to prevent out-of-bounds errors.

The membership, indeterminacy, and non-membership matrices for the region around (x, y) are extracted in lines 32 to 34. In line 36, the actual dimensions of these matrices are determined to account for potential edge clipping.

Lines 38 to 49 then adjust the kernel's size to match that of the extracted region, ensuring alignment during the operation.

The erosion operation is applied in lines 55 to 61, using the following formulas for neutrosophic morphological erosion:

$$\mu_{A\ominus B}(v) = \inf_{u\in\mathbb{U}} \max\left\{\mu_A(v+u), 1-\mu_B(u)\right\},$$
$$\sigma_{A\ominus B}(v) = \inf_{u\in\mathbb{U}} \max\left\{\sigma_A(v+u), 1-\sigma_B(u)\right\},$$
$$\omega_{A\ominus B}(v) = \sup_{u\in\mathbb{U}} \min\left\{\omega_A(v+u), \omega_B(u)\right\},$$

where $\mu_{A\ominus B}(v)$, $\sigma_{A\ominus B}(v)$, and $\omega_{A\ominus B}(v)$ represent the degrees of membership, indeterminacy, and non-membership, respectively, for the eroded image.

Finally, in lines 63 to 65, the computed membership, indeterminacy, and non-membership values are saved to `im_er` at (x, y). In line 66, the method returns the fully processed eroded neutrosophic image.

2.3. *Neutrosophic opening and closing*

The `opening(self, kernel)` method performs the neutrosophic opening operation on the image. This operation first applies the erosion operator, followed by the dilation operator, using the same *kernel*, which is also a neutrosophic image, passed as a parameter. The result is a neutrosophic image where small structures are removed, and larger shapes are preserved without altering their boundaries significantly. This method is especially useful for tasks where noise reduction or shape smoothing is required. The implementation is straightforward:

```
def opening(self, kernel):
    return self.erosion(kernel).dilation(kernel)
```

The `closing(self, kernel)` method, in contrast, performs the neutrosophic closing operation. This operation first applies the dilation operator to the image, followed by the erosion operator, using the same *kernel* provided. Closing is typically used to fill small holes and connect adjacent structures within an image without significantly altering their contours, making it useful in image enhancement tasks. The implementation is as follows:

```
def closing(self, kernel):
    return self.dilation(kernel).erosion(kernel)
```

The complete source code for this Python class has been released under the Open Source GNU General Public License version 3.0 (or GPL-3.0) and is freely accessible at the url github.com/lorenzoaffe/nsmorph.

3. **Examples and Observations**

In this section, we demonstrate the `NSmorph` class by using neutrosophic dilation, erosion, opening, and closing operations on an image affected by noise. These examples showcase how each neutrosophic morphological operator interacts with noisy data, specifically observing the effect on the image's brightness, clarity, and feature preservation. Such noise, often introduced as random variations in pixel values, can significantly compromise image quality in real-world applications.

```
from ns_morph import NSmorph
import cv2 as cv
```

```
3 import numpy as np
4 from matplotlib import pyplot as plt

6 path_image = "immagini/j_puntini.jpg"
7 image = NSmorph.load(path_image)
8 ns_image = NSmorph(image, 4)

10 path_kernel = "immagini/kernel_croce.jpg"
11 kernel = NSmorph.load(path_kernel)
12 ns_kernel = NSmorph(kernel, 1)

14 nsdil_image = ns_image.dilation(ns_kernel)

16 plt.suptitle("Neutrosophic dilation (M=membership, I=indeterminacy, NM=non-
       membership)")

18 plt.subplot(4,3,1)
19 plt.imshow(ns_image.getOrig(), cmap='gray')
20 plt.xticks([])
21 plt.yticks([])
22 plt.title("Original")

24 ...

26 plt.subplot(4,3,11)
27 plt.imshow(nsdil_image.getRepresentation(), cmap='gray')
28 plt.xticks([])
29 plt.yticks([])
30 plt.title("dilation")

32 plt.subplot(4,3,12)
33 plt.imshow(nsdil_image.getRepresentation(binary=True, limit_value=0.04), cmap=
       'gray')
34 plt.xticks([])
35 plt.yticks([])
36 plt.title("binarized dilation")

38 plt.show()
```

In lines 1 to 4, we import necessary modules and libraries, including `NSmorph`, OpenCV-Python, NumPy, and `pyplot` from Matplotlib. Lines 6 to 8 generate the neutrosophic image from the specified path using the static method `load(img_path)`. Similarly, lines 10 to 12 initialize the kernel. Line 14 performs neutrosophic dilation on the image with the kernel, storing the result in `nsdil_image`. From lines 18 to 36, the images are plotted, showing the original, the dilated, and the binarized dilated images.

Figures 1, 2, 3, and 4 present the results of applying NS-dilation, NS-erosion, NS-opening, and NS-closing respectively. Each figure includes grayscale images of the membership, indeterminacy, and non-membership components, as well as the interpolated neutrosophic image and a binarized version obtained through thresholding.

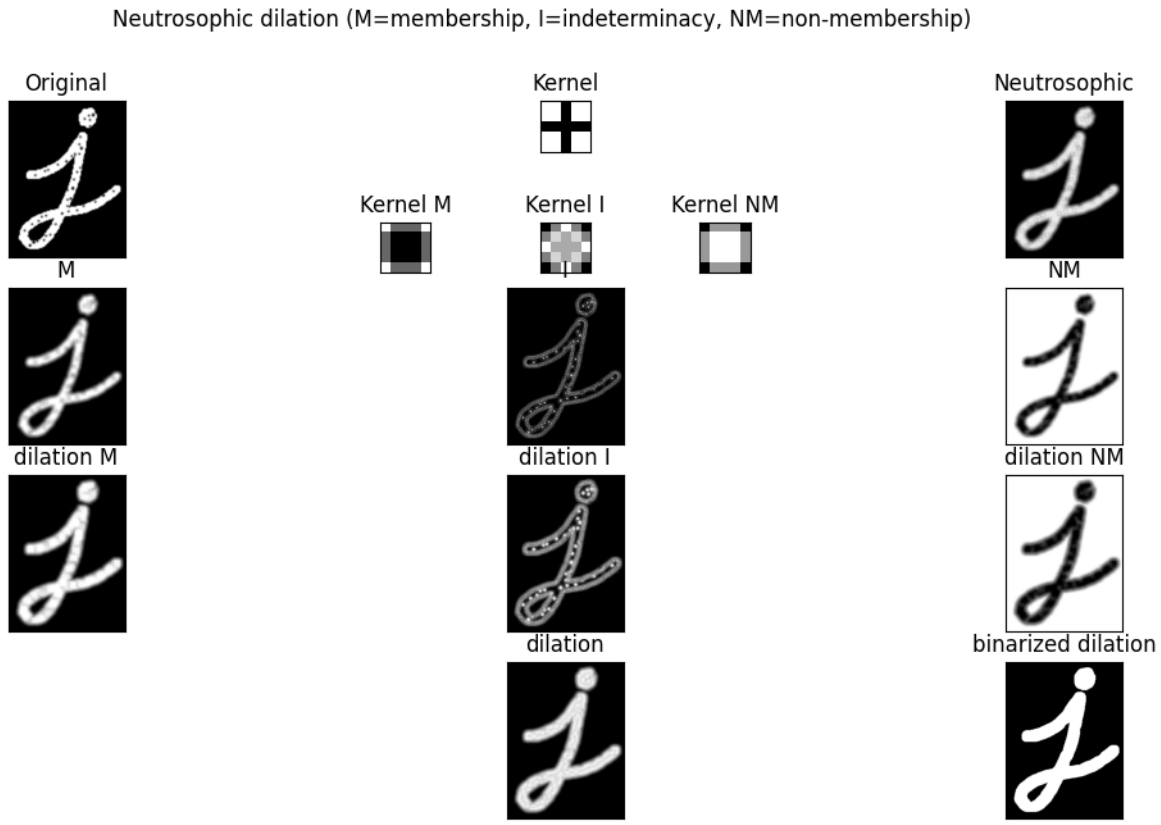

FIGURE 1. Example of the neutrosophic dilation of a neutrosophic image with image radius = 4 and kernel radius = 1

In Figure 1, we observe that the NS-dilation operator effectively smooths the shape of the image while filling in small holes. Notably, the indeterminacy component (I) retains the presence of noise, indicating areas where the operator's action is limited by the uncertainty inherent in the noisy regions.

Figure 2 shows the application of the NS-erosion operator, which reduces protrusions and expands holes. Similar to dilation, erosion captures noise effects, particularly in the membership (M) and non-membership (NM) components. This retention of noise highlights how neutrosophic operators can provide information on uncertainty while performing morphological transformations.

In Figure 3, the NS-opening operator, a sequence of erosion followed by dilation, preserves the indeterminacy and non-membership values in noisy areas but still manages to remove minor structures without disturbing the primary shape of the image. This operator is effective in refining shapes by reducing small-scale noise, emphasizing indeterminacy in regions with higher pixel variability.

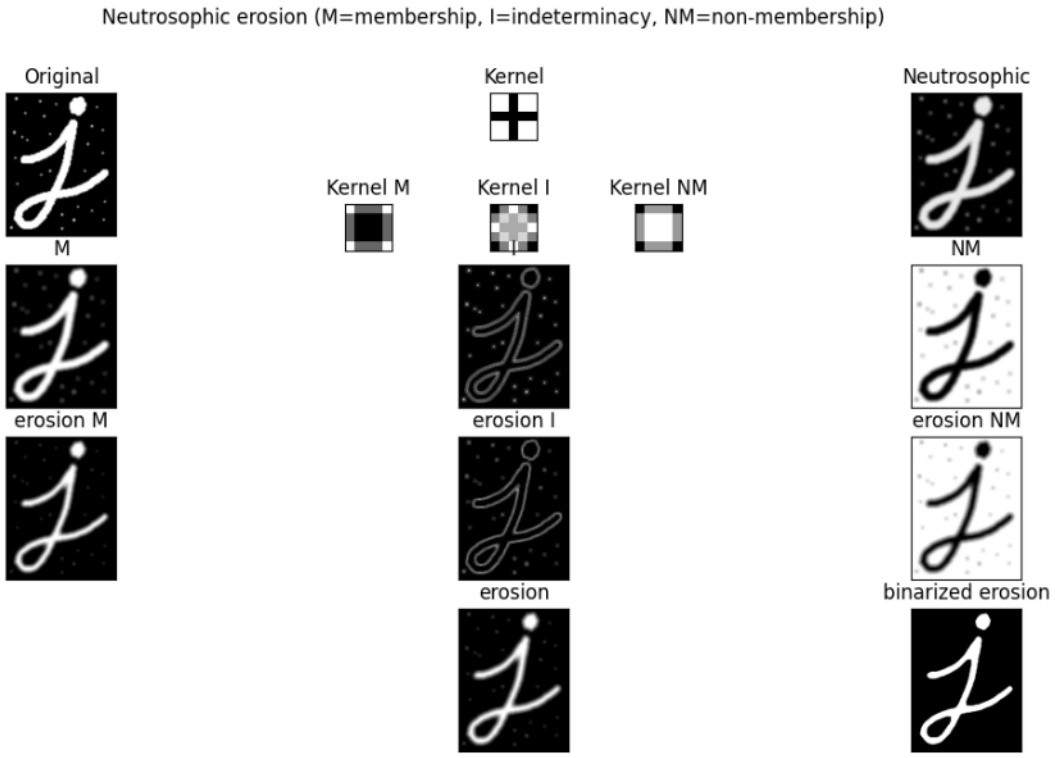

FIGURE 2. Example of the neutrosophic erosion of a neutrosophic image with image radius = 4 and kernel radius = 1

Finally, Figure 4 shows the NS-closing operation, which first applies dilation and then erosion. This operation fills small gaps and connects nearby features, with noise remaining visible in the indeterminacy component. The ability of NS-closing to smooth object boundaries without completely erasing uncertainty demonstrates its utility in applications where preserving ambiguous regions is important.

These examples illustrate the unique advantages of neutrosophic morphological operators, which maintain noise information through indeterminacy while effectively processing image structures. This dual capability of modification and noise retention makes neutrosophic morphology suitable for complex image processing tasks where uncertainty plays a significant role.

4. **Conclusions**

In this paper, we introduced `NSmorph`, a Python class designed for the manipulation of neutrosophic digital images through morphological operators specifically tailored to handle uncertainty and ambiguity. This class implements key neutrosophic operators, including NS-dilation, NS-erosion, NS-opening, and NS-closing, each serving as an extension of classical

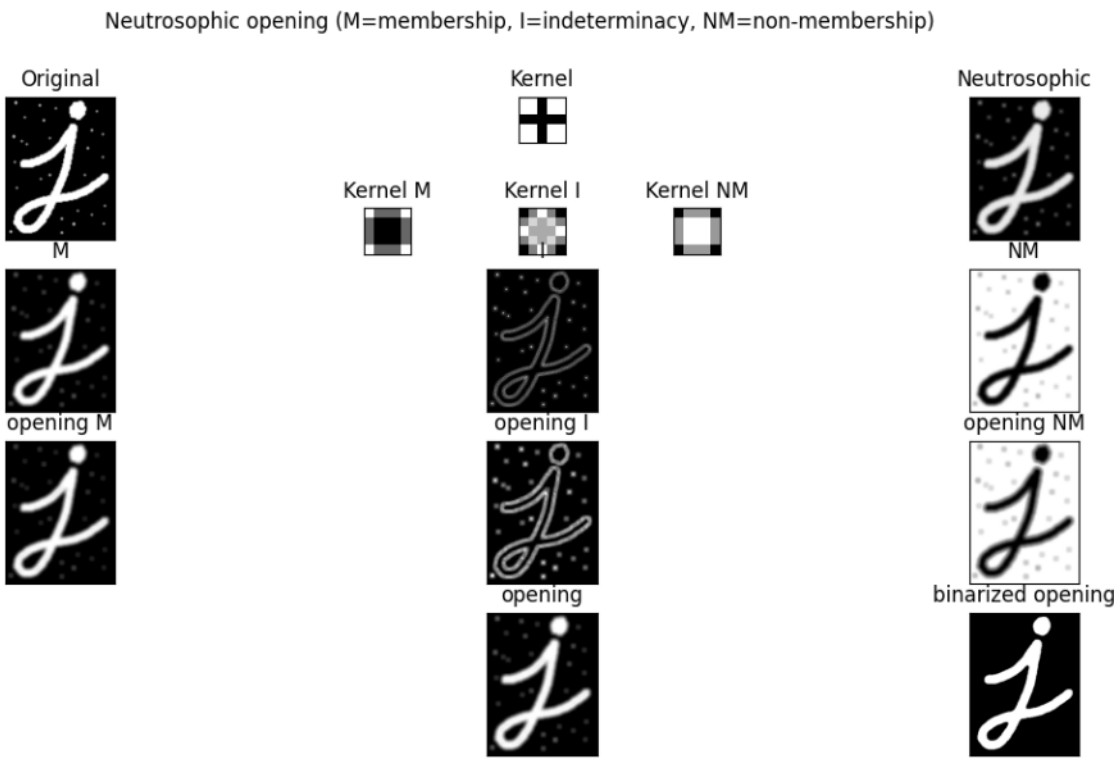

FIGURE 3. Example of the neutrosophic opening of a neutrosophic image with image radius = 4 and kernel radius = 1

morphology in a way that retains the core actions of these transformations while addressing noise and indeterminacy. The design of `NSmorph` includes a flexible constructor and a suite of methods that allow users to access, modify, and visualize the membership, indeterminacy, and non-membership components of an image, adding a layer of control that is essential for applications where uncertain data is prevalent.

The examples provided illustrate how neutrosophic operators differ from traditional morphology, showcasing their capacity to improve image processing outcomes in noisy environments. In particular, the neutrosophic dilation and erosion operations demonstrated a unique capability to maintain information on uncertainty in the indeterminacy component while performing standard morphological tasks, such as smoothing shapes and filling or reducing holes. This dual function of transforming image structures while retaining noise information in the indeterminacy channel provides a substantial advantage for applications that require both enhancement and detailed uncertainty analysis, such as medical imaging, remote sensing, and security surveillance.

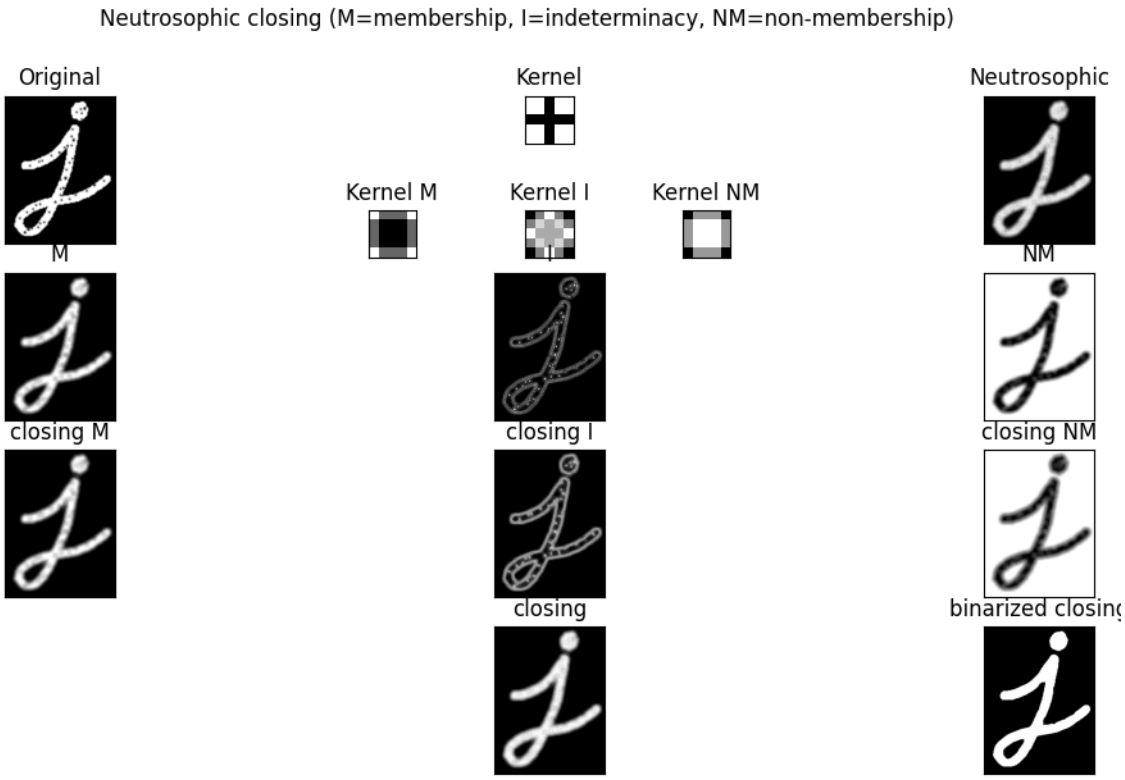

FIGURE 4. Example of the neutrosophic closing of a neutrosophic image with image radius = 4 and kernel radius = 1

Similarly, the neutrosophic opening and closing operations highlighted the class's capability to refine image structures. The NS-opening operator proved effective in reducing small-scale noise while sharpening key features, thus avoiding the loss of critical information. On the other hand, the NS-closing operator showed effectiveness in filling small gaps and connecting nearby structures without compromising the representation of ambiguous areas. These observations underscore how `NSmorph` enables users to adapt morphological operations to complex image processing needs, where standard morphological operators would typically smooth or discard valuable uncertain information.

Overall, the development of `NSmorph` contributes to the field of digital image processing by providing a robust tool for managing indeterminate data, preserving ambiguity, and enhancing the interpretability of images under uncertain conditions. The demonstrated effectiveness of neutrosophic morphological operators in retaining and processing noise information establishes `NSmorph` as a valuable asset for researchers and practitioners who require enhanced flexibility and control over image data affected by noise or uncertainty. Future work may explore

additional neutrosophic operators or adaptations of existing ones to further expand the class's utility across diverse image processing applications.

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References

1. Atanassov K.T. Intuitionistic Fuzzy Sets. *Fuzzy Sets and Systems* **20** (1), pp. 87-96, 1986.
2. Dougherty E.R. An Introduction to Morphological Image Processing. *SPIE Optical Engineering Press*, Bellingham, Washington (USA), 1992.
3. Dougherty E.R., Lotufo R.A. Hands-on Morphological Image Processing. *SPIE Press*, Bellingham, Washington (USA), 2003.
4. Broumi S., Talea M., Bakali A., Smarandache F. Single valued neutrosophic graphs. *Journal of New theory* **10**, pp. 86-101, 2016.
5. El-Ghareeb H.A. Novel Open Source Python Neutrosophic Package. *Neutrosophic Sets and Systems* **25**, pp. 136-160, 2019.
6. Mehmood A., Nadeem F., Nordo G., Zamir M., Park C.,Kalsoom H., Jabeen S., Khan M.Y. Generalized neutrosophic separation axioms in neutrosophic soft topological spaces. *Neutrosophic Sets and Systems* **32** (1), pp. 38–51, 2020.
7. Mehmood A., Nadeem F., Park C., Nordo G., Kalsoom H., Rahim Khan M., Abbas N. Neutrosophic soft alpha-open set in neutrosophic soft topological spaces. *Journal of Algorithms and Computation* **52**, pp. 37–66, 2020.
8. Nordo G., Mehmood A., Broumi S. Single Valued Neutrosophic Filters. *International Journal of Neutrosophic Science* **6**, pp. 8-21, 2020.
9. Nordo G., Jafari S., Mehmood A., Basumatary B. A Python Framework for Neutrosophic Sets and Mappings. *Neutrosophic Sets and Systems* **65**, pp. 199-236, 2024.
10. Salama A., El-Ghareeb H.A.. Manie A.M., Smarandache F. Introduction to develop some software programs for dealing with neutrosophic sets. *Neutrosophic Sets and Systems* **3**, pp. 51-52, 2019.
11. Salama A., Abd el-Fattah M., El-Ghareeb H.A.. Manie A.M. Design and Implementation of Neutrosophic Data Operations Using Object Oriented Programming. *International Journal of Computer Application* **4** (5), pp. 163-175, 2014.
12. Salama A.A., El-Hafeez S.A., El-Nakeeb A., El-Hassanein Eman M. Neutrosophic Approach for Mathematical Morphology. *Master's thesis*, Port Said University, 2018.
13. Serra J. Image Analysis and Mathematical Morphology. *Academic Press*, London (UK), 1982.
14. Serra J., Soille P. (Eds.) Mathematical Morphology and Its Applications to Image and Signal Processing. *Klwer Academic Publishers*, Dordrecht (Netherlands), 1994.
15. Shih F.Y. Image processing and Mathematical Morphology: Fundamentals and Applications. *CRC Press*, New York (USA), 2009.
16. Sleem A., Abdel-Baset M., El-henawy I. PyIVNS: A python based tool for Interval-valued neutrosophic operations and normalization. *SoftwareX* **12**, pp. 1-7, 2020.
17. Smarandache F. A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic. Rehoboth: American Research Press, 1999.

18. Wang H., Smarandache F., Zhang Y.Q., Sunderraman R. Single Valued Neutrosophic Sets. *Technical Sciences and Applied Mathematics*, pp. 10-14, 2012.
19. Wilson J.N, Ritter G.X. Handbook of Computer Vision Algorithms in Image Algebra (2nd edition). *CRC Press*, New York (USA), 2000.
20. Zadeh L.A. Fuzzy Sets, *Information and Control* **8** (3), pp. 338-353, 1965.

Fundamental group and complete parts of Neutrosophic Quadruple H_v-groups

Madeleine Al Tahan,[1] **Bijan Davvaz,**[2] **Florentin Smarandache,**[3] **Saba Al-Kaseasbeh**[4]

[1]Department of Mathematics, Abu Dhabi University, United Arab Emirates; altahan.madeleine@gmail.com
[2]Department of Mathematical Sciences, Yazd University, Yazd, Iran; davvaz@yazd.ac.ir
[3]Mathematics Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu
[4]Department of Mathematics, Tafila Technical University, Tafila, Jordan; saba.alkaseasbeh@gmail.com
*Correspondence: davvaz@yazd.ac.ir

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Abstract. It is well known that H_v-groups and groups are connected through regular relations. The purpose of this paper is to find a similar connection between neutrosophic H_v-groups and neutrosophic groups by using the concept of fundamental relations on Hv-groups. First, we characterize the complete parts of neutrosophic Hv-groups. Then we study their (strongly) regular relations. Finally, we find their fundamental group.

Keywords: neutrosophic quadruple number; neutrosophic quadruple H_v-group; fundamental group; complete part

1. Introduction

Fuzzy set theory [1] has many real-world applications, but it is not suitable for simulate an indeterminate issue in an abstract situation. By giving indeterminates representation, neutrosophic theory has advanced an important concept. One of the essential aspects in practically all problems in the real world is uncertainty or indeterminacy. Fuzzy theory is used to model uncertainty, whereas neutrosophic theory is employed to represent indeterminacy. In 1995, F. Smarandache created the concept of neutrosophy to represent problems involving indeterminates. For further background on neutrosophy and neutrosophic algebraic structures, see [2–6]. By assuming that the result of "interaction" between two elements of a non-empty set is a non-void set of elements, one is obviously generalizing the definition of a group. (and not only one element, as for groups), A hypergroup was first proposed in 1934 at the eighth congress

of Scandinavian mathematicians by the French mathematician Frederic Marty [7]. The rule that describes such a structure is known as a "hyperoperation", and the theory of algebraic structures with at least one multi-valued operation is recognized as the Hyperstructure Theory. Marty's motivation to introduce hypergroups is that the quotient of a group modulo any subgroup (not necessarily normal) is a hypergroup. This theory has been studied in the following decades and nowadays by many mathematicians. As a generalization of algebraic hyperstructures, Vougiouklis, in 1990, at the fourth A.H.A. congress [8,9] introduced the notion of H_v-structures.

Our paper presents a connection between hypergroups, fundamental groups and neutrosophy and it is constructed as follows: After an Introduction, in Section 2, we present some definitions related to (weak) hyperstructures that are used throughout the paper. In Section 3, we use the concept of neutrosophic H_v-group, defined by the authors in [10] and classify its complete parts and its (strongly) regular relations. Finally, in Section 4, we find the fundamental neutrosophic group of neutrosophic H_v-groups.

2. **Complete parts and regular relations in neurtosphic H_v-groups**

We use the notion of neutrosophic H_v-group, defined by the authors in [10] and classify its complete parts as well as its (strongly) regular relations. For basic definitions about algebraic hyperstructures we refer to [11–14]. In neutrosophy, $< X >$, $< antiX >$, and $< neutX >$ are paired two by two, as well as all three at once, to create the NeutroSynthesis. Neutrosophy lays out the universal relationships between $< X >$, $< antiX >$, and $< neutX >$. $< X >$ is the thesis, $< antiX >$ the antithesis, and $< neutX >$ the neutrothesis (neither $< X >$ nor $< antiX >$, but the neutrality in between them).

Definition 2.1. [4] Let B be a nonempty set. A neutrosophic quadruple B-number is an ordered quadruple (b_1, b_2T, b_3I, b_4F) where $b_1, b_2, b_3, b_4 \in B$ and T, I, F have their usual neutrosophic logic meanings.
The set of all neutrosophic quadruple B-numbers is denoted by NQ(B), that is,

$$NQ(B) = \{(b_1, b_42T, b_3I, b_4F) : b_1, b_2, b_3, b_4 \in B\}.$$

Let $(H\ , +)$ be an H_v-group with identity " 0" and define "$\oplus$" on $NQ(H\ )$ as follows:

$$(x_1, x_2T, x_3I, x_4F) \oplus (y_1, y_2T, y_3I, y_4F)$$
$$= \{(a, bT, cI, dF) :\ a \in x_1 + y_1, b \in x_2 + y_2, c \in x_3 + y_3, d \in x_4 + y_4\}.$$

Throughout this section, $(H, +)$ is an H_v-group with identity "0" and $0 + 0 = 0$.

Theorem 2.2. *[10] Let H be any set with a hyperoperation $+$. Then $(NQ(H), \oplus)$ is a neutrosophic quadruple H_v-group if and only if $(H, +)$ is an H_v-group with identity "$0 \in H$ " and $0 + 0 = 0$.*

Theorem 2.3. *[10] Let H be any set with a hyperoperation $+$. Then $(NQ(H), \oplus)$ is a neutrosophic quadruple hypergroup if and only if $(H, +)$ is a hypergroup with identity "$0 \in H$" and $0 + 0 = 0$.*

Theorem 2.4. *Let $(H, +)$ be an H_v-group with identity "0", $0 + 0 = 0$ and X be a non empty subset of $NQ(H)$. Then X is a complete part in $NQ(H)$ if and only if there exist $A_1, A_2, A_3, A_4 \subseteq H$ such that $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$ and that A_i is a complete part in H for $i = 1, 2, 3, 4$.*

Proof. Let X be a complete part in $NQ(H)$. Then there exist $A_1, A_2, A_3, A_4 \subseteq H$ such that $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$. We prove that A_1 is a complete part in H and the others are done in a similar manner. Let $x \in A_1 \cap P \neq \emptyset$. Then there exist $x_i \in H$ with $i = 1, 2, \ldots, k$ such that $x \in x_1 + \ldots + x_k$. Let $y \in A_2$, $z \in A_3$ and $w \in A_4$. It is clear that

$$(x, yT, zI, wF) \in X \cap ((x_1, yT, zI, wF) \oplus (x_2, 0T, 0I, 0F) \oplus \ldots \oplus (x_k, 0T, 0I, 0F)).$$

The latter and having X a complete part in $NQ(H)$ imply that $((x_1, yT, zI, wF) \oplus (x_2, 0T, 0I, 0F) \ldots \oplus (x_k, 0T, 0I, 0F) \subseteq X$. Consequently, we get $x_1 + \ldots + x_k \subseteq A_1$.

Conversely, let $A_1, A_2, A_3, A_4 \subseteq H$ be complete parts in H, $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$ and $(a, bT, cI, dF) \in X \cap ((x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF))$. Then $a \in A_1 \cap (x_1 + \ldots + x_k)$, $b \in A_2 \cap (y_1 + \ldots + y_k)$, $c \in A_1 \cap (z_1 + \ldots + z_k)$ and $d \in A_1 \cap (w_1 + \ldots + w_k)$. Having A_i a complete part in H for $i = 1, 2, 3, 4$ implies that $x_1 + \ldots + x_k \subseteq A_1$, $y_1 + \ldots + y_k \subseteq A_2$, $z_1 + \ldots + z_k \subseteq A_3$ and $w_1 + \ldots + w_k \subseteq A_4$. The latter implies that $(x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF) \subseteq X$. □

Corollary 2.5. *Let $(H, +)$ be an H_v-group with identity "0" and $0+0 = 0$ and A be a complete part in H. Then $NQ(A)$ is a complete part in $NQ(H)$.*

Proof. By setting $A_i = A$ for $i = 1, 2, 3, 4$ and $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$, Theorem 2.4 asserts that $NQ(A) = X$ is a complete part in $NQ(H)$. □

The authors in [15, 16] considered the set of arithmetic functions H and defined a hyperoperation $+$ on it as follows:

$$\alpha + \beta(n) = \{\alpha(d) + \beta(\frac{n}{d}) : d|n\}.$$

Let $0_\star(n) = 0$ for all $n \in \mathbb{N}$. It is clear that $0_\star + 0_\star = 0_\star$. The authors proved that $(H, +)$ is a hypergroup with identity $0_\star$ and characterized all complete parts in H as: $S = \bigcup_{r \in M} A_r$ where M is a non empty subset of the set of complex numbers $\mathbb{C}$ and $A_r = \{\alpha \in H : \alpha(1) = r\}$.

Proposition 2.6. *Let $(H,+)$ be the hypergroup of arithmetic functions under the above hyperoperation. Then $(NQ(H),\oplus)$ is a neutrosophic hypergroup.*

Proof. The proof follows from Theorem 2.3. □

Proposition 2.7. *Let $(H,+)$ be the hypergroup of arithmetic functions defined in [16] and X be a complete part in $NQ(H)$. Then there exist non empty subsets M_i of $\mathbb{C}$ with $i=1,2,3,4$ such that $X=\{(a,bT,cI,dF) : a\in S_1, b\in S_2, c\in S_3, d\in S_4\}$ and $S_i=\cup_{r\in M_i}A_r$ for $i=1,2,3,4$.*

Proof. The proof follows from Theorem 2.4. □

Corollary 2.8. *Let $(H,+)$ be the hypergroup of arithmetic functions defined in [16] and r be any complex number. Then $NQ(A_r)$ is a complete part in $NQ(H)$.*

Let $(H,+)$ be an H_v-group with identity "0", $0+0=0$ and R_i be a relation on H for $i=1,2,3,4$. We define ρ on $NQ(H)$ as follows:

$$(a,bT,cI,dF)\rho(a',b'T,c'I,d'F) \Leftrightarrow aR_1a', bR_2b', cR_3c', dR_4d'.$$

Proposition 2.9. *Let $(H,+)$ be an H_v-group with identity "0" and $0+0=0$. Then ρ is an equivalence relation on $NQ(H)$ if and only if R_i is an equivalence relation on H^* for $i=1,2,3,4$.*

Proof. The proof is straightforward. □

Theorem 2.10. *Let $(H,+)$ be an H_v-group with identity "0" and $0+0=0$. Then ρ is a regular relation on $NQ(H)$ if and only if R_i is a regular relation on H for $i=1,2,3,4$.*

Proof. Let ρ be a regular relation on $NQ(H)$ and $a,a'\in H$ with aR_1a'. Then $(a,0T,0I,0F)\rho(a',0T,0I,0F)$ (as $0R_i0$ for $i=2,3,4$). We need to show that $a+x\overline{R_1}a'+x$ and $x+a\overline{R_1}x+a'$ for all $x\in H$. We prove that $a+x\overline{R_1}a'+x$. Let $b\in a+x$. Then $(b,0T,0I,0F)\in(a,0T,0I,0F)\oplus(x,0T,0I,0F)$. Having $(a,0T,0I,0F)\rho(a',0T,0I,0F)$ and ρ a regular relation on $NQ(H)$ imply that

$$(a,0T,0I,0F)\oplus(x,0T,0I,0F)\overline{\rho}(a',0T,0I,0F)\oplus(x,0T,0I,0F).$$

The latter implies that there exist $(y,0T,0I,0F)\in(a',0T,0I,0F)\oplus(x,0T,0I,0F)$ such that $(z,0T,0I,0F)\rho(y,0T,0I,0F)$ for every $(z,0T,0I,0F)\in(a,0T,0I,0F)\oplus(x,0T,0I,0F)$. We get now that for every $z\in a+x$ there exists $y\in a'+x$ such that zR_1y. Thus, R_1 is a regular .

relation on H . In a similar manner, we can prove that R_i is a regular relation on H for $i = 2, 3, 4$.

Conversely, let R_i be a regular relation on H for $i = 1, 2, 3, 4$,

$$(a, bT, cI, dF)\rho(a', b'T, c'I, d'F) \text{ and } (x, yT, zI, wF) \in NQ(H).$$

We need to show that

$$(a, bT, cI, dF) \oplus (x, yT, zI, wF)\overline{\rho}(a', b'T, c'I, d'F) \oplus (x, yT, zI, wF).$$

Let $(e, fT, gI, hF) \in (a, bT, cI, dF) \oplus (x, yT, zI, wF)$. Then $e \in a + x$, $f \in b + y$, $g \in c + z$ and $h \in d + w$. Having aR_1a', bR_2b', cR_3c', dR_4d' and R_i a regular relation on H for $i = 1, 2, 3, 4$ imply that $a + x\overline{R_1}a' + x$, $b + y\overline{R_2}b' + y$, $c + z\overline{R_3}c' + z$, $d + w\overline{R_4}d' + w$. The latter implies that there exist $e' \in a' + x$, $f' \in b' + y$, $g' \in c' + z$, $h' \in d' + w$ such that eR_1e', fR_2f', gR_3g' and hR_4h'. We get now that $(e', f'T, g'I, h'F) \in (a', b'T, c'I, d'F) \oplus (x, yT, zI, wF)$ with $(e, fT, gI, hF)\rho(e', f'T, g'I, h'F)$. □

Theorem 2.11. *Let $(H, +)$ be an H_v-group with identity "0" and $0 + 0 = 0$. Then ρ is a strongly regular relation on $NQ(H)$ if and only if R_i is a strongly regular relation on H for $i = 1, 2, 3, 4$.*

Proof. Let ρ be a strongly regular relation on $NQ(H)$ and $a, a' \in H$ with aR_1a'. Then $(a, 0T, 0I, 0F)\rho(a', 0T, 0I, 0F)$ (as $0R_i0$ for $i = 2, 3, 4$). We need to show that $a + x\overline{\overline{R_1}}a' + x$ and $x + a\overline{\overline{R_1}}x + a'$ for all $x \in H$. We prove that $a + x\overline{\overline{R_1}}a' + x$ and the other is done in a similar manner. Let $b \in a + x$. Then $(b, 0T, 0I, 0F) \in (a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$. Having $(a, 0T, 0I, 0F)\rho(a', 0T, 0I, 0F)$ and ρ a strongly regular relation on $NQ(H)$ imply that $(a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)\overline{\overline{\rho}}(a', 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$. The latter implies that for all $(y, 0T, 0I, 0F) \in (a', 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$ and for all $(z, 0T, 0I, 0F) \in (a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$, we have $(z, 0T, 0I, 0F)\rho(y, 0T, 0I, 0F)$. We get now that for every $z \in a + x$ and for all $y \in a' + x$ such that zR_1y. Thus, R_1 is a strongly regular relation on H^*. In a similar manner, we can prove that R_i is a strongly regular relation on H for $i = 2, 3, 4$.

Conversely, let R_i be a strongly regular relation on H for $i = 1, 2, 3, 4$, $(a, bT, cI, dF)\rho(a', b'T, c'I, d'F)$ and $(x, yT, zI, wF) \in NQ(H)$. We need to show that $(a, bT, cI, dF) \oplus (x, yT, zI, wF)\overline{\overline{\rho}}(a', b'T, c'I, d'F) \oplus (x, yT, zI, wF)$. Let $(e, fT, gI, hF) \in (a, bT, cI, dF) \oplus (x, yT, zI, wF)$. Then $e \in a + x$, $f \in b + y$, $g \in c + z$ and $h \in d + w$. Having aR_1a', bR_2b', cR_3c', dR_4d' and R_i a strongly regular relation on H for $i = 1, 2, 3, 4$ imply that $a + x\overline{\overline{R_1}}a' + x$, $b + y\overline{\overline{R_2}}b' + y$, $c + z\overline{\overline{R_3}}c' + z$, $d + w\overline{\overline{R_4}}d' + w$. The latter implies that for all $e' \in a' + x$, $f' \in b' + y$, $g' \in c' + z$, $h' \in d' + w$, we have eR_1e', fR_2f', gR_3g' and hR_4h'. We get now that $(e', f'T, g'I, h'F) \in (a', b'T, c'I, d'F) \oplus (x, yT, zI, wF)$ with $(e, fT, gI, hF)\rho(e', f'T, g'I, h'F)$. □

Example 2.12. Let $\mathbb{Q}$ be the set of rational numbers, $(H,+)$ be the hypergroup of arithmetic functions and define the strongly regular equivalence relation R_i for $i=1,2,3,4$ on H as follows:

$$\alpha R_1 \gamma \Leftrightarrow \alpha(1) = \gamma(1) + q; q \in \mathbb{Q},$$

and for $i = 2,3,4$

$$\alpha R_i \gamma \Leftrightarrow \alpha(1) = \gamma(1).$$

Applying Theorem 2.11, we get ρ a strongly regular equivalence relation on $NQ(H)$, where ρ is defined as follows:

$$(\alpha, \beta T, \gamma I, \lambda F)\rho(\alpha', \beta' T, \gamma' I, \lambda' F)$$
$$\Leftrightarrow \alpha(1) = \alpha'(1) + q; q \in \mathbb{Q}, \beta(1) = \beta'(1), \gamma(1) = \gamma'(1), \lambda(1) = \lambda'(1).$$

3. Fundamental group of neutrosophic quadruple H_v-groups

In this part, we investigate the fundamental relation on neutrosophic H v-groups and determine its fundamental neutrosophic group..

In [3], Akinleye et. al. conducted their investigation of neutrosophic quadruple algebraic structures on quadruple numbers based on the set of real numbers. And they proved that if G is a group of real numbers then $NQ(G) = \{(g_1, g_2 T, g_3 I, g_4 F) : g_1, g_2, g_3, g_4 \in G\}$ is a neutrosophic group under the operation "$\oplus$" given by:

$$g_1, g_2 T, g_3 I, g_4 F) \oplus (g_1', g_2' T, g_3' I, g_4' F) = (g_1 + g_1', (g_2 + g_2')T, (g_3 + g_3')I, (g_4 + g_4')F).$$

The following theorem generalizes their result to any group.

Theorem 3.1. *Let G be a set with operation $+$ and $0 \in G$. Then $(NQ(G), \oplus)$ is a group if and only if $(G,+)$ is a group.*

Proof. The proof is similar to that of Theorem 2.1 in [3]. □

Proposition 3.2. *Let $(G,+)$ and $(G',+')$ be isomorphic groups. Then $(NQ(G), \oplus)$ and $(NQ(G'), \oplus')$ are isomorphic neutrosophic groups.*

Proof. Let $(G,+)$ and $(G',+')$ be isomorphic groups. Then there exists a group isomorphism $f : G \to G'$. Let $\phi : NQ(G) \to NQ(G')$ be defined as follows:

$$\phi((a, bT, cI, dF)) = (f(a), f(b)T, f(c)I, f(d)F).$$

One can easily see that ϕ is a group isomorphism. □

Definition 3.3. [13] For all $n > 1$, we define the relation β_n on a semihypergroup $(H, \circ)$ as follows:

$$x\beta_n y \text{ if there exist } a_1, \ldots, a_n \text{ in } H \text{ such that } \{x, y\} \subseteq \prod_{i=1}^{n} a_i$$

and we set $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H^*.

This relation was established by Koskas [17] and studied by many authors. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β.

Throughout this section, $\beta, \beta^\star$ are the relations on H and $\beta_N, \beta_N^\star$ are the relations on $NQ(H)$.

Theorem 3.4. *Let $(H, +)$ be an H_v-group with identity "0" and $0 + 0 = 0$ and let $a, a', b, b', c, c', d, d' \in H$. Then $(a, bT, cI, dF)\beta_N(a', b'T, c'I, d'F)$ if and only if $a\beta a'$, $b\beta b'$, $c\beta c'$ and $d\beta d'$.*

Proof. Let $(a, bT, cI, dF)\beta_N(a', b'T, c'I, d'F)$. Then there exist (x_i, y_iT, z_iI, w_iF) with $i = 1, \ldots, k$ such that $\{(a, bT, cI, dF), (a', b'T, c'I, d'F)\} \subseteq (x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF)$. The latter implies that $a, a' \in x_1 + \ldots + x_k$, $b, b' \in y_1 + \ldots + y_k$, $c, c' \in z_1 + \ldots + z_k$ and $d \in w_1 + \ldots + w_k$. Thus, $a\beta a'$, $b\beta b'$, $c\beta c'$ and $d, d'\beta d'$.

Conversely, let $a\beta a'$, $b\beta b'$, $c\beta c'$ and $d\beta d'$. Then there exist $k_1, k_2, k_3, k_4 \in \mathbb{N}$ and $x_1, \ldots, x_{k_1}$, $y_1, \ldots, y_{k_2}$, $z_1, \ldots, z_{k_3}$, $w_1, \ldots, w_{k_4} \in H$ such that $a, a' \in x_1 + \ldots + x_{k_1}$, $b, b' \in y_1 + \ldots + y_{k_2}$, $c, c' \in z_1 + \ldots + z_{k_3}$ and $d, d' \in w_1 + \ldots + w_{k_4}$. By setting $k = \max\{k_1, k_2, k_3, k_4\}$ and $x_i = 0$ for $k_1 < i \leq k$, $y_i = 0$ for $k_2 < i \leq k$, $z_i = 0$ for $k_3 < i \leq k$ and $w_i = 0$ for $k_4 < i \leq k$ and using the fact that $x \in 0 + x \cap x + 0$ for all $x \in H$, we get $a, a' \in x_1 + \ldots + x_k$, $b, b' \in y_1 + \ldots + y_k$, $c, c' \in z_1 + \ldots + z_k$ and $d, d' \in w_1 + \ldots + w_k$. The latter implies that $\{(a, bT, cI, dF), (a', b'T, c'I, d'F)\} \subseteq (x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF)$. Thus, $(a, bT, cI, dF)\beta_N(a', b'T, c'I, d'F)$. □

Theorem 3.5. *Let $(H, +)$ be an H_v^*-group with identity "0" and $0 + 0 = 0$ and let $a, a', b, b', c, c', d, d' \in H$. Then $(a, bT, cI, dF)\beta_N^\star(a', b'T, c'I, d'F)$ if and only if $a\beta^\star a'$, $b\beta^\star b'$, $c\beta^\star c'$ and $d\beta^\star d'$.*

Proof. Let $(a, bT, cI, dF)\beta_N^\star(a', b'T, c'I, d'F)$. Then there exist $(x_i, y_iT, z_iI, w_iF) \in NQ(H)$ with $i = 1, \ldots, k$ such that
$(a, bT, cI, dF)\beta_N(x_1, y_1T, z_1I, w_1F)$, $(x_i, y_iT, z_iI, w_iF)\beta_N(x_{i+1}, y_{i+1}T, z_{i+1}I, w_{i+1}F)$ for $i = 1, \ldots, k-1$ and $(x_k, y_kT, z_kI, w_kF)\beta_N(a', b'T, c'I, d'F)$. Theorem 3.4 implies that $a\beta x_1$, $x_i\beta x_{i+1}$ for $i = 1, \ldots, k-1$, $x_k\beta a'$, $b\beta y_1$, $y_i\beta y_{i+1}$ for $i = 1, \ldots, k-1$, $y_k\beta b'$, $c\beta z_1$, $z_i\beta z_{i+1}$ for $i = 1, \ldots, k-1$, $z_k\beta c'$, $d\beta w_1$, $w_i\beta w_{i+1}$ for $i = 1, \ldots, k-1$, $w_k\beta d'$. Thus, $a\beta^\star a'$, $b\beta^\star b'$, $c\beta^\star c'$ and $d\beta^\star d'$.

Conversely, let $a\beta^\star a'$, $b\beta^\star b'$, $c\beta^\star c'$ and $d\beta^\star d'$. Then there exist $x_i, y_i, z_i, w_i \in H$ such that $a\beta x_1$, $x_i\beta x_{i+1}$ for $i = 1, \ldots, k-1$, $x_k\beta a'$, $b\beta y_1$, $y_i\beta y_{i+1}$ for $i = 1, \ldots, l-1$, $y_l\beta b'$, $c\beta z_1$, $z_i\beta z_{i+1}$ for $i = 1, \ldots, m-1$, $z_m\beta c'$, $d\beta w_1$, $w_i\beta w_{i+1}$ for $i = 1, \ldots, s-1$, $w_s\beta d'$. By setting $t = \max\{k, l, m, s\}$ and $x_i = a'$ for $k < i \leq t$, $y_i = b'$ for $t < i \leq t$, $z_i = c'$ for $m < i \leq t$ and $w_i = d'$ for $s < i \leq t$. The latter implies that

$$(a, bT, cI, dF)\beta_N(x_1, y_1T, z_1I, w_1F), \quad (x_i, y_iT, z_iI, w_iF)\beta_N(x_{i+1}, y_{i+1}T, z_{i+1}I, w_{i+1}F),$$

for $i = 1, \ldots, t-1$ and

$$(x_t, y_tT, z_tI, w_tF)\beta_N(a', b'T, c'I, d'F).$$

Thus, $(a, bT, cI, dF)\beta_N^\star(a', b'T, c'I, d'F)$. □

Theorem 3.6. *Let $(H\ , +)$ be an H_v^*-group with identity "0" and $0 + 0 = 0$. Then $NQ(H)/\beta_N^\star \cong NQ(H/\beta^\star)$.*

Proof. Let $\phi : NQ(H)/\beta_N^\star \to NQ(H/\beta^\star)$ be defined as

$$\phi(\beta_N^\star((a, bT, cI, dF))) = (\beta^\star(a), \beta^\star(b)T, \beta^\star(c)I, \beta^\star(d)F).$$

Theorem 3.5 implies that ϕ is well-defined and one-to-one. Also, it is clear that ϕ is onto. We need to show that ϕ is a group homomorphism. Since

$$\beta_N^\star((a, bT, cI, dF)) \boxplus' \beta_N^\star((a', b'T, c'I, d'F)) = \beta_N^\star((x, yT, zI, wF))$$

where $(x, yT, zI, wF) \in (a, bT, cI, dF) \oplus (a', b'T, c'I, d'F) = (a + a', (b + b')T, (c + c')I, (d + d')F)$, it follows that $\phi(\beta_N^\star((a, bT, cI, dF)) \boxplus \beta_N^\star((a', b'T, c'I, d'F))) = \phi((x, yT, zI, wF)) = (\beta^\star(x), \beta^\star(y)T, \beta^\star(z)I, \beta^\star(w)F)$. Having $\beta^\star(x) = \beta^\star(a) \boxplus \beta^\star(a')$, $\beta^\star(y) = \beta^\star(b) \boxplus \beta^\star(b')$, $\beta^\star(z) = \beta^\star(c) \boxplus \beta^\star(c')$ and $\beta^\star(w) = \beta^\star(d) \boxplus \beta^\star(d')$ imply that $\phi(\beta_N^\star((a, bT, cI, dF)) \boxplus' \beta_N^\star((a', b'T, c'I, d'F))) = \phi(\beta_N^\star((a, bT, cI, dF)) \oplus' \phi(\beta_N^\star((a', b'T, c'I, d'F)))$. □

Theorem 3.7. *Let $(H, +)$ be an H_v^*-group with identity "0" and $0 + 0 = 0$. If G is the fundamental group of H^* (up to isomorphism) then $NQ(G)$ is the fundamental group of $NQ(H)$ (up to isomorphism).*

Proof. Since G is the fundamental group of H (up to isomorphism), it follows that $H/\beta^\star \cong G$. The latter and Proposition 3.2 imply that $NQ(H/\beta^\star) \cong NQ(G)$. Theorem 3.6 completes the proof. □

Example 3.8. Let $(H, +)$ be the hypergroup of arithmetic functions defined in [15] with the group of complex numbers $(\mathbb{C}, +)$ under standard addition as a fundamental group (up to isomorphism). Then $(NQ(\mathbb{C}), +)$ is the fundamental group of $NQ(H)$ up to isomorphism.

Corollary 3.9. *Let $(H,+)$ be an H_v-group with identity "0" and $0+0=0$. If H has a trivial fundamental group then $NQ(H)$ has a trivial fundamental group.*

Proof. The proof is straightforward by applying Corollary 3.7. □

Example 3.10. Let $H_1=\{0,1\}$ and define $(H_1,+_1)$ as follows:

$+_1$	0	1
0	0	1
1	1	H_1

Then $(NQ(H_1),\oplus)$ is a quadruple H_v-group. Since $0,1\in 1+_1 1$, it follows that $x\beta y$ for all $x,y\in H_1$. Thus, H_1 has trivial fundamental group. Corollary 3.9 asserts that $(NQ(H_1),\oplus)$ has trivial fundamental group.

Theorem 3.11. *Let $(H,+)$ be an H_v-group with identity "0" and $0+0=0$ and w_H be the heart of H . Then $NQ(w_H)$ is the heart of $NQ(H)$.*

Proof. Let $w_H=\{x\in H:\beta^\star(x)=\beta^\star(0)\}$ be the heart of H . Having $w_{NQ(H)}=\{(a,bT,cI,dF)\in NQ(H):\beta_N^\star((a,bT,cI,dF))=\beta_N^\star(0,0T,0I,0F)\}$ and applying Theorem 3.5, we get that $\beta^\star(a)=\beta^\star(b)=\beta^\star(c)=\beta^\star(d)=\beta^\star(0)$. Thus, $w_{NQ(H)}=\{(a,bT,cI,dF)\in NQ(H):a,b,c,d\in w_H\}=NQ(w_H)$. □

Example 3.12. Let $(H,+)$ be the hypergroup of arithmetic functions defined in [15]. Then $NQ(A_0)$ is the heart of $NQ(H)$.

4. Conclusion

This paper connected neutrosophic H_v-groups and neutrosophic groups by means of complete parts and regular relations. More precisely, it used the complete parts and the fundamental relation of H_v-groupls to reach its main results that are summarized in Theorems 4.6 and 4.7. The results were supported by non-trivial illustrative examples. For future research, it is interesting to find a connection between other types of neutrosophic H_v-structures and neutrosophic algebraic structures. One can investigate the connection between neutrosophic H_v-rings and neutrosophic rings or the connection between neutrosophic H_v-modules and neutrosophic modules.

References

1. L. A. Zadeh, *Fuzzy sets*, Inform and control 8 (1965), 338-353.
2. A. A. A. Agboola, B. Davvaz and F. Smarandache, *Neutrosophic quadruple algebraic hyperstructures*, Ann. Fuzzy Math. Inform. 14 (2017) 29-42.
3. S. A. Akinleye, F. Smarandache and A. A. A. Agboola, *On neutrosophic quadruple algebraic structures*, Neutrosophic Sets Syst. 12 (2016) 122-126.
4. Y. B. Jun , S-Z. Song, F. Smarandache and H.Bordbar, *Neutrosophic Quadruple BCK/BCI-Algebras*, Axioms 7(41), (2018), doi:10.3390/axioms7020041.
5. F. Smarandache, *Neutrosophy, neutrosophic probability, set, and logic*, American Research Press, USA, 1998.
6. F. Smarandache, *Neutrosophic quadruple numbers, refined neutrosophic quadruple numbers, absorbance law, and the multiplication of neutrosophic quadruple numbers*, Neutrosophic Sets and Systems 10 (2015) 96-98.
7. F. Marty, *Sur une generalization de la notion de group*, In 8*th* Congress Math. Scandenaves, (1934) 45-49.
8. T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press Monographs in Mathematics, Hadronic Press, Inc., Palm Harbor, FL, 1994.
9. T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, Algebraic hyperstructures and applications (Xanthi, 1990), 203-211, World Sci. Publ., Teaneck, NJ, 1991.
10. M. Al-Tahan and B. Davvaz, *On neutrosophic quadruple H_v^*-groups and their properties*, Southeast asian Bulletin of Mathematics, 44 (2020), 613-625.
11. P. Corsini and V. Leoreanu, *Applications of hyperstructures theory*, Advances in Mathematics, Kluwer Academic Publisher, 2003.
12. B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013. viii+200 pp.
13. B. Davvaz, *Semihypergroup Theory*, Elsevier/Academic Press, London, 2016. viii+156 pp.
14. B. Davvaz, V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, U.S.A., 2007.
15. M. Al-Tahan, B. Davvaz, *On the existence of hyperrings associated to arithmetic functions*, J.Number Theory 174 (2017) 136-149.
16. M. Al-Tahan and B. Davvaz, *Strongly regular relations of arithmetic functions*, Journal of Number Theory, 187 (2018) 391-402.
17. M. Koskas, *Groupoides, demi-hypergroupes et hypergroupes*, J. Math. Pure Appl., **(9) 49** (1970) 155-192.

Neutrosophic semi δ-preopen sets and neutrosophic semi δ-pre continuity

Mahima Thakur,[1] **F. Smarandache,**[2] **S. S. Thakur**[3]

[1]Department of Applied Mathematics, Jabalpur Engineering College Jabalpur, 482011, India; mahimavthakur@gmail.com

[2] Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA; fsmarandache@gmail.com

[3]Department of Applied Mathematics, Jabalpur Engineering College Jabalpur, 482011, India; satya_edu1974@yahoo.co.in

*Correspondence: mahimavthakur@gmail.com;

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Abstract. The purpose of this paper is to introduce the concepts of neutrosophic semi δ-preopen sets and neutrosophic semi δ-precontinuous mappings in neutrosophic topological spaces and obtain some of their properties and characterizations.

Keywords: Neutrosophic set, neutrosophic topology, neutrosophic semi δ-preopen sets and neutrosophic semi δ-precontinuity.

1. Introduction

As an elaboration of Zadeh's fuzzy sets [19] from 1965 and Atanassav's intuitionistic fuzzy sets [4] from 1983, Smarandache has proposed and described neutrosophic sets.He [13] defined neutrosophic set on a non empty set by considering three components, namely membership, Indeterminacy and non-membership whose sum lies between 0 and 3. Some more properties of neutrosophic sets are presented by Smarandache [13–15], Salama and Alblowi [11] , Lupiáñez [9]. Smarandache's Neutrosophic concepts have wide range of real time applications for the fields of Information systems, Computer science, Artificial Intelligence, Applied Mathematics and Decision making. In 2008, Lupiáñez [9] introduced the neutrosophic topology as a extension of intuitionistic fuzzy topology. Since 2008 many authors such as Lupiáñez [9, 10], Salama et.al. [11, 12] Karatas and Cemil [8], Acikgoz and his coworkers [1], Dhavaseelan et.al. [5], Al-Musaw [2], Dey and Ray [**?**] and others contributed in neutrosophic

topological spaces. Recently, many weak and strong forms of neurosophic open sets such as neutrosophic regular open [3], neutrosophic α-open [3], neutrosophic semi open [3,7],neutrosophic pre open [3,18],neutrosophic semi pre open [3,16],neutrosophic b-open [6], neutrosophic δ-open sets [17], neutrosophic δ-pre open [17] and weak and strong forms of neurosophic continuity such as neutrosophic semi continuity,and neurosophic precontinuity [3,18] and neurosophic α-continuity [3], and neurosophic semi pre continuity [3,16], neutrosophic b-continuity [6] have been investigated by different authors. In this paper we introduce a super class of above classes of neutrosophic open sets and a super class above mentioned neutrosophic continuity of mappings and studied their characterizations and properties.

2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel.

Definition 2.1. [13] A Neutrosophic set (**NS**) in X is a structure

$$A = \{< x, \mu_A(x), \varpi_A(x), \gamma_A(x) >: x \in X\}$$

where $\mu_A : X \to]^-0, 1^+[$, $\varpi A : X \to]^-0, 1^+[$, and $\gamma_A : X \to]^-0, 1^+[$ denotes the membership, indeterminacy, and non-membership of A satisfies the condition if $^-0 \leq \mu_A(x) + \varpi_A(x) + \gamma_A(x) \leq 3^+, \ \forall \ x \in X$.

In the real life applications in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the closed interval [0,1] and sum of membership, indeterminacy, and non-membership degrees of each element of universe of discourse lies between 0 and 3.

Definition 2.2. [12] Let X be a non empty set and the neutrosophic sets A and neutrosophic set B be in the form $A = \{< x, \mu_A(x), \varpi_A(x), \gamma_A(x) >: x \in X\}$, $B = \{< x, \mu_B(x), \varpi_B(x), \gamma_B(x) >: x \in X\}$ and let $\{A_i : i \in J\}$ be a arbitrary family of neutrosophic sets in X. Then:

(a) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, $\varpi_A(x) \leq \varpi_B(x)$, and $\gamma_A(x) \geq \gamma_B(x)$.
(b) $A = B$ if $A \subseteq B$ and $B \subseteq A$.
(c) $A^c = \{< x, \gamma_A(x), \varpi_A(x), \mu_A(x) >: x \in X\}$.
(d) $\cap A_i = \{< x, \wedge\mu_{A_i}(x), \wedge\varpi_{A_i}(x), \vee\gamma_{A_i}(x) >: x \in X\}$.
(e) $\cup A_i = \{< x, \vee\mu_{A_i}(x), \vee\varpi_{A_i}(x), \wedge\gamma_{A_i}(x) >: x \in X\}$.
(f) $\tilde{\mathbf{0}} = \{< x, 0, 0, 1 >: x \in X\}$ and $\tilde{\mathbf{1}} = \{< x, 1, 1, 0 >: x \in X\}$

Definition 2.3. [9, 11] A neutrosophic topology on a non empty set X is a family τ of neutrosophic sets in X, satisfying the following axioms:

(T_1) $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}} \in \tau$

(T_2) $G_1 \bigcap G_2 \in \tau$

(T_3) $G_1 \bigcup G_2 \in \tau$

In this case the pair (X, τ) is called a neutrosophic topological space and each neutrosophic set in τ is known as a neutrosophic open set in X. The complement A^c of a neutrosophic open set A is called a neutrosophic closed set in X.

Definition 2.4. [11] Let (X, τ) be a neutrosophic topological space and A be a neutrosophic set in X. Then the neutrosophic interior and neutrosophic closure of A are defined by:

Cl(A) = $\cap$ {K: K is a neutrosophic closed set such that A $\subseteq$ K }

Int(A) = $\cup$ {K: K is a neutrosophic open set such that K $\subseteq$ A }

Definition 2.5. [5] Let $\alpha, \eta, \beta \in [0, 1]$ and $0 \leq \alpha + \eta + \beta \leq 3$. A neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X is a neutrosophic set in X defined by

$$x_{(\alpha,\eta,\beta)}(y) = \begin{cases} (\alpha, \eta, \beta) & \text{if } y = x \\ (0, 0, 1) & \text{if } y \neq x \end{cases} \tag{1}$$

Definition 2.6. [1] Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point in X and $A = \{< x, \mu_A, \varpi_A, \gamma_A >: x \in X\}$ is a neutrosophic set in X. Then $x_{(\alpha,\eta,\beta)} \subseteq A$ if and only if $\alpha \subseteq \mu_A(x)$, $\eta \subseteq \varpi_A$, and $\beta \supseteq \nu_A(x)$.

Definition 2.7. [1] A nuetrosophic point $x_{(\alpha,\eta,\beta)}$ is said to be quasi-coincident (q-coincident, for short) with A, denoted by $x_{(\alpha,\eta,\beta)} q A$ iff $x_{(\alpha,\eta,\beta)} \not\subseteq A^c$. If $x_{(\alpha,\eta,\beta)}$ is not quasi-coincident with A, we denote by $\rceil(x_{(\alpha,\eta,\beta)} q A)$.

Definition 2.8. [1] Two neutrosophic set A and B of X said to be q-coincident (denoted by $A_q B$) if $A \not\subseteq B^c$.

Lemma 2.9. *[1] For any two neutrosophic sets A and B of X , $\rceil(A_q B) \Leftrightarrow A \subset B^c$. where $\rceil(A_q B)$ A is not q-coincident with B.*

Definition 2.10. [3] A neutrosophic set A of a NTS (X, τ) is called neutrosophic regular open set (resp. neutrosophic regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$).

Remark 2.11. [3] Every neutrosophic regular open (resp. neutrosophic regular closed) set is neutrosophic open (resp. neutrosophic closed), But the converse may not be true.

Definition 2.12. [17] The δ-interior (denoted by δint) of a neutrosophic set A of a NTS (X, τ) is the union of all neutrosophic regular open sets contained in A.

Definition 2.13. [17] The δ-closure (denoted by δcl) of a neutrosophic set A of a NTS (X, τ) is the intersection of all neutrosophic regular closed sets containing A.

Definition 2.14. [3, 6, 17] A neutrosophic set A of a NTS (X, τ) is called neutrosophic semiopen (resp. neutrosophic preopen ,, neutrosophic α-open, neutrosophic semi preopen ,]neutrosophic δ-open , neutrosophic δ-preopen, neutrosophic δ-semiopen , neutrosophic b-open if $A \subseteq cl(int(A))$ (resp. $A \subseteq int(cl(A))$, $A \subseteq int(cl(int(A)))$,$A \subseteq cl(int(cl(A)))$, $A = \delta int(A)$, $A \subseteq int(\delta cl(A))$, $A \subseteq cl(\delta int(A))$, $A \subseteq cl(int(A)) \cup int(cl(A)))$.

The family of all neutrosophic semiopen (resp. neutrosophic preopen, neutrosophic α-open, neutrosophic semi preopen, neutrosophic δ-open, δ-preopen, neutrosophic δ-semiopen, neutrosophic γ-open) sets of a NTS (X, τ) is denoted by $NSO(X)$ (resp. $NPO(X), N\alpha O(X), NSPO(X), N\delta O(X), N\delta PO(X), N\delta SO(X), NbO(X)$).

Definition 2.15. [3, 6, 17] A neutrosophic set A in a NTS (X, τ) is called neutrosophic neutrosophic semiclosed (resp. neutrosophic preclosed, neutrosophic α-closed, neutrosophic semi preclosed, neutrosophic δ-preclosed, neutrosophic δ-semiclosed, neutrosophic γ-closed) if $A^c \in NSO(X)$ (resp. $NPO(X), N\alpha O(X), NSPO(X), N\delta O(X), N\delta PO(X), N\delta SO(X), NbO(X)$).

Remark 2.16. [3, 16–18] Every neutrosophic δ-open (resp. neutrosophic δ-closed) set is neutrosophic open (resp. neutrosophic closed), every neutrosophic open (resp. neutrosophic closed) set is neutrosophic α-open (resp. neutrosophic α-closed), every neutrosophic α-open (resp. neutrosophic α-closed) set is neutrosophic semiopen (resp. neutrosophic semiclosed) as well as neutrosophic preopen (resp. neutrosophic preclosed) and every neutrosophic semiopen (resp. neutrosophic semiclosed) set and every neutrosophic preopen (resp. neutrosophic preclosed) set is neutrosophic semi-preopen (resp. neutrosophic semi-preclosed). But the separate converses may not be true.

Remark 2.17. [17] Every neutrosophic preopen (resp. neutrosophic preclosed) set is neutrosophic δ-preopen (resp. neutrosophic δ-preclosed) but the converse may not be true.

Remark 2.18. [6] Every neutrosophic semiopen (resp. neutrosophic semiclosed) and neutrosophic preopen (resp. neutrosophic preclosed) set is neutrosophic b-open (resp.neutrosophic b-closed), and every neutrosophic b-open (resp.neutrosophic b-closed)set is neutrosophic semi-preopen (resp.neutrosophic semi-preclosed)but the separate converses may not be true.

Definition 2.19. [1] Consider that f is a mapping from X to Y.

(a) Let A be a neutrosophic set in X with membership function $\mu_A(x)$, indeterminacy function $\varpi_A(x)$ and non-membership function $\sigma_A(x)$. The image of A under f, written as f(A), is a neutrosophic set of Y whose membership function, indeterminacy function and non-membership function are defined as

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\mu_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & \text{if } f^{-1}(y) = \phi, \end{cases}$$

$$\varpi_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\varpi_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & iff^{-1}(y) = \phi, \end{cases}$$

$$\gamma_{f(A)}(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \{\gamma_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & iff^{-1}(y) = \phi, \end{cases}$$

$\forall\, , y \in Y$. Where $f^{-1}(y) = \{x : f(x) = y\}$.

(b) Let B be a neutrosophic set in Y with membership function $\mu_B(y)$, indeterminacy function $\varpi_B(y)$ and non-membership function $\gamma_B(y)$. Then, the inverse image of B under f, written as $f^{-1}(B)$ is a neutrosophic set of X whose membership function, indeterminacy function and non-membership function are defined as $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, $\varpi_{f^{-1}(B)}(x) = \varpi_B(f(x))$, and $\gamma_{f^{-1}(B)}(x) = \gamma_B(f(x))$, respectively $\forall\, x \in X$.

Definition 2.20. [3,6,17] A mapping $f : (X, \tau) \to (Y, \sigma)$ is called neutrosophic semi continuous (resp. neutrosophic pre continuous , neutrosophic α-continuous,neutrosophic semi pre continuous, neutrosophic δ-pre continuous, neutrosophic b-continuous if $f^{-1}(A) \in NSO(X)$ (resp. $f^{-1}(A) \in NPO(X)$, $f^{-1}(A) \in N\alpha O(X)$, $f^{-1}(A) \in NSPO(X)$, $f^{-1}(A) \in N\delta PO(X)$, $f^{-1}(A) \in NbO(X)$) for each neutrosophic set $A \in \sigma$.

Remark 2.21. [3, 16, 17] Every neutrosophic continuous mappings is neutrosophic α-continuous, every neutrosophic α-continuous mapping is neutrosophic semi continuous and neutrosophic pre continuous , every neutrosophic semi continuous (resp. neutrosophic pre continuous) mapping is neutrosophic b-continuous and neutrosophic b-continuous mapping is neutrosophic semi pre continuous but the separate converses may not be true.The concepts of neutrosophic semi continuous and neutrosophic pre continuous mappings are independent.

Remark 2.22. [17] Every neutrosophic pre continuous mapping is neutrosophic δ-pre continuous but the converse may not be true.

3. Neutrosophic semi δ-preopen sets

In this section, we introduce the concept of neutrosophic semi δ-preopen set and study some of their properties in neutrosophic topological spaces.

Definition 3.1. A neutrosophic set A in a NTS (X, τ) is called:

(a) neutrosophic semi δ-preopen if there exists a neutrosophic δ-preopen set O such that $O \subseteq A \subseteq \delta cl(O)$.

(b) neutrosophic semi δ-preclosed if there exists a neutrosophic δ-preclosed set F such that $\delta int(F) \subseteq A \subseteq F$.

The family of all neutrosophic semi δ-preopen (resp. neutrosophic semi δ-preclosed) sets of a NTS (X, τ) is denoted by $NS\delta PO(X)$ (resp. $NS\delta PC(X)$).

Remark 3.2. Every neutrosophic δ-semiopen (resp. neutrosophic δ-semiclosed)set is neutrosophic semiopen (resp. neutrosophic semiclosed) but the converse may not be true.

Example 3.3. Let $X = \{a, b\}$ and neutrosophic sets A, B, O are defined as follows:

$$A = \{< a, 0.3, 0.5, 0.7 >, < b, 0.4, 0.5, 0.6 >\}$$

$$B = \{< a, 0.4, 0.5, 0.6 >, < b, 0.3, 0.5, 0.7 >\}$$

$$O = \{< a, 0.5, 0.5, 0.5 >, < b, 0.6, 0.5, 0.4 >\}$$

let $\tau = \{\tilde{0}, A, B, A \cup B, A \cap B, \tilde{1}\}$ be the neutrosophic topology on (X, τ). Then O is neutrosophic semiopen but not neutrosophic δ-semiopen.

Remark 3.4. Every neutrosophic δ-open (resp. neutrosophic δ-closed) set is neutrosophic δ-semiopen (resp. neutrosophic δ-semiclosed) but the converse may not be true. For, in the NTS (X, τ) of example (3.3), the neutrosophic set A is neutrosophic δ-semiopen but not neutrosophic δ-open.

Remark 3.5. The concepts of neutrosophic δ-semiopen and neutrosophic open sets are independent. For, in the NTS (X, τ) of example (3.3), the neutrosophic set O is neutrosophic δ-semiopen but not neutrosophic open and neutrosophic A is neutrosophic open but not neutrosophic δ-semiopen.

Theorem 3.6. *An neutrosophic set $A \in NS\delta PC(X)$ if and only if $A^c \in NS\delta PO(X)$.*

Remark 3.7. Every neutrosophic semi preopen (resp. neutrosophic semi preclosed) set and Every neutrosophic δ-preopen (resp. neutrosophic δ-preclosed) set is neutrosophic semi δ-preopen (resp. neutrosophic semi δ-preclosed). But the separate converse may not be true.

Example 3.8. Let $X = \{a, b\}$ and neutrosophic sets A, B, O, F are neutrosophic sets defined as follows:

$$A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.3, 0.5, 0.7 >\}$$

$$B = \{< a, 0.5, 0.5, 0.5 >, < b, 0.1, 0.5, 0.9 >\}$$

$$O = \{< a, 0.5, 0.5, 0.5 >, < b, 0.9, 0.5, 0.1 >\}$$

$$F = \{< a, 0.5, 0.5, 0.5 >, < b, 0.6, 0.5, 0.4 >\}$$

let $\tau = \{\tilde{0}, A, B, \tilde{1}\}$ be a neutrosophic topology on (X, τ), Then

(a) O is neutrosophic semi δ-preopen (resp. O^c neutrosophic semi preclosed) but not neutrosophic semi preopen (resp. neutrosophic semi preclosed).

(b) F is neutrosophic semi δ-preopen (resp. F^c is neutrosophic semi preclosed) but not neutrosophic δ-preopen (resp. neutrosophic δ-preclosed).

Remark 3.9. It is clear that from remark (2.11), (2.16), (2.17), (2.18), (3.2), (3.4), (3.5) and (3.7) that the following figure of implications is true.

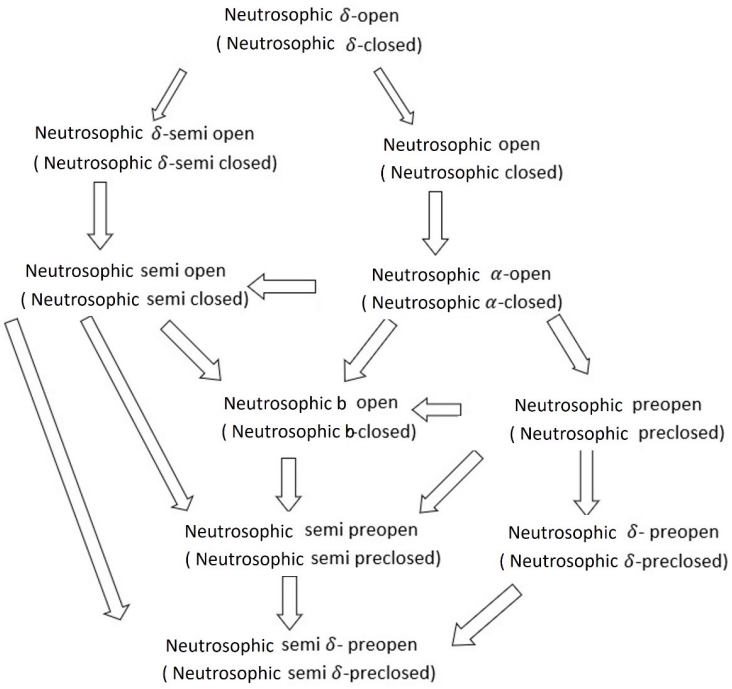

Figure 1

Theorem 3.10. *Let (X, τ) be a NTS . Then*

(a) *Any union of neutrosophic semi δ-preopen sets is neutrosophic semi δ-preopen.*

(b) *Any intersection of neutrosophic semi δ-preclosed sets is neutrosophic semi δ-preclosed.*

Theorem 3.11. *A neutrosophic set $A \in NS\delta PO(X)$ if and only if for every neutrosophic point $x_{(\alpha,\eta,\beta)} \in A$ there exists a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O \subseteq A$.*

Proof. If $A \in NS\delta PO(X)$ then we may take $O = A$ for every $x_{(\alpha,\eta,\beta)} \in A$.
Conversely. We have $A = \cup_{x_{(\alpha,\eta,\beta)} \in A} \{x_{(\alpha,\eta,\beta)}\} \subseteq \cup_{x_{(\alpha,\eta,\beta)} \in A} O \subseteq A$.The result now follows from the fact that any union of neutrosophic δ-preopen sets is neutrosophic δ-preopen. □

Theorem 3.12. *Let (X, τ) be a NTS .*

(a) *If $A \subseteq O \subseteq \delta cl(A)$ and $A \in NS\delta PO(X)$ then $O \in NS\delta PO(X)$,*
(b) *If $\delta int(B) \subseteq F \subseteq B$ and $B \in NS\delta PC(X)$ then $F \in NS\delta PC(X)$*

Proof. (a) Let $O_1 \in N\delta PO(X)$ such that $O_1 \subseteq A \subseteq \delta cl(O_1)$.
Clearly $O_1 \subseteq O$ and $A \subseteq \delta cl(O_1)$ implies that $\delta cl(A) \subseteq \delta cl(O_1)$. Consequently,
$O_1 \subseteq O \subseteq \delta cl(O_1)$. Hence $O \in NS\delta PO(X)$.
(b) Follows from (a). □

Lemma 3.13. *An neutrosophic set $A \in N\delta PO(X)$ if and only if there exists a neutrosophic set O such that $A \subseteq O \subseteq \delta cl(A)$.*

Proof. **Necessity** If $A \in N\delta PO(X)$ then $A \subseteq int(\delta cl(A))$.
Put $O = int(\delta cl(A))$ then O is a neutrosophic open and $A \subseteq O \subseteq \delta cl(A)$.
Sufficiency Let O be a neutrosophic open set such that $A \subseteq O \subseteq \delta cl(A)$, then $A \subseteq int(O) \subseteq int(\delta cl(A))$. Hence $A \in N\delta PO(X)$. □

Lemma 3.14. *Let Y be a neutrosophic subspace of NTS (X, τ) and A be a neutrosophic set in Y. If $A \in N\delta PO(X)$ then $A \in N\delta PO(Y)$.*

Proof. Since $A \in N\delta PO(X)$, By Lemma (3.13), there exists a neutrosophic set O in (X, τ) such that $A \subseteq O \subseteq \delta cl(A)$. Therefore $A \cap Y \subseteq O \cap Y \subseteq \delta cl(A) \cap Y = \delta cl_Y(A)$. It follows that $A \subseteq O \subseteq \delta cl_Y(A)$. Hence by Lemma (3.13), $A \in N\delta PO(Y)$. □

Theorem 3.15. *Let Y be a neutrosophic subspace of NTS (X, τ) and A be a neutrosophic set in Y. If $A \in NS\delta PO(X)$ then $A \in NS\delta PO(Y)$.*

Proof. Let $O \in N\delta PO(X)$ such that $O \subseteq A \subseteq cl(O)$. Then $O \cap Y \subseteq A \cap Y \subseteq cl(O) \cap Y$. It follows that $O \subseteq A \subseteq cl_Y(O)$. Now by Lemma (3.14), $O \in N\delta PO(Y)$ and hence $A \in NS\delta PO(Y)$. □

Theorem 3.16. *Let X and Y be NTS , such that X is product related to Y.*

(a) *If $A \in N\delta PO(X)$ and $O \in N\delta PO(Y)$, then $A \times O \in N\delta PO(X \times Y)$.*
(b) *N $A \in NS\delta PO(X)$ and $O \in NS\delta PO(Y)$, then $A \times O \in NS\delta PO(X \times Y)$.*

Proof. **(a)** N $A \in N\delta PO(X)$ and $O \in N\delta PO(Y)$. Then $A \times O \subseteq int(\delta cl(A)) \times int(\delta cl(O)) = int(\delta cl(A \times O))$.
(b) Let $F \subseteq A \subseteq \delta cl(F)$ and $\psi \subseteq O \subseteq \delta cl(\psi)$, $F \in N\delta PO(X)$ and $\psi \in N\delta PO(Y)$. Then $F \times \psi \subseteq A \times O \subseteq \delta cl(F) \times \delta cl(\psi) = \delta cl(A \times \psi)$. Now the result follows from (a). □

Definition 3.17. Let (X, τ) be a NTS and A be a neutrosophic set of X. Then the neutrosophic semi δ-preinterior (denoted by $s\delta pint$) and neutrosophic semi δ-preclosure (denoted by $s\delta pcl$) of A respectively defined as follows:
$s\delta pint(A) = \cup\{O : O \subseteq A; O \in NS\delta PO(X)\}$,
$s\delta pcl(A) = \cap\{O : O \supseteq A; O \in NS\delta PC(X)\}$.

The following theorem can be easily verified.

Theorem 3.18. *Let A and O be neutrosophic sets in a NTS (X, τ). Then:*

(a) $s\delta pcl(A) \subseteq cl(A)$
(b) $s\delta pcl(A)$ *is a neutrosophic semi δ-preclosed.*
(c) $A \in NS\delta PC(X) \Leftrightarrow A = s\delta pcl(A)$.
(d) $A \subseteq O \Rightarrow s\delta pcl(A) \subseteq s\delta pcl(O)$.
(e) $int(A) \subseteq s\delta pint(A)$.
(f) $s\delta pint(A)$ *is a neutrosophic semi preopen.*
(g) $A \in NS\delta PO(X) \Leftrightarrow A = s\delta pint(A)$.
(h) $A \subseteq O \Rightarrow s\delta pint(A) \subseteq s\delta pint(O)$.
(i) $s\delta pint(A^c) = (s\delta pcl(A))^c$.

Definition 3.19. Let A be a neutrosophic sets in a NTS (X, τ) and $x_{(\alpha,\eta,\beta)}$ is a neutrosophic point of X. Then A is called:

(a) neutrosophic semi δ-preneighborhood of $x_{(\alpha,\eta,\beta)}$ if there exists a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O \subseteq A$.
(b) neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$ if there exists a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q}O \subseteq A$.

Theorem 3.20. *An neutrosophic set $A \in NS\delta PO(X)$ if and only if for each neutrosophic point $x_{(\alpha,\eta,\beta)} \in A$, A is a neutrosophic semi δ-preneighborhood of $x_{(\alpha,\eta,\beta)}$.*

Theorem 3.21. *Let A be a neutrosophic sets in a NTS (X, τ). Then a neutrosophic point $x_{(\alpha,\eta,\beta)} \in s\delta pcl(A)$, if and only if every neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$ is quasi-coincident with A.*

Proof. **Necessity** Suppose $x_{(\alpha,\eta,\beta)} \in s\delta pcl(A)$ and if possible let there exists a neutrosophic semi δ-pre Q-neighborhood O of $x_{(\alpha,\eta,\beta)}$ such that $\rceil(O_qA)$. Then there exists a neutrosophic set

$O_1 \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q}O_1 \subseteq O_1 \subseteq O$ which show that $\rceil(O_{1q}A)$ and hence $A \subseteq O_1^c$. As $O_1^c \in NS\delta PC(X)$, $s\delta pcl(A) \subseteq O_1^c$. Since $x_{(\alpha,\eta,\beta)} \in O_1^c$, we obtain that $x_{(\alpha,\eta,\beta)} \notin s\delta pcl(A)$ which is a contradiction.

Sufficiency Suppose every neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$ is quasi-coincident with A. If $x_{(\alpha,\eta,\beta)} \notin s\delta pcl(A)$ then there exists a neutrosophic semi δ-preclosed set $O \supseteq A$ such that $x_{(\alpha,\eta,\beta)} \notin O$. So $O^c \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q}O^c$ and $\rceil(O_q^cA)$ a contradiction. □

Definition 3.22. A mappings $f : (X,\tau) \to (Y,\sigma)$ is said to be neutrosophic δ-preirresolute if $f^{-1}(A) \in NS\delta PO(X)$ for every neutrosophic set $A \in N\delta PO(Y)$.

Theorem 3.23. *If $f : (X,\tau) \to (Y,\sigma)$ is a neutrosophic δ-pre irresolute and neutrosophic open mapping, then $f^{-1}(A) \in NS\delta PO(X)$, for every $A \in NS\delta PO(Y)$.*

Proof. Let $A \in NS\delta PO(Y)$. Then there exists a neutrosophic set $O \in N\delta PO(X)$ such that $O \subseteq A \subseteq \delta cl(O)$. Therefore $f^{-1}(O) \subseteq f^{-1}(A) \subseteq f^{-1}(\delta cl(O))$ since f is neutrosophic open and δ-pre irresolute. $f^{-1}(O) \subseteq f^{-1}(A) \subseteq f^{-1}(\delta cl(O)) \subseteq \delta cl(f^{-1}(O))$ and $f^{-1}(O) \in NS\delta PO(X)$. Hence $f^{-1}(A) \in NS\delta PO(X)$. □

4. Neutrosophic semi δ-precontinuous mappings

Definition 4.1. A mappings $f : (X,\tau) \to (Y,\sigma)$ is said to be neutrosophic semi δ-precontinuous if $f^{-1}(A) \in NS\delta PO(X)$ for every neutrosophic set $A \in \sigma$.

Remark 4.2. Every neutrosophic δ-pre continuous (resp. neutrosophic semi precontinuous) mappings is neutrosophic semi δ-precontinuous but the converse may not be true.

Example 4.3. Let $X = \{a,b\}$ and $Y = \{p,q\}$ and neutrosophic sets A, B, O, F are defined as follows:

$$A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.3, 0.5, 0.7 >\}$$

$$B = \{< a, 0.5, 0.5, 0.5 >, < b, 0.1, 0.5, 0.9 >\}$$

$$O = \{< p, 0.5, 0.5, 0.5 >, < q, 0.9, 0.5, 0.1 >\}$$

$$F = \{< p, 0.5, 0.5, 0.5 >, < q, 0.6, 0.5, 0.4 >\}$$

let $\tau_1 = \{\tilde{0}, A, B, \tilde{1}\}$, $\tau_2 = \{\tilde{0}, O, \tilde{1}\}$ and $\tau_3 = \{\tilde{0}, F, \tilde{1}\}$. Then the mapping $f : (X,\tau_1) \to (Y,\tau_2)$ defined by $f(a) = p$, $f(b) = q$ is a neutrosophic semi δ-pre continuous but not neutrosophic semi precontinuous and the mapping $g : (X,\tau_1) \to (Y,\tau_3)$ defined by $g(a) = p$, $g(b) = q$ is a neutrosophic semi δ-precontinuous but not neutrosophic δ-pre continuous.

Remark 4.4. Remark (2.21), (2.22) and (4.2) reveals that the following figure of implications is true.

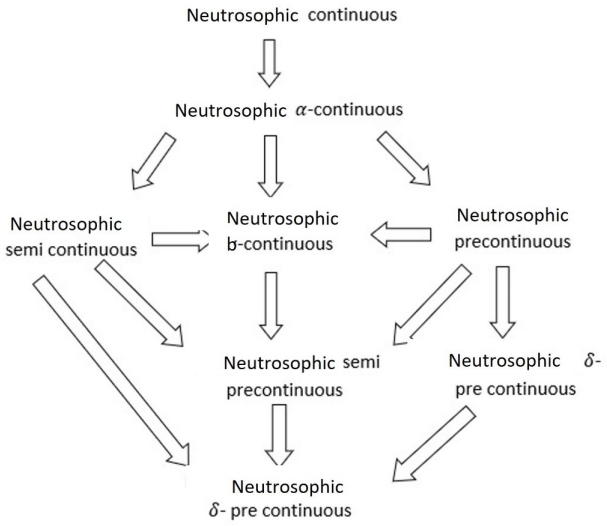

Figure 2

Theorem 4.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping from a NTS (X, τ) to NTS (Y, σ). Then the following statements are equivalent:*

(a) *f is neutrosophic semi δ-precontinuous.*

(b) *for every neutrosophic closed set A in Y, $f^{-1}(A) \in NS\delta PC(X)$.*

(c) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ in X and every neutrosophic open set A such that $f(x_{(\alpha,\eta,\beta)}) \in NS\delta PO(X)$ there is a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O$ and $f(O) \subseteq A$.*

(d) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every neighborhood A of $f(x_{(\alpha,\eta,\beta)}), f^{-1}(A)$ is a neutrosophic semi δ-pre neighborhood of $x_{(\alpha,\eta,\beta)}$.*

(e) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every neighborhood A of $f(x_{(\alpha,\eta,\beta)})$, there is a neutrosophic semi δ-pre neighborhood U of $x_{(\alpha,\eta,\beta)}$ such that $f(U) \subseteq A$.*

(f) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every neutrosophic open set A such that $f(x_{(\alpha,\eta,\beta)})_q A$, there is a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q} O$ and $f(O) \subseteq A$.*

(g) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every Q-neighborhood A of $f(x_{(\alpha,\eta,\beta)}), f^{-1}(A)$ is a neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$.*

(h) *for every neutrosophic point* $x_{(\alpha,\eta,\beta)}$ *of* X *and every Q-neighborhood* A *of* $f(x_{(\alpha,\eta,\beta)})$, *there is a neutrosophic semi pre Q-neighborhood* U *of* $x_{(\alpha,\eta,\beta)}$ *such that* $f(U) \subseteq A$.

(i) $f(s\delta pcl(A)) \subseteq cl(f(A))$, *for every neutrosophic set* A *of* X.

(j) $s\delta pcl(f^{-1}(O)) \subseteq f^{-1}(cl(O))$, *for every neutrosophic set* O *of* Y.

(k) $f^{-1}(int(0)) \subseteq s\delta pint(f^{-1}(O))$, *for every neutrosophic set* O *of* Y.

Proof. $(a) \Rightarrow (b)$ Obvious.

$(a) \Rightarrow (c)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neutrosophic open set in Y such that $f(x_{(\alpha,\eta,\beta)}) \in A$. Put $O = f^{-1}(A)$, then by (a), $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O$ and $f(O) \subseteq A$.

$(c) \Rightarrow (a)$ Let A be a neutrosophic open set in Y and $x_{(\alpha,\eta,\beta)} \in f^{-1}(A)$. Then $f(x_{(\alpha,\eta,\beta)}) \in A$. Now by (c) there is a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O$ and $f(O) \subseteq A$. Then $x_{(\alpha,\eta,\beta)} \in O \subseteq f^{-1}(A)$. Hence by theorem (3.11), $f^{-1}(A) \in NS\delta PO(X)$.

$(a) \Rightarrow (d)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neighborhood of $f(x_{(\alpha,\eta,\beta)})$. Then there is a neutrosophic open set U such that $f(x_{(\alpha,\eta,\beta)}) \in U \subseteq A$. Now $F^{-1}(U) \in NS\delta PO(X)$ and $x_{(\alpha,\eta,\beta)} \in f^{-1}(U) \subseteq f^{-1}(A)$. Thus $f^{-1}(A)$ is a neutrosophic semi pre neighborhood of $x_{(\alpha,\eta,\beta)}$ in X.

$(d) \Rightarrow (e)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neighborhood of $f(x_{(\alpha,\eta,\beta)})$. Then $U = f^{-1}(A)$ is a neutrosophic semi δ-pre neighborhood of $x_{(\alpha,\eta,\beta)}$ and $f(U) = f(f^{-1}(A)) \subseteq A$.

$(e) \Rightarrow (c)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neutrosophic open set such that $f(x_{(\alpha,\eta,\beta)}) \in A$. Then A is a neighborhood of $f(x_{(\alpha,\eta,\beta)})$. So there is neutrosophic semi δ-pre neighborhood U of $x_{(\alpha,\eta,\beta)}$ in X such that $x_{(\alpha,\eta,\beta)} \in U$ and $f(U) \subseteq A$. Hence there is a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O \subseteq U$ and so $f(O) \subseteq f(U) \subseteq A$.

$(a) \Rightarrow (f)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neutrosophic open set in Y such that $f(x_{(\alpha,\eta,\beta)})_q \in A$. Let $O = f^{-1}(A)$. Then $O \in NS\delta PO(X)$, $x_{(\alpha,\eta,\beta)q}O$ and $f(O) = f(f^{-1}(A)) \subseteq A$.

$(f) \Rightarrow (a)$ Let A be a neutrosophic open set in Y and $x_{(\alpha,\eta,\beta)} \in f^{-1}(A)$ clearly $f(x_{(\alpha,\eta,\beta)}) \in A$, choose the neutrosophic point $x^c_{(\alpha,\eta,\beta)}$ defined as

$$x^c_{(\alpha,\eta,\beta)}(z) = \begin{cases} (\beta,\eta,\alpha) & \text{if } z = x \\ (1,0,0) & \text{if } z \neq x \end{cases} \tag{2}$$

Then $f(x_{(\alpha,\eta,\beta)})_q A$ and so by (f), there exists a neutrosophic set $O \in NS\delta PO(X)$, such that $x_{(\alpha,\eta,\beta)q}O$ and $f(O) \subseteq A$.
Now $x^c_{(\alpha,\eta,\beta)q}O$ implies $x_{(\alpha,\eta,\beta)} \in O$.

Thus $x_{(\alpha,\eta,\beta)} \in O \subseteq f^{-1}(A)$. Hence by theorem (3.11) $f^{-1} \in NS\delta PO(X)$.

$(f) \Rightarrow (g)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a Q-neighborhood of $f(x_{(\alpha,\eta,\beta)})$. Then there is a neutrosophic open set A_1 in Y such that $f(x_{(\alpha,\eta,\beta)})_q A_1 \subseteq A$. By hypothesis there is a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q} O$ and $f(O) \subseteq A_1$. Thus $x_{(\alpha,\eta,\beta)q} O \subseteq f^{-1}(A_1) \subseteq f^{-1}(A)$.
Hence $f^{-1}(A)$ is a neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$.

$(g) \Rightarrow (h)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a Q-neighborhood of $f(x_{(\alpha,\eta,\beta)})$. Then $U = f^{-1}(A)$ is a neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$ and $f(U) = f(f^{-1}(A)) \subseteq A$.

$(g) \Rightarrow (f)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neutrosophic open set such that $f(x_{(\alpha,\eta,\beta)})_q \in A$. Then A is Q-neighborhood of $f(x_{(\alpha,\eta,\beta)})$. So there is a neutrosophic semi δ-pre Q-neighborhood δ of $x_{(\alpha,\eta,\beta)}$ such that $f(U) \subseteq A$. Now U being a neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$. Then there exists a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q} O \subseteq U$. Hence $x_{(\alpha,\eta,\beta)q} O$ and $f(O) \subseteq f(U) \subseteq A$.

$(b) \Leftrightarrow (i)$ Obvious.
$(h) \Leftrightarrow (j)$ Obvious.
$(j) \Leftrightarrow (k)$ Obvious. □

Theorem 4.6. *Let X, X_1, X_2 be a neutrosophic topological spaces and $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection of $X_1 \times X_2$ into X_i. Then if $f : X_1 \times X_2$ is a neutrosophic semi δ- pre continuous mapping, it follows that $p_i o f$ is also a neutrosophic semi δ- pre continuous mapping.*

Theorem 4.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping. If the graph mapping $g : X \to X \times Y$ of f is a neutrosophic semi δ-pre continuous, then f is a neutrosophic semi δ-pre continuous.*

Proof. Let O be a neutrosophic open set of Y. Then $\tilde{1} \times O$ is a neutrosophic open in $X \times Y$. Since g is a neutrosophic semi δ-pre continuous, $g^{-1}(\tilde{1} \times O) \in NS\delta PO(X)$. But $f^{-1}(O) = \tilde{1} \cap f^{-1}(O) = g^{-1}(\tilde{1} \times O)$, $f^{-1}(O) \in NS\delta PO(X)$. Hence f is a neutrosophic semi δ-pre continuous. □

Theorem 4.8. *Let X_i and X_i^* $(i = 1, 2)$ be a neutrosophic topological spaces such that X_1 is product related to X_2. If $f_i : X_i \to X_i^*$ $(i = 1, 2)$ is a neutrosophic semi δ-pre continuous, then $f_1 \times f_2 : X_1 \times X_2 \to X_i^* \times X_2^*$ is a neutrosophic semi δ-pre continuous.*

Proof. Let $A = \cup(A_\alpha \times O_\beta)$ where $A_\alpha^{'}s$ and $O_\beta^{'}s$ are neutrosophic open sets of X_1^* and X_2^* respectively, be a neutrosophic open sets of $X_1^* \times X_2^*$. We obtain $(f_1 \times f_2)^{-1}(A) = \cup\{f_1^{-1}(A_\alpha) \times f_2^{-1}(O_\beta)\}$. Since f_1 and f_2 are neutrosophic semi δ-pre continuous, $f_1^{-1}(A_\alpha) \in NS\delta PO(X_1)$ and $f_2^{-1}(O_\beta) \in NS\delta PO(X_2)$. Therefore by theorem (3.16)(b), $f_1^{-1}(A_\alpha) \times f_2^{-1}(O_\beta) \in NS\delta PO(X_1 \times X_2)$. Hence by theorem (3.10), $(f_1 \times f_2)^{-1}(A) \in NS\delta PO(X_1 \times X_2)$. □

References

1. Acikgoz A.; Cakalli H.; Esenbel F. ; Koˇcinac LJ.D.R., A quest of G-continuity in neutrosophic spaces, Math. Meth. Appl Sci., DOI: 10.1002/mma.7113.
2. Al-Musaw, A. F., On neutrosophic semi-regularization topological spaces Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 51–55. http://dx.doi.org/10.22075/ijnaa.2022.6232.
3. Arokiarani I.; Dhavaseelan R; Jafari S. ; Parimala M. On some new notions and functions in neutrosophic topological spaces, Neutrosophic Sets and Systems 16(1)(2017) ,16-19.
4. Atanassov,K., Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986), 87-96.
   bibitem DeyDey, S.; Ray, G. C. , Covering Properties via Neutrosophic b-open Sets. Neutrosophic Systems With Applications, 9(2023), 1-12. https://doi.org/10.61356/j.nswa.2023.66.
5. Dhavaseelan R.; Jafari S.; Ozel C.; Al Shumran M.A., Generalized neutrosophic contra-continuity, New Trends in Neutrosophic Theory and Applications-Volume II, Florentin Smarandache, Surapati Pramanik (Editors),Pons Editions Brussels, Belgium, EU 2018, 255-274.
6. Ebenanjar, P. E.; Immaculate J. J.; Wilfred, C. B., On Neutrosophic b-open sets in Neutrosophic topological space J. Phys.: Conf. Ser. 1139(2018) 012062, 1-5.
7. Iswarya,P. ; Bageerathi,K., On Neutrosophic Semi-open Sets in Neutrosophic Topological Spaces, IJMTT, (37)(3)(2016), 214-223.
8. Karatas, S. ; Cemil K., Neutrosophic Topology, Neutrosophic Sets and Systems 13(1) (2016). https://digitalrepository.unm.edu/nss_journal/vol13/iss1/12.
9. Lupiáñez F. G., On neutrosophic topology, The International Journal of Systems and Cybernetics, 37(6)(2008), 797-800.
10. Lupiáñez, F. G., On various neutrosophic topologies, The International Journal of Systems and Cybernetics, 38(6)(2009), 1009-1013.
11. Salama A. A. ; Alblowi,S. A., Neutrosophic set and neutrosophic topological spaces, IOSR Journal of Mathematics, Vol. 3(4)(2012), 31-35
12. Salama, A.A.; Broumi, S.; Alblowi, S. A. , Introduction to neutrosophic topological spatial region, possible application to gis topological rules. Inf. Eng. Electron. Bus. 2014, 6, 15–21.
13. Smarandache, F. , A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. American Research Press, Rehoboth, NM, 1999.
14. Smarandache, F., Neutrosophy and Neutrosophic Logics, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics, University of New Mexico, Gallup, NM 87301, USA(2002).

15. Smarandache F., Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Int. J. Pure Appl. Math., (24)(2005), 287-297.
16. Thomas, T. ; Anila. S , On Neutrosophic Semi-preopen Sets and Semi-preclosed Sets in a Neutrosophic Topological Space, International Journal of Scientific Research in Mathematical and Statistical Sciences, 5(5)(2018),138-143.
17. Vadivel A. ; Seenivasan, M.; John Sundar C., A Introduction to δ-open sets in a Neutrosophic Topological Spaces, Journal of Physics: Conference Series 1724 (2021) 012011. doi:10.1088/1742-6596/1724/1/012011.
18. Venkateswara Rao , V. ; Srinivasa Rao,Y., Neutrosophic Preopen sets and Preclosed sets in Neutrosophic Topological spaces, International journal of ChemTech Research, Vol 10(10)(2017),449-458
19. Zadeh, L. A., Fuzzy sets, Inform. and Control 8(1965), 338-353.

Neutrosophic Number Sequences: An introductory Study

Hasan Gökbaş[1], Selçuk Topal[2,*] and Florentin Smarandache [3]

[1]Bitlis Eren University, Faculty of Science and Arts, Department of Mathematics, Bitlis, Türkiye.
[2]Bitlis Eren University, Faculty of Science and Arts, Department of Mathematics, Bitlis, Türkiye.
[3]Mathematics Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA.

Emails: hgokbas@beu.edu.tr[1], s.topal@beu.edu.tr (selcuk.topal@yandex.com)[2], smarand@unm.edu[3]

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Abstract. In this paper, Neutrosophic definitions and properties of some special number sequences which are frequently found in the science literature, called Neutrosophic Number Sequences (NNSq) via Horadam sequence are studied for the first time. Especially for Neutrosophic Fibonacci (NFNq) and Neutrosophic Lucas (NLNq) number sequences, fundamental properties and identities such as Ruggles, Honsberger, Cassini, Catalan, d'Ocagne, and Tagiuri are given. In addition, Neutrosophic definitions of the sequences of Pell (NPNq), Pell-Lucas (NPLNq), Jacobsthal (NJNq), Jacobsthal-Lucas (NJLNq), Mersenne (NMNq), Mersenne-Lucas (NMLNq), Balancing (NBNq), and Lucas-Balancing (NLBNq) numbers are introduced. Besides defining these numbers and their sequences, since fuzzy and intuitionistic fuzzy sets are restrictions of neutrosophic sets, sequences of numbers within these sets are naturally and indirectly revealed.

Keywords: Neutrosophic number sequence; Neutrosophic Fibonacci number sequence; Neutrosophic Lucas number sequence; Neutrosophic Pell number sequence; Neutrosophic Jacobsthal number sequence; Neutrosophic Jacobsthal-Lucas number sequence; Neutrosophic Mersenne number sequence; Neutrosophic Mersenne-Lucas number sequence; Neutrosophic balancing number sequence; Neutrosophic Lucas-balancing number sequence; Horadam Form

1 Introduction

Neutrosophic Logic is a nascent field of study in which each proposition is estimated to have a rate of truth (percentage) in a subset of T, an uncertainty rate in a subset of I, and a falsity rate in a subset F. We utilize a subset of truth (or indeterminacy, or falsity), instead of a number only, since in many situations we are not capable of specifying the proportions of truth and of falsity strictly but to approximate them: for instance, a proposition is between $18-57\%$ true and between $61-79\%$ false, even worst: between $32-44\%$ or $41-51\%$ true (pursuant to several observers), and 56% or between $61-77\%$ false. The subsets are not essential intervals, but any sets (open or closed or half open/half-closed interval, discrete, continuous, intersections or unions of the previous sets, etc.) in keeping with the given proposition. Zadeh initiated the landscape by provide an insight for meaning and mathematical results from uncertainty situations (fuzzy).[1] Fuzzy sets brought a new dimension to the concept of classical set theory. Atanassov defined intuitionistic fuzzy sets (IFS) including membership and non-membership degrees.[2] Neutrosophy was proposed by Florentin as a computational method to the concept of neutrality.[3] Neutrosophic sets (NS) consider membership, non-membership, and indeterminacy degrees. Intuitionistic fuzzy sets are defined by the degree of membership and non-membership and, uncertainty degrees by the 1- (membership degree plus non-membership degree), while the degree of uncertainty is evaluated independently of the degree of membership and non-membership in neutrosophic sets. Here, membership, non-membership, and degree of uncertainty (uncertainty), such as degrees of accuracy and

falsity, can be evaluated according to the interpretation of the places to be used. It depends entirely on the subject area (the universe of discourse). This reveals a difference between neutrosophic set and intuitionistic fuzzy set. In this sense, the concept of neutrosophic is a possible solution and representation of problems in various fields. Mathematical differences between NS and IFL are as follows:

I) NS can discern absolute truth (truth in all possible worlds, according to Leibniz) from relative truth (truth in at least one world), because NS (absolute truth) = 1^+ while IFS (relative truth) $= 1$. This has practice in philosophy (see the Neutrosophy). The standard interval $[0, 1]$ used in IFS has been extended to the unitary non-standard interval $]^-0, 1^+[$ in NS. Parallel earmarks for absolute or relative falsehood, and absolute or relative indeterminacy are permitted in NS.

II) There is no limit on $\mathrm{T}, \mathrm{I}, \mathrm{F}$ other than they are subsets of $]^-0, 1^+[$, thus: $^-0 \leq \inf T + \inf I + \inf F \leq \sup \mathrm{T} + \sup \mathrm{I} + \sup \mathrm{F} \leq 3^+$in NL. This permissiveness allows dialetheist, para-consistent, and incomplete information to be described in NL, while these situations cannot be described in IFL since F (falsehood), T (truth), I (indeterminacy) are restricted either to $t+i+f = 1$ or to $t^2+f^2 \leq 1$, if $\mathrm{T}, \mathrm{I}, \mathrm{F}$ are all reduced to the points $\mathrm{t}, \mathrm{i}, \mathrm{f}$ respectively, or to $\sup \mathrm{T} + \sup \mathrm{I} + \sup \mathrm{F} = 1$ if $\mathrm{T}, \mathrm{I}, \mathrm{F}$ are subsets of $[0, 1]$ in IFL.

The Fibonacci sequence has delighted mathematicians and scientists alike for centuries with its beauty and its propensity to pop up in quite unexpected places. Leonardo de Pisa has not even guessed that the number sequences would be so adventurous with the rabbit problem. However, the Fibonacci numbers are surprisingly found in Pascal's triangle, Pythagorean triples, computer algorithms, graph theory, and many other areas of mathematics. They also occur in a variety of other fields such as physics, finance, architecture, computer sciences, color image processing, geostatistics, music, and art. There have been many studies in the literature about this special number sequence because of its numerous applications. There are many generalizations on this sequence some of which can be seen in.[4–14]

Table 1: Frequently studied special number sequences in mathematics literature

n	0	1	2	3	4	5	6	7	8	9	Formula
Fibonacci	0	1	1	2	3	5	8	13	21	34	F(n)=F(n-1)+F(n-2)
Lucas	2	1	3	4	7	11	18	29	47	76	L(n)=L(n-1)+L(n-2)
Pell	0	1	2	5	12	29	70	169	408	985	P(n)=2P(n-1)+P(n-2)
Pell-Lucas	2	2	6	14	34	82	198	478	1154	2786	PL(n)=2PL(n-1)+PL(n-2)
Jacobsthal	0	1	1	3	5	11	21	43	85	171	J(n)=J(n-1)+2J(n-2)
Jacobsthal-Lucas	2	1	5	7	17	31	65	127	257	511	JL(n)=JL(n-1)+2JL(n-2)
Mersenne	0	1	3	7	15	31	63	127	255	511	M(n)=3M(n-1)-2M(n-2)
Mersenne-Lucas	2	3	5	9	17	33	65	129	257	513	ML(n)=3ML(n-1)-2ML(n-2)
Balancing	0	1	6	35	204	1189	6930	40391	235416	1372105	B(n)=6B(n-1)-B(n-2)
Lucas-Balancing	1	3	17	99	577	3363	19601	114243	665857	3880899	BL(n)=6BL(n-1)-BL(n-2)

In a sequence of numbers containing a given rule, any three consecutive terms related to each other can be defined by the second-order linear difference equation. The relation between the terms of the sequence, where H_n is a sequence of numbers. If $AH_{n-1}+BH_{n-2}$, where A and $B \in \mathbb{Q}$, it can be defined by the second-order linear difference equation. The characteristic equation of this sequence can be expressed as $\gamma^2 - A\gamma - B = 0$. The characteristic roots of the equation $\gamma^2 - A\gamma - B = 0$ are found as $\gamma_1 = \frac{A+\sqrt{A^2+4B}}{2}$ and $\gamma_2 = \frac{A-\sqrt{A^2+4B}}{2}$. The general solution of this sequence is

$$H_n = \delta_1 \gamma_1^n + \delta_2 \gamma_2^n, \quad n \geq 1.$$

It is found δ_1 and δ_2 using initial conditions $H_1 = e$ and $H_2 = f$ of the sequence.

Fibonacci and Lucas numbers are defined by the recurrence relations, respectively. $F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1, n \geq 2$ and $L_{n+2} = L_{n+1} + L_n, L_0 = 2, L_1 = 1, n \geq 2$. The nth Fibonacci and Lucas number are formalized as $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2} \cdot F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$, this form of formulation is called the Binet's form.[15]

2 Preliminaries

In this section, we give the fundamental preliminaries of fuzzy Fibonacci numbers and Neutrosophic Triangular Numbers. Also, some basic concepts that are fundamentals of this paper will be covered. Definitions of a Single Valued Triangular Neutrosophic Number (SVTrN), its α-cut, β-cut, and γ-cut sets and arithmetical operations on them are the below.

Definition 2.1. [16] Let $\{F_n\}$ be Fibonacci sequence. For $n \geq 0$ integer, Fuzzy Fibonacci numbers are defined as

$$F^{\alpha}_{rn+l} = \left[F_{r(n-1)+l} + \alpha\left(F_{rn+l} - F_{r(n-1)+l}\right), F_{r(n+1)+l} - \alpha\left(F_{r(n+1)+l} - F_{rn+l}\right)\right]$$

where $r, l \in \mathbb{Z}^{+}$and $\alpha \in [0,1]$. When we take $r = 1$ and $l = 0$, several Fuzzy Fibonacci numbers are following;

$$F^{\alpha}_{0} = [1-\alpha, 1+\alpha], F^{\alpha}_{1} = [\alpha, 1], F^{\alpha}_{2} = [1, 2+\alpha], F^{\alpha}_{3} = [1+\alpha, 3+\alpha], \ldots$$

Since Fuzzy Fibonacci numbers include Fibonacci numbers, then Fuzzy Fibonacci numbers satisfy the similar recurrence relation with initial conditions $F^{\alpha}_{0} = [1-\alpha, 1+\alpha], F^{\alpha}_{1} = [\alpha, 1]$. That is

$$F^{\alpha}_{n} = F^{\alpha}_{n-1} + F^{\alpha}_{n-2}$$

for $n \geq 2$.

Definition 2.2. [16] Let $\{L_n\}$ be Lucas sequence. For $n \geq 0$ integer, Fuzzy Lucas numbers are defined as

$$L^{\alpha}_{rn+l} = \left[L_{r(n-1)+l} + \alpha\left(L_{rn+l} - L_{r(n-1)+l}\right), L_{r(n+1)+l} - \alpha\left(L_{r(n+1)+l} - L_{rn+l}\right)\right]$$

where $r, l \in \mathbb{Z}^{+}$and $\alpha \in [0,1]$. When we take $r = 1$ and $l = 0$, several Fuzzy Lucas numbers are following:

$$L^{\alpha}_{1} = [2-\alpha, 3-2\alpha], L^{\alpha}_{2} = [1+2\alpha, 4-\alpha], L^{\alpha}_{3} = [3+\alpha, 7-3\alpha], \ldots$$

Together with initial conditions $L^{\alpha}_{0} = [-1+3\alpha, 1+\alpha], L^{\alpha}_{1} = [2-\alpha, 3-2\alpha]$, Fuzzy Lucas numbers satisfy recurrence relation. That is

$$L^{\alpha}_{n} = L^{\alpha}_{n-1} + L^{\alpha}_{n-2}$$

for $n \geq 2$.

Definition 2.3. A Single Valued Triangular Neutrosophic Number (SVTrN) $\tilde{a} = \langle (a,b,c); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is a special neutrosophic set on the set of real numbers R, whose truth-membership, indeterminacy-membership and falsity-membership functions are respectively defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{(x-a)w_{\tilde{a}}}{b-a}, & a \leq x < b \\ \frac{(c-x)w_{\tilde{a}}}{c-b}, & b \leq x \leq c \\ 0, & \text{otherwise} \end{cases}$$

$$\vartheta_{\tilde{a}}(x) = \begin{cases} \frac{b-x+(x-a)u_{\tilde{a}}}{b-a}, & a \leq x < b \\ \frac{x-b+(c-x)u_{\tilde{a}}}{c-b}, & b \leq x \leq c \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_{\tilde{a}}(x) = \begin{cases} \frac{b-x+(x-a)y_{\tilde{a}}}{b-a}, & a \leq x < b \\ \frac{x-b+(c-x)y_{\tilde{a}}}{c-b}, & b \leq x \leq c \\ 0, & \text{otherwise} \end{cases}$$

The values $w_{\tilde{a}}, u_{\tilde{a}}$ and $y_{\tilde{a}}$ represent the truth-membership, indeterminacy-membership and falsity-membership, respectively, such that they satisfy the conditions $0 \leq w_{\tilde{a}} \leq 1, 0 \leq u_{\tilde{a}} \leq 1, 0 \leq y_{\tilde{a}} \leq 1$ and $0 \leq w_{\tilde{a}} + u_{\tilde{a}} + y_{\tilde{a}} \leq 3$. Additionally, if $a \geq 0$ and $c > 0$, then the SVTrN $\tilde{a} = \langle (a,b,c); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is called a positive SVTrN, denoted by $\tilde{a} > 0$. if $c \leq 0$ and $a < 0$, then the SVTrN $\tilde{a} = \langle (a,b,c); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is called a negative SVTrN, denoted by $\tilde{a} < 0$.

Definition 2.4. 2.2 A α-cut set of a SVTrN $\tilde{a} = \langle (a,b,c); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is a crisp subset of s set R, which is defined as $\tilde{a}_\alpha = \{x \mid \mu_{\tilde{a}}(x) \geq \alpha, 0 \leq \alpha \leq w_{\tilde{a}}\}$. $\tilde{a}_\alpha$ is a closed interval, denoted by $\tilde{a}_\alpha = [L_{\tilde{a}}(\alpha), R_{\tilde{a}}(\alpha)]$, α cut set of a SVTrN for truth-membership can be calculated as follows:

$$[L_{\tilde{a}}(\alpha), R_{\tilde{a}}(\alpha)] = \left[\frac{(w_{\tilde{a}} - \alpha)\, a + \alpha b}{w_{\tilde{a}}}, \frac{(w_a - \alpha)\, c + \alpha b}{w_{\tilde{a}}} \right].$$

Definition 2.5. A β-cut set of a SVTrN $\tilde{a} = \langle (a,b,c); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is a crisp subset of set R, which is defined as $\tilde{a}_\beta = \{x \mid \vartheta_{\tilde{a}}(x) \leq \beta, u_{\tilde{a}} \leq \beta \leq 1\}$. $\tilde{a}_\beta$ is a closed interval, denoted by $\tilde{a}_\beta = [L_{\tilde{a}}(\beta), R_{\tilde{a}}(\beta)]$, β cut set of a SVTrN for indeterminacy-membership can be calculated as follows:

$$[L_{\tilde{a}}(\beta), R_{\tilde{a}}(\beta)] = \left[\frac{(1-\beta)b + (\beta - u_{\tilde{a}})\, a}{1 - u_{\tilde{a}}}, \frac{(1-\beta)b + (\beta - u_{\tilde{a}})\, c}{1 - u_{\tilde{a}}} \right].$$

Definition 2.6. A γ-cut set of a SVTrN $\tilde{a} = \langle (a,b,c); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is a crisp subset of s set R, which is defined as $\tilde{a}_\gamma = \{x \mid \delta_{\tilde{a}}(x) \leq \gamma, y_{\tilde{a}} \leq \gamma \leq 1\}$. $\tilde{a}_\gamma$ is a closed interval, denoted by $\tilde{a}_\gamma = [L_{\tilde{a}}(\gamma), R_{\tilde{a}}(\gamma)]$, γ cut set of a SVTrN for falsity-membership can be calculated as follows:

$$[L_{\tilde{a}}(\gamma), R_{\tilde{a}}(\gamma)] = \left[\frac{(1-\gamma)b + (\gamma - y_{\tilde{a}})\, a}{1 - y_{\tilde{a}}}, \frac{(1-\gamma)b + (\gamma - y_{\tilde{a}})\, c}{1 - y_{\tilde{a}}} \right].$$

In a similar way to the arithmetical operations of the neutrosophic numbers, the arithmetical operations over SVTrNs might be defined by Deli and Subaş.[17] Obviously, if $w_{\tilde{a}} = 1, u_{\tilde{a}} = 0$ and $y_{\tilde{a}} = 0 \quad ; \tilde{a} = \langle (\underline{a}, a, \bar{a}); 1, 0, 0 \rangle$ and $\tilde{b} = \langle (\underline{b}, b, \bar{b}); 1, 0, 0 \rangle$ are the single valued triangular neutrosophic numbers, then the arithmetical operations generate equalities as follows:

i. $\tilde{a} + \tilde{b} = \langle (\underline{a} + \underline{b}, a + b, \bar{a} + \bar{b}); 1, 0, 0 \rangle$

ii. $\tilde{a} - \tilde{b} = \langle (\underline{a} - \bar{b}, a - b, \bar{a} - \underline{b}); 1, 0, 0 \rangle$

iii. $\tilde{a}\tilde{b} = \begin{cases} \langle (\underline{ab}, ab, \bar{a}\bar{b}); 1, 0, 0 \rangle, & \text{if } \tilde{a} > 0 \text{ and } \tilde{b} > 0, \\ \langle (\underline{a}\bar{b}, ab, \bar{a}\underline{b}); 1, 0, 0 \rangle, & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} > 0, \\ \langle (\bar{a}\bar{b}, ab, \underline{ab}); 1, 0, 0 \rangle, & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} < 0, \end{cases}$

iv. $\tilde{a}/\tilde{b} = \begin{cases} \langle (\underline{a}/\bar{b}, a/b, \bar{a}/\underline{b}); 1, 0, 0 \rangle, & \text{if } \tilde{a} > 0 \text{ and } \tilde{b} > 0, \\ \langle (\bar{a}/\bar{b}, a/b, \underline{a}/\underline{b}); 1, 0, 0 \rangle, & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} > 0, \\ \langle (\bar{a}/\underline{b}, a/b, \underline{a}/\bar{b}); 1, 0, 0 \rangle, & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} < 0, \end{cases}$

v. $\lambda\tilde{a} = \begin{cases} \langle (\lambda\underline{a}, \lambda\bar{a}, \lambda\bar{a}); 1, 0, 0 \rangle, & \text{if } \lambda > 0, \\ \langle (\lambda\bar{a}, \lambda a, \lambda\underline{a}); 1, 0, 0 \rangle, & \text{if } \lambda < 0, \end{cases}$

vi. $\tilde{a}^{-1} = \langle (1/\bar{a}, 1/a, 1/\underline{a}); 1, 0, 0 \rangle$

3 Neutrosophic Fibonacci and Lucas Numbers

In this section, Neutrosophic numbers including Fibonacci and Lucas number sequences with quadratic characteristic equations will be defined.

Definition 3.1. Let $\{F_n\}$ be Fibonacci sequence. For $n \geq 1$ integer, Neutrosophic Fibonacci numbers are defined as

$$F^{\alpha,\beta,\gamma}_{r(n-1)+l} = \big(\left[F_{r(n-1)+l} + \alpha \left(F_{rn+l} - F_{r(n-1)+l} \right), F_{r(n+1)+l} - \alpha \left(F_{r(n+1)+l} - F_{rn+l} \right) \right], [(1-\beta)F_{rn+l} + \beta F_{r(n-1)+l}, (1-\beta)F_{rn+l} + \beta F_{r(n+1)+l}], \left[(1-\gamma)F_{rn+l} + \gamma F_{r(n-1)+l}, (1-\gamma)F_{rn+l} + \gamma F_{r(n+1)+l} \right] \big)$$

where $r, l \in \mathbb{Z}^+, r(n-1) + l > 1; \alpha, \beta, \gamma \in [0,1]$ and $0 \leq \alpha + \beta + \gamma \leq 3$. When we take $r = 1$ and $l = 0$, several Neutrosophic Fibonacci numbers are following:

$$F_0^{\alpha,\beta,\gamma} = ([\alpha, 1], [1-\beta, 1], [1-\gamma, 1]); \quad F_1^{\alpha,\beta,\gamma} = ([1, 2-\alpha], [1, 1+\beta], [1, 1+\gamma]); \quad F_2^{\alpha,\beta,\gamma} = ([1+\alpha, 3-\alpha], [2-\beta, 2+\beta], [2-\gamma, 2+\gamma]); \ldots$$

Remark 3.2. The first number of a Neutrosophic Fibonacci sequence for $\alpha = 0.7, \beta = 0.23, \gamma = 0.07$ is $F_0^{0.7,0.23,0.07} = ([0.7, 1], [0.8, 1], [0.93, 1])$ can be seen (bold colored) in Table 2. Moreover, if we use only α, we obtain fuzzy Fibonacci number sequence and if we use α and γ s.t. $\alpha + \gamma \leq 1$, we obtain intuitionistic fuzzy Fibonacci number sequence.

Table 2: The first 6 elements of Neutrosophic Fibonacci Sequence for α, β, and γ cuts.

α	β	γ	F_0	F_1	F_2	F_3	F_4	F_5	F_0	F_1	F_2	F_3	F_4	F_5
			Left Truth						Right Truth					
1	0	1	1	1	2	3	5	8	1	1	2	3	5	8
0.9	0.05	0.9	0.9	1	1.9	2.9	4.8	7.7	1	1.1	2.1	3.2	5.3	8.5
0.8	0.2	0	0.8	1	1.8	2.8	4.6	7.4	1	1.2	2.2	3.4	5.6	9
0.7	**0.23**	**0.07**	**0.7**	1	1.7	2.7	4.4	7.1	**1**	1.3	2.3	3.6	5.9	9.5
0.6	0.6	0	0.6	1	1.6	2.6	4.2	6.8	1	1.4	2.4	3.8	6.2	10
0.5	0.5	0.5	0.5	1	1.5	2.5	4	6.5	1	1.5	2.5	4	6.5	10.5
0.4	0.4	0.04	0.4	1	1.4	2.4	3.8	6.2	1	1.6	2.6	4.2	6.8	11
0.3	0.3	1	0.3	1	1.3	2.3	3.6	5.9	1	1.7	2.7	4.4	7.1	11.5
0.2	0.2	0.2	0.2	1	1.2	2.2	3.4	5.6	1	1.8	2.8	4.6	7.4	12
			Left Indeterminacy						Right Indeterminacy					
			1	1	2	3	5	8	1	1	2	3	5	8
			1	1	2	3	5	8	1	1.05	2.05	3.1	5.15	8.25
			0.95	1	1.95	2.95	4.9	7.85	1	1.2	2.2	3.4	5.6	9
			0.8	1	1.8	2.8	4.6	7.4	**1**	1.23	2.23	3.46	5.69	9.15
			0.77	1	1.77	2.77	4.54	7.31	1	1.6	2.6	4.2	6.8	11
			0.4	1	1.4	2.4	3.8	6.2	1	1.5	2.5	4	6.5	10.5
			0.5	1	1.5	2.5	4	6.5	1	1.4	2.4	3.8	6.2	10
			0.6	1	1.6	2.6	4.2	6.8	1	1.3	2.3	3.6	5.9	9.5
			0.7	1	1.7	2.7	4.4	7.1	1	1.2	2.2	3.4	5.6	9
			Left Falsity						Right Falsity					
			0	1	1	2	3	5	1	2	3	5	8	13
			0.1	1	1.1	2.1	3.2	5.3	1	1.9	2.9	4.8	7.7	12.5
			1	1	2	3	5	8	1	1	2	3	5	8
			0.93	1	1.93	2.93	4.86	7.79	**1**	1.07	2.07	3.14	5.21	8.35
			1	1	2	3	5	8	1	1	2	3	5	8
			0.5	1	1.5	2.5	4	6.5	1	1.5	2.5	4	6.5	10.5
			0.96	1	1.96	2.96	4.92	7.88	1	1.04	2.04	3.08	5.12	8.2
			0	1	1	2	3	5	1	2	3	5	8	13
			0.8	1	1.8	2.8	4.6	7.4	1	1.2	2.2	3.4	5.6	9

Remark 3.3. Neutrosophic Fibonacci numbers satisfy the similar recurrence relation with initial conditions $F_0^{\alpha,\beta,\gamma}$ and $F_1^{\alpha,\beta,\gamma}$ since Neutrosophic Fibonacci numbers include Fibonacci numbers. That is

$$F_n^{\alpha,\beta,\gamma} = F_{n-1}^{\alpha,\beta,\gamma} + F_{n-2}^{\alpha,\beta,\gamma}, n \geq 2.$$

Definition 3.4. Let $\{L_n\}$ be Lucas sequence. For $n \geq 2$ integer, Neutrosophic Lucas numbers are defined as $L_{r(n-2)+l}^{\alpha,\beta,\gamma} = ([L_{r(n-1)+l} + \alpha(L_{rn+l} - L_{r(n-1)+l}), L_{r(n+1)+l} - \alpha(L_{r(n+1)+l} - L_{rn+l})], [(1-\beta)L_{rn+l} + \beta L_{r(n-1)+l}, (1-\beta)L_{rn+l} + \beta L_{r(n+1)+l}], [(1-\gamma)L_{rn+l} + \gamma L_{r(n-1)+l}, (1-\gamma)L_{rn+l} + \gamma L_{r(n+1)+l}])$

where $r, l \in \mathbb{Z}^+, r(n-1)+l > 1; \alpha, \beta, \gamma \in [0,1]$ and $0 \leq \alpha + \beta + \gamma \leq 3$. When we take $r = 1$ and $l = 0$, several Neutrosophic Lucas numbers are following:

$$L_0^{\alpha,\beta,\gamma} = ([1+2\alpha, 4-\alpha], [3-2\beta, 3+\beta], [3-2\gamma, 3+\gamma]); \quad L_1^{\alpha,\beta,\gamma} = ([3+\alpha, 7-3\alpha], [4-\beta, 4+3\beta], [4-\gamma, 4+3\gamma]); L_2^{\alpha,\beta,\gamma} = ([4+3\alpha, 11-4\alpha], [7-3\beta, 7+4\beta], [7-3\gamma, 7+4\gamma]); \ldots$$

Remark 3.5. Since Neutrosophic Lucas numbers include Lucas numbers, then Neutrosophic Lucas numbers satisfy the similar recurrence relation with initial conditions $L_0^{\alpha,\beta,\gamma}$ and $L_1^{\alpha,\beta,\gamma}$. That is

$$L_n^{\alpha,\beta,\gamma} = L_{n-1}^{\alpha,\beta,\gamma} + L_{n-2}^{\alpha,\beta,\gamma}, n \geq 2$$

Remark 3.6. The numbers consisting only of the truth-membership value of the Neutrosophic Fibonacci and Lucas numbers are called the Fuzzy Fibonacci and Lucas numbers respectively, where $\alpha \in [0,1]$ should be $0 \leq \alpha \leq 1$.

Remark 3.7. The numbers consisting of only the truth-membership and falsity-membership values of the Neutrosophic Fibonacci and Lucas numbers are called the Intuitionistic Fuzzy Fibonacci and Lucas numbers respectively, where $\alpha, \beta \in [0,1]$ should be $0 \leq \alpha + \beta \leq 1$.

Proposition 3.8. *$F_n^{\alpha,\beta,\gamma}$ be nth Neutrosophic Fibonacci number such that $n \geq 1$ integer. The Binet's formula of the $F_n^{\alpha,\beta,\gamma}$ numbers is*

$$F_n^{\alpha,\beta,\gamma} = \left(\left[\left(\frac{\gamma_1^{n+1}-\gamma_2^{n+1}}{\gamma_1-\gamma_2} + \alpha\frac{\gamma_1^{n}-\gamma_2^{n}}{\gamma_1-\gamma_2}\right), \left(\frac{\gamma_1^{n+3}-\gamma_2^{n+3}}{\gamma_1-\gamma_2} + \alpha\frac{\gamma_1^{n+1}-\gamma_2^{n+1}}{\gamma_1-\gamma_2}\right)\right], \left[\left(\frac{\gamma_1^{n+2}-\gamma_2^{n+2}}{\gamma_1-\gamma_2} - \beta\frac{\gamma_1^{n}-\gamma_2^{n}}{\gamma_1-\gamma_2}\right), \left(\frac{\gamma_1^{n+2}-\gamma_2^{n+2}}{\gamma_1-\gamma_2} + \beta\frac{\gamma_1^{n+1}-\gamma_2^{n+1}}{\gamma_1-\gamma_2}\right)\right], \left[\left(\frac{\gamma_1^{n+2}-\gamma_2^{n+2}}{\gamma_1-\gamma_2} - \gamma\frac{\gamma_1^{n}-\gamma_2^{n}}{\gamma_1-\gamma_2}\right), \left(\frac{\gamma_1^{n+2}-\gamma_2^{n+2}}{\gamma_1-\gamma_2} + \gamma\frac{\gamma_1^{n+1}-\gamma_2^{n+1}}{\gamma_1-\gamma_2}\right)\right]\right)$$

where $\gamma_1 = \frac{1+\sqrt{5}}{2}$ and $\gamma_2 = \frac{1-\sqrt{5}}{2}$ are roots of the characteristic equation of the Fibonacci numbers, $\alpha, \beta, \gamma \in [0,1]$ and $0 \leq \alpha + \beta + \gamma \leq 3$.

Proof: The proof by induction is applied on n. Similarly, Binet's formula for Neutrosophic Lucas numbers is obtained as

$$L_n^{\alpha,\beta,\gamma} = \left(\left[\left(\gamma_1^{n+1} + \gamma_2^{n+1} + \alpha\left(\gamma_1^{n} + \gamma_2^{n}\right)\right), \left(\gamma_1^{n+3} + \gamma_2^{n+3} + \alpha\left(\gamma_1^{n+1} + \gamma_2^{n+1}\right)\right)\right], \left[\left(\gamma_1^{n+2} + \gamma_2^{n+2} - \beta\left(\gamma_1^{n} + \gamma_2^{n}\right)\right), \left(\gamma_1^{n+2} + \gamma_2^{n+2} + \beta\right)\left(\gamma_1^{n+1} + \gamma_2^{n+1}\right)\right], \left[\left(\gamma_1^{n+2} + \gamma_2^{n+2} - \gamma\left(\gamma_1^{n} + \gamma_2^{n}\right)\right), \left(\gamma_1^{n+2} + \gamma_2^{n+2} + \gamma\left(\gamma_1^{n+1} + \gamma_2^{n+1}\right)\right)\right]\right).$$

Proposition 3.9. *$F_n^{\alpha,\beta,\gamma}$ be nth Neutrosophic Fibonacci number. The generating function formula of the $F_n^{\alpha,\beta,\gamma}$ numbers is*

$$h(t) = \frac{([2+\alpha, 5-2\alpha], [3-\beta, 3+2\beta], [3-\gamma, 3+2\gamma]) + ([1+\alpha, 3-\alpha], [2-\beta, 2+\beta], [2-\gamma, 2+\gamma])t}{1-t-t^2}$$

where $\alpha, \beta, \gamma \in [0,1]$ and $0 \leq \alpha + \beta + \gamma \leq 3$.

Proof: Let $h(t)$ be generating function for the Neutrosophic Fibonacci numbers as $h(t) = \sum_{n=1}^{\infty} F_n^{\alpha,\beta,\gamma} t^n$. Using $h(t)$, $-t(h)t$ and $-t^2h(t)$ we get the following equations, $-th(t) = \sum_{n=1}^{\infty} F_n^{\alpha,\beta,\gamma} t^{n+1}$ and $-t^2h(t) = -\sum_{n=1}^{\infty} F_n^{\alpha,\beta,\gamma} t^{n+2}$. After the needed calculations, the generating function for the Intuitionistic Fuzzy Fibonacci numbers is obtained as

$$h(t) = \frac{F_2^{\alpha,\beta,\gamma} + \left(F_3^{\alpha,\beta,\gamma} - F_2^{\alpha,\beta,\gamma}\right)t}{1-t-t^2}$$

$$h(t) = \frac{([2+\alpha, 5-2\alpha], [3-\beta, 3+2\beta], [3-\gamma, 3+2\gamma]) + ([1+\alpha, 3-\alpha], [2-\beta, 2+\beta], [2-\gamma, 2+\gamma])t}{1-t-t^2}.$$

.

The proof is completed. Similarly, generation function formula for Neutrosophic Lucas numbers is obtained as

$$m(t) = \frac{([4+3\alpha, 11-4\alpha], [7-3\beta, 7+4\beta], [7-3\gamma, 7+4\gamma]) + ([3+\alpha, 7-3\alpha], [4-\beta, 4+3\beta], [4-\gamma, 4+3\gamma])t}{1-t-t^2}$$

.

4 Some Properties of Neutrosophic Fibonacci and Lucas Numbers

In this section, we generalize some well-known identities of Fibonacci and Lucas numbers to Neutrosophic Fibonacci and Lucas numbers. We show that common identities and equalities in number theory and Koshy's book[15] are hold for Neutrosophic and Lucas Fibonacci Numbers. We notice that although some of the properties of formulas remain the same, the others change.

Proposition 4.1. *$F_n^{\alpha,\beta,\gamma}$ and $L_n^{\alpha,\beta,\gamma}$ be nth Neutrosophic Fibonacci and Lucas number such that $n \geq 1$ integer, respectively. Then, the following equalities hold:*

(a) $F_{n+2}^{\alpha,\beta,\gamma} - F_{n-2}^{\alpha,\beta,\gamma} = L_n^{\alpha,\beta,\gamma}$

(b) $F_{n+1}^{\alpha,\beta,\gamma} + F_{n-1}^{\alpha,\beta,\gamma} = L_n^{\alpha,\beta,\gamma}$

(c) $L_{n+1}^{\alpha,\beta,\gamma} + L_{n-1}^{\alpha,\beta,\gamma} = 5F_n^{\alpha,\beta,\gamma}$

(d) $L_{n+2}^{\alpha,\beta,\gamma} - L_{n-2}^{\alpha,\beta,\gamma} = 5F_n^{\alpha,\beta,\gamma}$

(e) $F_n^{\alpha,\beta,\gamma} + L_n^{\alpha,\beta,\gamma} = 2F_{n+1}^{\alpha,\beta,\gamma}$

Proof:

(a) $F_{n+2}^{\alpha,\beta,\gamma} - F_{n-2}^{\alpha,\beta,\gamma} = ([F_{n+3} + \alpha F_{n+2}, F_{n+5} - \alpha F_{n+3}], [F_{n+4} - \beta F_{n+2}, F_{n+4} + \beta F_{n+3}], [F_{n+4} - \gamma F_{n+2}, F_{n+4} + \gamma F_{n+3}]) - ([F_{n-1} + \alpha F_{n-2}, F_{n+1} - \alpha F_{n-1}], [F_n - \beta F_{n-2}, F_n + \beta F_{n-1}], [F_n - \gamma F_{n-2}, F_n + \gamma F_{n-1}])$. Since $F_{n+2} - F_{n-2} = L_n$,[15] then we get together with Definition 3.1 $F_{n+2}^{\alpha,\beta,\gamma} - F_{n-2}^{\alpha,\beta,\gamma} = L_n^{\alpha,\beta,\gamma}$.

Similarly, the other proofs can be obtained easily.

Proposition 4.2. ***(Ruggles's Identity)*** *$F_n^{\alpha,\beta,\gamma}$ be nth Neutrosophic Fibonacci number such that $m, n \geq 1$ integer. Then, the following equality holds*

$$F_{m-1}^{\alpha,\beta,\gamma} F_{n-m}^{\alpha,\beta,\gamma} + F_m^{\alpha,\beta,\gamma} F_{n-m+1}^{\alpha,\beta,\gamma} \cong (F_n, F_{n+2}, F_{n+4})$$
$$= ([F_n + \alpha F_{n+1}, F_{n+4} - \alpha F_{n+3}], [F_{n+2} - \beta F_n, F_{n+2} + \beta F_{n+4}], [F_{n+2} - \gamma F_n, F_{n+2} + \gamma F_{n+4}])$$

Proof: Let's prove the truth-membership value first.

$$\begin{aligned} F_m^{\alpha} F_{n-m+1}^{\alpha} + F_{m-1}^{\alpha} F_{n-m}^{\alpha} & \\ &= [F_{m+1} + \alpha F_m, F_{m+3} - \alpha F_{m+1}] [F_{n-m+2} + \alpha F_{n-m+1}, F_{n-m+4} - \alpha F_{n-m+2}] \\ &+ [F_m + \alpha F_{m-1}, F_{m+2} - \alpha F_m] [F_{n-m+1} + \alpha F_{n-m}, F_{n-m+3} - \alpha F_{n-m+1}] \\ &= [F_{m+1}F_{n-m+2} + F_m F_{n-m+1} + \alpha (F_{m+1}F_{n-m+1} + F_m F_{n-m} + F_m F_{n-m+2} + F_{m-1}F_{n-m+1}) \\ &+ \alpha^2 (F_m F_{n-m+1} + F_{m-1}F_{n-m}), F_{m+3}F_{n-m+4} + F_{m+2}F_{n-m+3} \\ &- \alpha (F_{m+3}F_{n-m+2} + F_{m+1}F_{n-m+4} + F_{m+2}F_{n-m+1} + F_m F_{n-m+3}) \\ &+ \alpha^2 (F_{m+1}F_{n-m+2} + F_m F_{n-m+1})] \end{aligned}$$

Since $F_{m-1}F_n + F_m F_{n+1} = F_{n+m}$,[15] then we have that

$$F_m^{\alpha} F_{n-m+1}^{\alpha} + F_{m-1}^{\alpha} F_{n-m}^{\alpha} = \left[F_n + 2\alpha F_{n-1} + \alpha^2 F_{n-2}, F_{n+4} - 2\alpha F_{n+2} + \alpha^2 F_n\right].$$

We put $\alpha = 0$ and $\alpha = 1$ in order to convert this Neutrosophic number to α-cut interval. Then we get the truth-membership value

$$F_m^{\alpha} F_{n-m+1}^{\alpha} + F_{m-1}^{\alpha,} F_{n-m}^{\alpha} \cong [F_n + \alpha F_{n+1}, F_{n+4} - \alpha F_{n+3}].$$

Similarly, we find the indeterminacy-membership value

$$F_m^{\beta}F_{n-m+1}^{\beta}+F_{m-1}^{\beta}F_{n-m}^{\beta}$$
$$=\left[(1-\beta)^2F_{n+4}+2(1-\beta)\beta F_{n+3}+\beta^2F_{n+2},(1-\beta)^2F_{n+4}+2(1-\beta)\beta F_{n+5}+\beta^2F_{n+6}\right].$$

We put $\beta=0$ and $\beta=1$ in order to convert this Neutrosophic number to β-cut interval. Then we get the indeterminacy-membership value

$$F_m^{\beta}F_{n-m+1}^{\beta}+F_{m-1}^{\beta}F_{n-m}^{\beta}\cong\left[F_{n+2}-\beta F_n,F_{n+2}+\beta F_{n+4}\right].$$

Finally, we find the falsity-membership value

$$F_m^{\gamma}F_{n-m+1}^{\gamma}+F_{m-1}^{\gamma}F_{n-m}^{\gamma}$$
$$=\left[(1-\gamma)^2F_{n+4}+2(1-\gamma)\gamma F_{n+3}+\gamma^2F_{n+2},(1-\gamma)^2F_{n+4}+2(1-\gamma)\gamma F_{n+5}+\gamma^2F_{n+6}\right].$$

We put $\gamma=0$ and $\gamma=1$ in order to convert this Neutrosophic number to γ-cut interval. Then we get the falsity-membership value

$$F_m^{\gamma}F_{n-m+1}^{\gamma}+F_{m-1}^{\gamma}F_{n-m}^{\gamma}\cong\left[F_{n+2}-\gamma F_n,F_{n+2}+\gamma F_{n+4}\right].$$

The same identity is obtained as

$$L_{m-1}^{\alpha,\beta,\gamma}L_{n-m}^{\alpha,\beta,\gamma}+L_m^{\alpha,\beta,\gamma}L_{n-m+1}^{\alpha,\beta,\gamma}\cong 5\left(F_n,F_{n+2},F_{n+4}\right)$$
$$=5\left(\left[F_n+\alpha F_{n+1},F_{n+4}-\alpha F_{n+3}\right],\left[F_{n+2}-\beta F_n,F_{n+2}+\beta F_{n+4}\right],\left[F_{n+2}-\gamma F_n,F_{n+2}+\gamma F_{n+4}\right]\right)$$

for Neutrosophic Lucas number.

Proposition 4.3. ***(Honsberger's Identity)*** *$F_n^{\alpha,\beta,\gamma}$ be nth Neutrosophic Fibonacci number such that $m,n\geq 1$ integer. Then, the following equality holds*

$$F_{m-1}^{\alpha,\beta,\gamma}F_n^{\alpha,\beta,\gamma}+F_m^{\alpha,\beta,\gamma}F_{n+1}^{\alpha,\beta,\gamma}\cong\left(F_{n+m},F_{n+m+2},F_{n+m+4}\right)$$
$$=\left(\left[F_{n+m}+\alpha F_{n+m+1},F_{n+m+4}-\alpha F_{n+m+3}\right],\left[F_{n+m+2}-\beta F_{n+m},F_{n+m+2}+\beta F_{n+m+4}\right],\left[F_{n+m+2}\right.\right.$$
$$\left.\left.-\gamma F_{n+m},F_{n+m+2}+\gamma F_{n+m+4}\right]\right).$$

Proof: Honsberger's identity is obtained by taking $n=n+m$ in the Ruggles's identity. The same identity is obtained as

$$L_{m-1}^{\alpha,\beta,\gamma}L_n^{\alpha,\beta,\gamma}+L_m^{\alpha,\beta,\gamma}L_{n+1}^{\alpha,\beta,\gamma}\cong 5\left(F_{n+m},F_{n+m+2},F_{n+m+4}\right)$$
$$=5\left(\left[F_{n+m}+\alpha F_{n+m+1},F_{n+m+4}-\alpha F_{n+m+3}\right],\left[F_{n+m+2}-\beta F_{n+m},F_{n+m+2}\right.\right.$$
$$\left.\left.+\beta F_{n+m+4}\right],\left[F_{n+m+2}-\gamma F_{n+m},F_{n+m+2}+\gamma F_{n+m+4}\right]\right)$$

for Neutrosophic Lucas number.

Proposition 4.4. *$F_n^{\alpha,\beta,\gamma}$ and $L_n^{\alpha,\beta,\gamma}$ be nth Neutrosophic Fibonacci and Lucas number such that $n\geq 1$ integer, respectively. Then, the following equalities hold:*

(a) $\left(F_n^{\alpha,\beta,\gamma}\right)^2+\left(F_{n+1}^{\alpha,\beta,\gamma}\right)^2\cong F_{2n+1}^{\alpha,\beta,\gamma}$

(b) $\left(L_n^{\alpha,\beta,\gamma}\right)^2+\left(L_{n+1}^{\alpha,\beta,\gamma}\right)^2\cong 5F_{2n+1}^{\alpha,\beta,\gamma}$

(c) $\left(F_{n+1}^{\alpha,\beta,\gamma}\right)^2-\left(F_{n-1}^{\alpha,\beta,\gamma}\right)^2\cong F_{2n}^{\alpha,\beta,\gamma}$

(d) $\left(L_{n+1}^{\alpha,\beta,\gamma}\right)^2-\left(L_{n-1}^{\alpha,\beta,\gamma}\right)^2\cong 5F_{2n}^{\alpha,\beta,\gamma}$

(e) $F_n^{\alpha,\beta,\gamma}L_n^{\alpha,\beta,\gamma}\cong F_{2n}^{\alpha,\beta,\gamma}$

Proof (a): Let's prove the truth-membership value first.

$$\begin{aligned}(F_n^{\alpha})^2 &= [F_{n+1}+\ \alpha F_n, F_{n+3}-\alpha F_{n+1}]\,[F_{n+1}+\alpha F_n, F_{n+3}-\alpha F_{n+1}]\\ &= \left[F_{n+1}^2+2\alpha F_{n+1}F_n+\alpha^2F_n^2, F_{n+3}^2-2\alpha F_{n+3}F_{n+1}+\alpha^2F_{n+1}^2\right]\end{aligned}$$

We put $\alpha = 0$ and $\alpha = 1$ in order to convert this Neutrosophic number to α-cut interval. Then we get the truth-membership value and

$$(F_n^{\alpha})^2 \cong \left(F_{n+1}^2, F_{n+2}^2, F_{n+3}^2\right) = F_{n+2}^2$$

$$\left(F_{n+1}^{\alpha}\right)^2 \cong \left(F_{n+2}^2, F_{n+3}^2, F_{n+4}^2\right) = F_{n+3}^2$$

Since $F_n^2 + F_{n+1}^2 = F_{2n+1}$,[15] then we get that

$$\left(F_n^{\alpha,\beta,\gamma}\right)^2 + \left(F_{n+1}^{\alpha,\beta,\gamma}\right)^2 \cong F_{2n+1}^{\alpha,\beta,\gamma}.$$

Similarly, we find the indeterminacy-membership and falsity-membership value. The other steps of proofs are easily obtained with the same way.

Proposition 4.5. *$F_n^{\alpha,\beta,\gamma}$ and $L_n^{\alpha,\beta,\gamma}$ be nth Neutrosophic Fibonacci and Lucas number such that $n \geq 1$ integer, respectively. Then, the following equalities hold:*

(a) $\sum_{i=1}^{n} F_i^{\alpha,\beta,\gamma} = F_{n+2}^{\alpha,\beta,\gamma} - F_2^{\alpha,\beta,\gamma}$

(b) $\sum_{i=1}^{n} F_{2i}^{\alpha,\beta,\gamma} = F_{2n+1}^{\alpha,\beta,\gamma} - F_1^{\alpha,\beta,\gamma}$

(c) $\sum_{i=2}^{n} F_{2i-1}^{\alpha,\beta,\gamma} = F_{2n}^{\alpha,\beta,\gamma} - F_1^{\alpha,\beta,\gamma}$

(d) $\sum_{i=1}^{n} \left(F_i^{\alpha,\beta,\gamma}\right)^2 \cong F_n^{\alpha,\beta,\gamma} F_{n+1}^{\alpha,\beta,\gamma}$

(e) $\sum_{i=1}^{n} L_i^{\alpha,\beta,\gamma} = L_{n+2}^{\alpha,\beta,\gamma} - L_2^{\alpha,\beta,\gamma}$

(f) $\sum_{i=1}^{n} L_{2i}^{\alpha,\beta,\gamma} = L_{2n+1}^{\alpha,\beta,\gamma} - L_1^{\alpha,\beta,\gamma}$

(g) $\sum_{i=2}^{n} L_{2i-1}^{\alpha,\beta,\gamma} = L_{2n}^{\alpha,\beta,\gamma} - L_2^{\alpha,\beta,\gamma}$

(h) $\sum_{i=1}^{n} \left(L_i^{\alpha,\beta,\gamma}\right)^2 \cong L_n^{\alpha,\beta,\gamma} L_{n+1}^{\alpha,\beta,\gamma} - L_2^{\alpha,\beta,\gamma} + L_1^{\alpha,\beta,\gamma}$

Proof (a): Let's prove the truth-membership value first.

$$\begin{aligned}\sum_{i=1}^{n} F_i^{\alpha} &= \left[\sum_{i=2}^{n+1} F_i + \alpha \sum_{i=1}^{n} F_i, \sum_{i=4}^{n+3} F_i - \alpha \sum_{i=2}^{n+1} F_i\right]\\ &= \left[\left(F_{n+3} - F_2 - F_1\right) + \alpha\left(F_{n+2} - F_2\right), \left(F_{n+5} - F_5\right) - \alpha\left(F_{n+3} - F_2 - F_1\right)\right]\\ &= \left[F_{n+3} + \alpha F_{n+2}, F_{n+5} - \alpha F_{n+3}\right] - \left[F_3 + \alpha F_2, F_5 - \alpha F_3\right] = F_{n+2}^{\alpha} - F_2^{\alpha}.\end{aligned}$$

Similarly, we find the indeterminacy-membership and falsity-membership value. The other steps of proof are easily obtained by following the same way.

Remark 4.6. Some special sequences well-known for the Fibonacci and Lucas sequences have also been calculated for the neutrosophic Fibonacci and Lucas numbers. The proofs of these equations are omitted. $F_n^{\alpha,\beta,\gamma}$ and $L_n^{\alpha,\beta,\gamma}$ be nth Neutrosophic Fibonacci and Lucas number such that $n \geq 1$ integer, respectively. Then, the following equalities hold:

(a) ***Tagiuri's Identity*** :

$$F_{m+k}^{\alpha,\beta,\gamma} F_{n-k}^{\alpha,\beta,\gamma} - F_m^{\alpha,\beta,\gamma} F_n^{\alpha,\beta,\gamma} \cong (-1)^{n-k}\left(-F_k F_{m-n+k}, 0, F_k F_{m-n+k}\right)$$

$$L_{m+k}^{\alpha,\beta,\gamma} L_{n-k}^{\alpha,\beta,\gamma} - L_m^{\alpha,\beta,\gamma} L_n^{\alpha,\beta,\gamma} \cong 5(-1)^{n-k+1}\left(-F_k F_{m-n+k}, 0, F_k F_{m-n+k}\right)$$

(b) ***d'Ocagne's Identity*** :

$$F_{m+1}^{\alpha,\beta,\gamma}F_{n-1}^{\alpha,\beta,\gamma} - F_m^{\alpha,\beta,\gamma}F_n^{\alpha,\beta,\gamma} \cong (-1)^{n-1}\left(-F_{m-n+1}, 0, F_{m-n+1}\right)$$
$$L_{m+1}^{\alpha,\beta,\gamma}L_{n-1}^{\alpha,\beta,\gamma} - L_m^{\alpha,\beta,\gamma}L_n^{\alpha,\beta,\gamma} \cong 5(-1)^n - \left(F_{m-n+1}, 0, F_{m-n+1}\right)$$

(c) ***Catalan's Identity*** :

$$F_{n+k}^{\alpha,\beta,\gamma}F_{n-k}^{\alpha,\beta,\gamma} - F_n^{\alpha,\beta,\gamma}F_n^{\alpha,\beta,\gamma} \cong (-1)^{n-k}\left(-F_k^2, 0, F_k^2\right)$$
$$L_{n+k}^{\alpha,\beta,\gamma}L_{n-k}^{\alpha,\beta,\gamma} - L_n^{\alpha,\beta,\gamma}L_n^{\alpha,\beta,\gamma} \cong 5(-1)^{n-k+1}\left(-F_k^2, 0, F_k^2\right)$$

(d) ***Cassini's Identity*** :

$$F_{n+1}^{\alpha,\beta,\gamma}F_{n-1}^{\alpha,\beta,\gamma} - F_n^{\alpha,\beta,\gamma}F_n^{\alpha,\beta,\gamma} \cong (-1)^{n-1}(-1,0,1)$$
$$L_{n+1}^{\alpha,\beta,\gamma}L_{n-1}^{\alpha,\beta,\gamma} - L_n^{\alpha,\beta,\gamma}L_n^{\alpha,\beta,\gamma} \cong 5(-1)^n(-1,0,1).$$

5 Other Neutrosophic Number Sequences

In this section, we focus to define numbers and their sequences of Neutrosophic and Neutrosophic-Lucas versions, and the recurrence relations with various initial conditions in the second book of Koshy.[18] Sequences of neutrosophic numbers whose characteristic equation of the second order can be defined within number sequences such as Pell,[18] Pell-Lucas,[18] Jacobsthal,[19] Jacobsthal-Lucas,[19] Mersenne,[20] Mersenne-Lucas,[21] Balancing,[22] and Lucas-Balancing.[22] Definitions of these neutrosophic number sequences are given below.

Definition 5.1. ***Neutrosophic Pell numbers*** are defined as follows:

$P_0^{\alpha,\beta,\gamma} = ([\alpha, 2-\alpha], [1-\beta, 1+\beta], [1-\gamma, 1+\gamma]); \quad P_1^{\alpha,\beta,\gamma} = ([1+\alpha, 5-3\alpha], [2-\beta, 2+3\beta], [2-\gamma, 2+$

$3\gamma]); P_2^{\alpha,\beta,\gamma} = ([2+3\alpha, 12-7\alpha], [5-3\beta, 5+7\beta], [5-3\gamma, 5+7\gamma]); \ldots$.

Remark 5.2. Neutrosophic Pell numbers satisfy the recurrence relation with initial conditions $P_0^{\alpha,\beta,\gamma}$ and $P_1^{\alpha,\beta,\gamma}$. That is $P_n^{\alpha,\beta,\gamma} = 2P_{n-1}^{\alpha,\beta,\gamma} + P_{n-\lambda,\gamma}^{\alpha,\beta}, n \geq 2$.

Definition 5.3. ***Neutrosophic Pell-Lucas numbers*** are defined as follows:

$PL_0^{\alpha,\beta,\gamma} = ([2, 6-4\alpha], [2, 2+4\beta], [2, 2+4\gamma]); \quad PL_1^{\alpha,\beta,\gamma} = ([2+4\alpha, 14-8\alpha], [6-$
$4\beta, 6+8\beta], [6-4\gamma, 6+8\gamma]); \quad PL_2^{\alpha,\beta,\gamma} = ([6+8\alpha, 34-20\alpha], [14-8\beta, 14+20\beta], [14-$

$8\gamma, 14+20\gamma])$;.

Remark 5.4. Neutrosophic Pell-Lucas numbers satisfy the recurrence relation with initial conditions $PL_0^{\alpha,\beta,\gamma}$ and $PL_1^{\alpha,\beta,\gamma}$. That is $PL_n^{\alpha,\beta,\gamma} = 2PL_{n-1}^{\alpha,\beta,\gamma} + PL_{n-2}^{\alpha,\beta,\gamma}, n \geq 2$.

Definition 5.5. ***Neutrosophic Jacobsthal numbers*** are defined as follows:

$J_0^{\alpha,\beta,\gamma} = ([\alpha, 1], [1-\beta, 1], [1-\gamma, 1]); \quad J_1^{\alpha,\beta,\gamma} = ([1, 3-2\alpha], [1, 1+$
$2\beta], [1, 1+2\gamma]); \quad J_2^{\alpha,\beta,\gamma} = ([1+2\alpha, 5-2\alpha], [3-2\beta, 3+2\beta], [3-2\gamma, 3+2\gamma]);$

.

Remark 5.6. Neutrosophic Jacobsthal numbers satisfy the recurrence relation with initial conditions $J_0^{\alpha,\beta,\gamma}$ and $J_1^{\alpha,\beta,\gamma}$. That is $J_n^{\alpha,\beta,\gamma} = J_{n-1}^{\alpha,\beta,\gamma} + 2J_{n-2}^{\alpha,\beta,\gamma}, n \geq 2$.

Definition 5.7. ***Neutrosophic Jacobsthal-Lucas numbers*** are defined as follows:

$JL_0^{\alpha,\beta,\gamma} = ([1+4\alpha, 7-2\alpha], [5-4\beta, 5+2\beta], [5-4\gamma, 5+2\gamma]); \quad JL_1^{\alpha,\beta,\gamma} = ([5+2\alpha, 17-10\alpha], [7-2\beta, 7+$
$10\beta], [7-2\gamma, 7+10\gamma]); JL_2^{\alpha,\beta,\gamma} = ([7+10\alpha, 31-14\alpha], [17-10\beta, 17+14\beta], [17-10\gamma, 17+14\gamma]); \ldots$

.

Remark 5.8. Neutrosophic Jacobsthal-Lucas numbers satisfy the recurrence relation with initial conditions $JL_0^{\alpha,\beta,\gamma}$ and $JL_1^{\alpha,\beta,\gamma}$. That is $JL_n^{\alpha,\beta,\gamma} = JL_{n-1}^{\alpha,\beta,\gamma} + 2JL_{n-2}^{\alpha,\beta,}, n \geq 2$.

Definition 5.9. ***Neutrosophic Balancing numbers*** are defined as follows:

$B_0^{\alpha,\beta,\gamma} = ([\alpha, 6-5\alpha], [1-\beta, 1+5\beta], [1-\gamma, 1+5\gamma]); B_1^{\alpha,\beta,\gamma} = ([1+5\alpha, 35-29\alpha], [6-5\beta, 6+29\beta], [6-5\gamma, 6+29\gamma]); \quad B_2^{\alpha,\beta,\gamma} = ([6+29\alpha, 204-169\alpha], [35-29\beta, 35+169\beta], [35-29\gamma, 35+169\gamma]); \ldots$

Remark 5.10. Neutrosophic Balancing numbers satisfy the recurrence relation with initial conditions $B_0^{\alpha,\beta,\gamma}$ and $B_1^{\alpha,\beta,\gamma}$. That is $B_n^{\alpha,\beta,\gamma} = 6B_{n-1}^{\alpha,\beta,\gamma} - B_{n-2}^{\alpha,\beta,\gamma}, n \geq 2$.

Definition 5.11. ***Neutrosophic Lucas-Balancing numbers*** are defined as follows:

$BL_0^{\alpha,\beta,\gamma} = ([1+2\alpha, 17-14\alpha], [3-2\beta, 3+14\beta], [3-2\gamma, 3+14\gamma]); \quad BL_1^{\alpha,\beta,\gamma} = ([3+14\alpha, 99-82\alpha], [17-14\beta, 17+82\beta], [17-14\gamma, 17+82\gamma]); \quad BL_2^{\alpha,\beta,\gamma} = ([17+82\alpha, 577-478\alpha], [99-82\beta, 99+478\beta], [99-82\gamma, 99+478\gamma]); \ldots$

Remark 5.12. Neutrosophic Lucas-Balancing numbers satisfy the recurrence relation with initial conditions $BL_0^{\alpha,\beta,\gamma}$ and $BL_1^{\alpha,\beta,\gamma}$. That is $BL_n^{\alpha,\beta,\gamma} = 6BL_{n-1}^{\alpha,\beta,\gamma} - BL_{n-2}^{\alpha,\beta,\gamma}, n \geq 2$.

Definition 5.13. ***Neutrosophic Mersenne numbers*** are defined as follows:

$M_0^{\alpha,\beta,\gamma} = ([\alpha, 3-2\alpha], [1-\beta, 1+2\beta], [1-\gamma, 1+2\gamma]); \quad M_1^{\alpha,\beta,\gamma} = ([1+2\alpha, 7-4\alpha], [3-2\beta, 3+4\beta], [3-2\gamma, 3+4\gamma]); M_2^{\alpha,\beta,\gamma} = ([3+4\alpha, 15-8\alpha], [7-4\beta, 7+8\beta], [7-4\gamma, 7+8\gamma]); \ldots$

Remark 5.14. Neutrosophic Mersenne numbers satisfy the recurrence relation with initial conditions $M_0^{\alpha,\beta,\gamma}$ and $M_1^{\alpha,\beta,\gamma}$. That is $M_n^{\alpha,\beta,\gamma} = 3M_{n-1}^{\alpha,\beta} - 2M_{n-2}^{\alpha,\beta}, n \geq 2$.

Definition 5.15. ***Neutrosophic Mersenne-Lucas numbers*** are defined as follows:

$ML_0^{\alpha,\beta,\gamma} = ([2+\alpha, 5-2\alpha], [3-\beta, 3+2\beta], [3-\gamma, 3+2\gamma]); \quad ML_1^{\alpha,\beta,\gamma} = ([3+2\alpha, 9-4\alpha], [5-2\beta, 5+4\beta], [5-2\gamma, 5+4\gamma]); ML_2^{\alpha,\beta,\gamma} = ([5+4\alpha, 17-8\alpha], [9-4\beta, 9+8\beta], [9-4\gamma, 9+8\gamma]); \ldots$

.

Remark 5.16. Neutrosophic Mersenne-Lucas numbers satisfy the recurrence relation with initial conditions $ML_0^{\alpha,\beta,\gamma}$ and $ML_1^{\alpha,\beta,\gamma}$. That is $ML_n^{\alpha,\beta,\gamma} = 3ML_{n-1}^{\alpha,\beta} - 2ML_{n-2}^{\alpha,\beta}, n \geq 2$.

6 A Generalization of Neutrosophic Number Sequences via Horadam

In this section, we introduce a generalization approach of Horadam[23] interpretation version of Neutrosophic number sequences. The well-known sequences, A and $B \in \mathbb{Q}$, defined by the second-order linear difference equation in the form of $H_n = AH_{n-1} + BH_{n-2}$, can be grouped according to their initial conditions as follows:

$H_n = AH_{n-1} + BH_{n-2}$ and $H_0 = 0, H_1 = 1$,

$H_n = AH_{n-1} + BH_{n-2}$ and $H_0 = 2, H_1 = A$,

$H_n = AH_{n-1} + BH_{n-2}$ and $H_0 = 1, H_1 = \frac{A}{2}$,

$H_n = AH_{n-1} + BH_{n-2}$ and $H_0 = 0, H_1 = \frac{A}{2}$.

Definition 6.1. Let $\{H_n\}$ be Horadam sequence. For $n \geq 0$ integer, *fuzzy number* sequences via Horadam are defined as

$$H_n^{\alpha,\beta,\gamma} = (H_{n-1}, H_n, H_{n+1}) = [H_{n-1} + \alpha(H_n - H_{n-1}), H_{n+1} - \alpha(H_{n+1} - H_n)]$$

where $\alpha \in [0, 1]$.

Table 3: Some well-known number sequences via Horadam Interpretation

A	B	H_n	Sequence Name	Initial Conditions H_0 H_1
1	1	$H_{n-1}+H_{n-2}$	Fibonacci	0 1
1	1	$H_{n-1}+H_{n-2}$	Lucas	2 1
2	1	$2H_{n-1}+H_{n-2}$	Pell	0 1
2	1	$2H_{n-1}+H_{n-2}$	Pell-Lucas	2 2
1	2	$H_{n-1}+2H_{n-2}$	Jacobsthal	0 1
1	2	$H_{n-1}+2H_{n-2}$	Jacobsthal-Lucas	2 1
3	-2	$3H_{n-1}-2H_{n-2}$	Mersenne	0 1
3	-2	$3H_{n-1}-2H_{n-2}$	Mersenne-Lucas	2 3
6	-1	$6H_{n-1}-H_{n-2}$	Balancing	0 1
6	-1	$6H_{n-1}-H_{n-2}$	Lucas-Balancing	1 3
2	1	$2H_{n-1}+H_{n-2}$	Modified-Lucas	1 1
p	q	$pH_{n-1}+qH_{n-2}$	Horadam	r s

Definition 6.2. Let $\{H_n\}$ be Horadam sequence. For $n \geq 0$ integer, *intuitionistic fuzzy* number sequences via Horadam are defined as

$H_n^{\alpha,\beta,\gamma} = (H_{n-1}, H_n, H_{n+1}) = ([H_{n-1} + \alpha(H_n - H_{n-1}), H_{n+1} - \alpha(H_{n+1} - H_n)], [H_n - \gamma(H_n - H_{n-1}), H_n + \gamma(H_{n+1} - H_n)])$ where $\alpha, \gamma \in [0,1]$ and $0 \leq \alpha + \gamma \leq 1$.

Definition 6.3. Let $\{H_n\}$ be Horadam sequence. For $n \geq 0$ integer, neutrosophic number sequences via Horadam are defined as

$H_n^{\alpha,\beta,\gamma} = (H_{n-1}, H_n, H_{n+1}) = ([H_{n-1} + \alpha(H_n - H_{n-1}), H_{n+1} - \alpha(H_{n+1} - H_n)], [H_n - \beta(H_n - H_{n-1}), H_n + \beta(H_{n+1} - H_n)], [H_n - \gamma(H_n - H_{n-1}), H_n + \gamma(H_{n+1} - H_n)])$ where $\alpha, \beta, \gamma \in [0,1]$ and $0 \leq \alpha + \beta + \gamma \leq 3$.

7 Applications

In this paper, we have studied interval valued neutrosophic sequence versions of many number sequences. Due to the usefulness and nature of number sequences, we have provided many alternatives for application to real-life problems. On the other hand, since each of the series we studied can correspond to a moment or situation, it can be easily applied to many fields such as forecasting, statistical methods, decision-making systems, mathematical economics. On the other hand, in terms of core mathematical research, many applications and features can be easily studied in areas such as mathematical analysis, matrix theory, theory of functions, graph theory, and sequence spaces on number sequences. In summary, we believe that this study opens a door for novel work, both theoretically and in terms of its application to real life problems.

8 Conclusions

In this paper, we have given a comprehensive introductory study of Neutrosophic number sequences as a guide, by exemplifying tables (in Appendix A) and studying basic number sequence properties. As a future direction, we plan to study other identities and properties of number sequences by increasing the diversity of these number sequences. Researchers, who are far from neutrosophic studies, might think that the scope of this paper is limited to neutrosophic at first glance, but since Neutrosophy is a nice fuzzy generalization, we offer a field of study for all fuzzy varieties and will continue to work with picture fuzzy, type fuzzy sets (like type-2 etc.) and extensions.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. Information and control, 8(3), 338-353.

[2] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. Fuzzy sets and Systems, 20(1), 87-96.

[3] Smarandache, F. (1998). Neutrosophy: Neutrosophic Probability, Set and Logic; American Research Press: Rehoboth, NM, USA.

[4] Adler, S. L. (1994). Quaternionic quantum mechanics and quantum fields. New York Oxford Univ. Press.

[5] Alves, F., Vieira, R. (2020). The Newton Fractal's Leonardo Sequence Study with the Google Colab. International Electronic Journal of Mathematics Education, 15(2), 1-9.

[6] Baez, J. (2001). The Octonians. Bull. Amer. Math. Soc., 145(39), 2, 145-205.

[7] Edson, M., Yayenie, O. (2009). A new generalization of Fibonacci sequences and the extended Binet's formula. Inyegers Electron. J. Comb. Number Theor., 9, 639-654.

[8] Falcon, S., Plaza, A. (2007) On the Fibonacci k-numbers. Chaos, Solitions & Fractals, 32(5), 1615-1624.

[9] George, A. H. (1969). Some formula for the Fibonacci sequence with generalization. Fibonacci Quart., 7, 113-130.

[10] Hamilton, W. R. (1844). Li on Quaternions; or on a New System of Imaginaries in Algebra. Philos. Mag. Ser. Taylor & Francis, 25(163), 10.

[11] Harman, C. J. (1981). Complex Fibonacci Numbers. The Fibonacci Quart., 19(1), 82-86.

[12] Horadam, A. F. (1961). A generalized Fibonacci sequence. Math. Mag., 68, 455-459.

[13] Ramirez, L. (2015). Some combinatorial properties of the k-Fibonacci and the k-Lucas quaternions. An. S t. Univ. Ovidius Constanta, Ser. Mat., 23(2), 201-212.

[14] Vajda, S. (1989). Fibonacci and Lucas Numbers and the Golden Section. Ellis Horwood Limited Publ., England.

[15] Koshy, T. (2001). Fibonacci and Lucas Numbers with Applications. A Wiley-Interscience Publication, New York.

[16] Irmak, N., & Demirtaş, N. (2019). Fuzzy Fibonacci and Fuzzy Lucas Numbers with their Properties. Mathematical Sciences and Applications E-Notes, 7(2), 218-224.

[17] Deli, I., Şubaş, Y. (2017). A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems, Int. J. Mach. Learn. & Cyber., 8, 1309-1322.

[18] Koshy, T. (2014). Pell and Pell-Lucas numbers with applications (Vol. 431). New York: Springer.

[19] Szynal-Liana, A., & Włoch, I. (2019). On Jacobsthal and Jacobsthal-Lucas hybrid numbers. In Annales Mathematicae Silesianae (Vol. 33, pp. 276-283).

[20] Catarino, P., Campos, H., & Vasco, P. (2016). On the Mersenne sequence. CM-Centro de Matemática.

[21] Saba, N., Boussayoud, A., & Kanuri, K. V. V. (2021). Mersenne Lucas numbers and complete homogeneous symmetric functions. Journal of mathematics and computer science, 24(2), 127-139.

[22] Ray, P. K. (2015). Balancing and Lucas-balancing sums by matrix methods. Mathematical Reports, 17(2), 225-233.

[23] Horadam, A. F. (1965). Basic properties of a certain generalized sequence of numbers. The Fibonacci Quarterly, 3(3), 161-176.

9 Appendix A

In Appendix A, in order to prevent the reader from drowning in the formulas, the calculated formulas for all the number sequences mentioned in the paper are given as tables.

Table 4: The first 6 elements of Neutrosophic Lucas Sequence for α, β, and γ cuts.

α	β	γ	L_0	L_1	L_2	L_3	L_4	L_5	L_0	L_1	L_2	L_3	L_4	L_5
			Left Truth						Right Truth					
1	0	1	3	4	7	11	18	29	3	4	7	11	18	29
0.9	0.05	0.9	2.8	3.9	6.7	10.6	17.3	27.9	3.1	4.3	7.4	11.7	19.1	30.8
0.8	0.2	0	2.6	3.8	6.4	10.2	16.6	26.8	3.2	4.6	7.8	12.4	20.2	32.6
0.7	0.23	0.07	2.4	3.7	6.1	9.8	15.9	25.7	3.3	4.9	8.2	13.1	21.3	34.4
0.6	0.6	0	2.2	3.6	5.8	9.4	15.2	24.6	3.4	5.2	8.6	13.8	22.4	36.2
0.5	0.5	0.5	2	3.5	5.5	9	14.5	23.5	3.5	5.5	9	14.5	23.5	38
0.4	0.4	0.04	1.8	3.4	5.2	8.6	13.8	22.4	3.6	5.8	9.4	15.2	24.6	39.8
0.3	0.3	1	1.6	3.3	4.9	8.2	13.1	21.3	3.7	6.1	9.8	15.9	25.7	41.6
0.2	0.2	0.2	1.4	3.2	4.6	7.8	12.4	20.2	3.8	6.4	10.2	16.6	26.8	43.4
			Left Indeterminacy						Right Indeterminacy					
			3	4	7	11	18	29	3	4	7	11	18	29
			3	4	7	11	18	29	3.05	4.15	7.2	11.4	18.6	29.9
			2.9	3.95	6.85	10.8	17.7	28.5	3.2	4.6	7.8	12.4	20.2	32.6
			2.6	3.8	6.4	10.2	16.6	26.8	3.23	4.69	7.92	12.6	20.5	33.1
			2.54	3.77	6.31	10.1	16.4	26.5	3.6	5.8	9.4	15.2	24.6	39.8
			1.8	3.4	5.2	8.6	13.8	22.4	3.5	5.5	9	14.5	23.5	38
			2	3.5	5.5	9	14.5	23.5	3.4	5.2	8.6	13.8	22.4	36.2
			2.2	3.6	5.8	9.4	15.2	24.6	3.3	4.9	8.2	13.1	21.3	34.4
			2.4	3.7	6.1	9.8	15.9	25.7	3.2	4.6	7.8	12.4	20.2	32.6
			Left Falsity						Right Falsity					
			1	3	4	7	11	18	4	7	11	18	29	47
			1.2	3.1	4.3	7.4	11.7	19.1	3.9	6.7	10.6	17.3	27.9	45.2
			3	4	7	11	18	29	3	4	7	11	18	29
			2.86	3.93	6.79	10.7	17.5	28.2	3.07	4.21	7.28	11.5	18.8	30.3
			3	4	7	11	18	29	3	4	7	11	18	29
			2	3.5	5.5	9	14.5	23.5	3.5	5.5	9	14.5	23.5	38
			2.92	3.96	6.88	10.8	17.7	28.6	3.04	4.12	7.16	11.3	18.4	29.7
			1	3	4	7	11	18	4	7	11	18	29	47
			2.6	3.8	6.4	10.2	16.6	26.8	3.2	4.6	7.8	12.4	20.2	32.6

Table 5: The first 6 elements of Neutrosophic Pell Sequence for α, β, and γ cuts.

α	β	γ	P_0	P_1	P_2	P_3	P_4	P_5	P_0	P_1	P_2	P_3	P_4	P_5
			Left Truth						Right Truth					
1	0	1	1	2	5	12	29	70	1	2	5	12	29	70
0.9	0.05	0.9	0.9	1.9	4.7	11.3	27.3	65.9	1.1	2.3	5.7	13.7	33.1	79.9
0.8	0.2	0	0.8	1.8	4.4	10.6	25.6	61.8	1.2	2.6	6.4	15.4	37.2	89.8
0.7	0.23	0.07	0.7	1.7	4.1	9.9	23.9	57.7	1.3	2.9	7.1	17.1	41.3	99.7
0.6	0.6	0	0.6	1.6	3.8	9.2	22.2	53.6	1.4	3.2	7.8	18.8	45.4	109.6
0.5	0.5	0.5	0.5	1.5	3.5	8.5	20.5	49.5	1.5	3.5	8.5	20.5	49.5	119.5
0.4	0.4	0.04	0.4	1.4	3.2	7.8	18.8	45.4	1.6	3.8	9.2	22.2	53.6	129.4
0.3	0.3	1	0.3	1.3	2.9	7.1	17.1	41.3	1.7	4.1	9.9	23.9	57.7	139.3

Left Indeterminacy						Right Indeterminacy					
1	2	5	12	29	70	1	2	5	12	29	70
0.95	1.95	4.85	11.65	28.15	67.95	1.05	2.15	5.35	12.85	31.05	74.95
0.8	1.8	4.4	10.6	25.6	61.8	1.2	2.6	6.4	15.4	37.2	89.8
0.77	1.77	4.31	10.39	25.09	60.57	1.23	2.69	6.61	15.91	38.43	92.77
0.4	1.4	3.2	7.8	18.8	45.4	1.6	3.8	9.2	22.2	53.6	129.4
0.5	1.5	3.5	8.5	20.5	49.5	1.5	3.5	8.5	20.5	49.5	119.5
0.6	1.6	3.8	9.2	22.2	53.6	1.4	3.2	7.8	18.8	45.4	109.6
0.7	1.7	4.1	9.9	23.9	57.7	1.3	2.9	7.1	17.1	41.3	99.7

Left Falsity						Right Falsity					
0	1	2	5	12	29	2	5	12	29	70	169
0.1	1.1	2.3	5.7	13.7	33.1	1.9	4.7	11.3	27.3	65.9	159.1
1	2	5	12	29	70	1	2	5	12	29	70
0.93	1.93	4.79	11.51	27.81	67.13	1.07	2.21	5.49	13.19	31.87	76.93
1	2	5	12	29	70	1	2	5	12	29	70
0.5	1.5	3.5	8.5	20.5	49.5	1.5	3.5	8.5	20.5	49.5	119.5
0.96	1.96	4.88	11.72	28.32	68.36	1.04	2.12	5.28	12.68	30.64	73.96
0	1	2	5	12	29	2	5	12	29	70	169

Table 6: The first 6 elements of Neutrosophic Pell-Lucas Sequence for α, β, and γ cuts.

α	β	γ	PL_0	PL_1	PL_2	PL_3	PL_4	PL_5	PL_0	PL_1	PL_2	PL_3	PL_4	PL_5
			Left Truth						Right Truth					
1	0	1	2	6	14	34	82	198	2	6	14	34	82	198
0.9	0.05	0.9	2	5.6	13.2	32	77.2	186.4	2.4	6.8	16	38.8	93.6	226
0.8	0.2	0	2	5.2	12.4	30	72.4	174.8	2.8	7.6	18	43.6	105.2	254
0.7	0.23	0.07	2	4.8	11.6	28	67.6	163.2	3.2	8.4	20	48.4	116.8	282
0.6	0.6	0	2	4.4	10.8	26	62.8	151.6	3.6	9.2	22	53.2	128.4	310
0.5	0.5	0.5	2	4	10	24	58	140	4	10	24	58	140	338
0.4	0.4	0.04	2	3.6	9.2	22	53.2	128.4	4.4	10.8	26	62.8	151.6	366
0.3	0.3	1	2	3.2	8.4	20	48.4	116.8	4.8	11.6	28	67.6	163.2	394
			Left Indeterminacy						Right Indeterminacy					
			2	6	14	34	82	198	2	6	14	34	82	198
			2	5.8	13.6	33	79.6	192.2	2.2	6.4	15	36.4	87.8	212
			2	5.2	12.4	30	72.4	174.8	2.8	7.6	18	43.6	105.2	254
			2	5.08	12.16	29.4	70.96	171.3	2.92	7.84	18.6	45.04	108.7	262.4
			2	3.6	9.2	22	53.2	128.4	4.4	10.8	26	62.8	151.6	366
			2	4	10	24	58	140	4	10	24	58	140	338
			2	4.4	10.8	26	62.8	151.6	3.6	9.2	22	53.2	128.4	310
			2	4.8	11.6	28	67.6	163.2	3.2	8.4	20	48.4	116.8	282
			Left Falsity						Right Falsity					
			2	2	6	14	34	82	6	14	34	82	198	478
			2	2.4	6.8	16	38.8	93.6	5.6	13.2	32	77.2	186.4	450
			2	6	14	34	82	198	2	6	14	34	82	198
			2	5.72	13.44	32.6	78.64	189.9	2.28	6.56	15.4	37.36	90.12	217.6
			2	6	14	34	82	198	2	6	14	34	82	198
			2	4	10	24	58	140	4	10	24	58	140	338
			2	5.84	13.68	33.2	80.08	193.4	2.16	6.32	14.8	35.92	86.64	209.2
			2	2	6	14	34	82	6	14	34	82	198	478

Table 7: The first 6 elements of Neutrosophic Jacobsthal Sequence for α, β, and γ cuts.

α	β	γ	J_0	J_1	J_2	J_3	J_4	J_5	J_0	J_1	J_2	J_3	J_4	J_5
			Left Truth						Right Truth					
1	0	1	1	1	3	5	11	21	1	1	3	5	11	21
0.9	0.05	0.9	0.9	1	2.8	4.8	10.4	20	1	1.2	3.2	5.6	12	23
0.8	0.2	0	0.8	1	2.6	4.6	9.8	19	1	1.4	3.4	6.2	13	25
0.7	0.23	0.07	0.7	1	2.4	4.4	9.2	18	1	1.6	3.6	6.8	14	28
0.6	0.6	0	0.6	1	2.2	4.2	8.6	17	1	1.8	3.8	7.4	15	30
0.5	0.5	0.5	0.5	1	2	4	8	16	1	2	4	8	16	32
0.4	0.4	0.04	0.4	1	1.8	3.8	7.4	15	1	2.2	4.2	8.6	17	34
0.3	0.3	1	0.3	1	1.6	3.6	6.8	14	1	2.4	4.4	9.2	18	36
			Left Indeterminacy						Right Indeterminacy					
			1	1	3	5	11	21	1	1	3	5	11	21
			0.95	1	2.9	4.9	10.7	20.5	1	1.1	3.1	5.3	11.5	22.1
			0.8	1	2.6	4.6	9.8	19	1	1.4	3.4	6.2	13	25.4
			0.77	1	2.54	4.54	9.62	18.7	1	1.46	3.46	6.38	13.3	26.1
			0.4	1	1.8	3.8	7.4	15	1	2.2	4.2	8.6	17	34.2
			0.5	1	2	4	8	16	1	2	4	8	16	32
			0.6	1	2.2	4.2	8.6	17	1	1.8	3.8	7.4	15	29.8
			0.7	1	2.4	4.4	9.2	18	1	1.6	3.6	6.8	14	27.6
			Left Falsity						Right Falsity					
			0	1	1	3	5	11	1	3	5	11	21	43
			0.1	1	1.2	3.2	5.6	12	1	2.8	4.8	10.4	20	41
			1	1	3	5	11	21	1	1	3	5	11	21
			0.93	1	2.86	4.86	10.58	20.3	1	1.14	3.14	5.42	11.7	23
			1	1	3	5	11	21	1	1	3	5	11	21
			0.5	1	2	4	8	16	1	2	4	8	16	32
			0.96	1	2.92	4.92	10.76	20.6	1	1.08	3.08	5.24	11.4	22
			0	1	1	3	5	11	1	3	5	11	21	43

Table 8: The first 6 elements of Neutrosophic Jacobsthal-Lucas Sequence for α, β, and γ cuts.

α	β	γ	JL_0	JL_1	JL_2	JL_3	JL_4	JL_5	JL_0	JL_1	JL_2	JL_3	JL_4	JL_5
			Left Truth						Right Truth					
1	0	1	5	7	17	31	65	127	5	7	17	31	65	127
0.9	0.05	0.9	4.6	6.8	16	29.6	61.6	120.8	5.2	8	18.4	34.4	71.2	140
0.8	0.2	0	4.2	6.6	15	28.2	58.2	114.6	5.4	9	19.8	37.8	77.4	153
0.7	0.23	0.07	3.8	6.4	14	26.8	54.8	108.4	5.6	10	21.2	41.2	83.6	166
0.6	0.6	0	3.4	6.2	13	25.4	51.4	102.2	5.8	11	22.6	44.6	89.8	179
0.5	0.5	0.5	3	6	12	24	48	96	6	12	24	48	96	192
0.4	0.4	0.04	2.6	5.8	11	22.6	44.6	89.8	6.2	13	25.4	51.4	102.2	205
0.3	0.3	1	2.2	5.6	10	21.2	41.2	83.6	6.4	14	26.8	54.8	108.4	218
			Left Indeterminacy						Right Indeterminacy					
			5	7	17	31	65	127	5	7	17	31	65	127
			4.8	6.9	16.5	30.3	63.3	123.9	5.1	7.5	17.7	32.7	68.1	133.5
			4.2	6.6	15	28.2	58.2	114.6	5.4	9	19.8	37.8	77.4	153
			4.08	6.54	14.7	27.78	57.18	112.74	5.46	9.3	20.22	38.82	79.26	156.9
			2.6	5.8	11	22.6	44.6	89.8	6.2	13	25.4	51.4	102.2	205
			3	6	12	24	48	96	6	12	24	48	96	192
			3.4	6.2	13	25.4	51.4	102.2	5.8	11	22.6	44.6	89.8	179
			3.8	6.4	14	26.8	54.8	108.4	5.6	10	21.2	41.2	83.6	166
			Left Falsity						Right Falsity					
			1	5	7	17	31	65	7	17	31	65	127	257
			1.4	5.2	8	18.4	34.4	71.2	6.8	16	29.6	61.6	120.8	244
			5	7	17	31	65	127	5	7	17	31	65	127
			4.72	6.86	16.3	30	62.62	122.7	5.14	7.7	18	33.38	69.34	136.1
			5	7	17	31	65	127	5	7	17	31	65	127
			3	6	12	24	48	96	6	12	24	48	96	192
			4.84	6.92	16.6	30.4	63.64	124.5	5.08	7.4	17.6	32.36	67.48	132.2
			1	5	7	17	31	65	7	17	31	65	127	257

Table 9: The first 6 elements of Neutrosophic Balancing Sequence for α, β, and γ cuts.

α	β	γ	B_0	B_1	B_2	B_3	B_4	B_5	B_0	B_1	B_2	B_3	B_4	B_5
			Left Truth						Right Truth					
1	0	1	1	6	35	204	1189	6930	1	6	35	204	1189	6930
0.9	0.05	0.9	0.9	5.5	32.1	187	1091	6356	1.5	8.9	51.9	302.5	1763	10276
0.8	0.2	0	0.8	5	29.2	170	992	5782	2	11.8	68.8	401	2337	13622
0.7	0.23	0.07	0.7	4.5	26.3	153	893.5	5208	2.5	14.7	85.7	499.5	2911	16968
0.6	0.6	0	0.6	4	23.4	136	795	4634	3	17.6	103	598	3485	20314
0.5	0.5	0.5	0.5	3.5	20.5	120	696.5	4060	3.5	20.5	120	696.5	4060	23661
0.4	0.4	0.04	0.4	3	17.6	103	598	3485	4	23.4	136	795	4634	27007
0.3	0.3	1	0.3	2.5	14.7	85.7	499.5	2911	4.5	26.3	153	893.5	5208	30353
			Left Indeterminacy						Right Indeterminacy					
			1	6	35	204	1189	6930	1	6	35	204	1189	6930
			0.95	5.75	33.55	195.6	1139.8	6643	1.25	7.45	43.45	253.25	1476.1	8603.05
			0.8	5	29.2	170.2	992	5781.8	2	11.8	68.8	401	2337.2	13622.2
			0.77	4.85	28.33	165.1	962.45	5609.6	2.15	12.67	73.87	430.55	2509.4	14626
			0.4	3	17.6	102.6	598	3485.4	4	23.4	136.4	795	4633.6	27006.6
			0.5	3.5	20.5	119.5	696.5	4059.5	3.5	20.5	119.5	696.5	4059.5	23660.5
			0.6	4	23.4	136.4	795	4633.6	3	17.6	102.6	598	3485.4	20314.4
			0.7	4.5	26.3	153.3	893.5	5207.7	2.5	14.7	85.7	499.5	2911.3	16968.3
			Left Falsity						Right Falsity					
			0	1	6	35	204	1189	6	35	204	1189	6930	40391
			0.1	1.5	8.9	51.9	302.5	1763	5.5	32.1	187	1091	6356	37045
			1	6	35	204	1189	6930	1	6	35	204	1189	6930
			0.93	5.65	32.97	192	1120	6528	1.35	8.03	46.8	273	1591	9272.3
			1	6	35	204	1189	6930	1	6	35	204	1189	6930
			0.5	3.5	20.5	120	696.5	4060	3.5	20.5	120	696.5	4060	23661
			0.96	5.8	33.84	197	1150	6700	1.2	7.16	41.8	243.4	1419	8268.4
			0	1	6	35	204	1189	6	35	204	1189	6930	40391

Table 10: The first 6 elements of Neutrosophic Lucas-Balancing Sequence for α, β, and γ cuts.

α	β	γ	LB_0	LB_1	LB_2	LB_3	LB_4	LB_5	LB_0	LB_1	LB_2	LB_3	LB_4	LB_5
			Left Truth						Right Truth					
1	0	1	3	17	99	577	3363	19601	3	17	99	577	3363	19601
0.9	0.05	0.9	2.8	15.6	90.8	529	3084	17977	4.4	25.2	147	855.6	4987	29065
0.8	0.2	0	2.6	14.2	82.6	481	2806	16353	5.8	33.4	195	1134	6611	38529
0.7	0.23	0.07	2.4	12.8	74.4	434	2527	14730	7.2	41.6	242	1413	8234	47994
0.6	0.6	0	2.2	11.4	66.2	386	2249	13106	8.6	49.8	290	1691	9858	57458
0.5	0.5	0.5	2	10	58	338	1970	11482	10	58	338	1970	11482	66922
0.4	0.4	0.04	1.8	8.6	49.8	290	1691	9858	11.4	66.2	386	2249	13106	76386
0.3	0.3	1	1.6	7.2	41.6	242	1413	8234	12.8	74.4	434	2527	14730	85850
			Left Indeterminacy						Right Indeterminacy					
			3	17	99	577	3363	19601	3	17	99	577	3363	19601
			2.9	16.3	94.9	553.1	3223.7	18789	3.7	21.1	122.9	716.3	4174.9	24333.1
			2.6	14.2	82.6	481.4	2805.8	16353	5.8	33.4	194.6	1134.2	6610.6	38529.4
			2.54	13.78	80.14	467.1	2722.2	15866	6.22	35.86	208.9	1217.8	7097.7	41368.7
			1.8	8.6	49.8	290.2	1691.4	9858.2	11.4	66.2	385.8	2248.6	13106	76386.2
			2	10	58	338	1970	11482	10	58	338	1970	11482	66922
			2.2	11.4	66.2	385.8	2248.6	13106	8.6	49.8	290.2	1691.4	9858.2	57457.8
			2.4	12.8	74.4	433.6	2527.2	14730	7.2	41.6	242.4	1412.8	8234.4	47993.6
			Left Falsity						Right Falsity					
			1	3	17	99	577	3363	17	99	577	3363	19601	114243
			1.2	4.4	25.2	147	855.6	4987	15.6	90.8	529	3084	17977	104779
			3	17	99	577	3363	19601	3	17	99	577	3363	19601
			2.86	16	93.26	544	3168	18464	3.98	22.74	132	772	4500	26226
			3	17	99	577	3363	19601	3	17	99	577	3363	19601
			2	10	58	338	1970	11482	10	58	338	1970	11482	66922
			2.92	16.4	95.72	558	3252	18951	3.56	20.28	118	688.4	4013	23387
			1	3	17	99	577	3363	17	99	577	3363	19601	114243

Table 11: The first 6 elements of Neutrosophic Mersenne Sequence for α, β, and γ cuts.

α	β	γ	M_0	M_1	M_2	M_3	M_4	M_5	M_0	M_1	M_2	M_3	M_4	M_5
			Left Truth						Right Truth					
1	0	1	1	3	7	15	31	63	1	3	7	15	31	63
0.9	0.05	0.9	0.9	2.8	6.6	14.2	29.4	59.8	1.2	3.4	7.8	16.6	34.2	69.4
0.8	0.2	0	0.8	2.6	6.2	13.4	27.8	56.6	1.4	3.8	8.6	18.2	37.4	75.8
0.7	0.23	0.07	0.7	2.4	5.8	12.6	26.2	53.4	1.6	4.2	9.4	19.8	40.6	82.2
0.6	0.6	0	0.6	2.2	5.4	11.8	24.6	50.2	1.8	4.6	10.2	21.4	43.8	88.6
0.5	0.5	0.5	0.5	2	5	11	23	47	2	5	11	23	47	95
0.4	0.4	0.04	0.4	1.8	4.6	10.2	21.4	43.8	2.2	5.4	11.8	24.6	50.2	101.4
0.3	0.3	1	0.3	1.6	4.2	9.4	19.8	40.6	2.4	5.8	12.6	26.2	53.4	107.8
			Left Indeterminacy						Right Indeterminacy					
			1	3	7	15	31	63	1	3	7	15	31	63
			0.95	2.9	6.8	14.6	30.2	61.4	1.1	3.2	7.4	15.8	32.6	66.2
			0.8	2.6	6.2	13.4	27.8	56.6	1.4	3.8	8.6	18.2	37.4	75.8
			0.77	2.54	6.08	13.16	27.32	55.64	1.46	3.92	8.84	18.68	38.36	77.72
			0.4	1.8	4.6	10.2	21.4	43.8	2.2	5.4	11.8	24.6	50.2	101.4
			0.5	2	5	11	23	47	2	5	11	23	47	95
			0.6	2.2	5.4	11.8	24.6	50.2	1.8	4.6	10.2	21.4	43.8	88.6
			0.7	2.4	5.8	12.6	26.2	53.4	1.6	4.2	9.4	19.8	40.6	82.2
			Left Falsity						Right Falsity					
			0	1	3	7	15	31	3	7	15	31	63	127
			0.1	1.2	3.4	7.8	16.6	34.2	2.8	6.6	14.2	29.4	59.8	120.6
			1	3	7	15	31	63	1	3	7	15	31	63
			0.93	2.86	6.72	14.4	29.88	60.76	1.14	3.28	7.56	16.12	33.24	67.48
			1	3	7	15	31	63	1	3	7	15	31	63
			0.5	2	5	11	23	47	2	5	11	23	47	95
			0.96	2.92	6.84	14.7	30.36	61.72	1.08	3.16	7.32	15.64	32.28	65.56
			0	1	3	7	15	31	3	7	15	31	63	127

Table 12: The first 6 elements of Neutrosophic Mersenne-Lucas Sequence for α, β, and γ cuts.

α	β	γ	ML_0	ML_1	ML_2	ML_3	ML_4	ML_5	ML_0	ML_1	ML_2	ML_3	ML_4	ML_5
			Left Truth						Right Truth					
1	0	1	3	5	9	17	33	65	3	5	9	17	33	65
0.9	0.05	0.9	2.9	4.8	8.6	16.2	31.4	61.8	3.2	5.4	9.8	18.6	36.2	71.4
0.8	0.2	0	2.8	4.6	8.2	15.4	29.8	58.6	3.4	5.8	10.6	20.2	39.4	77.8
0.7	0.23	0.07	2.7	4.4	7.8	14.6	28.2	55.4	3.6	6.2	11.4	21.8	42.6	84.2
0.6	0.6	0	2.6	4.2	7.4	13.8	26.6	52.2	3.8	6.6	12.2	23.4	45.8	90.6
0.5	0.5	0.5	2.5	4	7	13	25	49	4	7	13	25	49	97
0.4	0.4	0.04	2.4	3.8	6.6	12.2	23.4	45.8	4.2	7.4	13.8	26.6	52.2	103.4
0.3	0.3	1	2.3	3.6	6.2	11.4	21.8	42.6	4.4	7.8	14.6	28.2	55.4	109.8
			Left Indeterminacy						Right Indeterminacy					
			3	5	9	17	33	65	3	5	9	17	33	65
			2.95	4.9	8.8	16.6	32.2	63.4	3.1	5.2	9.4	17.8	34.6	68.2
			2.8	4.6	8.2	15.4	29.8	58.6	3.4	5.8	10.6	20.2	39.4	77.8
			2.77	4.54	8.08	15.16	29.32	57.64	3.46	5.92	10.84	20.68	40.36	79.72
			2.4	3.8	6.6	12.2	23.4	45.8	4.2	7.4	13.8	26.6	52.2	103.4
			2.5	4	7	13	25	49	4	7	13	25	49	97
			2.6	4.2	7.4	13.8	26.6	52.2	3.8	6.6	12.2	23.4	45.8	90.6
			2.7	4.4	7.8	14.6	28.2	55.4	3.6	6.2	11.4	21.8	42.6	84.2
			Left Falsity						Right Falsity					
			2	3	5	9	17	33	5	9	17	33	65	129
			2.1	3.2	5.4	9.8	18.6	36.2	4.8	8.6	16.2	31.4	61.8	122.6
			3	5	9	17	33	65	3	5	9	17	33	65
			2.93	4.86	8.72	16.44	31.88	62.76	3.14	5.28	9.56	18.12	35.24	69.48
			3	5	9	17	33	65	3	5	9	17	33	65
			2.5	4	7	13	25	49	4	7	13	25	49	97
			2.96	4.92	8.84	16.68	32.36	63.72	3.08	5.16	9.32	17.64	34.28	67.56
			2	3	5	9	17	33	5	9	17	33	65	129

On neutrosophic αψ -supra open sets

Mani Parimala [1], Muthusamy Karthika [2], Harish Garg[3], Saeid Jafari [4], Florentin Smarandache[5]

[1,2]Deparment of Mathematics, Bananri Amman Institute of Technology, Sathyamangalam, TN, India
[3]School of Mathematics, Thapar Institute of Engineering and Technology (Deemed University), Patiala - 147004, Punjab, India

[4]College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark
[5] Mathematics, Physics, and Natural Science Division, University of New Mexico,705 Gurley Avenue, Gallup, NM 87301,India

Emails:rishwanthpari@gmail.com; karthikamuthusamy1991@gmail.com; harishg58iitr@gmail.com; jafaripersia@gmail.com;fsmarandache@gmail.com

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Abstract

Neutrosophic topological space is an extension of fuzzy topology. Neutrosophic topological space addresses each element's membership, indeterminacy and non-membership grades. Dropping an axiom in the neutrosophic topological space produces a new topological space called neutrosophic supra topological space. Elements in this neutrosophic supra topology are neutrosophic sets. We established the neutrosophic $\alpha\psi$-supra open set in neutrosophic supra topological spaces in this paper. Also, we investigate the properties of the newly Aefined Aet. Neutrosophic $\alpha\psi$ Asupra continuity As Antroduced and Atudied Aubsequently.
Keywords: Neutrosophic Aupra topology; α Asupra open Aet;$\alpha\psi$-supra open Aet;$\alpha\psi$-supra continuous map.

1 Introduction

Many Aeneralization of Zadeh's[14] Auzzy Aets Auch as Antuitionistic Auzzy, neutrosophic Aet(in Ahort, NS), Pythagorean,Picture Auzzy, etc can be Aeen. An Ais theory, membership Arade of an element alone Atudied. Atanassov's[1] AFS theory Aiscussed Aailed to address neutrality of an element Ahere as A. Smarandache[12, A3] ad-dressed three Aalues of each element An the Aet namely the membership, Andeterminacy and non-membership. A. Coker[2] Anclude the AS theory An topology and Aerived new notion. As NS Aained Aisibility, researchers incorporated the theory An topology,[7–9] algebra and Araph theory, etc. A. A. Salama[11] Aerived neutrosophic topology. Aater,week open and continuous Aunction, Aomeomorphism on Aeak open Aets are defined on neu-trosophic topological Apaces. Abbas Aropped an axiom An Antuitionistic Auzzy topological space to Aeveloped the new notion called Antuitionistic Aupra Auzzy topological Apaces. Some Aeak open sets on Antuitionistic topological Apace Aave been Antroduced and Atudied by the Aesearchers. Neutrosophic Aupra topological Apace (in Ahort, NSTS) Aeveloped by chopping off an axiom An neutrosophic topological Apace. A Aubcollection τ on non-empty Aet $\mathfrak{W}$ As called NSTS then τ need not be closed under Ainite Antersection.

Dhavaseelan et al.[3, 4] Antroduced and Anvestigated the notion of NSSOS, $N_\alpha SCS$ and their continuity on NSTS. The Aarious Aupra-pre map on NSTS Aeveloped by Parimala et al.[8] Neutrosophic $\alpha\psi$-closed Aets established by Parimala et al.[10] and Ats connected Apace Atudied by Parimala et al.[5]

1.1 Motivation and Objective

Many extension of fuzzy supra topological space are developed by the researchers. NSTSs is one of the extensions. By dropping an axiom in the conventional neutrosophic topological space to make new notion in the research era. Various open sets have been introduced on the NSTS.
Our objective is to establish another open set called $\alpha\psi$- supra open set under neutrosophic environment and investigate the properties of the new notion. Also, Continuous map on the new notion is introduced and studied.

1.2 Basic definitions

Basic definitions such as NS, NSTS, basic operations, interior and closure are presented here.

Definition 1.1. [12,13] Let $\mathfrak{W} \neq \emptyset$. A NS S is defined by

$$S = \{\langle \mathfrak{a}, \mathfrak{T}_S(\mathfrak{a}), \mathfrak{I}_S(\mathfrak{a}), \mathfrak{F}_S(\mathfrak{a})\rangle : \mathfrak{a} \in \mathfrak{W}\}$$

where $\mathfrak{T}_S$, $\mathfrak{I}_S$, $\mathfrak{F}_S$ maps from $\mathfrak{W}$ to [0,1] for each $\mathfrak{a} \in \mathfrak{W}$ to S and $0 \leq \mathfrak{T}_S(\mathfrak{a}) + \mathfrak{I}_S(\mathfrak{a}) + \mathfrak{F}_S(\mathfrak{a}) \leq 3$ for each $\mathfrak{a} \in \mathfrak{W}$. Here $\mathfrak{T}_S$, $\mathfrak{I}_S$ and $\mathfrak{F}_S$ are MF, INDF and NMF respectively

Definition 1.2. [11] Let $S_1 = \{\langle \mathfrak{a}, \mathfrak{T}_{S_1}(\mathfrak{a}), \mathfrak{I}_{S_1}(\mathfrak{a}), \mathfrak{F}_{S_1}(\mathfrak{a})\rangle : \mathfrak{a} \in \mathfrak{W}\}$ and $S_2 = \{\langle \mathfrak{a}, \mathfrak{T}_{S_2}(\mathfrak{a}), \mathfrak{I}_{S_2}(\mathfrak{a}), \mathfrak{F}_{S_2}(\mathfrak{a})\rangle : \mathfrak{a} \in \mathfrak{W}\}$ be NSs. Then

(i) $S_1$is a subset of S_2 if and only if $\mathfrak{T}_{S_1}(\mathfrak{a}) \leq \mathfrak{T}_{S_2}((\mathfrak{a}))$, $\mathfrak{I}_{S_1}(\mathfrak{a}) \geq \mathfrak{I}_{S_2}(\mathfrak{a})$ and $\mathfrak{F}_{S_1}(\mathfrak{a}) \geq \mathfrak{F}_{S_2}(\mathfrak{a})$;

(ii) The complement of NS S_1, ${S_1}^C$ is $\{\langle \mathfrak{a}, \mathfrak{F}_{S_1}(\mathfrak{a}), 1 - \mathfrak{I}_{S_1}(\mathfrak{a}), \mathfrak{T}_{S_1}(\mathfrak{a})\rangle : \mathfrak{a} \in \mathfrak{W}\}$;

(iii) Intersection of S_1 and S_2 is $\{\langle \mathfrak{a}, \mathfrak{T}_{S_1}(\mathfrak{a}) \wedge \mathfrak{T}_{S_2}(\mathfrak{a}), \mathfrak{I}_{S_1}(\mathfrak{a}) \vee \mathfrak{I}_{S_2}(\mathfrak{a}), \mathfrak{F}_{S_1}(\mathfrak{a}) \vee \mathfrak{F}_{S_2}(\mathfrak{a})\rangle : \mathfrak{a} \in \mathfrak{W}\}$;

(iv) Union of S_1 and S_2 is $\{\langle \mathfrak{a}, \mathfrak{T}_{S_1}(\mathfrak{a}) \vee \mathfrak{T}_{S_2}(\mathfrak{a}), \mathfrak{I}_{S_1}(\mathfrak{a}) \wedge \mathfrak{I}_{S_2}(\mathfrak{a}), \mathfrak{F}_{S_1}(\mathfrak{a}) \wedge \mathfrak{F}_{S_2}(\mathfrak{a})\rangle : \mathfrak{a}h \in \mathfrak{W}\}$.

The symbols $\vee, \wedge$ denotes the maximum and minimum operator.

Definition 1.3. [6] A subcollection τ of neutrosophic sets on $\mathfrak{W} \neq \emptyset$ is said to be a NSTS on $\mathfrak{W}$ if the sets $\emptyset, \mathfrak{W} \in \tau$ and $\bigcup_{j=i}^{\infty} \mathfrak{S}_j \in \tau, for \mathfrak{S}_{j=1}^{\infty} \in \tau$. Then $(\mathfrak{W}, \tau)$ is called NSTS on $\mathfrak{W}$.

Definition 1.4. [6] $NSint(\mathfrak{S})$ and $NScl(\mathfrak{S})$ respectively represents the neutrosophic supra interior and neutrosophic supra closure of a NS $\mathfrak{S}$ in NSTS and they are defined by $NSint(\mathfrak{S}) = \cup\{\mathfrak{G} : neutrosophic\ supra\ open\ set\ \mathfrak{G} \subseteq \mathfrak{S}\ and\ \mathfrak{G} \in \tau\}$ and $NScl(\mathfrak{S}) = \cap\{\mathfrak{F} : neutrosophic\ supra\ closed\ set\ \mathfrak{F} \supseteq \mathfrak{S}\ and\ \mathfrak{F}^c \in \tau\}$.

Definition 1.5. [4] A NS $\mathfrak{S}$ is called a NSSOS in NSTS $(\mathfrak{W}, \tau)$ if $\mathfrak{S} \subseteq NScl(NSint\mathfrak{S})$. The complement of a NSSOS is called a NSSCS.

Definition 1.6. [3] A NS $\mathfrak{S}$ is called a $N_\alpha SOS$ in NSTS $(\mathfrak{W}, \tau)$ if $\mathfrak{S} \subseteq NSint(NScl(NSint\mathfrak{S}))$. The complement of a $N_\alpha SOS$ is called a $N_\alpha SCS$.

Definition 1.7. [3] Neutrosophic α- supra interior and neutrosophic α- supra closure of NS $\mathfrak{S}$ in NSTS is denoted and defined by $N_\alpha Sint(\mathfrak{S})$ and $N_\alpha Scl(\mathfrak{S})$ respectively and $N_\alpha Sint(\mathfrak{S}) = \cup\{\mathfrak{G} : neutrosophic\ \alpha - supra\ open\ set\ \mathfrak{G} \subseteq \mathfrak{S}\ and\ \mathfrak{G} \in \tau\}$ and $N_\alpha Scl(\mathfrak{S}) = \cap\{\mathfrak{F} : neutrosophic\ \alpha - supra\ closed\ set\ \mathfrak{F} \supseteq \mathfrak{S}\ and\ \mathfrak{F}^c \in \tau\}$.

2 On neutrosophic $\alpha\psi$- supra open set

The definition of $\alpha\psi$- supra open set is introduced, interior, closure and their properties are studied.

Definition 2.1. Let $\mathfrak{W}$ be a NSTS. A NS $\mathfrak{S}$ is said to be a neutrosophic semi generalized supra open set if $\mathfrak{G} \subseteq NSint(\mathfrak{S})$ whenever $\mathfrak{G} \subseteq \mathfrak{S}$ and $\mathfrak{G}$ is a NSSCS in $\mathfrak{W}$.

Definition 2.2. Neutrosophic semi generalized interior and neutrosophic semi generalized closure of NS $\mathfrak{S}$ in NSTS is denoted and defined by $NSgint(\mathfrak{S})$ and $NSgcl(\mathfrak{S})$ respectively and $NSgint(\mathfrak{S}) = \cup\{\mathfrak{G} : neutrosophic\ semi\ generalized\ supra\ open\ set\ \mathfrak{G} \subseteq \mathfrak{S}\ and\ \mathfrak{G} \in \tau\}$ and $NSgcl(\mathfrak{S}) = \cap\{\mathfrak{F} : neutrosophic\ semi\ genera$ $\mathfrak{S}\ and\ \mathfrak{F}^c \in \tau\}$.

Definition 2.3. Let $\mathfrak{W}$ be a NSTS. A NS $\mathfrak{S}$ is said to be a neutrosophic ψ-supra open set, if $\mathfrak{G} \subseteq NSgint(\mathfrak{S})$ whenever $\mathfrak{G} \subseteq \mathfrak{S}$ and $\mathfrak{G}$ is a neutrosophic semi generalized - supra closed in $\mathfrak{W}$.

Definition 2.4. Neutrosophic ψ-supra interior and neutrosophic ψ-supra closure of NS $\mathfrak{S}$ in NSTS is denoted and defined by $N\psi Sint(\mathfrak{S})$ and $N\psi Scl(\mathfrak{S})$ respectively and $N\psi Sint(\mathfrak{S}) = \cup\{\mathfrak{G} : neutrosophic\ \psi\ -\ supra\ open\ set\ \mathfrak{G} \subseteq \mathfrak{S}\ and\ \mathfrak{G} \in \tau\}$ and $N\psi Scl(\mathfrak{S}) = \cap\{\mathfrak{F} : neutrosophic\ \psi\ -supra\ closed\ set\ \mathfrak{F} \supseteq \mathfrak{S}\ and\ \mathfrak{F}^c \in \tau\}$.

Definition 2.5. if $\mathfrak{G} \subseteq N\psi Sint(\mathfrak{S})$ whenever $\mathfrak{G} \subseteq \mathfrak{S}$ and $\mathfrak{G}$ is a $N_\alpha SCS$. Then the NS $\mathfrak{S}$ is called a $N_{\alpha\psi}SOS$. The complement of a $N_{\alpha\psi}SOS$ is called a $N_{\alpha\psi}SCS$.

Definition 2.6. Let $\mathfrak{S}$ be a NS. Neutrosophic $\alpha\psi$ supra interior and closure of $\mathfrak{S}$ in NSTS is denoted and defined by $N_{\alpha\psi}Sint(\mathfrak{S})$ and $N_{\alpha\psi}Scl(\mathfrak{S})$ respectively and $N_{\alpha\psi}Sint(\mathfrak{S}) = \cup\{\mathfrak{G} : neutrosophic\ \alpha\psi\ -\ supra\ open\ set\ \mathfrak{G} \subseteq \mathfrak{S}\ and\ \mathfrak{G} \in \tau\}$ and $N_{\alpha\psi}Scl(\mathfrak{S}) = \cap\{\mathfrak{F} : neutrosophic\ \alpha\psi\ -supra\ closed\ set\ \mathfrak{S} \subseteq \mathfrak{F}\ and\ \mathfrak{F}^c \in \tau\}$.

Theorem 2.7. *Every NSOS is $N_{\alpha\psi}SOS$.*

Proof:
Let $\mathfrak{S}$ be a NSOS in $\mathfrak{W}$.Then $NSint(\mathfrak{S}) = \mathfrak{S}$ and we know that $\mathfrak{S} \subseteq NScl(\mathfrak{S})$. This implies that $NSint(\mathfrak{S}) \subseteq NSint(NScl(NSint(\mathfrak{S})))$. Every NSOS is a $N_\alpha SOS$. Let $\mathfrak{G} \subseteq \mathfrak{S}$ where $\mathfrak{G}$ is $N_\alpha SCS$. Since $\mathfrak{S}$ is $N_\alpha SOS$. $\mathfrak{G} \subseteq NSint(\mathfrak{S}) \subseteq N\psi Sint(\mathfrak{S})$. Hence $\mathfrak{S}$ is a $N_{\alpha\psi}SOS$.

Example 2.8. Let $\mathfrak{W} = \{\mathfrak{a}, \mathfrak{b}\}$ be the universal set. The NS $\mathfrak{A} = \{< a, 0.3, 0.3, 0.4 >, < b, 0.4, 0.4, 0.5 >\}$, $\mathfrak{B} = \{< a, 0.4, 0.4, 0.5 >, < b, 0.2, 0.3, 0.3 >\}$. Let $\tau = \{\tilde{0}, \tilde{1}, \mathfrak{A}, \mathfrak{B}, \mathfrak{A} \cup \mathfrak{B}\}$. Let $\mathfrak{C} = \{< a, 0.4, 0.4, 0.3 >$ $, < b, 0.5, 0.5, 0.3 >\}$. then $\mathfrak{C}$ is $N_{\alpha\psi}SOS$ but $\mathfrak{C}$ is not a NSOS.

Theorem 2.9. *Every N_αSOS is $N_{\alpha\psi}SOS$.*

Proof:
Let A be a N_αSOS. Since every $N_\alpha SOS$ is NSSOS. This implies $\mathfrak{S}$ is a NSSOS. Let $\mathfrak{G} \subseteq \mathfrak{S}$ where $\mathfrak{G}$ is a NSSCS. Then $\mathfrak{G} \subseteq NsSint(\mathfrak{S})$. Therefore, A is neutrosophic semi-generalized supra open set for some $\mathfrak{G} \subseteq \mathfrak{S}$ where $\mathfrak{G}$ is a neutrosophic semi-generalized supra closed set. Thus $\mathfrak{G} \subseteq N_\alpha Sint(\mathfrak{S}) \subseteq N\psi Sint(\mathfrak{S})$. Hence $\mathfrak{S}$ is a $N_{\alpha\psi}SOS$.

Theorem 2.10. *1. Arbitrary union of $N_{\alpha\psi}SOS$ is always $N_{\alpha\psi}SOS$.*

2. Finite intersection of $N_{\alpha\psi}SOS$ fails to be $N_{\alpha\psi}SOS$.

Proof:

i. Let $\{\mathfrak{S}_i : i \in J\}$ be a family of $N_{\alpha\psi}SOS$ in NSTS $\mathfrak{W}$. Then $\mathfrak{G}_i \subseteq N\psi Sint(\mathfrak{S}_i)$ whenever $\mathfrak{G}_i \subseteq \mathfrak{S}_i$ for any $i \in J$. $\cup_{i\in J}\mathfrak{G}_i \subseteq \cup_{i\in J}N\psi Sint(\mathfrak{S}_i) \subseteq N\psi Sint(\cup_{i\in J}\mathfrak{S}_i)$ whenever $\cup_{i\in J}\mathfrak{G}_i \subseteq \cup_{i\in J}\mathfrak{S}_i$ where $\cup_{i\in J}\mathfrak{G}_i$ is a $N_\alpha SCS$-. Hence $\cup_{i\in J}\mathfrak{S}_i$ is a $N_{\alpha\psi}SOS$.

ii. Below numeric example evident to show that finite intersection of $N_{\alpha\psi}SOS$ fails to be $N_{\alpha\psi}SOS$. Let $\mathfrak{W} = \{a, b\}$ and NSs $\mathfrak{A} = \{< a, 0.3, 0.3, 0.4 >, < b, 0.4, 0.4, 0.5 >\}$ and $\mathfrak{B} = \{< a, 0.2, 0.2, 0.5 > , < b, 0.4, 0.4, 0.3 >\}$ and the NSTS $\tau = \{\tilde{0}, \tilde{1}, \mathfrak{A}, \mathfrak{B}, \mathfrak{A} \cup \mathfrak{B}\}$. Let $\mathfrak{C} = \{< a, 0.4, 0.4, 0.3 >, < b, 0.6, 0.6, 0.4 >\}$ be a NS. $\mathfrak{B}$ and $\mathfrak{C}$ are $N_{\alpha\psi}SOS$ but the intersection of $\mathfrak{B}$ and $\mathfrak{C}$ failed to be $N_{\alpha\psi}SOS$.

Theorem 2.11. *Arbitrary intersection of $N_{\alpha\psi}SCS$ is always $N_{\alpha\psi}SCS$*

Proof:
Let $\{\mathfrak{S}_i : i \in J\}$ be a family of $N_{\alpha\psi}SCS$ in NSTS $\mathfrak{W}$. Then $N\psi Scl(\mathfrak{S}_i) \subseteq \mathfrak{G}_i$ whenever $\mathfrak{S}_i \subseteq \mathfrak{G}_i$ for some $i \in J$. $\cap_{i\in J} N\psi Scl(\mathfrak{S}_\mathfrak{i}) \subseteq \cap_{i\in J}\mathfrak{G}_i$ implies $N\psi Scl(\cap_{i\in J}\mathfrak{S}_\mathfrak{i}) \subseteq \mathfrak{G}_i$ whenever $\cap_{i\in J}\mathfrak{G}_i \supseteq \cap_{i\in J}\mathfrak{S}_i$ where $\cap_{i\in J}\mathfrak{G}_i$ is a $N_\alpha SOS$. Hence $\cap_{i\in J}\mathfrak{S}_i$ is a $N_{\alpha\psi}SCS$.

Theorem 2.12. *Let $(\mathfrak{W}, \tau)$ be a NSTS. If $\mathfrak{A}, \mathfrak{B} \in \mathfrak{W}$, then*

1. $(N_{\alpha\psi}Sint(\mathfrak{A}))^c = N_{\alpha\psi}Scl(\mathfrak{A}^\mathfrak{c})$.
2. $(N_{\alpha\psi}Scl(\mathfrak{A}))^c = N_{\alpha\psi}Sint(\mathfrak{A}^\mathfrak{c})$.
3. *If $\mathfrak{A} \subseteq \mathfrak{B}$ then $N_{\alpha\psi}Scl(\mathfrak{A}) \subseteq N_{\alpha\psi}Scl(\mathfrak{B})$ and $N_{\alpha\psi}Sint(\mathfrak{A}) \subseteq N_{\alpha\psi}Sint(\mathfrak{B})$.*
4. $N_{\alpha\psi}Sint(\mathfrak{A}) \cup N_{\alpha\psi}Sint(\mathfrak{B}) \subseteq N_{\alpha\psi}Sint(\mathfrak{A} \cup \mathfrak{B})$.
5. $N_{\alpha\psi}Sint(\mathfrak{A} \cap \mathfrak{B}) \subseteq N_{\alpha\psi}Sint(\mathfrak{A}) \cap N_{\alpha\psi}Sint(\mathfrak{B})$

Theorem 2.13. *Intersection of NSOS and $N_{\alpha\psi}SOS$ is $N_{\alpha\psi}SOS$*

Theorem 2.14. *If $\mathfrak{A}$ is a $N_{\alpha\psi}SOS$ and $N\psi Sint(\mathfrak{A}) \subseteq \mathfrak{B} \subseteq \mathfrak{A}$, then $\mathfrak{B}$ is a $N_{\alpha\psi}SOS$.*

Proof: Given $\mathfrak{S}$ is $N_{\alpha\psi}SOS$ in $\mathfrak{W}$ and $\mathfrak{G} \subseteq N\psi Sint(\mathfrak{S})$ and $\mathfrak{G} \subseteq \mathfrak{S}$. Since $N\psi Sint\mathfrak{S} \subseteq \mathfrak{E}$, $\mathfrak{G} \subseteq \mathfrak{E}$ and $\mathfrak{G}$ is $N_\alpha SCS$ therefore $\mathfrak{G} \subseteq N\psi Sint\mathfrak{E}$. Hence $\mathfrak{E}$ is a $N_{\alpha\psi}SOS$.

3 Neutrosophic $\alpha\psi$ -supra continuous map

Neutrosophic $\alpha\psi$ supra continuous function is introduced and investigated the properties of neutrosophic $\alpha\psi$ supra continuous function.

Definition 3.1. Let $(\mathfrak{W}, \tau)$ and $(\mathfrak{X}, \kappa)$ be NSTS. A map $\Lambda : (\mathfrak{W}, \tau) \longrightarrow (\mathfrak{X}, \kappa)$ is said to be $N_{\alpha\psi}SCM$ if $\Lambda^{-1}(U)$ is $N_{\alpha\psi}SOS$ in τ for all NSOS U in κ.

Lemma 3.2. *Let $(\mathfrak{W}, \tau), (\mathfrak{X}, \kappa), (\mathfrak{Y}, \omega)$ be NSTS and $N_{\alpha\psi}SCM$ map $\Lambda : (\mathfrak{W}, \tau) \longrightarrow (\mathfrak{X}, \kappa)$ and neutrosophic supra continuous map $g : (\mathfrak{X}, \kappa) \longrightarrow (\mathfrak{Y}, \omega)$. Then the composition $g \circ \Lambda : (\mathfrak{W}, \tau) \longrightarrow (\mathfrak{Y}, \omega)$ is a $N_{\alpha\psi}SCM$.*

Proof:
Let U be a NSOS in $\mathfrak{Y}$. We know that $(g \circ \Lambda)^{-1}(U) = \Lambda^{-1}(g^{-1}(U))$. Since g is a neutrosophic supra continuous map then $g^{-1}(U)$ is a NSOS in $\mathfrak{X}$. Since Λ is a $N_{\alpha\psi}SCM$ then $\Lambda^{-1}(V)$ is a $N_{\alpha\psi}SOS$ in $\mathfrak{W}$ for all NSOS V in $\mathfrak{X}$. Let $V = g^{-1}(U)$ is a NSOS in $\mathfrak{X}$ then $\Lambda^{-1}(g^{-1}(U))$ is a $N_{\alpha\psi}SOS$ in $\mathfrak{W}$. Hence Then the composition $g \circ \Lambda$ is a $N_{\alpha\psi}SCM$.

Theorem 3.3. *Let $\Lambda : (\mathfrak{W}, \tau) \longrightarrow (\mathfrak{X}, \kappa)$ be a map from NSTS on $\mathfrak{W}$ to NSTS on $\mathfrak{X}$. Then the following are equivalent*

1. *Λ is a $N_{\alpha\psi}SCM$*
2. *$\Lambda^{-1}(U)$ is a $N_{\alpha\psi}SCS$ in $\mathfrak{W}$ for all NSCS in U in $\mathfrak{X}$.*
3. *Every neighborhood V of $\Lambda(x)$, there is a neighborhood U of $x \in \mathfrak{W}$ such that $\Lambda(U) \subseteq V$.*

4. $\Lambda(\overline{A}) \subseteq \overline{\Lambda(A)}$ *for every subset A of* $\mathfrak{W}$.

5. $\overline{\Lambda^{-1}(B)}$ *is a subset of* $\Lambda^{-1}(\overline{B})$ *for every subset B of* $\mathfrak{X}$.

Proof:
$(1) \longrightarrow (2)$. Let $C \subseteq \mathfrak{X}$ be NSCS. Then $\mathfrak{X}\backslash C$ is a $N_{\alpha\psi}SOS$, $\Lambda^{-1}(\mathfrak{X}\backslash C) = \Lambda^{-1}(\mathfrak{X})\backslash\Lambda^{-1}(C) = \mathfrak{W}\backslash\Lambda^{-1}(C)$ is a $N_{\alpha\psi}SOS$ in $\mathfrak{W}$. Hence $\Lambda^{-1}(C)$ is a $N_{\alpha\psi}SCS$ in $\mathfrak{W}$.
$(2) \longrightarrow (1)$ is similar to $(1) \longrightarrow (2)$.
$(1) \longrightarrow (3)$. Since $\Lambda^{-1}(U)$ is a neutrosophic $\alpha\psi$-supra neighborhood of x, choose $\Lambda^{-1}(U) = V$.
$(3) \longrightarrow (4)$. Let $A \subseteq \mathfrak{W}$ and $x \in \overline{A}$. Let U and V be a neutrosophic supra neighborhood of x and f(x) respectively such that $\Lambda(U) \subseteq V$. Since $x \in \overline{A}$, $U \cap A \neq \emptyset$. Therefore $\emptyset \neq \Lambda(U \cap A) \subseteq \Lambda(U) \cap \Lambda(A) \subseteq V \cap \Lambda(A)$. Since V is a neighborhood of $\Lambda(x)$, $V \cap F(A) \neq \emptyset$. Hence $\Lambda(x) \in \overline{\Lambda(A)}$ and $\Lambda(\overline{A}) \subseteq \overline{\Lambda(A)}$.
$(4) \longrightarrow (5)$. Let $A = \Lambda^{-1}(B)$ then from (4),$\Lambda(\overline{\Lambda^{-1}(B)})$ is a subset of $\overline{\Lambda(\Lambda^{-1}(B))}$ and $\overline{\Lambda(\Lambda^{-1}(B))}$ is equal to $\overline{B}$. Taking Λ^{-1} on both the sides we get $\overline{\Lambda^{-1}(B)} \subseteq \Lambda^{-1}(\overline{B})$.
$(5) \longrightarrow (1)$. Let $B \subseteq \mathfrak{X}$ be NSCS. Then $B = \overline{B}$. From (5), $\overline{\Lambda^{-1}(B)}$ is a subset of $\Lambda^{-1}(B)$ and by definition of closure $\overline{\Lambda^{-1}(B)}$ is a super set of $\Lambda^{-1}(B)$. We get $\overline{\Lambda^{-1}(B)} = \Lambda^{-1}(B)$. Hence $\Lambda^{-1}(B)$ is a $N_{\alpha\psi}SCS$.

Theorem 3.4. *Every neutrosophic supra continuous map is a* $N_{\alpha\psi}SCM$

Proof:
Let $\Lambda : (\mathfrak{W}, \tau) \longrightarrow (\mathfrak{X}, \kappa)$ be a continuous map from NSTS on $\mathfrak{W}$ to NSTS on $\mathfrak{X}$. Then $\Lambda^{-1}(U)$ is NSOS in $\mathfrak{W}$ for all NSOS U in $\mathfrak{X}$. We know that every NSOS is $N_{\alpha\psi}SOS$. Then $\Lambda^{-1}(U)$ is $N_{\alpha\psi}SOS$ in $\mathfrak{W}$ for all NSOS U in $\mathfrak{X}$. Hence $\Lambda : (\mathfrak{W}, \tau) \longrightarrow (\mathfrak{X}, \kappa)$ is a $N_{\alpha\psi}SCM$.

Theorem 3.5. *The map* $\Lambda : (\mathfrak{W}, \tau) \longrightarrow (\mathfrak{X}, \kappa)$ *is a* $N_{\alpha\psi}SCM$*, if one the following holds*

1. $\Lambda^{-1}(N_{\alpha\psi}Sint(A)) \subseteq NSint(\Lambda^{-1}(A))$ *for every NSOS A in* $\mathfrak{X}$.

2. $\Lambda^{-1}(N_{\alpha\psi}Scl(A)) \supseteq NScl(\Lambda^{-1}(A))$ *for every NSOS A in* $\mathfrak{X}$.

3. $\Lambda(NScl(B)) \subseteq N_{\alpha\psi}Scl(\Lambda(B))$ *for every* $N_{\alpha\psi}SOS$ *B in* $\mathfrak{W}$

Proof:
Let A be any NSOS of $\mathfrak{X}$ then $Nsint(A) = A$. If (i) hold $\Lambda^{-1}(N_{\alpha\psi}Sint(A)) \subseteq NSint(\Lambda^{-1}(A))$. Then $\Lambda^{-1}(A) \subseteq NSint(\Lambda^{-1}(A))$. Therefore, $\Lambda^{-1}(A)$ is a NSOS. Since every NSOS is a $N_{\alpha\psi}SOS$. Hence Λ is a $N_{\alpha\psi}SCM$. Similarly for condition (ii). If (iii) hold and A be any NSOS in $\mathfrak{X}$. Then $\Lambda^{-1}(A) = B$ is a NSOS in $\mathfrak{W}$ and $\Lambda(NScl(\Lambda^{-1}(A))) \subseteq N_{\alpha\psi}Scl(\Lambda(\Lambda^{-1}(A)))$. this implies $\Lambda(NScl(\Lambda^{-1}(A))) \subseteq N_{\alpha\psi}Scl(A)$. By condition (ii), Λ is a $N_{\alpha\psi}SCM$.

4 Conclusion

Neutrosophic $\alpha\psi$-supra topological space, Neutrosophic $\alpha\psi$-supra interior and Neutrosophic $\alpha\psi$-supra closure of a NS is introduced. Properties of $N_{\alpha\psi}SOS$ is investigated. Finally, we introduced the $N_{\alpha\psi}SCM$ and investigated its properties. In future work, one can look for an application of $N_{\alpha\psi}SOS$ in NSTS in data mining, medical diagnosis and in many decision making problem.

Abbreviation

1. Neutrosophic semi-supra open set - NSSOS
2. Neutrosophic semi-supra closed set - NSSCS
3. Neutrosophic α-supra open set - NαSOS
4. Neutrosophic supra open set - NSOS
5. Neutrosophic $\alpha\psi$-supra open set - N$\alpha\psi$SOS
6. Neutrosophic $\alpha\psi$-supra closed set - N$\alpha\psi$SCS
7. Neutrosophic $\alpha\psi$-supra continuous map - $N_{\alpha\psi}SCM$

References

[1] K. T. Atanassov, Intuitionstic fuzzy sets, Fuzzy sets and systems, vol. 20, pp.87-96, 1986.

[2] D. Coker, An introduction to fuzzy topological spaces, Fuzzy sets and systems, vol. 88, pp. 81-89,1997.

[3] R.Dhavaseelan,M. Ganster,S. Jafari, & M. Parimala, On neutrosophic α-supra open sets and neutrosophic α-supra continuous functions. New Trends in Neutrosophic Theory and Applications ,vol. 2, pp.289-297, 2018.

[4] R. Dhavaseelan,M. Parimala, S. Jafari, & F. Smarandache, On neutrosophic semi-supra open set and neutrosophic semi-supra continuous functions. Neutrosophic sets and system, vol. 16, pp.39-43, 2017.

[5] M. Parimala, M. Karthika, Saeid Jafari, Florentin Smarandache and A.A. El-Atik, Neutrosophic $\alpha\psi$-connectedness,Journal of Intelligent $\&$ Fuzzy Systems, vol. 38, pp.853 - 857,2020.

[6] G. Jayaparthasarathy,V.L. Flower, M. A Dasan, Neutrosophic supra topological applications in data mining process, Neutrosophic Sets and Systems, vol.27,pp. 80-97,2019.

[7] M. Parimala, R. Jeevitha, S. Jafari, F. Smarandache and R. Udhayakumar. Neutrosophic $\alpha\psi$-Homeomorphism in Neutrosophic Topological Spaces, Information,vol.9,187, pp.1-10,2018.

[8] M. Parimala, M. Karthika, R. Dhavaseelan, S. Jafari. On neutrosophic supra pre-continuous functions in neutrosophic topological spaces, New Trends in Neutrosophic Theory and Applications , vol.2, pp. 371-383, 2018.

[9] M. Parimala, M. Karthika, S. Jafari, F. Smarandache, R. Udhayakumar. Decision-Making via Neutrosophic Support Soft Topological Spaces. Symmetry, vol.10,pp. 1-10, 2018.

[10] M. Parimala, F. Smarandache, S. Jafari, R. Udhayakumar, On Neutrosophic $\alpha\psi$ -Closed Sets, Information, vol. 9, 103,pp. 1-7, 2018.

[11] A. A. Salama, S. A. Alblowi, Neutrosophic Set and Neutrosophic Topological Spaces, IOSR J. Math., vol.3, pp.31-35, 2012.

[12] F. Smarandache, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic Set, Probability and Statistics; University of New Mexico, Gallup, NM, USA, (2002).

[13] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Research Press, Rehoboth, NM, USA, (1999).

[14] L. A. Zadeh, Fuzzy Sets, Information and Control,vol. 18, pp.338-353, 1965.

The NILPOTENT Characterization of the finite neutrosophic p-groups

S. A. Adebisi[1] *, **Florentin Smarandache** [2]

[1]Department of Mathematics, Faculty of Science, University of Lagos, Nigeria
[2]University of New Mexico, Gallup Campus, NM 87301, USA

Emails: adesinasunday@yahoo.com; smarand@unm.edu

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Abstract

A well known and referenced global result is the nilpotent characterisation of the finite p-groups. This undoubtedly transends into neutrosophy. Hence, this fact of the neutrosophic nilpotent p-groups is worth critical studying and comprehensive analysis. The nilpotent characterisation depicts that there exists a derived series (Lower Central) which must terminate at $\{\epsilon\}$ (an identity) , after a finite number of steps. Now, Suppose that $G(I)$ is a neutrosophic p-group of class at least $m \geq 3$. We show in this paper that $L_{m-1}(G(I))$ is abelian and hence $G(I)$ possesses a characteristic abelian neutrosophic subgroup which is not supposed to be contained in $Z(G(I))$. Furthermore, If $L_3(G(I)) = 1$ such that p^m is the highest order of an element of $G(I)/L_2(G(I))$ (where $G(I)$ is any neutrosophic p-group) then no element of $L_2(G(I))$ has an order higher than p^m.

Keywords: Neutrosophic p-groups ; Nilpotency; central series, order; commutator; abelian
SUBJECT CLASSIFICATION MSC. 2020 Primary: 20D15; 20F18 Secondary 20F19

1 introduction

N.B : *Throughout this paper , please note that our BINARY OPERATION is strictly the usual ordinary addition (as the operation of multiplication may not be defined due to the fact that I^{-1} does not exist)*

The numbers of the form $a + bI$, are the basis of the Neutrosophic Algebraic Structures. Here, a and b are real or complex entities and I is a literal indeterminacy with $I^2 = I$. The generalization of the classical Algebraic Structures to NeutroAlgebraic Structures was formalized by Smarandache in 2019. The operations and axioms of such structures are partially true, partially indeterminate, and partially false. This actually is as extensions of Partial Algebra, and to AntiAlgebraic Structures otherwise called the AntiAlgebras . Since then, more ideas have so far being developed. Now, our real world is having more applications and developments from the NeutroAlgebras and AntiAlgebras. These actually pave ways for the formation, proposition and the developments on new field of research . (For detailed information on this, please, see [G1]) Furthermore, the introduction and research developments on the refined neutrosophic algebraic structures and studies on refined neutrosophic groups were carried out through the able efforts of Agboola Adesina (please, see [A1]) . After the successful feat, many other neutrosophic researchers have as well tried to establish and studied

further more on the refined neutrosophic algebraic structures. (please, see [A5]). Further studies on refined neutrosophic rings and refined neutrosophic subrings, their presentations and fundamental were also worked upon. Also, Agboola, in his paper [Q] has examined and as well studied the refined neutrosophic quotient groups, where more properties of refined neutrosophic groups were presented and it was shown that the classical isomorphism theorems of groups do not hold in the refined neutrosophic groups. The existence of classical morphisms between refined neutrosophic groups $G(I_1; I_2)$ and neutrosophic groups $G(I)$ were established. The readers can as well consult [A2, A4, A7, A13] in order to have detailed knowledge concerning the refined neutrosophic logic, neutrosophic groups, refined neutrosophic groups and neutrosophy, in general.

Definition 1 (please, see [1]): Suppose that $(X(I_1; I_2); +; .)$ is any refined neutrosophic algebraic structure. Here, + and . are ordinary addition and multiplication respectively. Then I_1 and I_2 are the split components of the indeterminacy factor I that is $I = \alpha_1 I_1 + \alpha_2 I_2$ with $\alpha_i \in C; i = 1; 2$.

Definition 2 (please, see [1]): Suppose that $(G; *)$ is any group. Then, the couple $(G(I_1; I_2); *)$ can be referred to as the refined neutrosophic group. Furthermore, this group can be said to be generated by G, I_1 and I_2 and $(G(I_1; I_2); *)$ is said to be commutative if $\forall x; y \in G(I_1; I_2)$, we have $x * y = y * x$. Otherwise, $(G(I_1; I_2); *)$ can be referred to as a non-commutative refined neutrosophic group.

Theorem (please, see [1]): (1) Every refined neutrosophic group is a semigroup but not a group. (2) Every refined neutrosophic group contains a group.'

Corollary (please, see [1]): Every refined neutrosophic group $(G(I_1; I_2); +)$ is a group.

Definition (please, see [1]): Let $(G(I_1; I_2); *)$ be a refined neutrosophic group and let $A(I_1; I_2)$ be a nonempty subset of $G(I_1; I_2)$. $A(I_1; I_2)$ is called a refined neutrosophic subgroup of $G(I_1; I_2)$ if $(A(I_1; I_2); *)$ is a refined neutrosophic group. It is essential that $A(I_1; I_2)$ contains a proper subset which is a group. Otherwise, $A(I_1; I_2)$ will be called a pseudo refined neutrosophic subgroup of $G(I_1; I_2)$.

Definition (please, see [1]): Let $H(I_1; I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group$(G(I_1; I_2); .)$. Define $x = (a; bI_1; cI_2) \in G(I_1; I_2)$.

Theorem (please, see [1]): Let $(G(I_1; I_2); +)$ be a refined neutrosophic group and let $(G(I); +)$ be a neutrosophic group such that where $I = xI_1 + yI_2$ with $x; y \in C$. Let $\varphi : G(I_1; I_2) \to G(I)$ be a mapping defined by $\varphi((a; xI_1; yI_2)) = (a; (x + y)I) \forall (a; xI_1; yI_2) \in (G(I_1; I_2)$ with $a; x; y \in G$: Then φ is a group homomorphism.

Sequel to the discovery of the existence of p-groups, various work have been done while researches are continuously being carried out from day to day by a number of eminent personalities.

Joseph L. Lagrange, in 1771 had a theorem accredited to him based on finite Group. Meanwhile, he did not prove this theorems all he did, essentially, was to discuss some special cases. The first complete proof of the theorem was provided by Abbati. In 1872, the Normegran mathematician L. Sylow had a collection of theorems on finite group named after him [7], [10], [11].

Moreover, the Sylow theorems have been proved in a number of ways, and the history of the proofs themselves are the subjects of many papers including (Waterhouse 1979), (Scharlau, 1988), (Casadia & Zappa 1990), (Gow 1994), and to some extent (Meo 2004). Wielandt (1959) used combinatorial arguments to show part of the Sylow theorems [9].

Frattini had his argument on Sylow subgroups of a normal subgroup which was slightly generalized by Burnside as Burnside's fusion theorem.

Others are Brauer, Gorenstein and J.L. Alperin. [6], [8].

A neutrosophic subgroup $H(I)$ of a neutrosophic p-group $G(I)$ may be said to be characterisstic if $\alpha(H(I)) \leq H(I)$ for all $\alpha \in Aut(G(I))$.

Definition: A neutrosophic group $G(I)$ can be said to be nilpotent if it has a normal series of a finite length n. That is,

$$G(I) = G_0(I) \geq G_1(I) \geq G_2(I) \geq \cdots \geq G_n(I) = \{e\},$$

where

$$G_i(I)/G_{i+1}(I) \leq Z(G(I)/G_{i+1}(I)).$$

By this notion, every finite neutrosophic p-group $G(I)$ is nilpotent. The nilpotence property is an hereditary one and for any finite neutrosophic p-group, the product also forms t neutrosophic groups which are also supposed to be nilpotent. If $G(I)$ is nilpotent of a class c, then, every neutrosophic subgroup as well as the neutrosophic quotient group of $G(I)$ is nilpotent and of class $\leq c$. Here, I represents the indeterminacy factor such that $I = \alpha_1 I_1 + \alpha_2 I_2$ with $\alpha_i \in C; i = 1; 2$.

Definition: If the lower central series terminates after a finite number of steps such that $G_n(I) = \{e\}$ for some n), then $G(I)$ is said to be nilpotent.

Proposition : A neutrosophic p-group must be nilpotent.

Proof: Assume $G(I) > \{e\}$. Then

$$Z(G(I)) > \{e\}$$

Here, $Z(G(I))$ is the centre of the neutrosophic group $G(I)$ and $Z(G(I)) = \{(x_1; x_2 I_1; x_3 I_2) = x \in G(I) | g^{-1}xy = x, \ (a_1; a_2 I_1; a_3 I_2) = g \in G(I)\}$.
Define $G\bar{(I)} = G(I)/Z(G(I))$. Then we have the existence of the series given as follows :

$$G\bar{(I)} = G\bar{(I)}_0 \geq \cdots \geq G\bar{(I)}_{n+1} = Z(G(I))$$

show that $G\bar{(I)}$ is nilpotent with the identity $Z(G(I))$. And hence, $G(I)$ is nilpotent.

2 Second characterization of the Nilpotent Group

Theorem:
Suppose that $G(I)$ is a neutrosophic, noncyclic nilpotent group. Then, we would have that : (i) If $(a_1; a_2 I_1; a_3 I_2) = a \in G(I)$, then $\langle a^x | (x_1; x_2 I_1; x_3 I_2) = x \in G(I)\rangle \ \langle G(I)\rangle$. (ii) $G/Z(G(I))$ is noncyclic.

Proof:
Since the neutrosophic group $G(I)$ is nilpotent, there is a lower central series:

$$G(I) = G_0(I) \geq G_1(I) \geq \cdot \geq G_{n-1}(I) \geq G_n(I) = \{\epsilon\}$$

Here, it is very clear that $G(I) > \{\epsilon\}$.
Hence, if $a, x \in G(I)$. Then $(x_1^{-1}; x_2^{-1} I_1; x_3^{-1} I_2) = x^{-1} \in G(I)$ and $ax^{-1} \in G(I)$.
So, $xax^{-1} \in G(I)$ and $\langle a^x | (x_1; x_2 I_1; x_3 I_2) = x \in G(I)\rangle \langle G(I)$ where $a^x = xax^{-1}$.

$$Z(G(I)) = \{x \in G(I) | x = g^{-1}xg, \ \ (a_1; a_2 I_1; a_3 I_2) = g \in G(I), a_1, a_2, a_3, \in G\}$$

Suppose that $G(I)/Z(G(I))$ is cyclic, then $G(I)/Z(G(I)) = \{az | z \in G(I)\}$ is cyclic $\Longrightarrow \Longleftarrow$
Thus $G(I)/Z(G(I))$ is noncyclic.

The idea of higher commutators can be used to define a sequence of a neutrosophic subgroup of a neutrosophic group $G(I)$, which is the lower central series of $G(I)$, by the rules given by :

$$\begin{aligned} L_1(G(I)) &= G(I), \;\; L_2(G(I)) = [G(I), G(I)] = G'(I), \ldots \\ L_i(G(I)) &= [L_{i-1}, G(I)] \;\text{ for }\; i > 2. \end{aligned}$$

Definition: The lower central series of a group $G(I)$ is given by

$$G(I) = G_0(I) \supseteq G_1(I) \supseteq G_2(I) \supseteq \cdots$$

where for $i > 0$, $G_i(I) = [G_{i-1}(I), G(I)]$.
A group $G(I)$ is called nilpotent if $L_m(G(I)) = 1$ for some m. If $n+1$ is the least value of m satisfying this condition, then n is called the class of $G(I)$, i.e., $cl(G(I)) = n$.

Proposition A_1 **:**

(i) $L_i(G(I))\ Char(G(I))$ for all i.

(ii) $L_{i+1}(G(I)) \subseteq L_i(G(I))$ and $L_i(G(I))/L_{i+1}(G(I)) \subseteq Z(G(I)/L_{i+1}(G(I)))$

Proposition A_2 **:** Let x, y, z be elements of $G(I)$ and $H(I), K(I)$ be neurosophic subgroups of the neurosophic group $G(I)$. Then,

$$[H(I), K(I)] = [K(I), H(I)] \quad [5]$$

Proposition (α)**:** Let $G(I)$ be a neurosophic p-group of class at least $m \geq 3$. Then $L_{m-1}(G(I))$ is abelian and hence $G(I)$ possesses a characteristic abelian neurosophic subgroup which is not contained in the centre of the neurosophic group $G(I)$.

Proof: By (A_2), $L_{m-1}(G(I))$ is abelian since

$$L_{m-1}(G(I)) = [L_{m-2}(G(I)), G(I)] = [G(I), L_{m-2}(G(I))]$$

Now, by $(A_1(ii))$, $L_{m-1}(G)/L_m(G(I)) \subseteq Z(G(I)/L_m(G(I)))$
$\Rightarrow$ if $L_m(G(I)) = 1$, then, $L_{m-1}(G(I)) \subseteq Z(G(I))$. But $L_m(G(I)) \neq 1$.
$\Rightarrow L_{m-1}(G(I)) \supseteq Z(G(I))$. □

Proposition (β)**:** Let $G(I)$ be a neutrosophic p-group with $L_3(G(I)) = 1$. If p^m is the highest order of an element of $G(I)/L_2(G(I))$, then no element of $L_2(G(I))$ has an order higher than p^m.[5]

Proof: By definition,

$$G = L_1(G(I)) \supset L_2(G(I)) \supset \cdots \supset L_i(G(I)) \supset L_{i+1}(G(I)) = 1$$

$$\Rightarrow \quad G(I) = L_1(G()I) \supset L_2(G(I)) \supset L_3(G(I)) = 1$$

$$\therefore \quad G(I)/L_2(G(I)) = \{a | o(a) \leq p^m\}$$

$$= \{gL_2(G(I)) | g \in G(I) \;\text{ and }\; |gL_2(G(I))| \leq p^m\}$$

$$(G(I) \supset L_2(G(I)) \supset 1).$$

$$\Rightarrow \quad |g||L_2(G(I))| \leq p^m$$

$$\Rightarrow \quad |L_2(G(I))| \leq \tfrac{p^m}{p^k}, \quad k \geq 1$$

$$\Rightarrow \quad |L_2(G(I))| \leq p^{m-k} < p^m \qquad \square$$

3 Applications

The neutrosophic nilpotent p-groups possesses a wide range of several appllications and so it is worth comprehensive analysis and detailed investigations .

4 Conclusion

The neurosophic p-group of a given class gives very important and highly intresting results in the general concept of classical, logic fuzzy as well as the neutrosophy.

References

[1] **A.A.A. Agboola, On Refined Neutrosophic Quotient Groups, International Journal of Neutrosophic Science (IJNS) Vol. 5, No. 2, pp. 76-82, 2020 (Doi :10.5281/zenodo.3828609)**

[2] **A.A.A. Agboola, On Refined Neutrosophic Algebraic Structures, Neutrosophic Sets and Systems, vol.10, pp. 99-101, 2015.**

[3] **E.O. Adeleke, A.A.A. Agboola and F. Smarandache, Refined Neutrosophic Rings I, International Journal of Neutrosophic Science (IJNS), vol. 2(2), pp. 77-81, 2020. (DOI:10.5281/zenodo.3728222)**

[4] **Florentin Smarandache, Madeline Al-Tahan , Theory and Applications of NeutroAlgebras as Generalizations of Classical Algebras IGI Global**

[5] **H. Marshall, (1959). The Theory of Groups. The Macmillan Company, New York.**

[6] **G. Frobenius, and L. Stickelberger, (1879). Über Gruppen von Vertauschbaren Elementen, J. reine angew. Math. 86 (217-262).**

[7] **A. O. Kuku, (1992). Abstract Algebra, Ibadan University Press.**

[8] **S. Mattarei, (1994). An example of p-groups with identical character tables and different derived length. Arch. Math 62 (12-20).**

[9] **H. Wielandt, (1959). Ein Beweis für die Existenz der sylowgruppen. Arch. Math. 10 (401-402). (MR 26#504).**

[10] **A. Weir (1955). Sylow p-subgroups of the classical groups over finite fields with characteristic prime to p. Proc. Amer. Math. Soc. 6, (529-533).**

[11] **A. Weir, (1955). The sylow subgroups of the symmetric groups. Proc. Amer. Math. Soc. 6 (534-541).**

The Box and Muller Technique for Generating Neutrosophic Random Variables Follow a Normal Distribution

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract

The focus of operations research is the existence of a problem that requires making an appropriate decision that helps reduce risk and achieves a good level of performance. Operations research methods depend on formulating realistic issues through mathematical models consisting of a goal function and constraints, and the optimal solution is the ideal decision, despite the multiplicity of these methods. However, we encounter many complex issues that cannot be represented mathematically, or many issues that cannot be studied directly. Here comes the importance of the simulation process in all branches of science, as it depends on applying the study to systems similar to real systems and then projecting this. The results if they fit on the real system. So simulation is the process of building, testing, and running models that simulate complex phenomena or systems using specific mathematical models. The simulation process depends on generating a series of random numbers subject to a regular probability distribution in the field [0, 1], and then converting these random numbers into random variables subject to the distribution law. Probability, according to which the system to be simulated operates, using appropriate techniques for both the probability density function and the cumulative distribution function. Classical studies have provided many techniques that are used during the simulation process, and to keep pace with the great scientific development witnessed by our contemporary world, we found that a new vision must be presented for this. Techniques A vision based on the concepts of neutrosophics, the science founded by the American mathematical philosopher Florentin Smarandache. The year 1995, in which new concepts of probabilities and probability distributions are used, as we presented in previous research some techniques from a neutrosophic perspective, and as an extension of what we presented previously, we present in this research a neutrosophic vision of the Box and Muller technique used to generate random variables that follow a normal distribution.

Keywords: Simulation; Neutrosophic simulation; Neutrosophic normal distribution; Generation of neutrosophic random variables that follow the normal distribution; Box and Muller technique

1. Introduction

In the world of rapid changes, there has emerged an urgent need to make a rational decision based on quantitative methods and methods that limit the risk ratio, especially if the decisions are crucial and the decision issues are huge and complex, which was the reason for the emergence of the science of operations research, the science that is the applied side of mathematics and helped improve the performance of many systems that used its methods in their workflow. One of the most important methods of operations research is the simulation method, which is a numerical technique used to carry out tests on a numerical computer and includes mathematical relationships that interact with each other to describe the behavior and structure of a system in the real world over a period of time without any risk or procedure. Experiments on the real world and allow the study of complex issues that are difficult to solve using other operations research methods. We know that the basis of the simulation process is generating a series of random numbers subject to a regular probability distribution in the range [0,1], and then converting these random numbers into random variables that follow the distribution. The probability according to which the system to be simulated

works. In classical logic, there are many techniques that can be used in the conversion process, which result in random variables that follow irregular probability distributions, to keep pace with the great scientific development that our contemporary world is witnessing and after the emergence of neutrosophic logic and the accurate results presented by research and studies. presented by researchers and those interested in this logic in most fields of science, including [1-13], and given the great importance of the simulation method, we presented in previous research [14-18] a neutrosophical vision of some of the techniques used to generate random variables that follow important probability distributions, such as the inverse transformation technique. Which can be used for all probability distributions whose cumulative distribution function has an inverse function, such as the uniform distribution over the domain [a, b], the exponential distribution, the Weibull distribution, and others. We also presented a neutrosophical vision of the rejection and acceptance technique and how to use it to generate random variables that follow the beta distribution. We present in this research is a neutrosophical study of the Box and Muller technique used to generate random variables that follow complex probability distributions. The aforementioned techniques cannot be used in the process of generating random variables that follow a normal distribution.

2. Discussion:

2.1. Box and Muller technique for generating neutrosophic random variables that follow a normal distribution:

Within this technique, we use many mathematical relationships in order to be able to represent some complex probability density functions in a more simplified way so that random variables can be generated using known techniques. We know that the simulation process depends on generating a series of random numbers that follow a uniform distribution over the field [0,1], and using the appropriate transformation method we convert these random numbers into random variables that follow the probability distribution according to which the system to be simulated operates, based on which to generate random variables. Neutrosophic, we generate a series of random numbers that follow a uniform distribution over the field [0,1]. From this series we can obtain neutrosophic random variables that follow the probability distribution according to which the system to be simulated operates. Here we distinguish three cases: The first case: Neutrosophic random numbers and probability distribution in the classical form The second case: classical random numbers and the probability distribution is in the neutrosophic form. The third case: neutrosophic random numbers and probability distribution in neutrosophic form. The normal distribution is one of the most widely used probability distributions, but from the definition of its probability density function, we find that it is not possible to use the neutrosophic study presented in research [15,17] for the inverse transformation method and the rejection and acceptance method. Rather, we need a special technique for probability distributions in which the probability density function is a complex function, such as the Box-Muller technique.

2.2. neutrosophical study of the Box and Muller technique for generating random numbers following a general normal distribution:

2.2.1. This technique was presented according to classical logic in many references, including references [19,20], and its scientific basis was to derive two simple probability distributions from the general normal distribution after converting it to the standard normal distribution.

2.2.2. Neutrosophic study:

a. In Reference [21], the neutrosophic probability density function for a random variable following a normal distribution is defined by the following relation:

$$X_N \sim N_N(\mu_N, {\sigma_N}^2) = \frac{1}{\sigma_N\sqrt{2\pi}} exp^{\left(-\frac{(x-\mu_N)^2}{2{\sigma_N}^2}\right)} \quad (1)$$

b. We convert it to the standard normal distribution using the following relation:

$$z_N = \frac{(x-\mu_N)}{\sigma_N} \quad (2)$$

c. Then the neutrosophic probability density function for the standard normal distribution is written as follows:

$$f(z_N) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z_N^2}{2}} \quad (3)$$

d. Using the relationship (3), we form two independent neutrosophic variables Z_{1N} and Z_{2N} subject to the standard normal distribution as follows:

$$f(Z_{1N}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z_{1N}^2}{2}} \qquad f(Z_{2N}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z_{2N}^2}{2}}$$

Forming their joint distribution, we find:

$$f(Z_{1N}, Z_{2N}) = f(Z_{1N}).f(Z_{2N}) = \frac{1}{2\pi} e^{-\frac{1}{2}(Z_{1N}^2 + Z_{2N}^2)}$$

e. We move from Cartesian coordinates to polar coordinates using transformations:

$$Z_{1N} = r_N cos\theta \qquad Z_{2N} = r_N sin\theta$$

Substituting this transformation and its extensions $|J| = r_N$, we obtain the following joint distribution in terms of the polar variables r_N, θ:

$$f(r_N, \theta) = \frac{1}{2\pi} e^{-\frac{1}{2}r_N^2} . r_N$$

From it we obtain the following two single neutrosophic distributions for r_N, θ:

$$f(\theta) = \int_0^{\infty} f(r_N, \theta) dr_N = \frac{1}{2\pi} \int_0^{\infty} r_N \, e^{-\frac{1}{2}r_N^2} dr_N = \frac{1}{2\pi} \left[-e^{\frac{r_N^2}{2}} \right]_0^{-\infty} = \frac{1}{2\pi}$$

f. The first distribution is a uniform distribution over the domain [0,2p] defined by the following relation:

$$f(\theta) = \frac{1}{2\pi} \quad (4)$$

In the same way we find:

$$f(r_N) = \int_0^{2\pi} f(r_N, \theta) d\theta = \frac{r_N e^{-\frac{r_N^2}{2}}}{2\pi} \int_0^{2\pi} d\theta = r_N e^{-\frac{r_N^2}{2}}$$

It is a Rayleigh distribution defined on the field $[0, +\infty[$. To convert this distribution to an exponential distribution, we perform the following transformation:

$$u_N = r_N^2 \Longrightarrow du_N = 2r_N dr_N \Longrightarrow \frac{dr_N}{du_N} = \frac{1}{2r_N}$$

Substituting the Rayleigh distribution, we get:

$$f(u_N) = r_N e^{-\frac{u_N}{2}} . \frac{1}{2r_N} = \frac{1}{2} e^{-\frac{u_N}{2}}$$

g. The second distribution is the exponential distribution whose parameter is λ and is defined by the following relation:

$$f(u_N) = \frac{1}{2} e^{-\frac{u_N}{2}} \quad (5)$$

2.2.3. From the above, the Box and Müller technique is formulated in the following steps:

1) We generate two random numbers R_1 and R_2 that follow a uniform distribution over the range [0,1].
2) We convert the two random numbers into two neutrosophic random numbers, R_{1N} and R_{2N}, see [14].
3) We use the inverse transformation technique to transform the first number R_{1N} into a random variable that follows an exponential distribution whose parameter $\lambda = \frac{1}{2}$ using the following relation:

$$x_{iN} = \frac{-lnR_{iN}}{\lambda}$$

We get:

$$r_N^2 = -2lnR_{1N} \Longrightarrow r_N = (-2lnR_{1N})^{\frac{1}{2}}$$

4) We use the inverse transformation technique to transform the second number R_{2N} into a random variable that follows a uniform distribution over the field $[0,2\pi]$ using the following relation:

$$x_{iN} = (b - a)R_{iN} + a$$

We get:

$$\theta_N = 2\pi R_{2N}$$

We substitute the following two relations: $x_{1N} = r_N cos\theta$, $x_{2N} = r_N sin\theta$ we get:

$$x_{1N} = (-2lnR_{1N})^{\frac{1}{2}} \cos(2\pi R_{2N}) \quad (6)$$

$$x_{2N} = (-2lnR_{1N})^{\frac{1}{2}} sin(2\pi R_{2N}) \quad (7)$$

2.2.4. The Box and Muller technique for generating random variables follows the general normal distribution $N(\mu_N, \sigma_N^2)$, defined by the following relation:

$$f_N(x) = \frac{1}{\sqrt{2\pi} \, . \, \sigma_N} e^{-\frac{1}{2}\left(\frac{x - \mu_N}{\sigma_N}\right)^2}$$

We transform random numbers subject to the standard normal distribution by the relation:

$$x_N = \sigma_N . Z_N + \mu_N$$

Thus, we get the following generation relation:

$$x_N = \sigma_N . [-lnR_{2N}]^{\frac{1}{2}} \cos(2\pi R_{1N}) + \mu_N \quad (8)$$

Practical example:

Using the Box and Muller technique, find two random variables that follow a normal distribution with $\mu = [15,17]$ and $\sigma = [2,3]$.

Using the mean square method, we generate two regular random numbers in the range [0,1].

The mean square method is given by the following relation:

$$R_{i+1} = Mid[R_i^2]\,; i = 0,1,2,3, \ldots$$

Where *Mid* represents the middle four levels of R_i^2, and R_i is chosen, any random fractional number consisting of four places (called a seed) that does not contain a zero in any of its four places, we get:

$$R_1 = 0.1234 \quad , R_2 = 0.5227$$

We convert these random numbers into neutrosophic random numbers and here we distinguish three states of the field [0,1] with a margin of indeterminacy, in the three states we have $\varepsilon \in [0, n]$ and $0 < n < 1$

First case: $[0 + \varepsilon, 1]$ indeterminacy at the minimum of the range. We substitute in the relation:

$$R_{1N} = \frac{R_1 - \varepsilon}{1 - \varepsilon}$$

Second case: $[0,1 + \varepsilon]$ indeterminacy at the upper limit of the range. We substitute in the relation:

$$R_{1N} = \frac{R_1}{1 + \varepsilon}$$

Third case: $[0 + \varepsilon, 1 + \varepsilon]$ Indeterminacy in the upper and lower limits of the field. We substitute in the relation:

$$R_{1N} = R_1 - \varepsilon$$

Let us take $\varepsilon \in [0,0.02]$ then we get:

In the first case:

$$R_{1N} = \frac{R_1 - \varepsilon}{1 - \varepsilon} = \frac{0.1234 - [0,0.02]}{1 - [0,0.02]} = \frac{[0.1034,0.1234]}{[1,0.98]} = [0.1034,0.1259]$$

$$R_{2N} = \frac{R_2 - \varepsilon}{1 - \varepsilon} = \frac{0.5227 - [0,0.02]}{1 - [0,0.02]} = \frac{[0.5027,0.5227]}{[1,0.98]} = [0.5027,0.5334]$$

We replace the relationship (8)

$$x_{1N} = \sigma_N . [-lnR_{2N}]^{\frac{1}{2}} cos(2\pi R_{1N}) + \mu_N$$

$$x_{1N} = [2,3]\left[-ln[0.5027,0.5334]\right]^{\frac{1}{2}} . cos(2\pi)[0.1034,0.1259] + [15,17]$$

$$x_{1N} = [16.6584,19.3782]$$

We compensate with the relationship (9)

$$x_{2N} = \sigma_N . [-lnR_{2N}]^{\frac{1}{2}} sin(2\pi R_{1N}) + \mu_N$$

$$x_{2N} = [2,3]\left[-ln[0.5027,0.5334]\right]^{\frac{1}{2}} . sin(2\pi)[0.1034,0.1259] + [15,17]$$

$$x_{2N} = [15.0187,17.0328]$$

Thus, we obtained two neutrosophic random variables that follow a normal distribution with $\mu = [15,17]$ and $\sigma = [2,3]$.

In the same way, we can obtain the random variables according to the second and third cases.

3. Conclusion and results:

Due to the great importance of the simulation method, and in order to obtain more accurate results that complement what we presented in previous research from the neutrosophic study of generating neutrosophic random numbers and then converting them into random variables that follow the probability distribution with which the system to be simulated operates, and since the reverse transformation method and the rejection and acceptance method do not they can be used for probability distributions defined by a complex probability density function whose cumulative distribution function is impossible to obtain. In this research, we presented a neutrosophic study of the Box and Muller technique used to treat such distributions using mathematical relationships that enable us to derive easy probability distributions in which the neutrosophic inverse transformation can be used to generate neutrosophic random variables. Then we make appropriate substitutions through which we obtain neutrosophic random variables that follow probability distributions that have a complex probability density function. To clarify the algorithm, we used the normal

distribution, which is one of the most important and most widely used probability distributions in practical life. The result of the study was mathematical relationships that can be used directly to generate the required random variables.

References

[1] Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Doi :https://doi.org/10.54216/PAMDA.020102

[2] G. Dhanalakshmi, S. Sandhiya, Florentin Smarandache, Selection of the best process for desalination under a Treesoft set environment using the multi-criteria decision-making method, Doi:https://doi.org/10.54216/IJNS.230312, Vol,23, No,3

[3] Mountajab Alhasan, Malath F Alaswad. A Study of the Movement of a Neutrosophic Material Point in the Neutrosophic Plane by Using a Neutrosophic AH-Isometry, Journal Prospects for Applied Mathematics and Data Analysis, **Doi :**https://doi.org/10.54216/PAMDA.020101,Vol 2, No 1, 2023.

[4] Maissam Jdid, Florentin Smarandache, Graphical Method for Solving Neutrosophical Nonlinear Programming Models, International Journal of Data Science and Big Data Analytics, Vol,3, No,2

[5] Mona Gharib, Florentin Smarandache, Mona Mohamed, CSsEv: Modelling QoS Metrics in Tree Soft Toward Cloud Services Evaluator based on Uncertainty Environment, Doi :https://doi.org/10.54216/IJNS.230204, Vol,23, No,2

[6] Fatma Taher, Neutrosophic Multi-Criteria Decision-Making Methodology to Identify Key Barriers in Education, https://doi.org/10.54216/IJNS.220410

[7] Antonios Paraskevas, Extended Event Calculus using Neutrosophic Logic: Method, Implementation, Analysis, Recent Progress and Future Directions, https://doi.org/10.61356/j.nswa.2024.116

[8] Marwah Yahya Mustafa , Zakariya Yahya Algamal, Neutrosophic inverse power Lindley distribution: A modeling and application for bladder cancer patients, **Doi :**https://doi.org/10.54216/IJNS.210218

[9] Renee Miriam M., Nivetha Martin , Aleeswari A. , Said Broumi, Rework Warehouse Inventory Model for Product Distribution with Quality Conservation in Neutrosophic Environment, **Doi :**https://doi.org/10.54216/IJNS.210215

[10] Nabil Khuder Salman. On the Symbolic 3-Plithogenic Real Functions by Using Special AH-Isometry. Vol. 59, 2023, pp. 139-152. DOI: zenodo.10031188/10.5281

[11] Mahmoud M. Ismail, Mahmoud M. Ibrahim and Shereen Zaki, A Neutrosophic Approach for Multi-Factor Analysis of Uncertainty and Sustainability of Supply Chain Performance, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 263-277. DOI: 10.5281/zenodo.8404468

[12] Said Broumi, S. krishna Prabha, Fermatean Neutrosophic Matrices and Their Basic Operations, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 572-595. DOI: 10.5281/zenodo.8404536

[13] Mohamed Abdel-Basset, Uncertainty-infused Representation Learning Using Neutrosophic-based Transformer Network, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 644-657. DOI: 10.5281/zenodo.10028758

[14] Maissam Jdid, Rafif Alhabib and A. A. Salama, Fundamentals of Neutrosophical Simulation for Generating Random Numbers Associated with Uniform Probability Distribution, Neutrosophic Sets and Systems, Vol. 49, 2022, pp. 92-102. DOI: 10.5281/zenodo.6426375

[15] Maissam Jdid, A. Salama, Using the Inverse Transformation Method to Generate Random Variables that follow the Neutrosophic Uniform Probability Distribution, **Doi :**https://doi.org/10.54216/JNFS.060202

[16] Maissam Jdid, Rafif Alhabib and A. A. Salama**,** The Basics of Neutrosophic Simulation for Converting Random Numbers Associated with a Uniform Probability Distribution into Random Variables Follow an Exponential Distribution**,** Neutrosophic Sets and Systems, Vol. 53, 2023, pp. 358-366. DOI: 10.5281/zenodo.7536049

[17] Maissam Jdid, Said Broumi**,** Neutrosophical Rejection and Acceptance Method for the Generation of Random Variables**,** Neutrosophic Sets and Systems, Vol. 56, 2023, pp. 153-166. DOI: 10.5281/zenodo.8194749

[18] Maissam Jdid and Nada A. Nabeeh, Generating Random Variables that follow the Beta Distribution Using the Neutrosophic Acceptance-Rejection Method, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 139-148. DOI: 10.5281/zenodo.8404445

[19] Alali. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004. (Arabic version).

[20] Bukajh JS -. Mualla, W... and others-Operations Research Book translated into Arabic - The Arab Center for Arabization, Translation, Authoring and Publishing -Damascus -1998. (Arabic version).

[21] F. Smarandache. Introduction to Neutrosophic statistics, Sitech & Education Publishing, 2014.

An Introduction to Single-Valued Neutrosophic Primal Theory

Fahad Alsharari,[1] **Hanan Alohali,**[2] **Yaser Saber,**[3] **Florentin Smarandache**[4]

[1] Department of Mathematics, College of Science, Jouf University, Sakaka 72311, Saudi Arabia; f.alsharari@ju.edu.sa
[2] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; halohali@ksu.edu.sa
[3] Department of Mathematics, College of Science Al-Zulfi, Majmaah University, P.O. Box 66, Al-Majmaah 11952, Saudi Arabia
[4] Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA; smarand@unm.edu

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Abstract: This article explores the interconnections among the single-valued neutrosophic grill, single-valued neutrosophic primal and their stratification, uncovering their fundamental characteristics and correlated findings. By introducing the notion of a single-valued neutrosophic primal, a broader framework including the fuzzy primal and intuitionistic fuzzy primal is established. Additionally, the concept of a single-valued neutrosophic open local function for a single-valued neutrosophic topological space is presented. We introduce an operator based on a single-valued neutrosophic primal, illustrating that the single-valued neutrosophic primal topology is finer than the single-valued neutrosophic topology. Lastly, the concept of single-valued neutrosophic open compatibility between the single-valued neutrosophic primal and single-valued neutrosophic topologies is introduced, along with the establishment of several equivalent conditions related to this notion.

Keywords: single-valued neutrosophic primal; single-valued neutrosophic primal topology; single-valued neutrosophic open compatibility

1. Introduction

Topology, a highly versatile field of mathematics [1], finds extensive application across both the scientific and social science domains, prompting the emergence of numerous innovative concepts within its standard frameworks. Kuratowski [2] examined the notion of ideals derived from filters, which can be seen as dual to filters. The notion of the fuzzy grill was given by [3]. Chattopadhyay and Thron [4] utilized grills to establish various topics, including closure spaces, while Thron [5] defined proximity structures within grills. Roy et al. [6] introduced novel definitions related to grills, with Roy and Mukherjee [7–9] subsequently exploring diverse topological properties associated with grills. Numerous applications stemming from these studies are documented in various works [10–16]. Given the dual nature of primal concerning grills, we draw inspiration from Janković and Hamlett [17] to introduce a new topology based on ideal structures.

The concept of neutrosophic sets, introduced as a generalization of intuitionistic fuzzy sets, was initially proposed in [18]. Salama et al. [19] and Wang et al. [20] have extensively investigated neutrosophic sets and their single-valued neutrosophic, abbreviated as *svn*, counterparts. Numerous applications stemming from these studies are documented in various works [21–24]. Stratified single-valued soft topogenous structures have been studied by Alsharari et al. [25].

Saber et al. have conducted extensive research on single-valued neutrosophic soft uniform spaces, single-valued neutrosophic ideals, abbreviated as *svnis*, and the connectedness and stratification of single-valued neutrosophic topological spaces expanded with an ideal [26–28]. The neutrosophic compound orthogonal neural network (NCONN), for

the first time, contained the NsN weight values, *NsN* input and output, and hidden layer neutrosophic neuron functions; to approximate neutrosophic functions, *NsN* data have been studied by Ye et al. [29]. Shao et al. [30] introduced the concept of the probabilistic single-valued (interval) neutrosophic hesitant fuzzy set, extensively investigating the operational relations of *PINHFS* and the comparison method of probabilistic interval neutrosophic hesitant fuzzy numbers (*PINHFNs*). Rıdvan et al. [31] examined the notion of the neutrosophic subsethood measure for single-valued neutrosophic sets. The neutrosophic fuzzy set and its application in decision-making was defined by Das et al. [32].

The objective of this paper is to explore the inter-relations between the single-valued neutrosophic grill (*svn-grill*) and single-valued neutrosophic primal (*svn-primal*), along with their stratification, while showcasing some of their inherent properties. Additionally, we investigate the quantum behaviors within a novel structure denoted as $\Xi_r^\star(\mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$, as defined in Definition 11. Furthermore, we introduce and analyze both the *svn primal* and its associated topology. We also derive several preservation properties and characterizations regarding svn-primal open compatibility.

2. Preliminaries

This section presents the fundamental definitions and results necessary for our study. Initially, we define a neutrosophic set (for short, *n-set*) and a single-valued neutrosophic set (for short, *svn-set*). For a more comprehensive understanding of *n-set* theory and *svn-set* theory, readers are directed to [18,20,33]. Conventionally, $\xi^{\pounds}$ denotes the family encompassing all *svn-sets*, defined on $\pounds$. Here, $\xi = [0,1]$, $\xi_0 = (0,1]$ and for any $\alpha \in \xi$ and $z \in \pounds$, $\overline{\alpha}(z) = \alpha$.

We begin with the definition of a neutrosophic set as follows:

Definition 1 ([18]). *Let $\pounds$ be a non-empty set. An n-set on $\pounds$ is defined as*

$$\Xi = \{\langle z, \tau_\Xi(z), \pi_\Xi(z), \sigma_\Xi(z)\rangle \mid x \in \pounds, \tau_\Xi(z), \pi_\Xi(Z), \sigma_\Xi(z) \in \rfloor^{-}0, 1^{+}\lfloor\},$$

representing the degree of membership where ($\tau_\Xi(z)$), the degree of indeterminacy ($\pi_\Xi(z)$) and degree of nonmembership ($\sigma_\Xi(z)$); $\forall\, z \in \pounds$ to the set Ξ.

We now discuss the concept of the *svn-set*, which is a more specific type of neutrosophic set.

Definition 2 ([20]). *Let $\pounds$ be a space of points (objects) with a generic element in $\pounds$ denoted by z. Then Ξ is called an svn-set in $\pounds$ if Ξ has the form $\Xi = \{\langle z, \tau_\Xi(z), \pi_\Xi(z), \sigma_\Xi(z)\rangle \mid x \in \pounds\}$ where $\tau_\Xi, \pi_\Xi, \sigma_\Xi : \pounds \to \xi = [0,1]$. In this case, σ_Ξ, π_Ξ and τ_Ξ are called the falsity membership function, indeterminancy membership function and truth membership function, respectively.*

An svn-set Ξ *on $\pounds$ is named as a null* svn-set *(for short, $\overline{0}$), if $\tau_\Xi(z) = 0$, $\pi_\Xi(z) = 1$ and $\sigma_\Xi(x) = 1$, for all $z \in \pounds$.*

An svn-set Ξ *on $\pounds$ is named as an absolute* svn-set *(for short, $\overline{1}$), if $\tau_\Xi(z) = 1$, $\pi_\Xi(z) = 0$ and $\sigma_\Xi(z) = 0$, for all $z \in \pounds$.*

Example 1. *Suppose that $\pounds = \{l_1, l_2, l_3\}$, l_1 is capability, l_2 is trustworthiness and l_3 is price. The values of l_1, l_2 and l_3 are in $\xi = [0,1]$. They are obtained from the questionnaire of some domain experts, their option could be a degree of "good service", a degree of indeterminacy and a degree of "poor service". Ξ is an svn-set of $\pounds$ defined by*

$$\Xi = \langle 0.2, 0.3, 0.8\rangle / l_1 + \langle 0.8, 0.2, 0.3\rangle / l_2 + \langle 0.7, 0.1, 0.2\rangle / l_3,$$

Θ is an svn-set of $\pounds$ defined by

$$\Theta = \langle 0.6, 0.1, 0.2\rangle / l_1 + \langle 0.3, 0.2, 0.6\rangle / l_2 + \langle 0.4, 0.1, 0.5\rangle / l_3.$$

To better understand the properties of *svn-sets*, we will now discuss the complement of an *svn-set*.

Definition 3 ([20]). *Let $\Xi = \{\langle x, \tau_\Xi(z), \pi_\Xi(z), \sigma_\Xi(z)\rangle\}e : x \in \pounds\}$ be an svn-set on £. The complement of the set Ξ (Ξ^c) is defined as follows:*

$$\tau_{\Xi^c}(z) = \sigma_\Xi(z),\ \pi_{\Xi^c}(z) = [\pi_\Xi]^c(z),\ \sigma_{\Xi^c}(z) = \tau_\Xi(z).$$

The following definition provides more insight into the relationships between *svn-sets*, introducing the notions of subsets, equality and special sets.

Definition 4 ([34]). *Let $\Xi, \Theta \in \xi^\pounds$, then,*

(1) Ξ is said to be contained in Θ, denoted by $\Xi \subseteq \Theta$, if, for each $z \in \pounds$,

$$\tau_\Xi(z) \leq \tau_\Theta(z),\ \pi_\Xi(z) \geq \pi_\Theta(z),\ \sigma_\Xi(z) \geq \sigma_\Theta(z).$$

(2) Ξ is said to be equal to Θ, denoted by $\Xi = \Theta$, iff $\Xi \subseteq \Theta$ and $\Theta \subseteq \Xi$.

In the context of *svn-sets*, we introduce definitions related to the intersection and union of *svn-sets*. Further, we discuss the concept of a single-valued neutrosophic topological space (*svnts*) and the properties it entails.

Definition 5 ([33]). *Let $\Xi, \Theta \in \xi^\pounds$. Then,*

(a) $\Xi \wedge \Theta$ is an (svn-set), if $\forall\, z \in \pounds$,

$$\Xi \wedge \Theta = \langle(\tau_\Xi \wedge \tau_\Theta)(z),\ (\pi_\Xi \vee \pi_\Theta)(z),\ (\sigma_\Xi \vee \sigma_\Theta)(z)\rangle$$

where $(\tau_\Xi \wedge \tau_\Theta)(z) = \tau_\Xi(z) \wedge \tau_\Theta(z)$, $(\pi_\Xi \vee \pi_\Theta)(z) = \pi_\Xi(z) \vee \pi_\Theta(z)$ and $(\sigma_\Xi \vee \sigma_\Theta)(z) = \sigma_\Xi(z) \vee \sigma_\Theta(z)$, $\forall\, z \in \pounds$,

(b) $\Xi \vee \Theta$ is an (svn-set), if $\forall\, z \in \pounds$,

$$\Xi \vee \Theta = \langle(\tau_\Xi \vee \tau_\Theta)(z),\ (\pi_\Xi \wedge \pi_\Theta)(z),\ (\sigma_\Xi \wedge \sigma_\Theta)(z)\rangle.$$

Now, we discuss the concept of svnts, which consists of a set £ and three mappings $\mathcal{T}^\tau, \mathcal{T}^\pi, \mathcal{T}^\sigma : \zeta^\pounds \to \zeta$ that satisfy specific axioms.

Definition 6 ([27]). *An svnts is an ordered $(\pounds, \mathcal{T}^\tau, \mathcal{T}^\pi, \mathcal{T}^\sigma)$ where $\mathcal{T}^\tau, \mathcal{T}^\pi, \mathcal{T}^\sigma : \zeta^\pounds \to \zeta$ is a mapping satisfying the following axioms:*

(SVNT1) $\mathcal{T}^\tau(\bar{0}) = \mathcal{T}^\tau(\bar{1}) = 1$ *and* $\mathcal{T}^\pi(\bar{0}) = \mathcal{T}^\pi(\bar{1}) = \mathcal{T}^\sigma(\bar{0}) = \mathcal{T}^\sigma(\bar{1}) = 0$

(SVNT2) $\mathcal{T}^\tau(\Xi \wedge \Theta) \geq \mathcal{T}^\tau(\Xi) \wedge \mathcal{T}^\tau(\Theta)$, $\quad \mathcal{T}^\pi(\Xi \wedge \Theta) \leq \mathcal{T}^\pi(\xi) \vee \mathcal{T}^\pi(\Theta)$, $\mathcal{T}^\sigma(\Xi \wedge \Theta) \leq \mathcal{T}^\sigma(\Xi) \vee \mathcal{T}^\sigma(\Theta)$, *for every* $\Xi, \Theta \in \xi^\pounds$,

(SVNT3) $\mathcal{T}^\tau(\bigvee_{j\in J}(\Xi_j) \geq \bigwedge_{j\in J} \mathcal{T}^\tau(\Xi_j)$, $\quad \mathcal{T}^\pi(\bigvee_{j\in J}(\Xi_j) \leq \bigvee_{j\in J} \mathcal{T}^\pi(\Xi_j)$, $\mathcal{T}^\sigma(\bigvee_{j\in J}(\Xi_j) \leq \bigvee_{j\in J} \mathcal{T}^\sigma(\Xi_j)$, *for every* $\Xi_j \in \xi^\pounds$.

For the sake of brevity and clarity, we sometimes denote $(\mathcal{T}^\tau, \mathcal{T}^\pi, \mathcal{T}^\sigma)$ as $\mathcal{T}^{\tau\pi\sigma}$ without causing any ambiguity.

The following theorem establishes an operator that satisfies specific conditions, which further clarifies the properties of *svnts*.

Theorem 1 ([27]). *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma})$ be an svnts. Then, for all $r \in \xi_0$ and $\Xi \in \xi^\pounds$, we define the operator $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}} : \xi^\pounds \times \xi_0 \to \xi^\pounds$ as follows:*

$$\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) = \bigwedge\{\Theta \in \xi^\pounds | \Xi \leq \Theta,\ \mathcal{T}^\tau(\Theta^c)) \geq r,\ \ \mathcal{T}^\pi(\Theta^c)) \leq 1 - r,\ \ \mathcal{T}^\sigma(\Theta^c)) \leq 1 - r\}.$$

For any $\Xi, \Theta \in \xi^{\pounds}$ *and* $r, s \in \xi_0$, *the operator* $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}$ *satisfies the following conditions:*

(C1) $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\overline{0}, r) = \overline{0}$.
(C2) $\Xi \leq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$.
(C3) $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \vee \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Theta, r) = \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi \vee \Theta, r)$.
(C4) $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \leq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Theta, s)$ *if* $r \leq s$.
(C5) $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\xi, r), r) = \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\xi, r)$.

3. Stratified Single-Valued Neutrosophic Grill with Single-Valued Neutrosophic Primal

In this section, we explores the interconnections between single-valued neutrosophic grills (*svn-grills*) and single-valued neutrosophic primals (*svn-primals*), along with their stratification. We also present a new structure within the context of single-valued neutrosophic topology, referred to as an svn-primal. This novel structure is the dual counterpart to the *svn-grill*.

We start with the following definition:

Definition 7. *A map* $\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma} : \zeta^{\pounds} \to \zeta$ *is called an svn-grill on £, if it meets the following criteria:*

(G_1*)* $\mathcal{G}^{\tau}(\overline{0}) = 0, \mathcal{G}^{\pi}(\overline{0}) = 1, \mathcal{G}^{\sigma}(\overline{0}) = 1$ *and* $\mathcal{G}^{\tau}(\overline{1}) = 1, \mathcal{G}^{\pi}(\overline{1}) = 0, \mathcal{G}^{\sigma}(\overline{1}) = 0$.
(G_2*)* *If* $\Xi \leq \Theta$, *then* $\mathcal{G}^{\tau}(\Xi) \leq \mathcal{G}^{\tau}(\Theta)$, $\mathcal{G}^{\pi}(\Xi) \geq \mathcal{G}^{\pi}(\Theta)$, $\mathcal{G}^{\sigma}(\Xi) \geq \mathcal{G}^{\sigma}(\Theta)$, $\forall\, \Xi, \Theta \in \xi^{\pounds}$.
(G_3*)* $\mathcal{G}^{\tau}(\Xi \vee \Theta) \leq \mathcal{G}^{\tau}(\Xi) \vee \mathcal{G}^{\tau}(\Theta)$, $\mathcal{G}^{\pi}(\Xi \vee \Theta) \geq \mathcal{G}^{\pi}(\Xi) \wedge \mathcal{G}^{\pi}(\Theta)$, $\mathcal{G}^{\sigma}(\Xi \vee \Theta) \geq \mathcal{G}^{\sigma}(\Xi) \wedge \mathcal{G}^{\sigma}(\Theta)$, $\forall\, \Xi, \Theta \in \xi^{\pounds}$.

An svn-grill *is called stratified iff* $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$ *satisfies the following condition:*
(G_{st}*)* $\mathcal{G}^{\tau}(\Xi \vee \overline{\alpha}) \leq \mathcal{G}^{\tau}(\Xi) \vee \alpha$, $\mathcal{G}^{\pi}(\Xi \vee \overline{\alpha}) \geq \mathcal{G}^{\pi}(\Xi) \wedge \alpha$, $\mathcal{G}^{\sigma}(\Xi \vee \overline{\alpha}) \geq \mathcal{G}^{\sigma}(\Xi) \wedge \alpha, \forall\, \Xi \in \xi^{\pounds}$ *and* $\alpha \in \xi$.

For the sake of brevity and clarity, we will occasionally denote $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$ *as* $\mathcal{G}^{\tau\pi\sigma}$.

A single-valued neutrosophic grill topological space (svngts) is defined as the triple (£, $\mathcal{T}^{\tau\pi\sigma}$, $\mathcal{G}^{\tau\pi\sigma}$*).*

Theorem 2. *Let* $(\mathcal{G}^{\tau\pi\sigma})$ *be an svn-grill on £. Define* $\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st} : \zeta^{\pounds} \to \zeta$

$$\mathcal{G}^{\tau}_{st}(\Xi) = \bigwedge_{\left\{(\Xi_i, \overline{\alpha_i}) | i \in J\right\} \in \mathcal{K}(\Xi)} \left\{ \bigvee_{(\Xi_l, \overline{\alpha_l}) \in \left\{(\Xi_i, \overline{\alpha_i}) | i \in J\right\}} \mathcal{G}^{\tau}(\Xi_l) \vee \alpha_l \right\},$$

$$\mathcal{G}^{\pi}_{st}(\Xi) = \bigvee_{\left\{(\Xi_i, \overline{\alpha_i}) | i \in J\right\} \in \mathcal{K}(\Xi)} \left\{ \bigwedge_{(\Xi_l, \overline{\alpha_l}) \in \left\{(\Xi_i, \overline{\alpha_i}) | i \in J\right\}} \mathcal{G}^{\pi}(\Xi_l) \vee \alpha_l \right\},$$

$$\mathcal{G}^{\sigma}_{st}(\Xi) = \bigvee_{\left\{(\Xi_i, \overline{\alpha_i}) | i \in J\right\} \in \mathcal{K}(\Xi)} \left\{ \bigwedge_{(\Xi_l, \overline{\alpha_l}) \in \left\{(\Xi_i, \overline{\alpha_i}) | i \in J\right\}} \mathcal{G}^{\sigma}(\Xi_l) \vee \alpha_l \right\},$$

where $\mathcal{K}(\xi) = \left\{ \left\{ (\Xi_i, \overline{\alpha_i}) \mid i \in J, J\ is\ finite\ index\ set \right\} \right.$ *and* $\left. \Xi \leq \bigwedge_{i \in J} (\Xi_i \vee \overline{\alpha_i}) \right\}$. *Then,* $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$ *is the coarsest stratified svn-grill on £ which is finer than* $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$.

Proof. First, we will prove that $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$ is a stratified *svn-grill* on £.

(G_1) and (G_2) are straightforward.

(G_3) Assume that there exist $\Xi, \Theta \in \xi^{\pounds}$ such that

$$\mathcal{G}^{\tau}(\Xi \vee \Theta) \not\leq \mathcal{G}^{\tau}(\Xi) \vee \mathcal{G}^{\tau}(\Theta), \qquad \mathcal{G}^{\pi}(\Xi \vee \Theta) \not\geq \mathcal{G}^{\pi}(\Xi) \wedge \mathcal{G}^{\pi}(\Theta)$$

$$\mathcal{G}^{\sigma}(\Xi \vee \Theta) \not\geq \mathcal{G}^{\sigma}(\Xi) \wedge \mathcal{G}^{\sigma}(\Theta).$$

By the definition of $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$, there exist $\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\} \in \mathcal{K}(\Xi)$ and $\{(\Theta_\iota, \overline{\omega_\iota}) \mid \iota \in \Gamma\} \in \mathcal{K}(\Theta)$ such that

$$\mathcal{G}^{\tau}(\Xi \vee \Theta) \not\geq \left(\bigvee_{(\Xi_i, \overline{\alpha_i}) \in \left\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\right\}} \mathcal{G}^{\tau}(\Xi_t) \vee \alpha_t \right) \vee \left(\bigvee_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) \mid \iota \in \Gamma\}} \mathcal{G}^{\tau}(\Theta_j) \vee \omega_j \right).$$

$$\mathcal{G}^{\pi}(\Xi \vee \Theta) \not\leq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \left\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\right\}} \mathcal{G}^{\pi}(\Xi_t) \vee \alpha_t \right) \wedge \left(\bigwedge_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) \mid \iota \in \Gamma\}} \mathcal{G}^{\pi}(\Theta_j) \vee \omega_j \right).$$

$$\mathcal{G}^{\sigma}(\Xi \vee \Theta) \not\leq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \left\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\right\}} \mathcal{G}^{\sigma}(\Xi_t) \vee \alpha_t \right) \wedge \left(\bigwedge_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) \mid \iota \in \Gamma\}} \mathcal{G}^{\sigma}(\Theta_j) \vee \omega_j \right).$$

Set $k \in J \cup \Gamma$ such that

$$\Pi_k \vee \overline{\varrho_k} = \begin{cases} \Xi_k \vee \alpha_k, & \text{if } k \in J - (J \cap \Gamma), \\ \Theta_k \vee \omega_k, & \text{if } k \in \Gamma - (J \cap \Gamma), \\ (\Xi_k \vee \alpha_k) \vee (\Theta_k \vee \omega_k), & \text{if } k \in (J \cap \Gamma). \end{cases}$$

On the other hand,

$$\Xi \vee \Theta \leq \left(\bigwedge_{t \in J} (\Xi_t \vee \alpha_t) \right) \vee \left(\bigwedge_{\iota \in \Gamma} (\Theta_\iota \vee \omega_\iota) \right) = \bigwedge_{k \in (J \cap \Gamma)} (\Pi_k \vee \overline{\varrho_k}) \text{ and } \left\{ (\Pi_k, \overline{\varrho_k}) \mid k \in (J \cup \Gamma) \right\} \in \mathcal{K}(\Xi \vee \Theta).$$

Then, we have

$$\begin{aligned} \mathcal{G}^{\tau}_{st}(\Xi \vee \Theta) &\leq \bigvee_{(\Pi_v, \overline{\varrho_v}) \in \left\{(\Pi_k, \overline{\varrho_k}) \mid k \in (J \cup \Gamma)\right\}} \mathcal{G}^{\tau}(\Pi_v) \vee \varrho_v \\ &= \left(\bigvee_{(\Xi_i, \overline{\alpha_i}) \in \left\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\right\}} \mathcal{G}^{\tau}(\Xi_i) \vee \alpha_i \right) \vee \left(\bigvee_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) \mid \iota \in \Gamma\}} \mathcal{G}^{\tau}(\Theta_j) \vee \omega_j \right). \end{aligned}$$

$$\begin{aligned} \mathcal{G}^{\pi}_{st}(\Xi \vee \Theta) &\geq \bigwedge_{(\Pi_v, \overline{\varrho_v}) \in \left\{(\Pi_k, \overline{\varrho_k}) \mid k \in (J \cup \Gamma)\right\}} \mathcal{G}^{\tau}(\Pi_v) \vee \varrho_v \\ &= \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \left\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\right\}} \mathcal{G}^{\pi}(\Xi_i) \vee \alpha_i \right) \wedge \left(\bigwedge_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) \mid \iota \in \Gamma\}} \mathcal{G}^{\pi}(\Theta_j) \vee \omega_j \right). \end{aligned}$$

$$\begin{aligned} \mathcal{G}^{\sigma}_{st}(\Xi \vee \Theta) &\geq \bigwedge_{(\Pi_v, \overline{\varrho_v}) \in \left\{(\Pi_k, \overline{\varrho_k}) \mid k \in (J \cup \Gamma)\right\}} \mathcal{G}^{\tau}(\Pi_v) \vee \varrho_v \\ &= \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \left\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\right\}} \mathcal{G}^{\sigma}(\Xi_i) \vee \alpha_i \right) \wedge \left(\bigwedge_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) \mid \iota \in \Gamma\}} \mathcal{G}^{\sigma}(\Theta_j) \vee \omega_j \right). \end{aligned}$$

This is a contradiction. Thus, (G_3) holds.

(G_{st}) Assume that there exist $\Xi \in \xi^{\mathcal{L}}$ and $\alpha \in \xi$ such that

$$\mathcal{G}^{\tau}_{st}(\Xi \vee \overline{\alpha}) \not\leq \mathcal{G}^{\tau}_{st}(\Xi) \vee \alpha, \qquad \mathcal{G}^{\pi}_{st}(\Xi \vee \overline{\alpha}) \not\geq \mathcal{G}^{\pi}_{st}(\Xi) \wedge \alpha,$$

$$\mathcal{G}^{\sigma}_{st}(\Xi \vee \overline{\alpha}) \not\geq \mathcal{G}^{\sigma}_{st}(\Xi) \wedge \alpha.$$

By the definition of $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$, there exist $\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\} \in \mathcal{K}(\Xi)$ such that

$$\mathcal{G}^{\tau}_{st}(\Xi \vee \overline{\alpha}) \not\leq \left(\bigvee_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\tau}(\Xi_i) \vee \alpha_i \right) \vee \alpha$$

$$\mathcal{G}^{\pi}_{st}(\Xi \vee \overline{\alpha}) \not\geq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\pi}(\Xi_i) \vee \alpha_i \right) \wedge \alpha$$

$$\mathcal{G}^{\sigma}_{st}(\Xi \vee \overline{\alpha}) \not\geq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\sigma}(\Xi_i) \vee \alpha_i \right) \wedge \alpha$$

On the other hand, $\Xi \vee \overline{\alpha} \leq \bigwedge_{t \in (J \cap \Gamma)} (\Xi_t \vee \overline{\varrho_t})$ where $\varrho_t = \alpha_i \vee \alpha$; then, $\{(\Xi_t, \overline{\varrho_t}) \mid t \in J\} \in \mathcal{K}(\Xi \vee \overline{\alpha})$. Then, we have

$$\mathcal{G}^{\tau}_{st}(\Xi \vee \overline{\alpha}) \leq \left(\bigvee_{(\Xi_i, \overline{\varrho_i}) \in \{(\Xi_t, \overline{\varrho_t}) | t \in J\}} \mathcal{G}^{\tau}(\Xi_i) \vee \alpha_i \right) = \left(\bigvee_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\tau}(\Xi_i) \vee \alpha_i \right) \vee \alpha,$$

$$\mathcal{G}^{\pi}_{st}(\Xi \vee \overline{\alpha}) \geq \left(\bigwedge_{(\Xi_i, \overline{\varrho_i}) \in \{(\Xi_t, \overline{\varrho_t}) | t \in J\}} \mathcal{G}^{\pi}(\Xi_i) \vee \alpha_i \right) = \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\pi}(\Xi_i) \vee \alpha_i \right) \wedge \alpha,$$

$$\mathcal{G}^{\sigma}_{st}(\Xi \vee \overline{\alpha}) \geq \left(\bigwedge_{(\Xi_i, \overline{\varrho_i}) \in \{(\Xi_t, \overline{\varrho_t}) | t \in J\}} \mathcal{G}^{\sigma}(\Xi_i) \vee \alpha_i \right) = \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\sigma}(\Xi_i) \vee \alpha_i \right) \wedge \alpha.$$

This is a contradiction. Thus, $\mathcal{G}_{st}$ holds. Hence, $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$ is stratified.

Second, for any $\Xi \in \xi^{\pounds}$, there exists a collection $\{\overline{\alpha}\}$ with $\Xi \leq \Xi \vee \overline{\alpha}$ such that $\mathcal{G}^{\tau}_{st}(\Xi) \leq \mathcal{G}^{\tau}(\Xi)$, $\mathcal{G}^{\pi}_{st}(\Xi) \geq \mathcal{G}^{\pi}(\Xi)$ and $\mathcal{G}^{\sigma}_{st}(\Xi) \geq \mathcal{G}^{\sigma}(\Xi)$. Thus, $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$ is finer than $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$.

Finally, consider $(\mathcal{G}^{\star\tau}, \mathcal{G}^{\star\pi}, \mathcal{G}^{\star\sigma})$, a stratified *svn-grill* on $\pounds$ which is finer than $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$. And we will show that $\mathcal{G}^{\tau}_{st}(\Xi) \geq \mathcal{G}^{\star\tau}(\Xi)$, $\mathcal{G}^{\pi}_{st}(\Xi) \leq \mathcal{G}^{\star\pi}(\Xi)$ and $\mathcal{G}^{\sigma}_{st}(\Xi) \leq \mathcal{G}^{\star\sigma}(\Xi)$. Assume that $\Xi \in \xi^{\pounds}$ such that $\mathcal{G}^{\tau}_{st}(\Xi) \not\geq \mathcal{G}^{\star\tau}(\Xi)$, $\mathcal{G}^{\pi}_{st}(\Xi) \not\leq \mathcal{G}^{\star\pi}(\Xi)$ and $\mathcal{G}^{\sigma}_{st}(\Xi) \not\leq \mathcal{G}^{\star\sigma}(\Xi)$. By the definition of $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$, there exist $\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\} \in \mathcal{K}(\Xi)$ such that

$$\mathcal{G}^{\star\tau}(\Xi) \not\leq \left(\bigvee_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\tau}(\Xi_i) \vee \alpha_i \right),$$

$$\mathcal{G}^{\star\pi}(\Xi) \not\geq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\pi}(\Xi_i) \vee \alpha_i \right),$$

$$\mathcal{G}^{\star\sigma}(\Xi) \not\geq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\sigma}(\Xi_i) \vee \alpha_i \right).$$

On the other hand, $(\mathcal{G}^{\star\tau}, \mathcal{G}^{\star\pi}, \mathcal{G}^{\star\sigma})$ is stratified; then, we have

$$\begin{aligned}\mathcal{G}^{\star\tau}(\Xi) &\leq \mathcal{G}^{\star\tau}\left(\bigwedge_{t\in J}(\Xi_t \vee \overline{\alpha_t})\right)\\ &\leq \bigvee_{t\in J}\mathcal{G}^{\star\tau}(\Xi_t \vee \overline{\alpha_t})\\ &\leq \bigvee_{t\in J}(\mathcal{G}^{\star\tau}(\Xi_t) \vee \overline{\alpha_t})\\ &\leq \bigvee_{(\Xi_i,\overline{\alpha_i})\in\left\{(\Xi_t,\overline{\alpha_t})|t\in J\right\}} \mathcal{G}^{\tau}(\Xi_i) \vee \alpha_i\end{aligned}$$

Likewise, we can establish through a similar line of reasoning that

$$\mathcal{G}^{\star\pi}(\Xi) \geq \bigwedge_{(\Xi_i,\overline{\alpha_i})\in\left\{(\Xi_t,\overline{\alpha_t})|t\in J\right\}} \mathcal{G}^{\pi}(\Xi_i) \vee \alpha_i \quad \mathcal{G}^{\star\sigma}(\Xi) \geq \bigwedge_{(\Xi_i,\overline{\alpha_i})\in\left\{(\Xi_t,\overline{\alpha_t})|t\in J\right\}} \mathcal{G}^{\sigma}(\Xi_i) \vee \alpha_i.$$

This is a contradiction. Hence, $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$ is the coarsest stratified *svn-grill* on $\pounds$ which is finer than $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$. □

In order to better understand the notion of *svn-primal* mappings, let us provide some context for the following definition. Consider a non-empty set $\pounds$ and a mapping $\mathcal{P}^{\tau}, \mathcal{P}^{\pi}, \mathcal{P}^{\sigma} : \zeta^{\pounds} \to \zeta$. We will now introduce certain conditions that, when satisfied, characterize the mapping as an *svn-primal* on $\pounds$.

Definition 8. *Let $\pounds$ be a non-empty set. A mapping $\mathcal{P}^{\tau}, \mathcal{P}^{\pi}, \mathcal{P}^{\sigma} : \zeta^{\pounds} \to \zeta$ is said to be svn-primal on $\pounds$, if it meets the following conditions:*

(P1) $\mathcal{P}^{\tau}(\overline{1}) = 0, \mathcal{P}^{\pi}(\overline{1}) = 1, \mathcal{P}^{\sigma}(\overline{1}) = 1$ *and* $\mathcal{P}^{\tau}(\overline{0}) = 1, \mathcal{P}^{\pi}(\overline{0}) = 0, \mathcal{P}^{\sigma}(\overline{0}) = 0$.

(P2) If $\Xi \leq \Theta$, *then* $\mathcal{P}^{\tau}(\Theta) \leq \mathcal{P}^{\tau}(\Xi)$, $\mathcal{P}^{\pi}(\Theta) \geq \mathcal{P}^{\pi}(\Xi)$, $\mathcal{P}^{\sigma}(\Theta) \geq \mathcal{P}^{\sigma}(\Xi), \forall\, \Xi, \Theta \in \xi^{\pounds}$.

(P3) $\mathcal{P}^{\tau}(\Xi \wedge \Theta) \leq \mathcal{P}^{\tau}(\Xi) \vee \mathcal{P}^{\tau}(\Theta)$, $\mathcal{P}^{\pi}(\Xi \wedge \Theta) \geq \mathcal{P}^{\pi}(\Xi) \wedge \mathcal{P}^{\pi}(\Theta)$, $\mathcal{P}^{\sigma}(\Xi \wedge \Theta) \geq \mathcal{P}^{\sigma}(\Xi) \wedge \mathcal{P}^{\sigma}(\Theta), \forall\, \Xi, \Theta \in \xi^{\pounds}$. *Sometimes, we will write $\mathcal{P}^{\tau\pi\sigma}$ for $(\mathcal{P}^{\tau}, \mathcal{P}^{\pi}, \mathcal{P}^{\sigma})$.*

If $\mathcal{P}^{\tau\pi\sigma}$ and $\mathcal{P}^{\star\tau\pi\sigma}$ are svn-primals on $\pounds$, then "$\mathcal{P}^{\tau\pi\sigma}$ is finer than $\mathcal{P}^{\star\tau\pi\sigma}$ or ($\mathcal{P}^{\star\tau\pi\sigma}$ is coarser than $\mathcal{P}^{\tau\pi\sigma}$)" denoted by $\mathcal{P}^{\tau\pi\sigma} \leq \mathcal{P}^{\star\tau\pi\sigma}$ if and only if

$$\mathcal{P}^{\tau}(\Xi) \leq \mathcal{P}^{\star\tau}(\Theta), \quad \mathcal{P}^{\pi}(\Xi) \geq \mathcal{P}^{\star\pi}(\Theta), \quad \mathcal{P}^{\sigma}(\Xi) \geq \mathcal{P}^{\star\sigma}(\Theta), \forall, \Xi, \Theta \in \xi^{\pounds}.$$

The triable $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ is called a single-valued neutrosophic primal topological space (svnpts). For $\alpha \in \zeta_0$, $(\pounds, \mathcal{T}^{\tau\pi\sigma}_{\alpha}, \mathcal{P}^{\tau\pi\sigma}_{\alpha})$ is primal topological space (pts) in [35].

Remark 1. *The terms (P_2) and (P_3), in Definition 10, correspond to the next condition*

$$\mathcal{P}^{\tau}(\Xi \wedge \Theta) = \mathcal{P}^{\tau}(\Xi) \vee \mathcal{P}^{\tau}(\Theta), \quad \mathcal{P}^{\pi}(\Xi \wedge \Theta) \neq \mathcal{P}^{\pi}(\Xi) \wedge \mathcal{P}^{\pi}(\Theta), \quad \mathcal{P}^{\sigma}(\Xi \wedge \Theta) \neq \mathcal{P}^{\sigma}(\Xi) \wedge \mathcal{P}^{\sigma}(\Theta)$$

Example 2. *Suppose that $\pounds = \{l_1, l_2, l_3\}$; define the svn-set $\Theta \in \xi^{\pounds}$ as follows*

$$\Theta = \langle (0.4, 0.4, 0.4), (0, 0, 0), (0, 0, 0) \rangle$$

Define the mappings $\mathcal{P}^{\tau}, \mathcal{P}^{\pi}, \mathcal{P}^{\sigma} : \xi^{\pounds} \to \xi$ as follows:

$$\mathcal{P}^{\tau}(\Xi) = \begin{cases} 1, & \text{if } \Xi = \overline{0}, \\ \frac{1}{4}, & \text{if } \Xi = \Theta, \\ \frac{1}{2}, & \text{if } \overline{0} < \Xi < \Theta, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{P}^{\pi}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ \frac{3}{4}, & \text{if } \Xi = \Theta, \\ \frac{1}{2}, & \text{if } \overline{0} < \Xi < \Theta, \\ 1, & \text{otherwise,} \end{cases}$$

$$\mathcal{P}^{\sigma}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ \frac{1}{2}, & \text{if } \Xi = \Theta, \\ \frac{1}{4}, & \text{if } \overline{0} < \Xi < \Theta, \\ 1, & \text{otherwise,} \end{cases}$$

Then, $\mathcal{P}^{\tau\pi\sigma}$ is svn-primal on £.

Theorem 3. *Let $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$ be an svng on £. Then, the collection $\{\Theta \in \xi^{£} : \mathcal{G}^{\tau}(\Theta^{c}) \geq r, \mathcal{G}^{\pi}(\Theta^{c}) \leq 1 - r, \mathcal{G}^{\sigma}(\Theta^{c}) \leq 1 - r, r \in \xi_0\}$ is an svn-primal on £.*

Proof. (P_1) Since $\mathcal{G}^{\tau}(\overline{0}) = 0, \mathcal{G}^{\pi}(\overline{0}) = 1, \mathcal{G}^{\sigma}(\overline{0}) = 1$ and $\mathcal{G}^{\tau}(\overline{1}) = 1, \mathcal{G}^{\pi}(\overline{1}) = 0, \mathcal{G}^{\sigma}(\overline{1}) = 0$ implies that $\mathcal{P}^{\tau}(\overline{1}) = 0, \mathcal{P}^{\pi}(\overline{1}) = 1, \mathcal{P}^{\sigma}(\overline{1}) = 1$ and $\mathcal{P}^{\tau}(\overline{0}) = 1, \mathcal{G}^{\pi}(\overline{0}) = 0, \mathcal{G}^{\sigma}(\overline{0}) = 0$.

(P_2) Let $\Xi \leq \Theta$; then, $\Theta^{c} \leq \Xi^{c}$ and, thus,

$$r \leq \mathcal{G}^{\tau}(\Theta^{c}) \leq \mathcal{G}^{\tau}(\Xi^{c}), \;\; 1 - r \geq \mathcal{G}^{\pi}(\Theta^{c}) \geq \mathcal{G}^{\pi}(\Xi^{c}), \;\; 1 - r \geq \mathcal{G}^{\sigma}(\Theta^{c}) \geq \mathcal{G}^{\sigma}(\Xi^{c}).$$

Hence, $\mathcal{P}^{\tau}(\Theta) \leq \mathcal{P}^{\tau}(\Xi), \mathcal{P}^{\pi}(\Theta) \geq \mathcal{P}^{\pi}(\Xi), \;\; \mathcal{P}^{\sigma}(\Theta) \geq \mathcal{P}^{\sigma}(\Xi)$.

(P_3) Suppose

$$\mathcal{P}^{\tau}(\Xi \wedge \Theta) \not\leq \mathcal{P}^{\tau}(\Xi) \vee \mathcal{P}^{\tau}(\Theta), \qquad \mathcal{P}^{\pi}(\Xi \wedge \Theta) \not\geq \mathcal{P}^{\pi}(\Xi) \wedge \mathcal{P}^{\pi}(\Theta),$$

$$\mathcal{P}^{\sigma}(\Xi \wedge \Theta) \not\geq \mathcal{P}^{\sigma}(\Xi) \wedge \mathcal{P}^{\sigma}(\Theta).$$

Then, there exists $r \in \xi_0$ such that

$$\mathcal{P}^{\tau}(\Xi \wedge \Theta) \geq r \geq \mathcal{P}^{\tau}(\Xi) \vee \mathcal{P}^{\tau}(\Theta), \qquad \mathcal{P}^{\pi}(\Xi \wedge \Theta) \leq 1 - r \leq \mathcal{P}^{\pi}(\Xi) \wedge \mathcal{P}^{\pi}(\Theta),$$

$$\mathcal{P}^{\sigma}(\Xi \wedge \Theta) \leq 1 - r \leq \mathcal{P}^{\sigma}(\Xi) \wedge \mathcal{P}^{\sigma}(\Theta).$$

Since $\mathcal{P}^{\tau}(\Xi \wedge \Theta) \geq r, \mathcal{P}^{\pi}(\Xi \wedge \Theta) \leq 1 - r$ and $\mathcal{P}^{\sigma}(\Xi \wedge \Theta) \leq 1 - r$, we have $\mathcal{G}^{\tau}([\Xi \wedge \Theta]^{c}) \geq r$, $\mathcal{P}^{\pi}([\Xi \wedge \Theta]^{c}) \leq 1 - r$ and $\mathcal{G}^{\sigma}([\Xi \wedge \Theta]^{c}) \leq 1 - r$ implies that

$$\mathcal{G}^{\tau}(\Xi^{c} \vee \Theta^{c}) \geq r, \;\; \mathcal{G}^{\pi}(\Xi^{c} \vee \Theta^{c}) \leq 1 - r, \;\; \mathcal{G}^{\sigma}(\Xi^{c} \vee \Theta^{c}) \leq 1 - r.$$

From the definition of $\mathcal{G}^{\tau\pi\sigma}$, we have

$$\mathcal{G}^{\tau}(\Xi^{c}) \vee \mathcal{G}^{\tau}(\Theta^{c}) \geq \mathcal{G}^{\tau}(\Xi^{c} \vee \Theta^{c}) \geq r, \qquad \mathcal{G}^{\pi}(\Xi^{c}) \wedge \mathcal{G}^{\pi}(\Theta^{c}) \leq \mathcal{G}^{\pi}(\Xi^{c} \vee \Theta^{c}) \leq 1 - r,$$

$$\mathcal{G}^{\sigma}(\Xi^{c}) \wedge \mathcal{G}^{\sigma}(\Theta^{c}) \leq \mathcal{G}^{\sigma}(\Xi^{c} \vee \Theta^{c}) \leq 1 - r.$$

Since $\mathcal{G}^{\tau}(\Xi^{c}) \vee \mathcal{G}^{\tau}(\Theta^{c}) \geq r$, $\mathcal{G}^{\pi}(\Xi^{c}) \wedge \mathcal{G}^{\pi}(\Theta^{c}) \leq 1 - r$ and $\mathcal{G}^{\sigma}(\Xi^{c} \vee \Theta^{c}) \leq 1 - r$, we obtain $\mathcal{G}^{\tau}(\Theta^{c}) \geq r$, $\mathcal{G}^{\pi}(\Theta^{c}) \leq 1 - r$, $\mathcal{G}^{\sigma}(\Theta^{c}) \leq 1 - r$ and $\mathcal{G}^{\tau}(\Xi^{c}) \geq r$, $\mathcal{G}^{\pi}(\Xi^{c}) \leq 1 - r$, $\mathcal{G}^{\sigma}(\Xi^{c}) \leq 1 - r$ implies that $\mathcal{P}^{\tau}(\Theta) \geq r$, $\mathcal{P}^{\pi}(\Theta) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Theta) \leq 1 - r$ and $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$. Hence,

$$\mathcal{P}^{\tau}(\Xi) \vee \mathcal{P}^{\tau}(\Theta) \geq r, \qquad \mathcal{P}^{\pi}(\Xi) \wedge \mathcal{P}^{\pi}(\Theta) \leq 1 - r,$$

$$\mathcal{P}^{\sigma}(\Xi) \wedge \mathcal{P}^{\sigma}(\Theta) \leq 1 - r.$$

This is a contradiction. Consequently, (P_3) holds. □

Proposition 1. *Let $\{\mathcal{P}_j^{\tau\pi\sigma}\}_{j \in J}$ be a collection of svn-primals on £. Then, their union $\bigvee_{i \in J} \mathcal{P}_j^{\tau\pi\sigma}$ is also an* svn-primal *on £.*

Proof. Directly from Definition 7. □

4. Single-Valued Neutrosophic Primal Open Local Function in Šostak Sense

In this section, we investigate the concept of svn-primal open local functions within the context of Šostak's sense. Our primary focus is to explore the properties of these functions and their relationship with svnts and neutrosophic primals. Through a series of definitions, theorems and discussions, we aim to provide a comprehensive understanding of this unique concept and its implications in the domain of neutrosophic topology. The introductory results presented here lay the foundation for further exploration of this fascinating topic, shedding light on the details of svn structures in the sense of Šostak.

Definition 9. *Let $n, m, v \in \xi_0$ and $n+m+v \leq 3$. A single-valued neutrosophic point (svn-point) $y_{n,m,v}$ is an svn-set in $\xi^{\pounds}$ for each $z \in \Theta$, defined by*

$$y_{n,m,v}(z) = \begin{cases} (n,m,v), & \text{if } y = z, \\ (0,1,1), & \text{if } y \neq z, \end{cases}$$

An svn-point $y_{n,m,v}$ *is said to belong to an* svn-set $\Xi = \langle z, \tau_{\Xi}(z), \pi_{\Xi}(z), \sigma_{\Xi}(z)\rangle \in \xi^{\pounds}$ *, denoted by $y_{n,m,v} \in \Xi$ iff $n \leq \tau_{\Xi}(z)$, $m \geq \pi_{\Xi}(z)$ and $v \geq \sigma_{\Xi}(z)$. We indicate the set of all* svn-points *in £ as (svn-point (£)).*

For each $y_{n,m,v} \in$ svn-point$(\pounds)$ and $\Theta \in \xi^{\omega}$ we shall write $y_{n,m,v}$ quasi-coincident with Ξ, denoted by $y_{n,m,v} q \Xi$, if

$$n + \tau_{\Xi}(z) > 1, \quad n + \pi_{\Xi}(z) \leq 1, \quad v + \sigma_{\Xi}(z) \leq 1.$$

For all $\Theta, \Xi \in \xi^{\pounds}$ we shall write Ξ q θ to mean that Ξ is quasi-coincident with Θ if there exists $z \in \pounds$ such that

$$\tau_{\Xi}(z) + \tau_{\Theta}(z) > 1, \quad \pi_{\Xi}(z) + \pi_{\Theta}(z) \leq 1, \quad \sigma_{\Xi}(z) + \sigma_{\Theta}(z) \leq 1.$$

Definition 10. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma})$ be an svnts, for all $\Xi \in \xi^{\pounds}$, $y_{n,m,v} \in$svn-point $(\pounds)$ and $r \in \xi_0$. Then, Ξ is said to be an r-open $\mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}$-neighborhood (r-OQN) of $y_{n,m,v}$, defined as follows*

$$\mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r) = \{\Xi \in \xi^{\pounds} | y_{n,m,v} q \Xi, \ \mathcal{T}^{\tau}(\Xi)) \geq r, \quad \mathcal{T}^{\pi}(\Xi)) \leq 1-r, \quad \mathcal{T}^{\sigma}(\Xi)) \leq 1-r\}.$$

Lemma 1. *An* svn-point $y_{n.m.v} \in \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Theta, r)$ *iff every* r-O$\mathcal{Q}N$ *of $y_{n,n,v}$ is quasi-coincident with Ξ.*

Definition 11. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ be an svnpts, for each $r \in \xi_0$ and $\Xi \in \xi^{\pounds}$. Then, the single-valued neutrosophic primal open local function $\Xi_r^{\star}(\mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ of Ξ is the union of all svn-points $y_{n,m,v}$ such that if $\Theta \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ and $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1-r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1-r$, then there is at least one $z \in \pounds$, for which $\tau_{\Xi}(z) + \tau_{\Theta}(z) - 1 > \tau_{\Pi}(z)$, $\pi_{\Xi}(z) + \pi_{\Theta}(z) - 1 \leq \pi_{\Pi}(z)$, $\sigma_{\Xi}(z) + \sigma_{\Theta}(z) - 1 \leq \sigma_{\Pi}(z)$.*

In this article, we will write $\Xi_r^{\star}$ for $\Xi_r^{\star}(\mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ without any ambiguity.

Example 3. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ be an svnpts. The simplest svn-primal on £ is $\mathcal{P}_0^{\tau}, \mathcal{P}_0^{\pi}, \mathcal{P}_0^{\sigma} : \zeta^{\pounds} \to \zeta$ where*

$$\mathcal{P}_0^{\tau}(\Xi) = \begin{cases} 1, & \text{if } \Xi = \overline{0}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{P}_0^{\pi}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ 1, & \text{otherwise}, \end{cases}$$

$$\mathcal{P}_0^{\sigma}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ 1, & \text{otherwise,} \end{cases}$$

If $\mathcal{P}^{\tau\pi\sigma} = \mathcal{P}_0^{\tau\pi\sigma}$, *then, for each* $\Xi \in \xi^{\pounds}$, $r \in \xi_0$, *we have* $\Xi_r^{\star} = \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$.

Theorem 4. *Let* $(\pounds, \mathcal{T}^{\tau\pi\sigma})$ *be an svnts and* $\mathcal{P}_1^{\tau\pi\sigma}$, $\mathcal{P}_2^{\tau\pi\sigma}$ *be two svn-primals on* $\pounds$. *Then,* $\forall$ $\Xi, \Theta \in \xi^{\pounds}$ *and* $r \in \xi_0$, *we have*

(1) *If* $\Xi \leq \Theta$, *then* $\Xi_r^{\star} \leq \Theta_r^{\star}$.
(2) *If* $\mathcal{P}_1^{\tau} \leq \mathcal{P}_2^{\tau}$, $\mathcal{P}_1^{\pi} \geq \mathcal{P}_2^{\pi}$ *and* $\mathcal{P}_1^{\sigma} \geq \mathcal{P}_2^{\sigma}$, *then* $\Xi_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma}) \geq \Xi_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})$.
(3) $\Xi_r^{\star} = \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi_r^{\star}, r) \leq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$.
(4) *If* $(\Xi_r^{\star})_r^{\star} \leq \Xi_r^{\star}$.
(5) *If* $\mathcal{P}^{\tau}(\Theta) \geq r$, $\mathcal{P}^{\pi}(\Theta) \leq 1 - r$ *and* $\mathcal{P}^{\sigma}(\Theta) \leq 1 - r$, *then* $(\Xi \vee \Theta)_r^{\star} = \Xi_r^{\star} \vee \Theta_r^{\star} = \Xi_r^{\star}$.
(6) $(\Xi_r^{\star} \vee \Theta_r^{\star}) = (\Xi \vee \Theta)_r^{\star}$.
(7) $\mathcal{T}^{\tau}(\Theta) \geq r$, $\mathcal{T}^{\pi}(\Theta) \leq 1 - r$ *and* $\mathcal{T}^{\sigma}(\Theta) \leq 1 - r$, *then* , $(\Theta \wedge \Xi_r^{\star}) \leq (\Theta \wedge \Xi)_r^{\star}$.
(8) $(\Xi_r^{\star} \wedge \Theta_r^{\star}) \geq (\Xi \wedge \Theta)_r^{\star}$.

Proof. (1) Let $\Xi \in \xi^{\pounds}$ and $r \in \xi_0$ such that $\Xi_r^{\star} \not\leq \Theta_r^{\star}$. Then, there exists $z \in \pounds$ and $n, m, v \in \xi_0$ such that

$$\tau_{\Xi_r^{\star}}(z) \geq n > \tau_{\Theta_r^{\star}}(z), \qquad \pi_{\Xi_r^{\star}}(z) < m \leq \pi_{\Theta_r^{\star}}(z), \qquad \sigma_{\Xi_r^{\star}}(z) < v \leq \sigma_{\Theta_r^{\star}}(z). \tag{1}$$

Since $\tau_{\Theta_r^{\star}}(z) < n$, $\pi_{\Theta_r^{\star}}(z) \geq m$ and $\sigma_{\Theta_r^{\star}}(z) \geq v$, there exists $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ with $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1 - r$, such that for every $x \in \pounds$, we have,

$$\tau_{\mathcal{A}}(x) + \tau_{\Theta}(z) - 1 \leq \tau_{\Pi}(x), \quad \pi_{\mathcal{A}}(x) + \pi_{\Theta}(x) - 1 > \pi_{\Pi}(x), \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Theta}(x) - 1 > \sigma_{\Pi}(x).$$

Since $\Xi \leq \Theta$, we have

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(x), \quad \pi_{\mathcal{A}}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi}(x), \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi}(x).$$

So, $\tau_{\Xi_r^{\star}}(z) < n$, $\pi_{\Xi_r^{\star}}(z) \geq m$ and $\sigma_{\Xi_r^{\star}}(z) \geq v$, and this is a contradiction for Equation (1). Hence, $\Xi_r^{\star} \leq \Theta_r^{\star}$.

(2) Suppose that $\Xi_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma}) \not\geq \Xi_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})$. Then, there exist $z \in \pounds$ and $n, m, v \in \xi_0$ such that

$$\tau_{\Xi_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z) < n \leq \tau_{\Xi_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z),$$

$$\pi_{\Xi_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z) \geq m > \pi_{\Xi_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z), \tag{2}$$

$$\sigma_{\Xi_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z) \geq v > \sigma_{\Xi_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z).$$

Since $\tau_{\Xi_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z) < n$, $\pi_{\Xi_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z) \geq m$ and $\sigma_{\Xi_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z) \geq v$, there exists $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ with $\mathcal{P}_1^{\tau}(\Pi) \geq r$, $\mathcal{P}_1^{\pi}(\Pi) \leq 1 - r$, $\mathcal{P}_1^{\sigma}(\Pi) \leq 1 - r$, such that for every $x \in \pounds$, we have

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(x), \quad \pi_{\mathcal{A}}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi}(x), \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi}(x).$$

Since, $\mathcal{P}_2^{\tau}(\Pi) \geq \mathcal{P}_1^{\tau}(\Pi) \geq r$, $\mathcal{P}_2^{\pi}(\Pi) \leq \mathcal{P}_1^{\pi}(\Pi) \leq 1 - r$, $\mathcal{P}_2^{\sigma}(\Pi) \leq \mathcal{P}_1^{\sigma}(\Pi) \leq 1 - r$, we have

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(x), \quad \pi_{\mathcal{A}}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi}(x), \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi}(x).$$

Hence, $\tau_{\Xi_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z) < s$, $\pi_{\Xi_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z) \geq m$ and $\sigma_{\Xi_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})}(z) \geq v$, and this is a contradiction for Equation (2). Thus, $\Xi_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma}) \geq \Xi_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}, \mathcal{T}^{\tau\pi\sigma})$.

(3) Assume that $\Xi_r^\star \not\leq \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi, r)$; then, there exist $z \in \pounds$ and $n, m, v \in \xi_0$ such that

$$\tau_{\Xi_r^\star}(z) \geq n > \tau_{\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi,r)}(z), \quad \pi_{\Xi_r^\star}(z) < m \leq \pi_{\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi,r)}, \quad \sigma_{\Xi_r^\star}(z) < v \leq \sigma_{\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi,r)}(z). \tag{3}$$

Since $\tau_{\Xi_r^\star}(z) \geq n$, $\pi_{\Xi_r^\star}(z) < m$ and $\pi_{\Xi_r^\star}(z) < v$, we have $y_{n,m,v} \in \Xi_r^\star$. So, there is at least one $x \in \pounds$, for each $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}\tau\pi\sigma}(y_{n,m,v}, r)$ with $\mathcal{P}^\tau(\Pi) \geq r$, $\mathcal{P}^\pi(\Pi) \leq 1 - r$, $\mathcal{P}^\sigma(\Pi) \leq 1 - r$ such that

$$\tau_{\mathcal{A}}(x) + \tau_\Xi(x) > \tau_\Pi(z) + 1, \quad \pi_{\mathcal{A}}(x) + \pi_\Xi(x) \leq \pi_\Pi(z) + 1, \sigma_{\mathcal{A}}(x) + \sigma_\Xi(x) \leq \sigma_\Pi(x) + 1.$$

From Lemma 1, we have $y_{n,m,v} \in \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi, r)$. This is a contradiction for Equation (3). Thus, $\Xi_r^\star \leq \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi, r)$.

We will now prove this relationship $\Xi_r^\star \geq \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi_r^\star, r)$. Assume that $\Xi_r^\star \not\geq \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi_r^\star, r)$; then, there exist $z \in \pounds$ and $n, m, v \in \xi_0$ such that

$$\tau_{\Xi_r^\star}(z) < n \leq \tau_{\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi_r^\star,r)}(z), \quad \pi_{\Xi_r^\star}(z) \geq m > \pi_{\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi_r^\star,r)}(z), \quad \sigma_{\Xi_r^\star}(z) \geq v > \sigma_{\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi_r^\star,r)}(z). \tag{4}$$

Since $\tau_{\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi_r^\star,r)}(z) \geq n$, $\pi_{\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi_r^\star,r)}(z) < m$ and $\sigma_{\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi_r^\star,r)}(z) < v$, we have $y_{n,m,v} \in \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi_r^\star, r)$. So, there exists at least one $x \in \pounds$ with $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}\tau\pi\sigma}(y_{n,m,v}, r)$ such that

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi_r^\star}(x) > 1, \quad \pi_{\mathcal{A}}(x) + \pi_{\Xi_r^\star}(x) \leq 1, \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Xi_r^\star}(x) \leq 1.$$

Thus, $\Xi_r^\star(x) \neq 0$. Suppose that $n_1 = \tau_{\Xi_r^\star}(x)$, $m_1 = \pi_{\Xi^\star}(x)$ and $v_1 = \sigma_{\Xi_r^\star}(x)$. Then, $x_{n_1,m_1,v_1} \in \Xi_r^*$ and $n_1 + \tau_{\mathcal{A}}(x) > 1$, $m_1 + \pi_{\mathcal{A}} \leq 1$, $v_1 + \sigma_{\mathcal{A}}(x) \leq 1$, so that $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}\tau\pi\sigma}(x_{n_1,m_1,v_1}, r)$. Now, $x_{n_1,m_1,v_1} \in \Xi_r^*$ implies there is at least one $z^{'} \in \pounds$ such that $\tau_{\mathcal{B}}(z^{'}) + \tau_\Xi(z^{'}) - 1 > \tau_\Pi(z^{'})$, $\pi_{\mathcal{B}}(z^{'}) + \pi_\Xi(z^{'}) - 1 \leq \pi_\Pi(z^{'})$, $\sigma_{\mathcal{B}}(z^{'}) + \sigma_\Xi(z^{'}) - 1 \leq \sigma_\Pi(z^{'})$ $\forall$, $\mathcal{P}^\tau(\Pi) \geq r$, $\mathcal{P}^\pi(\Pi) \leq 1 - r$, $\mathcal{P}^\sigma(\Pi) \leq 1 - r$ and $\mathcal{B} \in \mathcal{Q}_{\mathcal{T}\tau\pi\sigma}(x_{n_1,m_2,v_1}, r)$. This is also true for $\mathcal{A}$. So, there is at least one $z^{''} \in \pounds$ such that $\tau_{\mathcal{A}}(z^{''}) + \tau_\Xi(z^{''}) - 1 > \tau_\Pi(z^{'})$, $\pi_{\mathcal{A}}(z^{''}) + \pi_\Xi(z^{'}) - 1 \leq \pi_\Pi(z^{''})$, $\sigma_{\mathcal{A}}(z^{''}) + \sigma_\Xi(z^{''}) - 1 \leq \sigma_\Pi(z^{'})$. Since $\mathcal{A}$ is an arbitrary and $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}\tau\pi\sigma}(y_{n,m,v}, r)$, then $\tau_{\Xi_r^\star}(z) > n$, $\pi_{\Xi_r^\star}(z) \leq m$, $\sigma_{\Xi_r^\star}(z) \leq v$. This contradicts Equation (4). Thus, $\Xi_r^\star \geq \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi_r^\star, r)$.

(4) Using (3) we obtain that $(\Xi_r^\star)_r^\star = \mathcal{C}_{\mathcal{T}\tau\pi\sigma}((\Xi_r^\star)_r^\star, r) \leq \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi_r^\star, r) \leq \Xi_r^\star$.

(5) Straightforward.

(6) ($\Rightarrow$) Since $\Xi, \Theta \leq \Xi \vee \Theta$. By (1), we have $\Xi_r^\star \leq (\Xi \vee \Theta)_r^\star$ and $\Theta_r^\star \leq (\Xi \vee \Theta)_r^\star$. Thus, $\Xi_r^\star \vee \Theta_r^\star \leq (\Xi \vee \Theta)_r^\star$.

($\Leftarrow$) Let $\Xi_r^\star \vee \Theta_r^\star \not\geq (\Xi \vee \Theta)_r^\star$; then, there exist $z \in \pounds$ and $n.m.v \in \xi_0$ such that

$$\begin{gathered}
\tau_{(\Xi_r^\star \vee \Theta_r^\star)}(z) < n \leq \tau_{(\Xi \cup \Theta)_r^\star}(z), \\
\pi_{(\Xi_r^\star \vee \Theta_r^\star)}(z) \geq m > \pi_{(\Xi \cup \Theta)_r^\star}(z), \\
\sigma_{(\Xi_r^\star \vee \Theta_r^\star)}(z) \geq v > \sigma_{(\Xi \cup \Theta)_r^\star}(z).
\end{gathered} \tag{5}$$

Since $\tau_{(\Xi_r^\star \vee \Theta_r^\star)}(z) < n$, $\pi_{(\Xi_r^\star \vee \Theta_r^\star)}(z) \geq m$, $\sigma_{(\Xi_r^\star \vee \Theta_r^\star)}(z) \geq v$, we obtain $\tau_{\Xi_r^\star}(z) < n$, $\pi_{\Xi_r^\star}(z) \geq m$, $\sigma_{\Xi_r^\star}(z) \geq v$ or $\tau_{\Theta_r^\star}(z) < n$, $\pi_{\Theta_r^\star}(z) \geq m$, $\sigma_{\Theta_r^\star}(z) \geq v$. So, there exists $\mathcal{A}_1 \in \mathcal{Q}_{\mathcal{T}\tau\pi\sigma}(y_{n,m,v}, r)$ such that for each $x \in \pounds$ and for some $\mathcal{P}^\tau(\Pi_1) \geq r$, $\mathcal{P}^\pi(\Pi_1) \leq 1 - r$, $\mathcal{P}^\sigma(\Pi_1) \leq 1 - r$ we have,

$$\begin{gathered}
\tau_{\mathcal{A}_1}(x) + \tau_\Xi(x) - 1 \leq \tau_{\Pi_1}(x), \\
\pi_{\mathcal{A}_1}(x) + \pi_\Xi(x) - 1 > \pi_{\Pi_1}(x), \\
\sigma_{\mathcal{A}_1}(x) + \sigma_\Xi(x) - 1 > \sigma_{\Pi_1}(x).
\end{gathered}$$

Similarly, there exists $\mathcal{A}_2 \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ such that for each $x \in \pounds$ and for some $\mathcal{P}^{\tau}(\Pi_2) \geq r$, $\mathcal{P}^{\pi}(\Pi_2) \leq 1-r$, $\mathcal{P}^{\sigma}(\Pi_2) \leq 1-r$ we have,

$$\tau_{\mathcal{A}_2}(x) + \tau_{\Xi}(x) - 1 \leq \tau_{\Pi_2}(x),$$

$$\pi_{\mathcal{A}_2}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi_2}(x),$$

$$\sigma_{\mathcal{A}_2}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi_2}(x).$$

Since $\mathcal{A} = \mathcal{A}_1 \wedge \mathcal{A}_2 \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ and by (P_3), we obtain $\mathcal{P}^{\tau}(\Pi_1 \wedge \Pi_2) \geq r$, $\mathcal{P}^{\pi}(\Pi_1 \wedge \Pi_2) \leq 1-r$, $\mathcal{P}^{\sigma}(\Pi_1 \wedge \Pi_2) \leq 1-r$. Thus, for each $x \in \xi$,

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi\vee\Theta}(x) - 1 \leq \tau_{\Pi_1\wedge\Pi_2}(x),$$

$$\pi_{\mathcal{A}}(x) + \pi_{\Xi\vee\Theta}(x) - 1 > \pi_{\Pi_1\wedge\Pi_2}(x),$$

$$\sigma_{\mathcal{A}}(x) + \sigma_{\Xi\vee\Theta}(x) - 1 > \sigma_{\Pi_1\wedge\Pi_2}(x).$$

Hence, $\tau_{(\Xi\cup\Theta)^{\star}_r}(z) < n$, $\pi_{(\Xi\cup\Theta)^{\star}_r}(z) \geq m$, $\sigma_{(\Xi\cup\Theta)^{\star}_r}(z) \geq v$. This is a conflict with Equation (5). Thus, $\Xi^{\star}_r \vee \Theta^{\star}_r \geq (\Xi \vee \Theta)^{\star}_r$.

(7) and (8) are obvious. □

Example 4. *Suppose that* $\pounds = \{l_1, l_2, l_3\}$*; define the* svns $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \Theta \in \xi^{\pounds}$ *as follows*

$$\mathcal{A}_1 = \langle (0.8, 0.8, 0.8), (0.8, 0.8, 0.8), (0.8, 0.8, 0.8) \rangle,$$

$$\mathcal{A}_2 = \langle (0.6, 0.6, 0.6), (0.6, 0.6, 0.6), (0.6, 0.6, 0.6) \rangle,$$

$$\mathcal{A}_3 = \langle (0.5, 0.5, 0.5), (0.5, 0.5, 0.5), (0.5, 0.5, 0.5) \rangle,$$

$$\Theta = \langle (0.2, 0.2, 0.2), (0, 0, 0), (0, 0, 0) \rangle.$$

Define the mappings $\mathcal{T}^{\tau}, \mathcal{T}^{\pi}, \mathcal{T}^{\sigma} : \xi^{\pounds} \to \xi$ *and* $\mathcal{P}^{\tau}, \mathcal{P}^{\pi}, \mathcal{P}^{\sigma} : \xi^{\pounds} \to \xi$ *as follows:*

$$\mathcal{T}^{\tau}(\Xi) = \begin{cases} 1, & \text{if } \Xi = \overline{0}, \\ 1, & \text{if } \Xi = \overline{1}, \\ \frac{1}{2}, & \text{if } \Xi = \mathcal{A}_1, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{P}^{\tau}(\Xi) = \begin{cases} 1, & \text{if } \Xi = \overline{0}, \\ \frac{1}{4}, & \text{if } \Xi = \Theta, \\ \frac{1}{2}, & \text{if } \overline{0} < \Xi < \Theta, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{T}^{\pi}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ 0, & \text{if } \Xi = \overline{1}, \\ \frac{1}{2}, & \text{if } \Xi = \mathcal{A}_2, \\ 1, & \text{otherwise,} \end{cases} \qquad \mathcal{P}^{\pi}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ \frac{3}{4}, & \text{if } \Xi = \Theta, \\ \frac{1}{2}, & \text{if } \overline{0} < \Xi < \Theta, \\ 1, & \text{otherwise,} \end{cases}$$

$$\mathcal{T}^{\sigma}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ 0, & \text{if } \Xi = \overline{1}, \\ \frac{1}{2}, & \text{if } \Xi = \mathcal{A}_3, \\ 1, & \text{otherwise,} \end{cases} \qquad \mathcal{P}^{\sigma}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ \frac{2}{3}, & \text{if } \Xi = \Theta, \\ \frac{1}{2}, & \text{if } \overline{0} < \Xi < \Theta, \\ 1, & \text{otherwise,} \end{cases}$$

Let $\mathcal{B} = \langle (0.4, 0.4, 0.4), (0.4, 0.4, 0.4), (0.4, 0.4, 0.4) \rangle$*. Then,* $\overline{0} = (\mathcal{B}^{\star}_{\frac{1}{2}})^{\star}_{\frac{1}{2}} \neq \mathcal{B}^{\star}_{\frac{1}{2}} = \overline{0.2}$.

Theorem 5. *Let* $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ *be an svnpts and* $\{\Xi_i : i \in J\} \subset \xi^{\pounds}$*. Then:*

(1) $(\bigvee(\Xi_i)^{\star}_r : i \in J) \leq (\bigvee \Xi_i : i \in J)^{\star}_r$.
(2) $(\bigwedge \Xi_i : i \in J)^{\star}_r \leq (\bigwedge(\Xi_i)^{\star}_r : i \in J)$.

Proof. (1) Since $\Xi_i \leq \bigvee \Xi_i$, for all $i \in J$, by Theorem 4(1), we obtain $(\Xi_i)_r^\star \leq (\bigvee \Xi_i)_r^\star$, for every $i \in J$. Thus,

$$(\bigvee (\Xi_i)_r^\star : \ i \in J) \leq (\bigvee \Xi_i : \ i \in J)_r^\star.$$

(2) Since $\bigwedge \Xi_i \leq \mathcal{A}_i$, and by (1) in Theorem 4, we have $(\bigwedge \Xi_i)_r^\star \leq (\Xi_i)_r^\star$, $\forall\, i \in J$. Hence,

$$(\bigwedge (\Xi_i)_r^\star : \ i \in J) \geq (\bigwedge \Xi_i : \ i \in J)_r^\star.$$

□

Remark 2. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ be an svnpts and $\Xi \in \xi^{\pounds}$; we can define*

$$\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) = \Xi \cup \Xi_r^\star, \quad \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) = \Xi \wedge ((\Xi^c)_r^\star)^c.$$

Clearly, $\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}$ is an svn-closure operator and $(\mathcal{T}^\tau(\mathcal{P}^\tau), \mathcal{T}^\pi(\mathcal{P}^\pi), \mathcal{T}^\sigma(\mathcal{P}^\sigma))$ is the svnt generated by $\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}$, i.e.,

$$\mathcal{T}^\star(\mathcal{I})(\Xi) = \bigvee \{r \mid \ \mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi^c, r) = \Xi^c\}.$$

Now, if $\mathcal{P}^{\tau\pi\sigma} = \mathcal{P}_0^{\tau\pi\sigma}$, $\forall\, \Xi \in \xi^{\pounds}$, then $\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) = \Xi \cup \Xi_r^\star = \Xi \vee \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) = \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$. So, $\mathcal{T}^{\tau\star}(\mathcal{P}_0^\tau) = \mathcal{T}^\tau$, $\mathcal{T}^{\pi\star}(\mathcal{P}_0^\pi) = \mathcal{T}^\pi$ and $\mathcal{T}^{\sigma\star}(\mathcal{P}_0^\sigma) = \mathcal{T}^\sigma$.

Theorem 6. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ be an svnpts and $\Xi \in \xi^{\pounds}$ and $r \in \xi_0$ and $\Xi \in \xi^{\pounds}$. Then*

(1) $\mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi \vee \Theta, r) \leq \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \vee \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Theta, r)$.
(2) $\mathcal{I}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \leq \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \leq \Xi \leq \mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \leq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$.
(3) $\mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi \wedge \Theta, r) = \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \wedge \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Theta, r)$.
(4) $\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi^c, r) = [\mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)]^c$ *and* $[\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)]^c = \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi^c, r)$.

Proof. Straightforward from $\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}$, $\mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}$ and $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}$. □

Theorem 7. *Let $(\pounds, \mathcal{T}_1^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ and $(\pounds, \mathcal{T}_2^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ be svnptss and $\mathcal{T}_1^\tau \leq \mathcal{T}_2^\tau$, $\mathcal{T}_1^\pi \geq \mathcal{T}_2^\pi$, $\mathcal{T}_1^\sigma \geq \mathcal{T}_2^\sigma$. Then,*

(1) $\Xi_r^\star(\mathcal{T}_2^\tau, \mathcal{P}^\tau) \leq \Xi_r^\star(\mathcal{T}_1^\tau, \mathcal{P}^\tau)$, $\Xi_r^\star(\mathcal{T}_2^\pi, \mathcal{P}^\pi) \geq \Xi_r^\star(\mathcal{T}_1^\pi, \mathcal{P}^\pi)$ *and* $\Xi_r^\star(\mathcal{T}_2^\sigma, \mathcal{P}^\sigma) \geq \Xi_r^\star(\mathcal{T}_1^\sigma, \mathcal{P}^\sigma)$.
(2) $\mathcal{T}_1^{\star\tau}(\mathcal{P}^\tau) \leq \mathcal{T}_2^{\star\tau}(\mathcal{P}^\tau)$, $\mathcal{T}_1^{\star\pi}(\mathcal{P}^\pi) \geq \mathcal{T}_2^{\star\pi}(\mathcal{P}^\pi)$, $\mathcal{T}_1^{\star\sigma}(\mathcal{P}^\sigma) \geq \mathcal{T}_2^{\star\sigma}(\mathcal{P}^\sigma)$.

Proof. (1) Let

$$\Xi_r^\star(\mathcal{T}_2^\tau, \mathcal{P}^\tau) \not\leq \Xi_r^\star(\mathcal{T}_1^\tau, \mathcal{P}^\tau), \quad \Xi_r^\star(\mathcal{T}_2^\pi, \mathcal{P}^\pi) \not\geq \Xi_r^\star(\mathcal{T}_1^\pi, \mathcal{P}^\pi),$$

$$\Xi_r^\star(\mathcal{T}_2^\sigma, \mathcal{P}^\sigma) \not\geq \Xi_r^\star(\mathcal{T}_1^\sigma, \mathcal{P}^\sigma),$$

then, there exist $z \in \pounds$ and $n, m, v \in \xi_0$ such that

$$\tau_{\Xi_r^\star(\mathcal{T}_2^\tau, \mathcal{P}^\tau)}(z) \geq n > \tau_{\Xi_r^\star(\mathcal{T}_1^\tau, \mathcal{P}^\tau)}(z),$$

$$\pi_{\Xi_r^\star(\mathcal{T}_2^\pi, \mathcal{P}^\pi)}(z) < m \leq \pi_{\Xi_r^\star(\mathcal{T}_1^\pi, \mathcal{P}^\pi)}(z), \tag{6}$$

$$\sigma_{\Xi_r^\star(\mathcal{T}_2^\sigma, \mathcal{P}^\sigma)}(z) < v \leq \sigma_{\Xi_r^\star(\mathcal{T}_1^\sigma, \mathcal{P}^\sigma)}(z).$$

Since $\tau_{\Xi_r^\star(\mathcal{T}_1^\tau, \mathcal{P}^\tau)}(z) < n$, $\pi_{\Xi_r^\star(\mathcal{T}_1^\pi, \mathcal{P}^\pi)}(z) \geq m$, $\sigma_{\Xi_r^\star(\mathcal{T}_1^\sigma, \mathcal{P}^\sigma)}(z) \geq v$, there exists $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}_1^{\tau\pi\sigma}}(y_{n,m,v}, r)$ with $\mathcal{P}^\tau(\Pi) \geq r$, $\mathcal{P}^\pi(\Pi) \leq 1 - r$, $\mathcal{P}^\sigma(\Pi) \leq 1 - r$, such that for each $x \in \pounds$,

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(x), \quad \pi_{\mathcal{A}}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi}(x), \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi}(x).$$

Since $\mathcal{T}_1^\tau \leq \mathcal{T}_2^\tau$, $\mathcal{T}_1^\pi \geq \mathcal{T}_2^\pi$, $\mathcal{T}_1^\sigma \geq \mathcal{T}_2^\sigma$, we have $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}_2^{\tau\pi\sigma}}(y_{n,m,v}, r)$. Hence, $\tau_{\Xi_r^\star(\mathcal{T}_2^\tau, \mathcal{P}^\tau)}(z) < n$, $\pi_{\Xi_r^\star(\mathcal{T}_2^\pi, \mathcal{P}^\pi)}(z) \geq m$, $\sigma_{\Xi_r^\star(\mathcal{T}_2^\sigma, \mathcal{P}^\sigma)}(z) \geq v$. This is a contradiction of Equation (6).

(2) Similar to part (1). □

Theorem 8. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}_1^{\tau\pi\sigma})$ and $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}_2^{\tau\pi\sigma})$ be svnptss and $\mathcal{P}_1^\tau \leq \mathcal{P}_2^\tau$, $\mathcal{P}_1^\pi \geq \mathcal{P}_2^\pi$, $\mathcal{P}_1^\sigma \geq \mathcal{P}_2^\sigma$. Then,*

(1) $\Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \geq \Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^\pi)$, $\Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^\pi) \leq \Xi_r^\star(\mathcal{P}_2^\pi, \mathcal{T}^\pi)$ *and* $\Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \leq \Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^\tau)$.

(2) $\mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau) \leq \mathcal{T}^{\star\tau}(\mathcal{P}_2^\tau)$, $\mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi) \geq \mathcal{T}^{\star\pi}(\mathcal{P}_2^\pi)$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma) \geq \mathcal{T}^{\star\sigma}(\mathcal{P}_2^\sigma)$.

Proof. The same method as the proof of Theorem 7. □

Theorem 9. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma})$ be an svnts and $\mathcal{P}_1^{\tau\pi\sigma}$, $\mathcal{P}_2^{\tau\pi\sigma}$ be two svn-primals on £. Then, $\forall\, \Xi \in \xi^{\pounds}$ and $r \in \xi_0$,*

(1)

$$\Xi_r^\star(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau, \mathcal{T}^\tau) = \Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \vee \Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^\tau),$$
$$\Xi_r^\star(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi, \mathcal{T}^\pi) = \Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^\pi) \wedge \Xi_r^\star(\mathcal{P}_2^\pi, \mathcal{T}^\pi),$$
$$\Xi_r^\star(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma, \mathcal{T}^\sigma) = \Xi_r^\star(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma) \wedge \Xi_r^\star(\mathcal{P}_2^\sigma, \mathcal{T}^\sigma).$$

(2)

$$\Xi_r^\star(\mathcal{P}_1^\tau \vee \mathcal{P}_2^\tau, \mathcal{T}^\tau) = \Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^{\star\tau}(\mathcal{P}_2^\tau)) \wedge \Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau))),$$
$$\Xi_r^\star(\mathcal{P}_1^\pi \vee \mathcal{P}_2^\pi, \mathcal{T}^\pi) = \Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^{\star\pi}(\mathcal{P}_2^\pi)) \vee \Xi_r^\star(\mathcal{P}_2^\pi, \mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi)),$$
$$\Xi_r^\star(\mathcal{P}_1^\sigma \vee \mathcal{P}_2^\sigma, \mathcal{T}^\sigma) = \Xi_r^\star(\mathcal{P}_1^\sigma, \mathcal{T}^{\star\sigma}(\mathcal{P}_2^\sigma)) \vee \Xi_r^\star(\mathcal{P}_2^\sigma, \mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma)).$$

Proof. (1) Let

$$\Xi_r^\star(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau, \mathcal{T}^\tau)) \not\leq \Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \vee \Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^\tau),$$

$$\Xi_r^\star(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi, \mathcal{T}^\pi) \not\geq \Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^\pi) \wedge \Xi_r^\star(\mathcal{P}_2^\pi, \mathcal{T}^\pi),$$

$$\Xi_r^\star(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma, \mathcal{T}^\sigma) \not\geq \Xi_r^\star(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma) \wedge \Xi_r^\star(\mathcal{P}_2^\sigma, \mathcal{T}^\sigma).$$

Then there exist $z \in \pounds$ and $n, m, v \in \xi_0$ s.t.

$$\tau_{\Xi_r^\star(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau, \mathcal{T}^\tau))}(z) \geq n > \tau_{\Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau)}(z) \vee \tau_{\Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^\tau)}(z),$$

$$\pi_{\Xi_r^\star(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi, \mathcal{T}^\pi)}(z) < m \leq \pi_{\Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^\pi)}(z) \wedge \pi_{\Xi_r^\star(\mathcal{P}_2^\pi, \mathcal{T}^\pi)}(z), \tag{7}$$

$$\sigma_{\Xi_r^\star(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma, \mathcal{T}^\sigma)}(z) < v \leq \sigma_{\Xi_r^\star(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma)}(z) \wedge \sigma_{\Xi_r^\star(\mathcal{P}_2^\sigma, \mathcal{T}^\sigma)}(z).$$

Since $[\tau_{\Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau)}(z) \vee \tau_{\Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^\tau)}(z)] < n$, $[\pi_{\Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^\pi)}(z) \wedge \pi_{\Xi_r^\star(\mathcal{P}_2^\pi, \mathcal{T}^\pi)}(z)] \geq m$, $[\sigma_{\Xi_r^\star(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma)}(z) \wedge \sigma_{\Xi_r^\star(\mathcal{P}_2^\sigma, \mathcal{T}^\sigma)}(z)] \geq v$, we obtain $\tau_{\Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau)}(z) < n$, $\pi_{\Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^\pi)}(z) \geq m$, $\sigma_{\Xi_r^\star(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma)}(z) \geq v$ and $\tau_{\Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^\tau)}(z) < n$, $\pi_{\Xi_r^\star(\mathcal{P}_2^\pi, \mathcal{T}^\pi)}(z) \geq m$, $\sigma_{\Xi_r^\star(\mathcal{P}_2^\sigma, \mathcal{T}^\sigma)}(z) \geq v$.

First, by taking the first part, $\tau_{\Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau)}(z) < n$, $\pi_{\Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^\pi)}(z) \geq m$, $\sigma_{\Xi_r^\star(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma)}(z) \geq v$, there will be $\mathcal{A}_1 \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ with $\mathcal{P}_1^\tau(\Pi_1) \geq r$, $\mathcal{P}_1^\pi(\Pi_1) \leq 1 - r$, $\mathcal{P}_1^\sigma(\Pi_1) \leq 1 - r$, such that $\forall$, $x \in \pounds$,

$$\tau_{\mathcal{A}_1}(x) + \tau_\Xi(x) - 1 \leq \tau_{\Pi_1}(x),$$

$$\pi_{\mathcal{A}_1}(x) + \pi_\Xi(x) - 1 > \pi_{\Pi_1}(x),$$

$$\sigma_{\mathcal{A}_1}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi_1}(x).$$

Secondly, by taking the second part, $\tau_{\Xi_r^\star(\mathcal{P}_2^\tau,\mathcal{T}^\tau)}(z) < n,\ \pi_{\Xi_r^\star(\mathcal{P}_2^\pi,\mathcal{T}^\pi)}(z) \geq m,\ \sigma_{\Xi_r^\star(\mathcal{P}_2^\sigma,\mathcal{T}^\sigma)}(z) \geq v$, there will be $\mathcal{A}_2 \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ with $\mathcal{P}_2^\tau(\Pi_2) \geq r$, $\mathcal{P}_2^\pi(\Pi_2) \leq 1-r$, $\mathcal{P}_2^\sigma(\Pi_2) \leq 1-r$, such that $\forall$, $x \in \pounds$,

$$\tau_{\mathcal{A}_2}(x) + \tau_{\Xi}(x) - 1 \leq \tau_{\Pi_2}(x),$$

$$\pi_{\mathcal{A}_2}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi_2}(x),$$

$$\sigma_{\mathcal{A}_2}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi_2}(x).$$

Thus, $\forall\, y \in \pounds$, we obtain

$$\tau_{(\mathcal{A}_1 \wedge \mathcal{A}_2)}(x) + \tau_{\Xi}(x) - 1 > \tau_{(\Pi_1 \wedge \Pi_2)}(x),$$

$$\pi_{(\mathcal{A}_1 \vee \mathcal{A}_2)}(x) + \pi_{\Xi}(x) - 1 \leq \pi_{(\Pi_1 \vee \Pi_2)}(x),$$

$$\sigma_{(\mathcal{A}_1 \vee \mathcal{A}_2)}(x) + \sigma_{\Xi}(x) - 1 \leq \sigma_{(\Pi_1 \vee \Pi_2)}(x).$$

Since $(\mathcal{A}_1 \wedge \mathcal{A}_2) \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ and

$$(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau)(\Pi_1 \wedge \Pi_2) \geq r, \quad (\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi)(\Pi_1 \vee \Pi_2) \leq 1-r,$$

$$(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi)(\Pi_1 \vee \Pi_2) \leq 1-r,$$

we obtain that $\tau_{\Xi_r^\star(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau,\mathcal{T}^\tau))}(z) \leq n$, $\pi_{\Xi_r^\star(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi,\mathcal{T}^\pi)}(z) > m$, $\sigma_{\Xi_r^\star(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma,\mathcal{T}^\sigma)}(z) > v$ and this is a contradiction of Equation (7). So,

$$\Xi_r^\star(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau, \mathcal{T}^\tau)) \leq \Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \vee \Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^\tau),$$

$$\Xi_r^\star(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi, \mathcal{T}^\pi) \geq \Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^\pi) \wedge \Xi_r^\star(\mathcal{P}_2^\pi, \mathcal{T}^\pi),$$

$$\Xi_r^\star(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma, \mathcal{T}^\sigma) \geq \Xi_r^\star(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma) \wedge \Xi_r^\star(\mathcal{P}_2^\sigma, \mathcal{T}^\sigma).$$

On the other side $\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau \leq \mathcal{P}_1^\tau$, $\mathcal{P}_1^\pi \vee \mathcal{P}_2^\pi \geq \mathcal{P}_1^\pi$, $\mathcal{P}_1^\sigma \vee \mathcal{P}_2^\sigma \geq \mathcal{P}_1^\sigma$ and $\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau \leq \mathcal{P}_2^\tau$, $\mathcal{P}_1^\pi \vee \mathcal{P}_2^\pi \geq \mathcal{P}_2^\pi$, $\mathcal{P}_1^\sigma \vee \mathcal{P}_2^\sigma \geq \mathcal{P}_2^\sigma$, so by Theorem 4(2),

$$\Xi_r^\star(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau) \geq \Xi_r^\star(\mathcal{P}_1^\tau) \vee \Xi_r^\star(\mathcal{P}_2^\tau), \quad \Xi_r^\star(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi) \leq \Xi_r^\star(\mathcal{P}_1^\tau) \wedge \Xi_r^\star(\mathcal{P}_2^\pi),$$

$$\Xi_r^\star(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma) \leq \Xi_r^\star(\mathcal{P}_1^\sigma) \wedge \Xi_r^\star(\mathcal{P}_2^\sigma).$$

Hence, $\Xi_r^\star(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau, \mathcal{T}^\tau) = \Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \vee \Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^\tau)$, $\Xi_r^\star(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi, \mathcal{T}^\pi) = \Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^\pi) \wedge \Xi_r^\star(\mathcal{P}_2^\pi, \mathcal{T}^\pi)$ and $\Xi_r^\star(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma, \mathcal{T}^\sigma) = \Xi_r^\star(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma) \wedge \Xi_r^\star(\mathcal{P}_2^\sigma, \mathcal{T}^\sigma)$.

(2) Similar to part (1). □

From the previous theory, we can see the important result that $(\mathcal{T}^{\star\tau}(\mathcal{P}^\tau), \mathcal{T}^{\star\pi}(\mathcal{P}^\pi), \mathcal{T}^{\star\sigma}(\mathcal{P}^\star))$ (for short $\mathcal{T}^\star(\mathcal{P})$) and $([\mathcal{T}^{\star\tau}(\mathcal{P}^\tau)]^\star(\mathcal{P}^\tau), [\mathcal{T}^{\star\pi}(\mathcal{P}^\pi)]^\star((\mathcal{P}^\pi))$, $[\mathcal{T}^{\star\sigma}(\mathcal{P}^\star)]^\star(\mathcal{P}^\sigma))$ (for short $\mathcal{T}^{\star\star}$) are equal for any *svn-primals* on $\pounds$.

Theorem 10. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ be an svnpts. Then, for each $\Xi \in \xi^{\pounds}$ and $r \in \xi_0$,*

(1) $\Xi_r^\star(\mathcal{P}^\tau) = \Xi_r^\star(\mathcal{P}^\tau, \mathcal{T}^{\star\tau})$, $\Xi_r^\star(\mathcal{P}^\pi) = \Xi_r^\star(\mathcal{P}^\pi, \mathcal{T}^{\star\pi})$ *and* $\Xi_r^\star(\mathcal{P}^\sigma) = \Xi_r^\star(\mathcal{P}^\sigma, \mathcal{T}^{\star\sigma}))$.

(2) $(\mathcal{T}^{\star\tau}(\mathcal{P}^{\tau}) = \mathcal{T}^{\star\star\tau}, \mathcal{T}^{\star\pi}(\mathcal{P}^{\pi}) = \mathcal{T}^{\star\star\pi}, \mathcal{T}^{\star\sigma}(\mathcal{P}^{\pi})) = \mathcal{T}^{\star\star\sigma}$.

Proof. By putting $\mathcal{P}_1^{\tau} = \mathcal{P}_2^{\tau}$, $\mathcal{P}_1^{\pi} = \mathcal{P}_2^{\pi}$ and $\mathcal{P}_1^{\sigma} = \mathcal{P}_2^{\sigma}$ in the second part of the above theorem, we obtain the required proof. □

Theorem 11. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma})$ be an svnts and $\mathcal{P}_1^{\tau\pi\sigma}$, $\mathcal{P}_2^{\tau\pi\sigma}$ be two svn-primals on £. Then, $\forall$ $r \in \xi_0$ and $\Xi \in \xi^{\pounds}$, $\mathcal{T}^{\star\tau}(\mathcal{P}_1^{\tau} \wedge \mathcal{P}_2^{\tau}) = \mathcal{T}^{\star\tau}(\mathcal{P}_1^{\tau}) \wedge \mathcal{T}^{\star\tau}(\mathcal{P}_2^{\tau})$, $\mathcal{T}^{\star\pi}(\mathcal{P}_1^{\pi} \wedge \mathcal{P}_2^{\pi}) = \mathcal{T}^{\star\pi}(\mathcal{P}_1^{\pi}) \vee \mathcal{T}^{\star\pi}(\mathcal{P}_2^{\pi})$ and $\mathcal{T}^{\star\sigma}(\mathcal{P}_1^{\sigma} \wedge \mathcal{P}_2^{\sigma}) = \mathcal{T}^{\star\sigma}(\mathcal{P}_1^{\sigma}) \vee \mathcal{T}^{\star\sigma}(\mathcal{P}_2^{\sigma})$.*

Proof. ($\Rightarrow$): Since $\mathcal{P}_1^{\tau} \geq \mathcal{P}_1^{\tau} \wedge \mathcal{P}_2^{\tau}$, $\mathcal{P}_1^{\pi} \leq \mathcal{P}_1^{\pi} \vee \mathcal{P}_2^{\pi}$, $\mathcal{P}_1^{\sigma} \leq \mathcal{P}_1^{\sigma} \vee \mathcal{P}_2^{\sigma}$ and $\mathcal{P}_2^{\tau} \geq \mathcal{P}_1^{\tau} \wedge \mathcal{P}_2^{\tau}$, $\mathcal{P}_2^{\pi} \leq \mathcal{P}_1^{\pi} \vee \mathcal{P}_2^{\pi}$, $\mathcal{P}_2^{\sigma} \leq \mathcal{P}_1^{\sigma} \vee \mathcal{P}_2^{\sigma}$, by Theorem 8(2) we obtain $\mathcal{T}^{\star\tau}(\mathcal{P}_1^{\tau}) \geq \mathcal{T}^{\star\tau}(\mathcal{P}_1^{\tau} \wedge \mathcal{P}_2^{\tau})$, $\mathcal{T}^{\star\pi}(\mathcal{P}_1^{\pi}) \leq \mathcal{T}^{\star\pi}(\mathcal{P}_1^{\pi} \vee \mathcal{P}_2^{\pi})$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_1^{\sigma}) \leq \mathcal{T}^{\star\sigma}(\mathcal{P}_1^{\sigma} \vee \mathcal{P}_2^{\sigma})$ and $\mathcal{T}^{\star\tau}(\mathcal{P}_2^{\tau}) \geq \mathcal{T}^{\star\tau}(\mathcal{P}_1^{\tau} \wedge \mathcal{P}_2^{\tau})$, $\mathcal{T}^{\star\pi}(\mathcal{P}_2^{\pi}) \leq \mathcal{T}^{\star\pi}(\mathcal{P}_1^{\pi} \vee \mathcal{P}_2^{\pi})$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_2^{\sigma}) \leq \mathcal{T}^{\star\sigma}(\mathcal{P}_1^{\sigma} \vee \mathcal{P}_2^{\sigma})$. Hence,

$$\mathcal{T}^{\star\tau}(\mathcal{P}_1^{\tau} \wedge \mathcal{P}_2^{\tau}) \leq \mathcal{T}^{\star\tau}(\mathcal{P}_1^{\tau}) \wedge \mathcal{T}^{\star\tau}(\mathcal{P}_2^{\tau}),$$

$$\mathcal{T}^{\star\pi}(\mathcal{P}_1^{\pi} \wedge \mathcal{P}_2^{\pi}) \geq \mathcal{T}^{\star\pi}(\mathcal{P}_1^{\pi}) \vee \mathcal{T}^{\star\pi}(\mathcal{P}_2^{\pi}),$$

$$\mathcal{T}^{\star\sigma}(\mathcal{P}_1^{\sigma} \wedge \mathcal{P}_2^{\sigma}) \geq \mathcal{T}^{\star\sigma}(\mathcal{P}_1^{\sigma}) \vee \mathcal{T}^{\star\sigma}(\mathcal{P}_2^{\sigma}).$$

($\Leftarrow$): Suppose that $(\mathcal{T}^{\star\tau}(\mathcal{P}_1^{\tau}) \wedge \mathcal{T}^{\star\tau}(\mathcal{P}_2^{\tau}))(\Xi) \geq r$, $(\mathcal{T}^{\star\pi}(\mathcal{P}_1^{\pi}) \vee \mathcal{T}^{\star\pi}(\mathcal{P}_2^{\pi}))(\Xi) \leq 1 - r$, $(\mathcal{T}^{\star\sigma}(\mathcal{P}_1^{\sigma}) \vee \mathcal{T}^{\star\sigma}(\mathcal{P}_2^{\sigma})(\Xi) \leq 1 - r$. Then, $\mathcal{T}^{\star\tau}(\mathcal{P}_1^{\tau})(\Xi) \geq r$, $\mathcal{T}^{\star\pi}(\mathcal{P}_1^{\pi})(\Xi) \leq 1 - r$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_1^{\sigma})(\Xi) \leq 1 - r$ and $\mathcal{T}^{\star\tau}(\mathcal{P}_2^{\tau})(\Xi) \geq r$, $\mathcal{T}^{\star\pi}(\mathcal{P}_2^{\pi})(\Xi) \leq 1 - r$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_2^{\sigma})(\Xi) \leq 1 - r$; that means $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi^c, r) = \Xi^c \vee [\Xi^c]_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}) = \Xi^c$ and $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi^c, r) = \Xi^c \vee [\Xi^c]_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}) = \Xi^c$. Thus, $[\Xi^c]_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}) \leq \Xi^c$ and $[\Xi^c]_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}) \leq \Xi^c$. So, $[\Xi^c]_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma}) \vee [\Xi^c]_r^{\star}(\mathcal{P}_2^{\tau\pi\sigma}) \leq \Xi^c$. From Theorem 9(1), we have $[\Xi^c]_r^{\star}(\mathcal{P}_1^{\tau\pi\sigma} \wedge \mathcal{P}_2^{\tau\pi\sigma}) \leq \Xi^c$. Therefore, $\mathcal{T}^{\star\tau}(\mathcal{P}_1^{\tau} \wedge \mathcal{P}_2^{\tau})(\Xi) \geq r$, $\mathcal{T}^{\star\pi}(\mathcal{P}_1^{\pi} \vee \mathcal{P}_2^{\pi})(\Xi) \leq 1 - r$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_1^{\sigma} \vee \mathcal{P}_2^{\sigma})(\Xi) \leq 1 - r$. This completes the proof. □

Definition 12. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma})$ be an svnts with $\mathcal{P}^{\tau\pi\sigma}$ an svn-primal on £. Then, $\mathcal{T}^{\tau\pi\sigma}$ is called single-valued neutrosophic primal open compatible with $\mathcal{P}^{\tau\pi\sigma}$, indicated by $\mathcal{T}^{\tau\pi\sigma} \models \mathcal{P}^{\tau\pi\sigma}$, if $\forall$ $\Xi \in \xi^{\pounds}$, $y_{n,m,v} \in \Xi$ and $\Pi \in \xi^{\pounds}$ with $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1 - r$, there exists $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ such that*

$$\tau_{\mathcal{A}}(z) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(z), \quad \pi_{\mathcal{A}}(z) + \pi_{\Xi}(z) - 1 > \pi_{\Pi}(z), \quad \sigma_{\mathcal{A}}(z) + \sigma_{\tilde{\xi}}(z) - 1 > \sigma_{\Pi}(z),$$

holds for every $z \in \pounds$, then $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$.

Definition 13. *Let $\{\Theta_i : i \in \Gamma\} \sqsubseteq \xi^{\pounds}$ such that $\Theta_i q \Xi$, $\forall$ $i \in \Gamma$ and $\Xi \in \xi^{\pounds}$. Then, $\{\Theta_i : i \in \Gamma\}$ is called an r-single-valued neutrosophic quasi-cover (for short, r-svnq-cover) of Ξ iff, $\forall$ $z \in \pounds$,*

$$\tau_{\Xi}(z) + \tau_{\vee_{i\in\Gamma}(\Theta_i)}(z) \geq 1, \quad \pi_{\Xi}(z) + \pi_{\vee_{i\in\Gamma}(\Theta_i)}(z) < 1, \quad \sigma_{\Xi}(z) + \sigma_{\vee_{i\in\Gamma}(\Theta_i)}(z) < 1.$$

Further, let $(\pounds, \mathcal{T}^{\tau\pi\sigma})$ be an *svnts*, for any $\mathcal{T}^{\tau}(\Theta_i) \geq r$, $\mathcal{T}^{\pi}(\Theta_i) \leq 1 - r$, $\mathcal{T}^{\sigma}(\Theta_i) \leq 1 - r$. Then, this *r-svnq-cover* will be called a single-valued neutrosophic quasi open-cover (for short, *svnqo-cover*) of Ξ.

Theorem 12. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma})$ be an* svnts *with* svn-primal *$\mathcal{P}^{\tau\pi\sigma}$ on £. Then, the following conditions are equivalent*

(1) $\mathcal{T}^{\tau\pi\sigma} \models \mathcal{P}^{\tau\pi\sigma}$.

(2) *If for each $\Xi \in \xi^{\pounds}$ has a svnqo-cover of $\{\Theta_i : i \in \Gamma\}$ such that $\tau_{\mathcal{A}}(z) + \tau_{\Theta_i}(z) - 1 \leq \tau_{\Pi}(z)$ $\pi_{\mathcal{A}}(z) + \pi_{\Theta_i}(z) - 1 > \pi_{\Pi}(z)$, $\sigma_{\mathcal{A}}(z) + \sigma_{\Theta_i}(z) - 1 > \sigma_{\Pi}(z)$, for every $z \in \pounds$ and for some $\mathcal{T}^{\tau}(\Pi) \geq r$, $\mathcal{T}^{\pi}(\Pi) \leq 1 - r$, $\mathcal{T}^{\sigma}(\Pi) \leq 1 - r$, then $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$.*

(3) *For any $\Xi \in \xi^{\pounds}$, $\Xi \wedge \Xi_r^{\star} = \overline{0}$ implies $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$.*

(4) *For any* $\Xi \in \xi^{\pounds}$, $\mathcal{P}^{\tau}(\widehat{\Xi}) \geq r$, $\mathcal{P}^{\pi}(\widehat{\Xi}) \leq 1-r$, $\mathcal{P}^{\sigma}(\widehat{\Xi}) \leq 1-r$, *where* $\widehat{\Xi} = \bigvee y_{n,m,v}$ *such that* $y_{n,m,v} \in \Xi$ *but* $y_{n,m,v} \notin \Xi_r^{\star}$.

(5) *For all* $\mathcal{T}^{\star\tau}(\Xi^c) \geq r$, $\mathcal{T}^{\star\pi}(\Xi^c) \leq 1-r$, $\mathcal{T}^{\star\sigma}(\Xi^c) \leq 1-r$, *we have* $\mathcal{P}^{\tau}(\widehat{\Xi}) \geq r$, $\mathcal{P}^{\pi}(\widehat{\Xi}) \leq 1-r$, $\mathcal{P}^{\sigma}(\widehat{\Xi}) \leq 1-r$.

(6) *For each* $\Xi \in \xi^{\pounds}$, *if* Ξ *contains no* $\Theta \neq \overline{0}$ *with* $\Theta \leq \Theta_r^{\star}$, *then* $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1-r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1-r$.

Proof. $1 \Rightarrow 2$: Let $\{\Theta_i : i \in \Gamma\}$ be an *svnqo-cover* of $\Xi \in \xi^{\pounds}$ such that, $\forall\, i \in \Gamma$, $\tau_{\mathcal{A}}(z) + \tau_{\Theta_i}(z) - 1 \leq \tau_{\Pi}(z)$, $\pi_{\mathcal{A}}(z) + \pi_{\Theta_i}(z) - 1 > \pi_{\Pi}(z)$, $\sigma_{\mathcal{A}}(z) + \sigma_{\Theta_i}(z) - 1 > \sigma_{\Pi}(z)$, for any $z \in \pounds$ and for some $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1-r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1-r$. Thus, as $\{\Theta_i : i \in \Gamma\}$ is an *svnqo-cover* of Ξ, for each $y_{n,m,v} \in \Xi$, there exists at least one $\Theta_{i\circ}$ such that $y_{n,m,v} q \Theta_{i\circ}$ and for each $z \in \pounds$, $\tau_{\mathcal{A}}(z) + \tau_{\Theta_{i\circ}}(z) - 1 \leq \tau_{\Pi}(z)$, $\pi_{\mathcal{A}}(z) + \pi_{\Theta_{i\circ}}(z) - 1 > \pi_{\Pi}(z)$, $\sigma_{\mathcal{A}}(z) + \sigma_{\Theta_{i\circ}}(z) - 1 > \sigma_{\Pi}(z)$ for some $\mathcal{P}^{\sigma}(\Xi) \leq 1-r$, $\mathcal{T}^{\tau}(\Pi) \geq r$, $\mathcal{T}^{\pi}(\Pi) \leq 1-r$, $\mathcal{T}^{\sigma}(\Pi) \leq 1-r$. Significantly, $\Theta_{i\circ} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$. From (1), we obtain $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1-r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1-r$.

$2 \Rightarrow 1$: It is clear from this that the family of $\{\Theta_i : i \in \Gamma\}$ contains at least one $\Theta_{i\circ} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$, such that every *svn-point* of Ξ constitutes a *svnqo-cover* of Ξ.

$1 \Rightarrow 3$: Let $\Xi \wedge \Xi_r^{\star} = \overline{0}$, for each $z \in \xi$, $y_{n,m,v} \in \Xi$ implies $y_{n,m,v} \notin \Xi_r^{\star}$. Then, there exists $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ and $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1-r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1-r$, such that for all $z \in \pounds$

$$\tau_{\mathcal{A}}(z) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(z), \;\; \pi_{\mathcal{A}}(z) + \pi_{\Xi}(x) - 1 > \pi_{\Pi}(z), \;\; \sigma_{\mathcal{A}}(x) + \sigma_{\Xi}(z) - 1 > \sigma_{\Pi}(x).$$

Since $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$, from (1), we have $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1-r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1-r$.

$3 \Rightarrow 1$: For each $y_{n,m,v} \in \Xi$, there exists an $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ such that, for any $z \in \pounds$,

$$\tau_{\mathcal{A}}(z) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(z), \quad \pi_{\mathcal{A}}(z) + \pi_{\Xi}(z) - 1 > \pi_{\Pi}(z), \quad \sigma_{\mathcal{A}}(z) + \sigma_{\Xi}(z) - 1 > \sigma_{\Pi}(z),$$

for some $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1-r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1-r$, implies $y_{n,m,v} \notin \Xi_r^{\star}$. Firstly, there are two cases: either $\Xi_r^{\star} = \overline{0}$ or $\Xi_r^{\star} \neq \overline{0}$ but $n > \tau_{\Xi_r^{\star}}(z) \neq \overline{0}$, $m \leq \pi_{\Xi_r^{\star}}(z) \neq \overline{0}$, $v \leq \sigma_{\Xi_r^{\star}}(z) \neq \overline{0}$. Assume, if possible, $y_{n.m.v} \in \Xi$ such that $n > \tau_{\Xi_r^{\star}}(z) \neq \overline{0}$, $m \leq \pi_{\Xi_r^{\star}}(z) \neq \overline{0}$, $v \leq \sigma_{\Xi_r^{\star}}(z) \neq \overline{0}$. Let $n_1 = \tau_{\Xi_r^{\star}}(z)$, $m_1 = \pi_{\Xi_r^{\star}}(z)$, $v_1 = \sigma_{\Xi_r^{\star}}(z)$. Then, $y_{n_1,m_1,v_1} \in \Xi_r^{\star}(z)$. Secondly, $y_{n_1,m_1,v_1} \in \Xi$. Thus, for all $\mathcal{B} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$, $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1-r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1-r$, there is at least one $z \in \pounds$ such that

$$\tau_{\mathcal{B}}(z) + \tau_{\Xi}(z) - 1 > \tau_{\Pi}(z), \quad \pi_{\mathcal{B}}(z) + \pi_{\Xi}(z) - 1 \leq \pi_{\Pi}(z), \quad \sigma_{\mathcal{B}}(z) + \sigma_{\Xi}(z) - 1 \leq \sigma_{\Pi}(z).$$

Since $y_{n_1,m_1,v_1} \in \Xi$, this contradicts the assumption for each *svn-point* of Ξ. So, $\Xi_r^{\star} = \overline{0}$. That means $y_{n,m,v} \in \Xi$ implies $y_{n,m,v} \notin \Xi_r^{\star}$. Now, this is true for each $\Xi \in \xi^{\pounds}$. So, $\Xi \vee \Xi_r^{\star} = \overline{0}$. Thus, by (3), we obtain $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1-r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1-r$ implies $\mathcal{T}^{\tau\pi\sigma} \models \mathcal{P}^{\tau\pi\sigma}$.

$3 \Rightarrow 4$: Let $y_{n,m,v} \in \widehat{\Xi}$. Then, $y_{n,m,v} \in \Xi$ but $y_{n,m,v} \notin \Xi_r^{\star}$. So, there exists an $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ such that $\forall\, z \in \pounds$,

$$\tau_{\mathcal{A}}(z) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(z), \quad \pi_{\mathcal{A}}(z) + \pi_{\Xi}(z) - 1 > \pi_{\Pi}(z), \quad \sigma_{\mathcal{A}}(z) + \sigma_{\Xi}(z) - 1 > \sigma_{\Pi}(z),$$

for some $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1-r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1-r$. Since $\widehat{\Xi} \leq \Xi$,

$$\tau_{\mathcal{A}}(z) + \tau_{\widehat{\Xi}}(z) - 1 \leq \tau_{\Pi}(z), \quad \pi_{\mathcal{A}}(z) + \pi_{\widehat{\Xi}}(z) - 1 > \pi_{\Pi}(z), \quad \sigma_{\mathcal{A}}(z) + \sigma_{\widehat{\Xi}}(z) - 1 > \sigma_{\Pi}(z),$$

for some $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1-r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1-r$. Thus, $y_{n,m,v} \notin \widehat{\Xi_r^{\star}}$ implies that $\widehat{\Xi_r^{\star}} = \overline{0}$ or $\widehat{\Xi_r^{\star}} \neq \overline{0}$ but $n > \tau_{\widehat{\Xi_r^{\star}}}(z)$, $m \leq \pi_{\widehat{\Xi_r^{\star}}}(z)$, $v \leq \sigma_{\widehat{\Xi_r^{\star}}}(z)$. Let $y_{s_1,m_1,v_1} \in$

svn-point$(\pounds)$ such that $n_1 \leq \tau_{\overbrace{\Xi_r^\star}}(z) < n$, $m_1 > \pi_{\overbrace{\Xi_r^\star}}(z) \geq m$, $v_1 > \sigma_{\overbrace{\Xi_r^\star}}(z) \geq v$; this means $y_{n_1,m_1,v_1} \in \overbrace{\Xi_r^\star}$. Then, $\forall\ \mathcal{B} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n_1,m_1,v_1}, r)$ and for every $\mathcal{P}^\tau(\Pi) \geq r$, $\mathcal{P}^\pi(\Pi) \leq 1-r$, $\mathcal{P}^\sigma(\Pi) \leq 1-r$, there is at least one $z \in \pounds$ such that

$$\tau_{\mathcal{B}}(z) + \tau_{\overbrace{\Xi}}(z) - 1 > \tau_\Pi(z), \quad \pi_{\mathcal{B}}(z) + \pi_{\overbrace{\Xi}}(z) - 1 \leq \pi_\Pi(z), \quad \sigma_{\mathcal{B}}(z) + \sigma_{\overbrace{\Xi}}(z) - 1 \leq \sigma_\Pi(z).$$

Since $\overbrace{\Xi} \leq \Xi$, then for any $\mathcal{B} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n_1,m_1,v_1}, r)$ and $\mathcal{P}^\tau(\Pi) \geq r$, $\mathcal{P}^\pi(\Pi) \leq 1-r$, $\mathcal{P}^\sigma(\Pi) \leq 1-r$, there is at least one $z \in \pounds$ such that

$$\tau_{\mathcal{B}}(z) + \tau_\Xi(z) - 1 > \tau_\Pi(z), \quad \pi_{\mathcal{B}}(z) + \pi_\Xi(z) - 1 \leq \pi_\Pi(z), \quad \sigma_{\mathcal{B}}(z) + \sigma_\Xi(z) - 1 \leq \sigma_\Pi(z).$$

This implies $y_{n_1,m_1,v_1} \in \Xi_r^\star$. But $n_1 < n$, $m_1 \geq m$, $v_1 \geq v$ and $y_{n,m,v} \in \overbrace{\Xi}$ implies $y_{n_1,m_1,v_1} \in \overbrace{\Xi}$ and hence $y_{n_1,m_1,v_1} \notin \overbrace{\Xi_r^\star}$. This is a contradiction. Thus, $\Xi_r^\star = \overline{0}$, so that $y_{n,m,v} \in \overbrace{\Xi}$ implies $y_{n,m,v} \notin \overbrace{\Xi_r^\star}$ with $\overbrace{\Xi_r^\star} = \overline{0}$. Thus, $\overbrace{\Xi} \wedge \overbrace{\Xi_r^\star} = \overline{0}$ for every $\Xi \in \xi^{\pounds}$. Hence, by (3), $\mathcal{P}^\tau(\overbrace{\Xi}) \geq r$, $\mathcal{P}^\pi(\overbrace{\Xi}) \leq 1-r$, $\mathcal{P}^\sigma(\overbrace{\Xi}) \leq 1-r$.

$4 \Rightarrow 5$: The same method as the proof of $3 \Rightarrow 4$.

$4 \Rightarrow 6$: Let $\Xi \in \xi^{\pounds}$, Ξ contains no $\Theta \neq \overline{0}$ with $\Theta \leq \Theta_r^\star$. Then, $\forall\ \Xi \in \xi^{\pounds}$, $\Xi = \overbrace{\Xi} \vee (\Xi \wedge \Xi_r^\star)$. By Theorem 4(5), we have $\Xi_r^\star = [\overbrace{\Xi} \vee (\Xi \wedge \Xi_r^\star)]_r^\star = \overbrace{\Xi_r^\star} \vee [\Xi \wedge \Xi_r^\star]_r^\star$. Now, by (4), we obtain $\mathcal{P}^\tau(\overbrace{\Xi}) \geq r$, $\mathcal{P}^\pi(\overbrace{\Xi}) \leq 1-r$, $\mathcal{P}^\sigma(\overbrace{\Xi}) \leq 1-r$; then, $\overbrace{\Xi_r^\star} = \overline{0}$. Thus, $[\Xi \wedge \Xi_r^\star]_r^\star = \Xi_r^\star$ but $\Xi \wedge \Xi_r^\star \leq \Xi_r^\star$, then $\Xi \wedge \Xi_r^\star \leq (\Xi \wedge \Xi_r^\star)_r^\star$. This contradicts the hypothesis if $\overline{0} \neq \Theta \leq \Xi$ with $\Theta \leq \Theta_r^\star$. Hence, $\Xi \wedge \Xi_r^\star = \overline{0}$, so, $\Xi = \overbrace{\Xi}$ by (4), we obtain that $\mathcal{P}^\tau(\Xi) \geq r$, $\mathcal{P}^\pi(\Xi) \leq 1-r$, $\mathcal{P}^\sigma(\Xi) \leq 1-r$.

$6 \Rightarrow 4$: Since $\forall\ \Xi \in \xi^{\pounds}$, $\Xi \wedge \Xi_r^\star = \overline{0}$. Hence, by (6), as Ξ contains no non-empty single-valued neutrosophic subset Θ with $\Theta \leq \Theta_r^\star$, $\mathcal{P}^\tau(\Xi) \geq r$, $\mathcal{P}^\pi(\Xi) \leq 1-r$, $\mathcal{P}^\sigma(\Xi) \leq 1-r$.

$5 \Rightarrow 1$: For every $\Xi \in \xi^{\pounds}$, $y_{n,m,v} \in \Xi$, there exists $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ such that

$$\tau_{\mathcal{A}}(z) + \tau_\Xi(z) - 1 \leq \tau_\Pi(z), \quad \pi_{\mathcal{A}}(z) + \pi_\Xi(z) - 1 > \pi_\Pi(z), \quad \sigma_{\mathcal{A}}(z) + \sigma_\xi(z) - 1 > \sigma_\Pi(z),$$

holds for every $z \in \pounds$ and for some $\mathcal{P}^\tau(\Xi) \geq r$, $\mathcal{P}^\pi(\Xi) \leq 1-r$, $\mathcal{P}^\sigma(\Xi) \leq 1-r$. This implies $y_{n,m,v} \notin \Xi_r^\star$. Let $\Theta = \Xi \vee \Xi_r^\star$. Then, $\Theta_r^* = (\Xi \vee \Xi_r^\star)_r^\star = \Xi_r^\star \vee (\Xi_r^\star)_r^\star = \Xi_r^\star$ by Theorem 4(4). So, $\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Theta, r) = \Theta \vee \Theta_r^\star = \Theta$. That means $\mathcal{T}^{\star\tau}(\Theta^c) \geq r$, $\mathcal{T}^{\star\pi}(\Theta^c) \leq 1-r$, $\mathcal{T}^{\star\sigma}(\Theta^c) \leq 1-r$. Thus, by (5), we obtain that $\mathcal{P}^\tau(\Theta) \geq r$, $\mathcal{P}^\pi(\Theta) \leq 1-r$, $\mathcal{P}^\sigma(\Theta) \leq 1-r$.

Again, For any $y_{s,m,v} \in$ *svn-point*$(\pounds)$, $y_{s,m,v} \notin \overbrace{\Theta_r^\star}$ implies $y_{s,m,v} \in \Theta$ but $y_{s,m,v} \notin \Theta_r^\star = \Xi_r^\star$. So, $\Theta = \Xi \vee \Xi_r^\star$, $y_{s,m,v} \in \Xi$. Now, by the hypothesis about Ξ, we have for every $y_{s,m,v} \in \Xi_r^\star$. So, $\overbrace{\Theta} = \Xi$. Hence, $\mathcal{P}^\tau(\Theta) \geq r$, $\mathcal{P}^\pi(\Theta) \leq 1-r$, $\mathcal{P}^\sigma(\Theta) \leq 1-r$. Thus, $\mathcal{T}^{\tau\pi\sigma} \models \mathcal{P}^{\tau\pi\sigma}$. □

Theorem 13. *Let* $(\pounds, \mathcal{T}^{\tau\pi\sigma})$ *be an* svnts *with* svn-primal $\mathcal{P}^{\tau\pi\sigma}$ *on* $\pounds$. *Then, the following cases are equivalent and implied* $\mathcal{T}^{\tau\pi\sigma} \models \mathcal{P}^{\tau\pi\sigma}$.

(1) *For each* $\Xi \in \xi^{\pounds}$, $\Xi \wedge \Xi_r^\star = \overline{0}$ *implies* $\Xi_r^* = \overline{0}$.

(2) *For each* $\Xi \in \xi^{\pounds}$, $\overbrace{\xi_r^\star} = \overline{0}$.

(3) *For each* $\Xi \in \xi^{\pounds}$, $\Xi \wedge \Xi_r^\star = \Xi_r^\star$.

Proof. The proof follows a similar line of reasoning to that of Theorem 12. □

5. Conclusions

In this paper, first, we investigated the complex area of stratification of svn grills and determined some of their fundamental features. The links between svn grills and svn-primals were investigated. We also introduced and explored the concept of svn-primal open local functions in the context of Šostak's sense. By extending the notions of svn sets and related topological structures, we have presented a novel approach to understanding the properties and relationships within this unique framework.

Our investigation began by defining svn-sets and their corresponding notions. Building upon these fundamental definitions, we introduced svnts and explored various operations and properties within these spaces, such as the neutrosophic closure and interior. A central contribution of this work has been the introduction and exploration of svn-primals and their associated operators. We have provided several equivalent conditions characterizing the compatibility of svnts with neutrosophic primals. Additionally, we discussed the properties of primal open local functions, their relationship with svnts, and the induced operators.

In conclusion, the results presented in this paper contribute to the growing field of neutrosophic topology by offering a deeper understanding of svn structures in Šostak's sense. The properties and correlations investigated here establish the framework for further studies in this field, opening options for future investigations and applications of these innovative notions in various fields.

In terms of related and future research directions, it is of great interest to investigate the connections between our findings and the advancements in the field of neural networks (Gu and Sheng [36]; Gu et al. [37]; Deng et al. [38]; Gu et al. [39]). Furthermore, the possible connections to multidimensional systems and signal processing need further investigation, as illustrated by Wang et al. [40] and Xiong et al. [41].

By bridging the gap between svn structures and related domains, we might promote collaboration across disciplines and discover new applications for our findings. This not only enriches the discipline of neutrosophic topology, but also benefits the larger scientific community by providing new insights and fresh approaches to difficult problems.

For forthcoming papers

The theory can be extended in the following normal methods.

1—Basic concepts of neutrosophic metric topological spaces can be studied using the notion of svn-primal present in this article;

2—Examine the connected, separation axioms and soft closure spaces in the context of svn-primals.

Abbreviations

The following abbreviations are used in this manuscript:

n-set	neutrosophic set
svn-set	single-valued neutrosophic set
svnits	single-valued neutrosophic ideals
svnts	single-valued neutrosophic topological spaces
svngts	single-valued neutrosophic grill topological spaces
svnpts	single-valued neutrosophic primal topological spaces
svnqo-cover	single-valued neutrosophic quasi open-cover
svns-primal	single-valued neutrosophic primal
$\mathcal{T}^{\tau\pi\sigma} \models \mathcal{P}^{\tau\pi\sigma}$	single-valued neutrosophic primal open compatible
svn-point	single-valued neutrosophic point

References

1. Willard, S. *General Topology*; Courier Corporation: North Chelmsford, MA, USA, 2012.
2. Kuratowski, K. *Topology: Volume I*; Elsevier: Amsterdam, The Netherlands, 2014.
3. Abbas, S.E.; Hebeshi, M.A. Construction of stratified L-fuzzy topological structures. *Appl. Math. Lett.* **2011**, *24*, 1063–1066. [CrossRef]
4. Chattopadhyay, K.C.; Thron, W.J. Extensions of closure spaces. *Can. J. Math.* **1977**, *29*, 1277–1286. [CrossRef]
5. Thron, W.J. Proximity structures and grills. *Math. Ann.* **1973**, *206*, 35–62. [CrossRef]
6. Roy, B.; Mukherjee M.N.; Ghosh, S.K. On a new operator based on a grill and its associated topology. *Arab. J. Math.* **2008**, *14*, 21–32.
7. Roy, B.; Mukherjee, M.N. On a typical topology induced by a grill. *Soochow J. Math.* **2007**, *33*, 771–786.
8. Roy, B.; Mukherjee, M.N. Concerning topologies induced by principal grills. *Analele Stiint. Ale Univ. Mat.* **2009**, *55*, 285–294.
9. Roy, B.; Mukherjee, M.N. On a type of compactness via grills. *Mat. Vesn.* **2007**, *59*, 113–120.
10. Thangamariappan, R.; Renukadevi, V. Topology generated by cluster systems. *Math. Vesn.* **2015**, *67*, 174–184.
11. Azzam, A.A.; Hussein, S.S.; Osman, H.S. Compactness of topological spaces with grills. *Ital. J. Pure Appl. Math.* **2020**, *44*, 198–207.
12. Talabeigi, A. On the Tychonoff's type theorem via grills. *Bull. Iran. Math. Soc.* **2016**, *42*, 37–41.
13. Boroojerdian, N.; Talabeigi, A. One-point λ-compactification via grills. *Iran. J. Sci. Technol. Trans. Sci.* **2017**, *41*, 909–912. [CrossRef]
14. Sulimana, S.S.; Esmaeela, R.B. On some topological concepts via grill. *Int. J. Nonlinear Anal. Appl.* **2022**, *13*, 3765–3772. [CrossRef]
15. Azzam, A.A. Grill Nano topological spaces with grill Nano generalized closed sets. *J. Egypt. Math. Soc.* **2017**, *25*, 164–166. [CrossRef]
16. Hosny, R.A. δ-sets with grill. *Int. Math. Forum* **2012**, *7*, 2107–2113.
17. Janković, D.; Hamlett, T.R. New topologies from old via ideals. *Am. Math. Mon.* **1990**, *97*, 295–310. [CrossRef]
18. Smarandache, F. *A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics*, 6th ed.; InfoLearnQuest: Ann Arbor, MI, USA, 2007. Available online: https://philarchive.org/rec/SMAAUF-2 (accessed on 10 February 2019).
19. Salama, A.A.; Alblowi, S.A. Neutrosophic set and neutrosophic topological spaces. *IOSR J. Math.* **2012**, *3*, 31–35. [CrossRef]
20. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. Single valued neutrosophic sets. *Multispace Multistruct.* **2010**, *4*, 410–413.
21. Rodrigo, P.A.; Maheswari, S. Neutrosophic gsα*-Open and Closed Maps in Neutrosophic Topological Spaces. *Neutrosophic Syst. Appl.* **2023**, *8*, 42–49. [CrossRef]
22. Ye, J. Single valued neutrosophic cross-entropy for multicriteria decision making problems. *Appl. Math. Modell.* **2014**, *38*, 1–3. [CrossRef]
23. Dhar, R. Compactness and Neutrosophic Topological Space via Grills. *Neutrosophic Syst. Appl.* **2023**, *2*, 1–7. [CrossRef]
24. Dey, S.; Ray, G.C. Separation Axioms in Neutrosophic Topological Spaces. *Neutrosophic Syst. Appl.* **2023**,*2*, 38–54. [CrossRef]
25. Alsharari, F.; Saber, Y.M.; Alohali, H.; Alqahtani, M.H.; Ebodey, M.; Elmasry, T.; Alsharif, J.; Soliman, A.F.; Smarandache, F.; Sikander, F. On stratified single-valued soft topogenous structures. *Heliyon* **2024**, *10*, e27926. [CrossRef]
26. Saber, Y.M.; Alohali, H.; Elmasry, T.; Smarandache, F. On single-valued neutrosophic soft uniform spaces. *AIMS Math.* **2024**, *9*, 412–439. [CrossRef]
27. Saber, Y.M.; Alsharari, F.; Smarandache, F. On Single-valued neutrosophic ideals in Šostak sense. *Symmetry* **2020**, *12*, 193. [CrossRef]
28. Saber, Y.M.; Alsharari, F.; Smarandache, F.; Abdel-Sattar, A. Connectedness and stratification of single-valued neutrosophic topological spaces. *Symmetry* **2020**, *12*, 1464. [CrossRef]
29. Ye, J.; Cui, W. Neutrosophic Compound Orthogonal Neural Network and Its Applications in Neutrosophic Function Approximation. *Symmetry* **2019**, *11*, 147. [CrossRef]
30. Shao, S.; Zhang, X.; Li, Y.; Bo, C. Probabilistic Single-Valued (Interval) Neutrosophic Hesitant Fuzzy Set and Its Application in Multi-Attribute Decision Making. *Symmetry* **2018**, *10*, 419. [CrossRef]

31. Rıdvan, S.; Ahmet, K. Subsethood measure for single valued neutrosophic sets. *J. Intell. Fuzzy Syst.* **2015**, *29*, 525–530.
32. Das, S.; Roy, B.K.; Kar, M.B.; Kar, S.; Pamučar, D. Neutrosophic fuzzy set and its application in decision making. *J. Ambient. Intell. Humaniz. Comput.* **2020**, *11*, 5017–5029. [CrossRef]
33. Yang, H.L.; Guo, Z.L.; Liao, X. On single valued neutrosophic relations. *J. Intell. Fuzzy Syst.* **2016**, *30*, 1045–1056. [CrossRef]
34. Ye, J.A. multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2450–2466. [CrossRef]
35. Acharjee, S.; Özkoç, M.; Issaka, F.Y. Primal topological spaces. *arXiv* **2022**, arXiv:2209.12676.
36. Gu, B.; Sheng, V.S. A robust regularization path algorithm for v support vector classification. *IEEE. Trans. Neural. Netw. Learn. Syst.* **2017**, *28*, 1241–1248. [CrossRef] [PubMed]
37. Gu, B.; Sun, X.; Sheng, V.S. Structural minimax probability machine. *IEEE. Trans. Neural. Netw. Learn. Syst.* **2017**, *28*, 1646–1656. [CrossRef] [PubMed]
38. Deng, W.; Zhao, H.M.; Yang, X.H.; Xiong, J.X.; Sun, M.; Li, B. Study on an improved adaptive PSO algorithm for solving multi-objective gate assignment. *Appl. Soft. Comput.* **2017**, *59*, 288–302. [CrossRef]
39. Gu, B.; Sheng, V.S.; Tay, K.Y.; Romano, W.; Li, S. Incremental support vector learning for ordinal regression. *IEEE. Trans. Neural. Netw. Learn. Syst.* **2015**, *26*, 1403–1416. [CrossRef] [PubMed]
40. Wang, B.; Gu, X.; Ma, L.; Yan, S. Temperature error correction based on BP neural network in meteorological WSN. *Int. J. Sens. Netw.* **2017**, *23*, 265–278. [CrossRef]
41. Xiong, L.; Xu, Z.; Shi, Y.Q. An integer wavelet transform-based scheme for reversible data hiding in encrypted images. *Multidim. Syst. Sign. Process.* **2018**, *29*, 1191–1202. [CrossRef]

PCM with Linguistic Contradiction Degree Representations in Decision making on Academic Stress causing Factors

N. Angel[1], P. Pandiammal[2], N. Ramila Gandhi [3], Nivetha Martin[4,*], Florentin Smarandache[5]

[1] School of Mathematics, Madurai Kamaraj University, Madurai, Tamil Nadu, India
[2]Department of Mathematics, GTN college of Arts & Science, Dindigul, Tamilnadu, India
[3] Department of Mathematics, PSNA College of Education and Technology, Dindigul
[4] Department of Mathematics, Arul Anandar College(Autonomous),Karumathur, Tamil Nadu, India
[5] Emeritus Professor University of New Mexico, Mathematics, Physics, and Natural Science Division, USA

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Abstract

Plithogenic Cognitive Map (PCM) is the generalized form of Cognitive maps that has recently ebbed into the field of decision-making. The first developed PCM model comprises of factors, connection matrix with numeric contradiction degree between the factors. In this research work a PCM model with linguistic contradiction degree representations between the core and sub factors is developed to make the decision-making more comprehensive. The model formulated in this research work is illustrated with the factors causing academic stress to the students of digital educational system. Personal, Social, Economic and Institutional are considered as the core factors and the contradiction degree in linguistic sense is considered with respect to each of these core factors and ten sub factors. The obtained results on comparing with conventional models are highly promising and this model will certainly set new benchmarks of a comprehensive decision-making model.

Keywords: PCM; linguistic variable; contradiction degree; academic stress

1. Introduction

Plithogenic sets developed by Smarandache perform prominent role in the process of decision making. The plithogenic representations moderate the complicated phases involved in making optimal decisions. The demonstration of a plithogenic set (P) as a pentuple of the form (P, a, V,d,c) facilitates easy depiction of data. The origin and development of plithogeny is more closely associated with the handling of attributes (a) and set of attribute values (V) together with the degree of appurtenance (d) and contradiction (c). The plithogenic set is more comprehensive as it accommodates various kinds of representations namely crisp, fuzzy, intuitionistic and neutrosophic. Researchers have applied these plithogenic sets to simplify the process of making decision involving several criteria and alternatives. Abdel et al [1] in manufacturing industries, banking sectors, disaster management, supplier selection, Grida et al [20] in IOT based supply chain, Ulutas et al [37] in logistic risks, Ahmad et al [3] in novel MCDM model for COVID diagnosis, and also in materials selection, food processing methods.

Researchers have also combined the theory of plithogeny with other decision making methods to evolve a more generalized plithogenic decision making models. For instance the decision making models based on Plithogenic graphs, Plithogenic Hypersoft sets, Plithogenic hypergraph, Plithogenic soft sets are developed by extending plithogenic representations to the existing domains of decision making. The integrated plithogenic models are also widely applied in several decision making scenarios. Ahmad et al [2] in AI based decision making Covid 19, Ozcil et al [31] in MAIRCA

method, Rana et al [35] in frequency matrix multi attribute decision making technique, Gayen et al [15] introduced plithogenic hypersoft subgroup, Gomez et al [17] applied plithogenic sets in the field of education and so on.

Another significant decision making model developed on integrating plithogenic sets with cognitive maps is Plithogenic Cognitive Maps (PCM). Nivetha and Smarandache laid the foundation for the conceptualization of PCM. This integrated decision making model is applied by the researchers in making optimal decisions. Martin et al [27] developed new plithogenic cognitive maps, Priya et al [34] developed induced plithogenic cognitive maps, instigating spiritual intelligence in youth.

Generally, a plithogenic cognitive map is defined as a directed graph with nodes and edges to mirror the factors of the problems and their relationship with one another. The plithogenic connection matrix comprises of the degree of appurtenance representing the degree of influence of one factor over another. The most influential factor is determined by finding the limit point of the dynamical system by considering the degree of contraction between the factors. But on deep investigation over the aspects of assigning contradiction degree between the factors several questions arises. In the above developed PCM by Nivetha and Smarandache, the influence and the association between the factors is determined by placing the factors in ON position. If a factor is placed in ON position, then it is assumed to be the dominant factor and the other factors that are in OFF position are assigned the contradiction degree in sequential manner with respect to the dominant factor. For instance, if there are three factors say F1, F2 and F3. If the factor F1 is placed in ON position, then the factor F1 is considered to be dominant and henceforth the contradiction degree of F2 and F3 with respect to F1 are 1/3 and 2/3 respectively. The construction of PCM model based on such a way of assigning of contradiction degree with respect to dominant factors have few shortcomings as follows

- The way of assigning contradiction degree lacks realism.
- The factors considered for making decisions are more alike as they belong to the same category and also, they lack differentiation.
- The numerical degree of contradiction may not be promising at all instances of decision making.

To handle the above stated limitations, PCM model with linguistic contradiction degree is developed in this research work, also in addition to the factors of the decision making problem, the core factors to which these factors are associated are also considered and the contradiction degree of the factors are assigned with respect to the factors but with respect to the core factors to make the representations more meaningful and pragmatic.

The research work is organized into the following sections, section 2 presents the state of art of research of the applications of different representations of Cognitive Map models namely crisp, fuzzy, intuitionistic and neutrosophic with respect to the academic problems of the learners. The Section 3 describes the steps involved in PCM model with linguistic contradiction degree. Section 4 applies the proposed model to the decision making problem on academic stress. Section 5 discusses the results and concludes the work.

2. Literature Review

This section presents the conceptualizations of crisp, fuzzy, intuitionistic, neutrosophic and plithogenic cognitive maps and their application with special reference to the field of education. Cognitive Maps are the primary decision-making tool used in making decisions related to many social entities. Cognitive Maps was initially originated by Robert Alexrod [7] .Fuzzy Cognitive Maps (FCM) developed by Kosko [23] are widely used in the many fields like medicine, Education, agriculture, politics, social awareness, technology, psychology, and many other fields too. These type of cognitive maps are further extended to intuitionistic fuzzy cognitive maps (IFCM). The notion of intuitionistic sets introduced by Atanssov [6] comprises of membership and non-membership values. IFCM are applied in the field of medicine, agriculture, architecture, decision making entities, technology and so on. These IFCMs are also extended to Neutrosophic cognitive Maps (NCM) by Vasantha and Smarandache. The neutrosophic sets introduced by Smarandache comprises truth, intermediate and falsity membership values. NCMs are applied in the fields of medicine, day to day life entities, education, psychology, economics and many other fields too.

Researchers have applied these decision making approaches in the field of education especially to investigate the academic problems associated with learning, learners and teachers. The decision making based on Cognitive Maps are applied. Gorelova et al [19] in studying the interrelationship between the factors associated with effective education, Banathy [8] in framing the educational system for future generations.

The decision making based on the fuzzy cognitive maps in the field of education are applied. Cole J R et al [11] developed FCM in education, Baron et al [9] in assessment of learning in the environment, Luo X et al [24] in game based learning, Merlin [30] developed augmented fuzzy cognitive maps in education and applied FCM in AJEL type learning in students, Mansouri et al [25] to predict student performance, Anusha et al [4] in analysing the reasons for stress in college students, Devadoss A V et al [12] in environmental education for next generation learners, Prakash P et al [33] in causes for aversion to mathematics by engineering students, Glykas et al [16] in the theories, methodologies and tools, Dias et al [14] in higher education blended learning environment, Asghari et al [5] presented Iran's future higher education scenarios, Pena Ayala et al [32] made an approach of FCM in student centered education, Gordaliza et al [18] used FCM to support complex learning environment, Chiang et al [10] used FCM in supporting underrepresented students and so on.

NCMs are also applied in the field of education. Devadoss et al [13] in studying the quality of primary education, Kalaichelvi et al [21] in studying the problems faced by girl students in the education, Thiruppathi et al [36] made a study in suicide problems using NCM and also gave a solution for the above said problem, Mary et al [29] made a comparative study using NCM on the factors of quality training on student and teachers and so on.

3. Methodology

This section describes the step involved in determining the fixed point of the Plithogenic dynamical system.

Step I: The decision making problem is first well defined and the core factors, sub factors (Fi) encircled with the problem along with the association between the sub factors are initially decided based on the expert's opinion. Inaddition to the sub factors, the core factors are also considered to which each of the factors say Fi are subjected to.

Step 2: The sub factors of the problem are taken as the nodes say P1, P2 ,… Pn. The relationship between the sub factors are represented using the crisp degree of appurtenance which assumes crisp values say {-1,0,1}, fuzzy degree of appurtenance that assumes values from [-1,1], intuitionistic degree of appurtenance that assigns values from $\rho([0,1])^2$and neutrosophic degree of appurtenance that takes values from $\rho([0,1])^3$.

Step 3: The connection matrix representing the associational impact between the factors is termed as Plithogenic crisp, Plithogenic fuzzy, Plithogenic intuitionistic and Plithogenic neutrosophic based on the nature of the degree of appurtenance.

Step 4: An instantaneous vector is of the form Z (t) = (a1,a2,…,an) is initially considered, in which the ai's assume either of the value 0 or 1 to indicate the ON or OFF position of the factors. An instantaneous vector of the form (10000..00) represents the ON position of the first factor and it is passed on to the connection matrix. The resultant vector is obtained using plithogenic operators together with the linguistic contradiction degree of the factors Fi with respect to the core factors. The resultant vector thus obtained is updated using threshold function by assigning the value 1 if the ai's values are greater than 1, 0 if the values of ai's are lesser than 1.

Plithogenic operators are defined as follows:

$$f \wedge_P g = (1 - c)[f \wedge_F g] + c[f \vee_F g]$$

where c represents the contradiction degree and

$\wedge_F$ represents the t_{norm} given by $f \wedge_F g = fg$

$\vee_F$ represents the t_{conorm} given by $f \vee_F g = f + g - fg$

The linguistic variable are quantified using fuzzy numbers of conveniences.

Step 5: The Step 4 is repeated until two consecutive resultant vectors are equal

i.e $|Z(t+1) - Z(t)| \leq \epsilon$

Let us consider a decision making problem on weight management involving three core factors say CF1,CF2 and CF3 and five sub-factors say SF1,SF2,SF3,SF4 and SF5.

In this case the strategies or the measures of weight management are taken into consideration. The main objective is to determine the most significant and most influential strategy.

The core factors that are considered in this context are

CF1 Biological

CF2 Behavioural

CF3 Psychological

The strategies that are considered as the sub factors are as follows

SF1 Refraining from stressful factors

SF2 Avoidance of Junk Food items

SF3 Intake of supplementary nutrients to maintain hormone levels

SF4 Developing a positive tendency of maintaining ideal weight

SF5 Leading a healthy lifestyle by affiliating closely with nature

The association between the sub-factors are presented as a connection matrix of the form

	SF1	SF2	SF3	SF4	SF5
SF1	0	0.2	0.3	0.5	0.7
SF2	0.5	0	0.8	0.7	0.5
SF3	0.3	0.7	0	0.8	0.8
SF4	0.7	0.7	0.5	0	0.7
SF5	0.8	0.7	0.7	0.5	0

The contradiction degree of the sub factors with respect to the core factors are presented in the following table

	SF1	SF2	SF3	SF4	SF5
CF1	0.5	0.2	0.2	0.7	0.5
CF2	0.5	0.7	0.8	0.5	0.5
CF3	0.2	0.5	0.7	0.3	0.5

ON Position of Vectors	Core Factors	SF1	SF2	SF3	SF4	SF5
(1 0 0 0 0)	CF1	1	0.67	0.67	0.85	0.85
	CF2	1	0.88	0.92	0.86	0.86
	CF3	1	0.78	0.86	0.77	0.85

4. Application to Decision making on Academic Stress of the Learners

The emerging problem of the students is the competition among themselves which leads to stress in them. The competitive world leads the children today to a stressful life. The factor causing academic stress among the students are studied by

educationalist. Majority of the factors are subjected to Personal, Social, Institutional and Economical. Moreover, these factors were analyzed using statistical tools and not using the concept of cognitive maps of any kind. Based on the above said factors, the following are considered as the nodes of Plithogenic Cognitive maps. These nodes are the sub-factors of the decision-making problem subjected to the four core factors of Personal, Economic, Social and Institutional

The following stress causing sub-factors are taken into consideration in constructing a PCM model

1. Peer Competition (P1)
2. Mismanagement of time (P2)
3. Confined evaluation system (P3)
4. Phobia of examinations (P4)
5. Parental pressure (P5)
6. Anxiety of future (P6)
7. Lack of encouragement (P7)
8. Poor family background (P8)
9. Weak bond between student and teacher (P9)
10. Infeasible Learning environment (P10)

Case (i) PCM with Linguistic contradiction degree with respect to Core Factors

To draw conclusions on the interrelationship between the factors, the contradiction degree between the attributes is given by linguistic variables and they are further quantified by triangular numbers as given in Table 4.2

Table 1: Quantifications of Linguistic Variables

Linguistic Representation of Contradiction Degree	**Triangular Quantifications**	**Crisp value**
Very High	(0.7,0.8,0.9)	0.8
High	(0.6,0.7,0.8)	0.7
Moderate	(0.4,0.5,0.6)	0.5
Low	(0.4,0.3,0.2)	0.3
Very Low	(0.3,0.2,0.1)	0.2

Also the contradiction degree of the sub-factors is assumed with respect to each of the core factors as follows.

The contradiction degree of the attributes with respect to the personal aspect of a student is given as in Table 4.3

Table 2: Contradiction degree of the sub-factors with respect to the core factor Personal

P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
M	M	VH	L	H	VL	H	VH	VH	VH
0.5	0.5	0.8	0.3	0.7	0.2	0.7	0.8	0.8	0.8

Let us take P1 to be in ON position and the other attributes in OFF position.

Let us consider the instantaneous state vector as X = (1 0 0 0 0 0 0 0 0 0)

$X *_P P(E)$ = (0.5 0.75 0.96 0.86 0.85 0.6 0.76 0.84 0.84 0.84)

→ (1 0.75 0.96 0.86 0.85 0.6 0.76 0.84 0.84 0.84) = X1

$X1 *_P P(E)$ = (0.88 0.83 0.96 0.86 0.9248 0.8128 0.9156 0.941 0.9152 0.9088)

→ (1 0.83 0.96 0.86 0.93 0.81 0.92 0.94 0.92 0.91) = X2

$X2 *_P P(E)$ = (0.88 0.91 0.96 0.86 0.9496 0.8608 0.9428 0.9618 0.9344 0.9408)

→ (1 0.91 0.96 0.86 0.95 0.86 0.94 0.96 0.93 0.94) = X3

X3*$_P$ P(E) = (0.88 0.92 0.96 0.8706 0.9564 0.8756 0.9462 0.967 0.9408 0.9472)

→ (1 0.92 0.96 0.87 0.96 0.88 0.95 0.97 0.94 0.95) = X4

X4*$_P$ P(E) = (0.88 0.925 0.96 0.8772 0.9598 0.883 0.9496 0.9696 0.944 0.9504)

→ (1 0.93 0.96 0.88 0.96 0.88 0.95 0.97 0.94 0.95) = X5

X5*$_P$ P(E) = (0.88 0.925 0.96 0.8838 0.9598 0.883 0.9496 0.9696 0.944 0.9504)

→ (1 0.93 0.96 0.88 0.96 0.88 0.95 0.97 0.94 0.95) = X6

X5 = X6

Thus the fixed points of the system is obtained. The result obtained shall be interpreted as follows: the interrelationship between the factor P1 and other factors with respect to personal core factor is reflected in the fixed point. The degree of association of the factor P1 with factors P8, P3, P5 are very high and that of with the factors P4 and P6 is high. The same shall be repeated by keeping other factors in ON position.

By repeating the same procedure for other core factors the results obtained are tabulated in Table 4.4.

Table 3: Interrelationship score values between the sub-factors with respect to the core factors

ON Position of Vectors	**Core Factors**	**P1**	**P2**	**P3**	**P4**	**P5**	**P6**	**P7**	**P8**	**P9**	**P10**
(1000000000)	P	1	0.93	0.96	0.88	0.96	0.88	0.95	0.97	0.94	0.95
	E	1	0.94	0.96	0.96	0.84	0.88	0.92	0.81	0.93	0.81
	S	1	0.93	0.9	0.95	0.81	0.79	0.82	0.83	0.83	0.72
	I	1	0.92	0.84	0.91	0.97	0.89	0.86	0.97	0.78	0.84
(0100000000)	P	0.88	1	0.94	0.93	0.96	0.88	0.95	0.97	0.94	0.95
	E	0.94	1	0.95	0.98	0.85	0.88	0.92	0.83	0.93	0.82
	S	0.9	1	0.89	0.97	0.82	0.84	0.86	0.84	0.84	0.73
	I	0.89	1	0.81	0.95	0.97	0.9	0.86	0.97	0.81	0.84
(0010000000)	P	0.9	0.93	1	0.88	0.96	0.88	0.95	0.97	0.94	0.95
	E	0.96	0.94	1	0.96	0.86	0.87	0.92	0.82	0.93	0.81
	S	0.94	0.93	1	0.95	0.84	0.84	0.82	0.83	0.83	0.72
	I	0.9	0.92	1	0.91	0.97	0.9	0.86	0.97	0.78	0.84
(0001000000)	P	0.88	0.93	0.96	1	0.96	0.89	0.96	0.97	0.96	0.95
	E	0.95	0.94	0.96	1	0.86	0.88	0.92	0.84	0.94	0.82
	S	0.9	0.94	0.9	1	0.84	0.8	0.84	0.86	0.86	0.75
	I	0.89	0.92	0.84	1	0.97	0.89	0.86	0.97	0.84	0.84
(0000100000)	P	0.9	0.93	0.93	0.88	1	0.88	0.95	0.98	0.94	0.95
	E	0.96	0.94	0.95	0.96	1	0.88	0.92	0.92	0.93	0.86
	S	0.94	0.94	0.88	0.95	1	0.84	0.86	0.93	0.83	0.79
	I	0.9	0.92	0.78	0.91	1	0.9	0.86	0.98	0.78	0.85
(0000010000)	P	0.9	0.94	0.93	0.89	0.96	1	0.97	0.97	0.95	0.95
	E	0.96	0.95	0.95	0.97	0.84	1	0.95	0.84	0.93	0.82
	S	0.94	0.96	0.88	0.96	0.82	1	0.93	0.86	0.84	0.75
	I	0.9	0.95	0.79	0.93	0.97	1	0.93	0.97	0.79	0.84

(0000001000)	P	0.88	0.95	0.93	0.9	0.96	0.92	1	0.97	0.96	0.95
	E	0.94	0.97	0.95	0.97	0.84	0.93	1	0.82	0.94	0.81
	S	0.91	0.98	0.88	0.96	0.81	0.92	1	0.84	0.86	0.73
	I	0.88	0.97	0.8	0.94	0.97	0.95	1	0.97	0.84	0.84
(0000000100)	P	0.89	0.93	0.93	0.88	0.97	0.88	0.95	1	0.94	0.96
	E	0.94	0.94	0.95	0.96	0.93	0.88	0.92	1	0.93	0.9
	S	0.91	0.93	0.88	0.95	0.92	0.84	0.82	1	0.83	0.84
	I	0.89	0.92	0.78	0.91	0.98	0.9	0.86	1	0.78	0.86
(0000000010)	P	0.88	0.94	0.93	0.89	0.96	0.9	0.97	0.97	1	0.95
	E	0.94	0.95	0.95	0.97	0.84	0.9	0.95	0.82	1	0.81
	S	0.89	0.96	0.88	0.96	0.81	0.87	0.93	0.84	1	0.73
	I	0.89	0.95	0.79	0.93	0.97	0.92	0.93	0.97	1	0.84
(0000000001)	P	0.88	0.93	0.92	0.88	0.96	0.88	0.95	0.97	0.94	1
	E	0.94	0.94	0.95	0.96	0.84	0.88	0.92	0.81	0.93	1
	S	0.89	0.94	0.88	0.95	0.81	0.82	0.86	0.83	0.83	1
	I	0.89	0.92	0.78	0.91	0.97	0.89	0.86	0.97	0.78	1

Case (ii) : PCM with Linguistic contradiction degree with respect to Factors

The obtained results in Table 4.4 are compared with the results obtained using the approach of PCM developed in [60]

Let us determine the approach of PCM to our problem which considers the linguistic contradiction degree between only the sub-factors without the intervention of the core factors. Let us take P1 to be in ON position and the other factors to be in OFF state. The contradiction degree of the node P1 with respect to the other nodes are as represented in Table 4.5.

Table 4: Contradiction degree of the node P1 with respect to the other nodes

P1	**P2**	**P3**	**P4**	**P5**	**P6**	**P7**	**P8**	**P9**	**P10**
0	M	L	L	M	M	H	H	H	M
0	0.5	0.3	0.3	0.5	0.5	0.7	0.7	0.7	0.5

Now consider the instantaneous state vector as X = (1 0 0 0 0 0 0 0 0 0)

$X*_P$ P(E) = (0 0.75 0.86 0.86 0.75 0.75 0.76 0.76 0.76 0.6)

→ (1 0.75 0.86 0.86 0.75 0.75 0.76 0.76 0.76 0.6) = X1

$X1*_P$ P(E) = (0.688 0.83 0.86 0.86 0.83 0.83 0.8884 0.8868 0.8868 0.78)

→ (1 0.83 0.86 0.86 0.83 0.83 0.89 0.89 0.89 0.78) = X2

$X2*_P$ P(E) = (0.688 0.895 0.86 0.86 0.895 0.895 0.9326 0.9122 0.8982 0.845)

→ (1 0.9 0.86 0.86 0.9 0.9 0.93 0.91 0.9 0.85) = X3

$X3*_P$ P(E) = (0.72 0.915 0.86 0.864 0.905 0.915 0.936 0.936 0.9134 0.855)

→ (1 0.92 0.86 0.86 0.91 0.92 0.94 0.94 091 0.86) = X4

$X4 *_P P(E)$ = (0.736 0.92 0.86 0.8772 0.92 0.92 0.9428 0.9394 0.9172 0.87)

→ (1 0.92 0.86 0.88 0.92 0.92 0.92 0.94 0.94 0.92 0.87) = X5

$X5 *_P P(E)$ = (0.736 0.92 0.86 0.8772 0.92 0.92 0.9428 0.9428 0.9172 0.87)

→ (1 0.92 0.86 0.88 0.920.92 0.94 0.94 0.92 0.87) = X6

X5 = X6

In the above obtained fixed point the factor P1 has closer association with the factors P7 and P8, but it is not possible to make interpretations on the nature of association. The results obtained by considering each of the factors are presented in Table 4.6

Table 5: Fixed Point of the ON position of the factors of PCM

ON position of vectors	**P1**	**P2**	**P3**	**P4**	**P5**	**P6**	**P7**	**P8**	**P9**	**P10**
(1000000000)	1	0.92	0.86	0.88	0.92	0.92	0.94	0.94	0.92	0.87
(0100000000)	0.92	1	0.88	0.95	0.95	0.92	0.94	0.95	0.92	0.92
(0010000000)	0.9	0.93	1	0.88	0.95	0.86	0.88	0.95	0.79	0.83
(0001000000)	0.82	0.89	0.86	1	0.9	0.93	0.88	0.9	0.9	0.9
(0000100000)	0.86	0.88	0.89	0.86	1	0.86	0.86	0.95	0.89	0.92
(0000010000)	0.86	0.92	0.89	0.88	0.95	1	0.93	0.95	0.87	0.92
(0000001000)	0.87	0.93	0.89	0.88	0.92	0.93	1	0.94	0.86	0.82
(0000000100)	0.91	0.9	0.91	0.86	0.93	0.86	0.9	1	0.9	0.9
(0000000010)	0.88	0.88	0.87	0.94	0.95	0.88	0.93	0.95	1	0.88
(0000000001)	0.9	0.94	0.8	0.89	0.9	0.9	0.9	0.9	0.8	1

5 Discussion

The average values of the interrelational impacts in the first case is presented in Table 5.1 and the differences between the values obtained in Table 4.6 between each of the interrelational impacts are presented in Table 5.2

Table 6: Average values of the Interrelationship score values between the sub-factors with respect to the core factors

ON position of vectors	**P1**	**P2**	**P3**	**P4**	**P5**	**P6**	**P7**	**P8**	**P9**	**P10**
(1000000000)	1	0.93	0.92	0.93	0.9	0.86	0.89	0.9	0.87	0.83
(0100000000)	0.9	1	0.9	0.96	0.9	0.88	0.9	0.9	0.88	0.84
(0010000000)	0.93	0.93	1	0.93	0.9	0.87	0.89	0.9	0.87	0.83
(0001000000)	0.91	0.93	0.92	1	0.91	0.87	0.9	0.91	0.9	0.84
(0000100000)	0.93	0.93	0.89	0.93	1	0.88	0.9	0.95	0.87	0.86
(0000010000)	0.93	0.95	0.89	0.94	0.9	1	0.95	0.91	0.88	0.84
(0000001000)	0.9	0.97	0.89	0.94	0.9	0.93	1	0.9	0.9	0.83
(0000000100)	0.91	0.93	0.89	0.93	0.95	0.88	0.89	1	0.87	0.89

(0000000010)	0.9	0.95	0.89	0.94	0.9	0.9	0.95	0.9	1	0.83
(0000000001)	0.9	0.93	0.88	0.93	0.9	0.9	0.9	0.9	0.87	1

Table 7: Differences between the values with respect to case (i) & case (ii)

ON position of vectors	**P1**	**P2**	**P3**	**P4**	**P5**	**P6**	**P7**	**P8**	**P9**	**P10**
(1000000000)	0	0.01	0.06	0.05	-0.02	-0.06	-0.05	-0.04	-0.05	-0.04
(0100000000)	-0.02	0	0.02	0.01	-0.05	-0.04	-0.04	-0.05	-0.04	-0.08
(0010000000)	0.03	0	0	0.05	-0.05	0.01	0.01	-0.05	0.08	0
(0001000000)	0.09	0.04	0.06	0	0.01	-0.06	0.02	0.01	0	-0.06
(0000100000)	0.07	0.05	0	0.07	0	0.02	0.04	0	-0.02	-0.06
(0000010000)	0.07	0.03	0	0.06	-0.05	0	0.02	-0.04	0.01	-0.08
(0000001000)	0.03	0.04	0	0.06	-0.02	0	0	-0.04	0.04	0.01
(0000000100)	0	0.03	-0.02	0.07	0.02	0.02	-0.01	0	-0.03	-0.01
(0000000010)	0.02	0.07	0.02	0	-0.05	0.02	0.02	-0.05	0	-0.05
(0000000001)	0	-0.01	0.08	0.04	0	0	0	0	0.07	0

On observing the differences, it is very evident that the results obtained in both the cases are promising, but in the latter case the nature of the interrelation cannot be determined whereas in the first case the nature of the interrelation is very explicit and henceforth the decision making is made more comprehensive.

Conclusion

This paper proposes a modified version of plithogenic cognitive maps with linguistic contradiction degree. The newly developed PCM model is more advantageous in comparison to the earlier formulated PCM. The extended PCM model is highly compatible in making interpretations on the associations between the factors in relation with the core factors of the decision making problem. The comparative analysis substantiates the efficacy of the constructed model. As an extension of this research work, the PCM based on contradictions using linguistic variable shall be associated with other decision making methods.

References

[1] Abdel-Basset, M., Ding, W., Mohamed, R., & Metawa, N. (2020). An integrated plithogenic MCDM approach for financial performance evaluation of manufacturing industries. *Risk Management*, *22*, 192-218.

[2] Ahmad, M. R., & Afzal, U. (2022). Mathematical modeling and AI based decision making for COVID-19 suspects backed by novel distance and similarity measures on plithogenic hypersoft sets. *Artificial Intelligence in Medicine*, *132*, 102390.

[3] Ahmad, M. R., Saeed, M., Afzal, U., & Yang, M. S. (2020). A novel MCDM method based on plithogenic hypersoft sets under fuzzy neutrosophic environment. *Symmetry*, *12*(11), 1855.

[4] Anusha, G., & Ramana, P. V. (2015). Analysis of Reasons for Stress on College Students using Combined Disjoint Block Fuzzy Cognitive Maps (CDBFCM). *International Journal For Research In Emerging Science And Technology*, *2*, 16-21.

[5] Asghari, S., & Akbarpour Shirazi, M. (2023). Presenting Iran's future higher education scenarios using fuzzy cognitive maps. *Research and Planning in Higher Education*, *24*(1), 1-26.
[6] Atanassov, K. (2016). Intuitionistic fuzzy sets. *International journal bioautomation*, *20*, 1.
[7] Axelrod, R. (Ed.). (2015). *Structure of decision: The cognitive maps of political elites*. Princeton university press.
[8] Banathy, B. H. (1991). Cognitive mapping of educational systems for future generations. *World Futures: Journal of General Evolution*, *31*(1), 5-17.
[9] Barón, H. B., Crespo, R. G., Pascual Espada, J., & Martínez, O. S. (2015). Assessment of learning in environments interactive through fuzzy cognitive maps. *Soft Computing*, *19*, 1037-1050.
[10] Chiang, D. F., Guerrero, S. A., & Sexton, E. C. (2023). Supporting Underrepresented Students in Health Sciences: Using a Fuzzy Cognitive Mapping Approach.
[11] Cole, J. R., & Persichitte, K. A. (2000). Fuzzy cognitive mapping: Applications in education. *International journal of intelligent systems*, *15*(1), 1-25.
[12] Devadoss, A. V., Anand, M. C. J., & Bellarmin, A. J. (2013). A Study of Quality in Primary Education Combined Disjoint Block Neutrosophic Cognitive Maps (CDBNCM). In *Indo-Bhutan International Conference On Gross National Happiness* (Vol. 2, pp. 256-261).
[13] Devadoss, A. V., Anand, M. C. J., Felix, A., & Alexandar, S. (2015). A analysis of environmental education for the next generation using combined disjoint block fuzzy cognitive maps (CDBFCMS). In *Indo-Bhutan international conference on gross national happiness* (pp. 262-268).
[14] Dias, S. B., Hadjileontiadou, S. J., Hadjileontiadis, L. J., & Diniz, J. A. (2015). Fuzzy cognitive mapping of LMS users' quality of interaction within higher education blended-learning environment. *Expert systems with Applications*, *42*(21), 7399-7423.
[15] Gayen, S., Smarandache, F., Jha, S., Singh, M. K., Broumi, S., & Kumar, R. (2020). *Introduction to plithogenic hypersoft subgroup*. Infinite Study.
[16] Glykas, M. (Ed.). (2010). *Fuzzy cognitive maps: Advances in theory, methodologies, tools and applications* (Vol. 247). Springer Science & Business Media.
[17] Gómez, G. Á., Moya, J. V., Ricardo, J. E., & Sánchez, C. B. V. (2020). *Evaluating Strategies of Continuing Education for Academics Supported in the Pedagogical Model and Based on Plithogenic Sets* (Vol. 37). Infinite Study.
[18] Gordaliza, J. A., & Flórez, R. E. V. (2013). Using fuzzy cognitive maps to support complex environmental issues learning. In *Proceedings of New Perspectives in Science Education Conference,*.
[19] Gorelova, G. V., LYABACH, N. N., & KUIZHEVA, S. K. (2017). Application of Cognitive Modeling in the Study of the Interrelations between the Educational system and Society. *Revista ESPACIOS*, *38*(56).
[20] Grida, M., Mohamed, R., & Zaied, A. N. H. (2021). *A novel plithogenic MCDM framework for evaluating the performance of IoT based supply chain*. Infinite Study.
[21] Kalaichelvi, A., & Gomathy, L. (2011). Application of neutrosophic cognitive maps in the analysis of the problems faced by girl students who got married during the period of study. *Int. J. of Mathematical Sciences and Applications*, *1*(3).
[22] Kandasamy, W. V., & Smarandache, F. (2003). *Fuzzy cognitive maps and neutrosophic cognitive maps*. Infinite Study.
[23] Kosko, B. (1986). Fuzzy cognitive maps. *International journal of man-machine studies*, *24*(1), 65-75.
[24] Luo, X., Wei, X., & Zhang, J. (2009, October). Game-based learning model using fuzzy cognitive map. In *Proceedings of the first ACM international workshop on Multimedia technologies for distance learning* (pp. 67-76).
[25] Mansouri, T., ZareRavasan, A., & Ashrafi, A. (2021). A learning fuzzy cognitive map (LFCM) approach to predict student performance. *Journal of Information Technology Education: Research*, *20*, 221-243.
[26] Martin, N. (2022). Plithogenic SWARA-TOPSIS Decision Making on Food Processing Methods with Different Normalization Techniques. *Advances in Decision Making*, 69.
[27] Martin, N., & Smarandache, F. (2020). Plithogenic cognitive maps in decision making. *Infinite Study*.
[28] Martin, N., Priya, R., & Smarandache, F. (2021). *New Plithogenic sub cognitive maps approach with mediating effects of factors in COVID-19 diagnostic model*. Infinite Study.
[29] Mary, M. F. J., & Merlin, M. M. M. *A Comparative Study Using Neutrosophic Cognitive Map and Triangular Fuzzy Cognitive Map for Analyzing The Factors for Quality Training of Elementary Education Student Teachers In Tamilnadu*. Infinite Study.

[30] Merlin, M. (2015). Application of augmented fuzzy cognitive map in education. *International Journal in IT & Engineering*, *3*(3), 39-54.

[31] Özçil, A., Tuş, A., Öztaş, G. Z., Adalı, E. A., & Öztaş, T. (2021). The novel integrated model of plithogenic sets and MAIRCA method for MCDM problems. In *Intelligent and Fuzzy Techniques: Smart and Innovative Solutions: Proceedings of the INFUS 2020 Conference, Istanbul, Turkey, July 21-23, 2020* (pp. 733-741). Springer International Publishing.

[32] Peña-Ayala, A., & Sossa-Azuela, J. H. (2013). Decision making by rule-based fuzzy cognitive maps: an approach to implement student-centered education. In *Fuzzy cognitive maps for applied sciences and engineering: From fundamentals to extensions and learning algorithms* (pp. 107-120). Berlin, Heidelberg: Springer Berlin Heidelberg.

[33] Prakash, P., Jerlin, E., & Fernandes, B. (2014). A study on the causes for aversion to mathematics by engineering students using fuzzy cognitive maps (FCMs). *International Journal of Innovative Research in Science, Engineering and Technology*, *3*(3), 10143-10150.

[34] Priya, R., Martin, N., & Kishore, T. E. (2022, May). Plithogenic cognitive analysis on instigating spiritual intelligence in smart age youth for humanity redemption. In *AIP Conference Proceedings* (Vol. 2393, No. 1, p. 020181). AIP Publishing LLC.

[35] Rana, S., Qayyum, M., Saeed, M., Smarandache, F., & Khan, B. A. (2019). *Plithogenic fuzzy whole hypersoft set, construction of operators and their application in frequency matrix multi attribute decision making technique*. Infinite Study.

[36] Thiruppathi, P., Saivaraju, N., & Ravichandran, K. S. (2010). A study on suicide problem using combined overlap block neutrosophic cognitive maps. *International Journal of Algorithms, Computing and Mathematics*, *3*(4), 22-28.

[37] Ulutaş, A., Meidute-Kavaliauskiene, I., Topal, A., & Demir, E. (2021). Assessment of collaboration-based and non-collaboration-based logistics risks with plithogenic SWARA method. *Logistics*, *5*(4), 82.

Selection of the best process for desalination under a Treesoft set environment using the multi-criteria decision-making method

G. Dhanalakshmi[1] , S. Sandhiya[1]*, Florentin Smarandache[2]

[1]Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies, Chennai, Tamilnadu, India

[2]Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA.
Emails: dlklal.govardhan@gmail.com; sandhyasundarr@gmail.com; smarand@unm.edu

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Abstract

Multi-criteria decision-making (MCDM), which has been called a revolution in the field, is one of the most exact methods for making decisions. Multicriteria decision-making (MCDM) is the process of selecting options by considering multiple criteria to determine which is best. A multitude of applications in engineering, design, and finance are possible with the tools and methods derived from MCDM. Application-oriented problems with multiple criteria involve ambiguous and more inaccurate options, to deal with this ambiguity Smarandache introduced Treesoft sets, which are an extension of hypersoft sets. So, in this paper, we will consider a real-life application-oriented problem "Desalination process" under the treesoft sets environment and find the best method for desalination using one of the MCDM methods.

Keywords: Desalination; Multi-criteria Decision Making; Treesoft Set; Neutrosophic set

1. Introduction

One of the most valuable resources on the earth is water. More than 40% of people worldwide are already impacted by its scarcity, according to the UN a figure that prompts concern and motivates the pursuit of remedies. Desalination is one of these, and it is not novel either. It involves using physical and chemical methods to extract the minerals mostly salt from saltwater. In the upcoming years, increasing desalination plant capacity while lowering environmental impact will be crucial. Since 70% of our world is covered in water, it is easy to assume that there is an abundance of it. Nevertheless, fresh water is limited in the world (it makes up only 3% of total water), and two-thirds of that is either frozen solid or unstable. Approximately 2.7 billion people experience scarcity at least one month of the year, while 1.1 billion people worldwide lack access to clean water. Desalination can be beneficial in these situations because, ironically, a lot of places where fresh water scarcity occurs are near the sea.The method of desalination is used to get rid of the dissolved mineral salts in water. This currently one of the most popular methods for extracting fresh water for use in agriculture or human consumption when applied to seawater. Reverse osmosis, solar distillation, electrodialysis, nanofiltration, and gas hydrate are some of the commonly used methods in this desalination process.

Decision-making is a multifaceted intellectual process that involves solving problems while taking into account various factors to achieve a desired outcome. This process may be rational or irrational and can use implicit and explicit assumptions, which may be impacted by biological, cultural, physiological, and social factors. The complexity of decision-making is influenced by these factors, as well as by the level of risk and authority involved. In today's world, computer equipment can assist in automatically calculating and estimating solutions to decision-making problems by using mathematical equations, manifold statistics, mathematics, and economic theories Multicriteria decision-making (MCDM) aims to find the best option by considering several criteria during the selection process. Various tools and techniques from MCDM can be applied in different sectors, such as finance, engineering, and robotics. The scientific tools for managing incomplete data are believed to be periodic mathematics (PM), rough set theory (RST), fuzzy set theory (FST), and probability set theory (PST), which are particularly relevant to these issues, regardless of the missing data range.

Fuzzy sets were first proposed by Zadeh to manage uncertainty, leading to their usefulness in various tasks like pattern detection, medical diagnosis, and decision-making. As a result of their popularity, numerous generalizations of fuzzy sets, such as rough sets, soft sets, intuitionistic fuzzy sets, bipolar fuzzy sets, bipolar soft sets, m-polar soft sets, hypersoft sets, and many others, have been introduced. Soft sets (Ss) are an intriguing expansion of fuzzy sets with additional vagueness and ambiguity features at the highest level of incompleteness. Soft sets deal with uncertain, fuzzy, and unspecified elements. In every decision-making scenario, the role of parameters is relevant, but when parameters are unclear, experts must evaluate them. In such cases, uncertainty is assessed by assigning a fuzzy membership grade, or a fuzzy soft set, which is an extension of soft sets.

There are various circumstances in which attributes need to be further classified into attribute-valued disjoint sets, and fuzzy soft sets are seen to be appropriate parametrization technique to deal with vagueness and uncertainty. Smarandache originated the idea of hypersoft sets (HSSs) from the notion of soft sets by substituting a multiparameter (sub-attribute) function defined on the cartesian product of n attributes for the one-parameter function. Compared to SS, the established HSS is more adaptable and better suited for contexts where decisions are made. Additionally, he introduced sophisticated HSS extensions, such as fuzzy HSS, intutionistic fuzzy HSS, and crisp HSS. HSS theory and its extensions have been gaining surprising traction in recent years. Numerous researchers examined advanced distinguishing operators in addition to traits derived from HSS and its expansions.

The hypersoft sets, a soft set extension, used a multi-argument approximation function to solve the shortcomings of the current structures for attribute-valued disjoint sets. Eventually, MultiSoft sets were added to soft sets to address ambiguous circumstances in real-world issues. Also, treesoft sets, which are extremely close to hypersoft sets, have been an extension of MultiSoft sets in recent times. Treesoft sets are concerned with parameters, sub-parameters, sub-sub-parameters, and so on, while hypersoft sets deal with parameters and sub-parameters. One of the unique characteristics of treesoft sets is their ability to have numerous sub-sub-parameters, denoted as level 0, level 1, level 2,…,level n. Uncertainties in real-world problems will be greatly diminished with the use of treesoft sets. In this paper One of the MCDM method, the ELECTRE method is used. ELECTRE (Elimination and choice Translating Reality) approach has been used to solve issues in many different fields because how well it ranks the alternatives in terms of precedence among the criteria. Benayoun et al. first introduced the ELECTRE method, which has then expanded into many other methods. Numerous domains, including risk assessment, supplier selection, and multiple-criteria decision making have effectively used ELECTRE approaches.

2. Preliminaries :

2.1 Linguistic variable: A linguistic variable is defined in terms of a base variable, whose values are real numbers within a specific range. A base variable in the usual sense, as exemplified by any physical variable(e.g., temperature, pressure, electric current, magnetic flux, etc.) as well as any other numerical variable (e.g., interest rate, blood count, age, performance, etc.)

Table 1: Linguistic variable

Linguistic Terms	umerical range
Very Small	[0.0-0.25]
Small	[0.25-0.45]
Medium	[0.45-0.65]
Large	[0.65-0.85]
Very Large	[0.85-1.0]

2.2 Fuzzy set: Let U be the universe whose generic element be denoted by u. A fuzzy set F in U is a function $F: U \to [0,1]$. We frequently use μ_F for the function F and say that the fuzzy set F is characterized by its membership function $\mu_F: U \to [0,1]$ which associates with each u in U, a real number $\mu_F(u)$ in $[0,1]$. The value $\mu_F(u)$ at u represents the grade of membership of u in F and is interpreted as the degree to which u belongs to F. Thus, the closer the value of $\mu_F(u)$ is to 1, the more u belongs to F.

2.3 Soft set: Let U be a universal set, $P(U)$ be the power set of U and A be the set of all parameters. A soft set (δ, A) on U is defined by:

$(\delta, A) = \{(a, \delta(a)): a \in A, \delta(a) \in P(U)\}$ where $\delta: A \to P(U)$

2.4 Hypersoft sets: Let U be the universal set and $P(U)$ be the power set of U. Consider $a_1, a_2, a_3, \dots, a_n$, for $n \geq 1$ be n well defined attributes, whose corresponding sub-attributes of a_i are respectively the sets $A_1, A_2, A_3, \dots, A_n$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ and $i, j \in \{1,2,3, \dots, n\}$ then the pair $(\delta, A_1 \times A_2 \times \dots \times A_n)$ is said to be hypersoft set over X where $\delta: E_1 \times E_2 \times \dots \times E_n \to P(U)$.

2.5 MultiSoft sets: Let U be the universal set and H be a non-empty subset of U. And $P(H)$ is the power set of H. Let $A_1, A_2, \dots, A_n$ be $n \geq 2$ sets of attributes (parameters) whose intersection $A_1 \cap A_2 \cap \dots \cap A_n = \emptyset$. Let $A = A_1 \cup A_2 \cup \dots \cup A_n$ and $P(A)$ be the powerset of A. Then $F: P(A) \to P(H)$ is a Multisoft set over H.

2.6 Treesoft sets: Let U be the universal set and H a non-empty subset of U, with $P(H)$ the powerset of H. Let A be the set of attributes, $A = \{A_1, A_2, \dots, A_n\}$, for integer $n \geq 1$, where $A_1, A_2, \dots, A_n$ are attributes of first level (since they have one-digit indexes). Each attribute $A_i, 1 \leq i \leq n$, is formed by sub-attributes:

$$A_1 = \{A_{1,1}, A_{1,2}, \dots\}$$

$$A_2 = \{A_{2,1}, A_{2,2}, \dots\}$$

.

.

.

$$A_n = \{A_{n,1}, A_{n,2}, \dots\}$$

Where $A_{i,j}$ are sub-attributes (or attributes of second level) (since they have two-digit indexes). Again, each sub-attribute $A_{i,j}$ is formed by sub-sub-attributes (or attributes of third level): $A_{i,j,k}$ and so on, as much refinement as needed into each application, up to sub-sub-…-sub-attributes or attributes of $m - level$ (or having m digits into indexes): $A_{i_1,i_2,\dots,i_m}$ Therefore, a graph-tree is formed, that we denote as Tree(A) , whose root is A (considered as level zero), then nodes of $level\ 1, level\ 2, \dots, level\ m$, we call it as leaves of the graph-tree, all terminal nodes (nodes that have no descendants). Then the treesoft set is:

$$F: P\big(Tree(A)\big) \to P(H)$$

Tree(A) is the set of all nodes and leaves (from level 1to level m) of the graph-tree, and $P(Tree(A))$ is the powerset of the Tree(A). All node sets of the TreeSoft set of $level\ m$ are:

$$Tree(A) = \{A_{i_1} | i_1 = 1,2, \dots\} \cup \{A_{i_{1,i_2}} | i_1, i_2 = 1,2, \dots\} \cup \dots \cup \{A_{i_1,i_2,\dots,i_m} | i_1, i_2, \dots, i_m = 1,2 \dots\}$$

The first set is formed by the nodes of $level 1$, second set by the nodes of $level\ 2$, third set by the nodes of $level\ 3$, and so on, the last set is formed by the nodes of $level\ m$. If the graph-tree has only two levels $(m = 2)$, then the TreeSoft set is reduced to a MultiSoft set.

Example 1: Consider the set $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ be the cancer diagnosed patients, and $P(S)$ the powerset of S with the corresponding attributes $H = \{H_1, H_2, H_3\}$

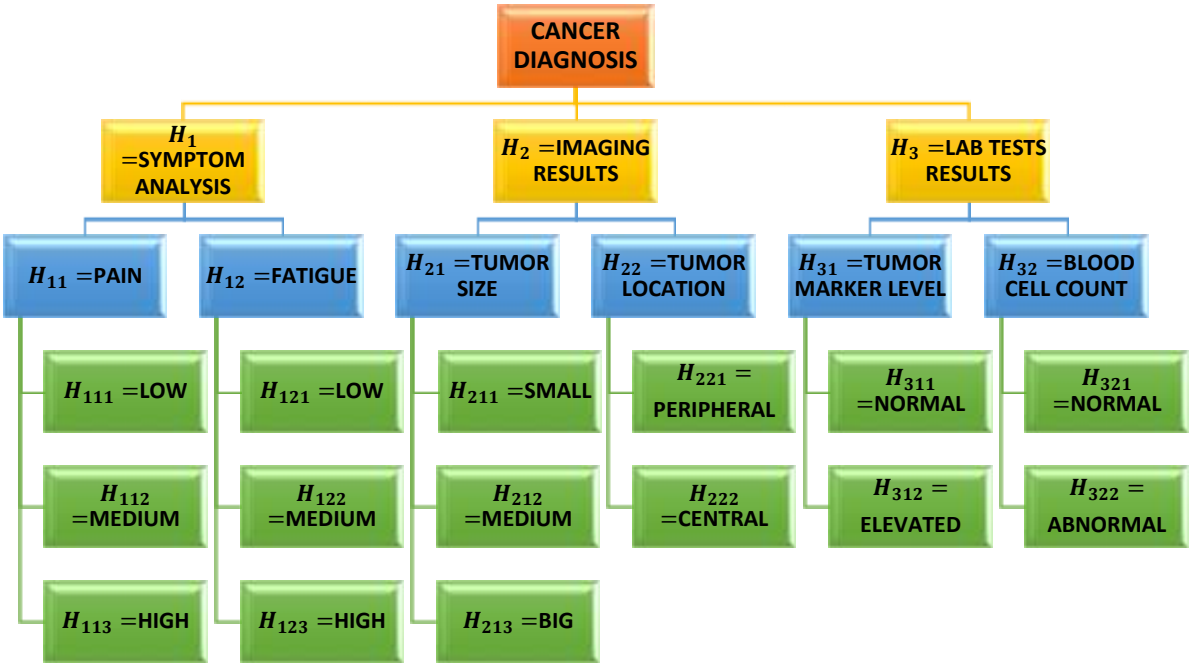
Now lets's assume that the function F gets the following values:

$$F(Fatigue, Tumor\ location, Blood\ cell\ count, Abnormal) = \{s_4, s_5, s_8, \}$$

$$F(Fatigue, Tumor\ location, Blood\ cell\ count, normal) = \{s_1, s_2\}$$

$$F(Fatigue, Tumor\ location, Blood\ cell\ count) = all\ patients\ having\ these\ alternatives\ come\ iunder\ this\ category$$

$$F(Fatigue, Tumor\ location, Blood\ cell\ count, abnormal) \cup F(Fatigue, Tumor\ location, Blood\ cell\ count, normal) = \{s_4, s_5, s_8, s_1, s_2\}$$



3. MCDM Method

ELECTRE (Elimination and Choice Translating reality) MCDM method: The ELECTRE approach is notable outranking models that are broadly applicable for comparing performances across a range of factors for various MCDM issues. The Engineering and Management departments have already started using ELECTRE as a decision-making tool. Benayoun et al.'s ELECTRE techniques, which was first developed, and then rooted out to ELECTRE I, II, III, IV, and IS methods are a popular family of outranking techniques for MCDM issues, which have seen lot of positive and beneficial developments.

Step 1: Develop the Decision matrix $X = [x_{ij}]_{n\times m}$

Step 2: Normalized matrix and weighted normalized matrix can be determined using the below formulae:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} {x_{ij}}^2}} \quad (1)$$

$$V_{ij} = w_{ij} \sum_{i=1}^{m} r_{ij} \quad (2)$$

Step 3: Find the concordance and discordance interval sets

The concordance interval set is used to describe the dominance query

$c_{ab} = \{j | x_{aj} \geq x_{bj}\}$, where $j = 1,2,3 \ldots n$ (3)

The discordance interval sets is given by

$d_{ab} = \{j | x_{aj} < x_{bj}\}$, where $j = 1,2,3 \ldots n$ (4)

Step 4: Calculation of concordance interval matrix

$$c_{ab} = \sum_{j \in c_{ab}} w_j \ for\ j = 1,2,3 \ldots n \quad (5)$$

Step 5: Determine the discordance interval matrix

$$d(a,b) = \frac{j \in D_{ab}{}^{max}\left|v_{aj} - v_{bj}\right|}{j \in J; m,n \in I^{max}\left|v_{mj} - v_{nj}\right|} \quad (6)$$

Step 6: Calculation of concordance index matrix

$$\bar{c} = \sum_{a=1}^{m}\sum_{b=1}^{m}\frac{c(a,b)}{m(m-1)} \quad (7)$$

Based on the threshold value $\bar{c}$ concordance dominance matrix E can be determined as follows

$$E = \begin{cases} e(a,b) = 1 \; if \; c(a,b) \geq \bar{c} \\ e(a,b) = 0 \; if \; c(a,b) < \bar{c} \end{cases} \quad (8)$$

Step 7: Determine the discordance index matrix

The preference of dissatisfaction can be measured by discordance index:

$$\bar{d} = \sum_{a=1}^{m}\sum_{b=1}^{m}\frac{d(a,b)}{m(m-1)} \quad (9)$$

Based on the discordance index mentioned above, the discordance matrix F is given by :

$$F = \begin{cases} f(a,b) = 1 \; if \; d(a,b) \leq \bar{d} \\ f(a,b) = 0 \; if \; d(a,b) > \bar{d} \end{cases} \quad (10)$$

Step 8: Calculate the net superior and net inferior value

Let c_a and d_a be the net superior and net inferior value

The c_a is given by:

$$c_a = \sum_{b=1}^{n} c(a,b) - \sum_{b=1}^{n} c(b,a) \quad (11)$$

On the contrary d_a is used to determine the number of inferiority ranking of the alternatives:

$$d_a = \sum_{b=1}^{n} d(a,b) - \sum_{b=1}^{n} d(b,a) \quad (12)$$

Step 9: Calculate the ranking of the alternatives and find the best alternative.

4. Formulation of the problem: Let A and B respectively be the Environmental head and Technical Manager; they are the decision makers, and they will decide the best desalination process from the below methods for converting sea water into drinking water, with the following weight criterions given by {0.2,0.13,0.16,0.11,0.21,0.19}

$D = \{d_1, d_2, d_3, d_4, d_5\}$ be the desalination process, were,

$$d_1 = Reverze\ Osmosis, d_2 = Solar\ Distillation, d_3 = Electro\ dialysis,$$

$$d_4 = Nano\ filtration, d_5 = Gas\ hydrate\ formation$$

and the corresponding attributes be

$$e_1 = Costs, e_2 = Chemical\ reactions, e_3 = Process\ of\ desalination,$$

$$e_4 = Resources, e_5 = Health\ aspects, e_6 = Public\ Support$$

with the corresponding sub-attributes

$$e_1 = E_1 = \begin{Bmatrix} e_{11} = economic\ costs, \\ e_{12} = environmental\ costs \end{Bmatrix};\ e_2 = E_2 = \begin{Bmatrix} e_{21} = adverse\ effects, \\ e_{22} = non - adverse\ effects \end{Bmatrix}$$

$$e_3 = E_3 = \begin{Bmatrix} e_{31} = membrane\ process, \\ e_{32} = thermal\ process \end{Bmatrix};\ e_4 = E_4 = \begin{Bmatrix} e_{41} = manpower, \\ e_{42} = equipment \end{Bmatrix}$$

$$e_5 = E_5 = \{e_{51} = iodine\ deficieny\};\ e_6 = E_6 = \begin{Bmatrix} e_{61} = high\ fresh\ water\ scarcity \\ e_{62} = low\ fresh\ water\ scarcity \end{Bmatrix}$$

And the corresponding sub-sub-attributes are given below:

$$e_{11} = \begin{Bmatrix} e_{111} = energy\ cost, \\ e_{112} = land\ cost \end{Bmatrix};\ e_{12} = \begin{Bmatrix} e_{121} = destruction\ of\ food\ chain, \\ e_{122} = residual\ salt \end{Bmatrix};$$

$$e_{21} = \{e_{211} = more\ of\ chemicals\};\ e_{22} = \{e_{221} = purity\ of\ water\};$$

$$e_{31} = \{e_{311} = thin\ sheet\ of\ filter\};\ e_{32} = \{e_{321} = heat\ energy\};$$

$$e_{41} = \begin{Bmatrix} e_{411} = researchers, \\ e_{412} = technicians \end{Bmatrix};\ e_{42} = \begin{Bmatrix} e_{421} = electrical, \\ e_{422} = electronical \end{Bmatrix};$$

$$e_{51} = \{e_{511} = removal\ of\ iodine\ deficiency\};$$

$$e_{61} = \{e_{611} = high\ public\ support\};\ e_{62} = \{e_{621} = less\ fresh\ water\ scarcity\}$$

Let the function be $\varphi: E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \to P(S)$

Now let us assume the relation

$$\varphi(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6) = \varphi\begin{pmatrix} environmental(e_{12}), purity\ of\ water(e_{221}), membrane\ process\ (e_{31}), electrical(e_{421}), \\ iodine\ deficieny\ (e_{51}), high\ public\ support\ (e_{611}). \end{pmatrix}$$

These are some of the actual requirements for selecting the best method from desalination process, the decision makers give their opinions in matrix form separately as:

$$A = \begin{matrix} \varphi(d_1) = \{e_{12}(0.7), e_{221}(0.8), e_{31}(0.7), e_{421}(0.6), e_{51}(0.5), e_{611}(0.6)\} \\ \varphi(d_2) = \{e_{12}(0.7), e_{221}(0.3), e_{31}(0.4), e_{421}(0.5), e_{51}(0.1), e_{611}(0.2)\} \\ \varphi(d_3) = \{e_{12}(0.1), e_{221}(0.4), e_{31}(0.6), e_{421}(0.2), e_{51}(0.3), e_{611}(0.6)\} \\ \varphi(d_4) = \{e_{12}(0.5), e_{221}(0.3), e_{31}(0.4), e_{421}(0.2), e_{51}(0.7), e_{611}(0.4)\} \\ \varphi(d_5) = \{e_{12}(0.8), e_{221}(0.1), e_{31}(0.3), e_{421}(0.2), e_{51}(0.6), e_{611}(0.2)\} \end{matrix}$$

$$B = \begin{matrix} \omega(d_1) = \{e_{12}(0.9), e_{221}(0.7), e_{31}(0.5), e_{421}(0.6), e_{51}(0.3), e_{611}(0.5)\} \\ \omega(d_2) = \{e_{12}(0.2), e_{221}(0.5), e_{31}(0.3), e_{421}(0.2), e_{51}(0.7), e_{611}(0.4)\} \\ \omega(d_3) = \{e_{12}(0.6), e_{221}, (0.3)e_{31}(0.4), e_{421}(0.1), e_{51}(0.3), e_{611}(0.4)\} \\ \omega(d_4) = \{e_{12}(0.5), e_{221}(0.4), e_{31}(0.2), e_{421}(0.6), e_{51}(0.5), e_{611}(0.8)\} \\ \omega(d_5) = \{e_{12}(0.1), e_{221}(0.8), e_{31}(0.4), e_{421}(0.3), e_{51}(0.4), e_{611}(0.5)\} \end{matrix}$$

Step 1: Construct the decision matrix

$$A = \begin{matrix} 0.7 & 0.8 & 0.7 & 0.6 & 0.5 & 0.6 \\ 0.7 & 0.3 & 0.4 & 0.5 & 0.1 & 0.2 \\ 0.1 & 0.4 & 0.6 & 0.2 & 0.3 & 0.6 \\ 0.5 & 0.3 & 0.4 & 0.2 & 0.7 & 0.4 \\ 0.8 & 0.1 & 0.3 & 0.2 & 0.6 & 0.2 \end{matrix}$$

$$B = \begin{matrix} 0.9 & 0.7 & 0.5 & 0.6 & 0.3 & 0.5 \\ 0.2 & 0.5 & 0.3 & 0.2 & 0.7 & 0.4 \\ 0.6 & 0.3 & 0.4 & 0.1 & 0.3 & 0.4 \\ 0.5 & 0.4 & 0.2 & 0.6 & 0.5 & 0.8 \\ 0.1 & 0.8 & 0.4 & 0.3 & 0.4 & 0.5 \end{matrix}$$

Step 2: By using an accuracy function (average of the matrices) both A and B has been reduced to a single matrix as follows:

$$Q = \begin{matrix} 0.8 & 0.75 & 0.6 & 0.6 & 0.4 & 0.55 \\ 0.45 & 0.2 & 0.35 & 0.35 & 0.4 & 0.3 \\ 0.35 & 0.35 & 0.5 & 0.15 & 0.3 & 0.4 \\ 0.5 & 0.35 & 0.3 & 0.35 & 0.6 & 0.6 \\ 0.45 & 0.45 & 0.35 & 0.25 & 0.5 & 0.35 \end{matrix}$$

Step 3: The normalized and weighted normalized matrix are found using equation (1) and (2) mentioned above:

$$Q = \begin{matrix} 0.134 & 0.095 & 0.099 & 0.077 & 0.083 & 0.103 \\ 0.076 & 0.025 & 0.058 & 0.052 & 0.083 & 0.056 \\ 0.059 & 0.044 & 0.082 & 0.019 & 0.062 & 0.075 \\ 0.084 & 0.044 & 0.049 & 0.045 & 0.125 & 0.112 \\ 0.076 & 0.057 & 0.058 & 0.032 & 0.104 & 0.065 \end{matrix}$$

Step 4: Based on the above conditions in equation (3) and (4) and applying the equations (5) and (6) the concordance and discordance interval matrices are calculated and obtained as follows:

Concordance interval Matrix

$$Q_1 = \begin{matrix} 0 & 1 & 1 & 0.6 & 0.79 \\ 0.21 & 0 & 0.52 & 0.27 & 0.47 \\ 0 & 0.48 & 0 & 0.29 & 0.35 \\ 0.4 & 0.73 & 0.84 & 0 & 0.71 \\ 0.21 & 0.89 & 0.65 & 0.29 & 0 \end{matrix}$$

Discordance interval matrix

$$Q_2 = \begin{matrix} 0 & 0 & 0 & 0.819 & 0.354 \\ 1 & 0 & 0.766 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0.147 & 0.528 & 0 & 0.272 \\ 1 & 0.61 & 0.594 & 1 & 0 \end{matrix}$$

Step 5: Again, by applying equation (7), (8), (9) and (10) we obtain the concordance index matrix and discordance index matrix

Concordance index matrix

$$Q_1 = \begin{matrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{matrix}$$

Discordance index matrix

$$Q_2 = \begin{matrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{matrix}$$

Step 6: Finally, the net superior and net inferior values are found using the equations (11) and (12), then the ranking is found:

ESALINATION PROCESS	**ET SUPERIOR ALUE**	**ANK**	**ET INFERIOR ALUE**	**ANK**
$d_1\ (Reverse\ Osmosis)$	2.57	1	-2.827	1
$d_2\ (Solar\ Distillation)$	-1.63	4	2.009	4
$d_3(Electro\ dialysis)$	-1.89	5	2.112	5
$d_4(Nano\ filtration)$	1.23	2	-1.872	2
$d_5\ (Gas\ Hydrate\ formation$	-0.28	3	0.578	3

From the net superior value of Concordance matrix and net inferior value of Discordance matrix and by ranking of the alternatives we can conclude that **Reverse Osmosis** method is ranked in the first position when compared to all other desalination methods.

4. Conclusion

In an effort to discuss the Treesoft set environment, a real-world problem involving selecting the optimal desalination procedure using one of the MCDM approaches is taken up in this study. When choosing the process, the related attributes are also given priority. The ELECTRE approach has already been used to enhance performance in problems with multiple criteria outranking. Additionally, we may solve more ambiguous and multi-attribute problems by utilising Treesoft sets. Treesoft sets can be used in all other fuzzy environments Intutionistic, Pythagorean, Neutrosophic, and so forth in future research.

References

[1] Abbas, M, Murtaza.G & Smarandache,F. "Basic operations on hypersoft sets and hypersoft point". Neutrosophic sets and systems, vol.35, no. 1, pp. 407-421 (2020).

[2] Atanassov, K. Intuitionistic fuzzy sets. "Fuzzy sets and systems", vol. 20,0 no.1, pp. 87-96 (1986). http://dx.doi.org/10.1016/S0165-0114(86)80034-3

[3] Cagman, N., Enginoglu, S., & Citak,F. "Fuzzy soft set theory and its applications", Iranian journal of fuzzy systems, vol.8, no.3, pp.137-147(2011). 10.22111/IJFS.2011.292

[4] Khalid Abd, Kazem Abhary, Romeo Marian. "Application of Fuzzy Logic to Multi-Objective Scheduling Problems in Robotic Flexible Assembly Cells". Automation Control and Intelligent Systems. vol. 1, No. 3, 2013, pp. 34-41. 10.11648/j.acis.20130103.11

[5] Kong, Z., Wang., L., Wu, Z. "Application of fuzzy soft set in decision making problems based on grey theory" . J. Comput. Appl. Math.,vol. 236, pp. 1521-1530(2011). https://doi.org/10.1016/j.cam.2011.09.016

[6] Maji, P.K., Biswas, R., & Roy, A.R. "Fuzzy soft sets". Journal of fuzzy mathematics. vol. 9, pp. 589-602 (2001).

[7] Maji, P.K, Biswas R and Roy A.R. "Soft set theory", Computer. Math. Appl., vol. 45, pp. 555-562(2003). https://doi.org/10.1016/S0898-1221(03)00016-6

[8] Smarandache, F. ' Extension of soft sets to hypersoft set, and then to plithogenic hypersoft set, " Neutrosophic Sets and system, vol. 22, pp. 168-170,(2018). 10.5281/zenodo.1408740

[9] Smarandache, F. " Practical applications of IndetermSoft sets and IndetermHypersoft sets and Introduction to Treesoft set as an extension of Multisoft set, " Neutrosophic Sets and System, vol.51, pp. 941-947, (2022). 10.5281/zenodo.7154238

[10] Smarandache, F. "New types of Soft sets, HyperSoft sets, IndetermSoft sets, IndetermHyperSoft set, and TreeSoft set", Neutrosophic Sets and System, vol.8 (2023) **https://doi.org/10.61356/j.nswa.2023.41**.

[11] Yildiray Celik and Sultan Yamak, "Fuzzy soft set theory applied to medical diagnosis using fuzzy arithmetic operations" vol.1, no. 82 (2013). https://doi.org/10.1186/1029-242X-2013-82

[12] Zadeh L.A. "Fuzzy sets, Information and Control" vol. 8, pp. 338-353 (1965). https://doi.org/10.1016/S0019-9958(65)90241-X

CSsEv: Modelling QoS Metrics in Tree Soft Toward Cloud Services Evaluator based on Uncertainty Environment

Mona Gharib[1], Florentin Smarandache[2], Mona Mohamed[3],*

[1]Department of Mathematics, Faculty of Science, Zagazig University, 44519 Zagazig, Egypt

[2]University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA

[3]Higher Technological Institute, 10th of Ramadan City 44629, Egypt

Emails: monagharib@zu.edu.eg; smarand@unm.edu; mona.fouad@hti.edu.eg

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Abstract

Cloud computing (ClC) has become a more popular computer paradigm in the preceding few years. Quality of Service (QoS) is becoming a crucial issue in service alteration because of the rapid growth in the number of cloud services. When evaluating cloud service functioning using several performance measures, the issue becomes more complex and non-trivial. It is therefore quite difficult and crucial for consumers to choose the best cloud service. The user's choices are provided in a quantifiable manner in the current methods for choosing cloud services. Hence, this study attempts to achieve this objective through construction. decision-making framework so-called cloud services evaluator (CSsEv). The main indicator and its sub-indicators are formed as nodes at levels(n) in tree soft sets (TSSs). Thereafter Single Value Neutrosophic Sets (SVNSs) as branch of neutrosophic sets which conjunction with the Multi-Criteria Decision Making (MCDM) technique to facilitate analysis and evaluation process for the available Cloud services providers. Hence, entropy is employed to obtain indicators and sub_indicators' weights and Complex Proportional Assessment utilizes these weights to facilitate the decision process of selecting optimal ClSPs.

Keywords: Cloud Computing (ClC); tree soft sets (TSSs); Quality of Service (QoS); Single Value Neutrosophic Sets (SVNSs); Multi-Criteria Decision Making (MCDM)

1. Introduction

Cloud computing (ClC) is a concept that draws from several cutting-edge inventions and has certain characteristics with previous computing technologies [1]. This viewpoint is predicated on [2] which described ClC as a new paradigm that allows users to access self-service, on-demand computer services (software and hardware) over the Internet, regardless of the system or position. In the same vein scholars in[3] where ClC is similar to other utilities, ClC is a new technology that distributes computing resources over the web as a service to cloud customers. Utilizing ClC paradigm [4] enables ubiquitous, practical, and on-demand network access through sharing a pool of quickly accessible configurable computer resources (such as networks, servers, storage, apps, and services). In contrast to traditional paradigms, ClC offers significant advantages since CCT services are more promptly and easily accessible around the clock with relatively cheap investment expenses [5]. Due to [6] where three service tiers included in ClC concept are (1) infrastructure as a service (IaaS) that offers assistance for users of the cloud,(2) platform as a service (Pass) renders a platform for application development, whereas SaaS gives users access to ready-made applications, and (3) software as a service (SaaS) empowers users with a range of services. ClC's use and development have thus increased dramatically to make use of ClC's benefits. This is considered a robust motivator for the exponential increase in cloud service providers (ClSPs). For instance [1] Google, IBM, Microsoft, Amazon GoGrid…etc as ClSPs, Accordingly [7], this incentive presented difficulty for cloud customers in locating the finest ClSP for their desired and essential needs. Study of [8] overcoming this issue as cloud consumer must determine what services are necessary for Quality of Service (QoS) for selecting optimal ClSP.

Another study [6] discussed the perspectives that embraced perspective of [8] through determining a set of QoS parameters as security , availability, response time…etc to measure the performance of cloud services to select optimal ClSP amongst variety of ClSPs. For instance, cloud service measurement initiative consortium (CSMIC) considered one of perspectives has been clarified by [6] which determined a set of quality of service (QoS) indicators, dubbed the service measurement index (SMI), for gauging different aspects of a cloud service.

The issue of ClSP selection releases through deploying MCDM techniques. Due to [9]-[10] these techniques are facilitating the process of decision making for solving complex problems. Another motivator for deploying these techniques is highlighted in [11] where a wide variety of assessment criteria for different cloud services from several cloud service providers (ClSPs) must be taken into consideration in order to identify which ClSP best suits the demands of a cloud user.

Also, such problem of selection characterized with uncertainty and vague. Hence, MCDM techniques have been hybridized with other uncertainty techniques in [12] that have ability to treat with such issue.

Zadeh proposed fuzzy set (FSs) [13] to deal with ambiguities and untrustworthy data utilized in decision-making. FSs are considering membership function, and an intuitionistic fuzzy sets (IFSs) advanced version of FSs is proposed by Atanassov for affording the chance to address far more difficult decision-making issues [14]. Due to the ability of IFSs to consider membership function and non-membership function. Also, indeterminacy degree beside membership and non-membership taken into consideration through Neutrosophic theory. This theory is introduced by Smarandache [15] so, this theory is more flexible comparing with FSs. Also, Smarandache [16] proposed new version of soft set as hyper soft and tree soft.

The issue of choosing a cloud service has received a lot of attention in the past year. The literature's current methodology seldom ever considers the association between QoS criteria [17].

In this scenario, we are representing the relation and association between QoS criteria through leveraging tree soft sets (TSSs) for the first time in this field. Also, MCDM techniques are interacting with Neutrosophic theory to strength MCDM in uncertainty and vague situation during decision making in ClSP selection process.

Hence, we are suggesting decision making framework where the objective of this framework is summarizing in a few points:

1. The association and relation between QoS's indicators are formed in TSSs through representing main indicators in first nodes in primary level and other indicators formed in next nodes included in later levels as sub-indicators.
2. The formed indicators and sub-indicators are contributed to evaluate ClSPs and select optimal one.
3. MCDM techniques utilized in vague environment toward evaluation process through interacting with Neutrosophic theory and the result of this interaction produces decision making framework so called cloud services evaluator (CSsEv).
4. Applying constructed CSsEv in real case study to validate accuracy of this framework.

2. Theoretical Background

Herein, we attempted to aggregate the various scholars' perspectives in prior studies. The objective of this section has been achieved through covering the following pillars.

2.1 First pillar: Analyzing and evaluating QoS's indicators based on MCDM.

Scholars concentrated on evaluating CSPs' performance as well as developing frameworks for locating suitable cloud services[6]. Hence, the selection problem has been extensively solved in literature using the MCDM techniques. For instance [18] applied Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) in complex decision making process for ranking large number of alternatives. TOPSIS's extensive use in resolving decision-making issues in the literature has led to its use in ranking cloud services relative to other MCDM techniques like complex proportional assessment (COPRAS) and multi-attributive border approximation area comparison (MABAC)[6]. Evidence of this is the study of [19] claimed that multi-criteria optimization and compromise solution VIKOR is less

efficient than TOPSIS. scholars in [20] explored the advantages of using entropy in addition to AHP for the suggested weighting technique.

2.2 Second pillar: Boosting MCDM for decision making process in ambiguity environment.

Decision-makers can freely express their opinions about truth, indeterminacy, and falsity membership values in a neutrosophic theory, and each membership value can stand alone. Hence neutrosophic has been combined with several MCDM techniques [6]-[21] for assisting decision-makers in clearing up any uncertainty in their thinking. Accordingly, Neutrosophic set theory has grown in significance in many decision-making scenarios because it gives decision-makers the freedom to rank the options in terms of language. For instance [22] employed the decision-making trial and evaluation laboratory approach (DEMATEL) in conjunction with neutrosophic set theory to address the transport service provider selection problem. Also, this theory, in particular, uses an interval value neutrosophic set in [23] to address uncertainty, whereas CODAS is utilized to determine the optimal placement for a wind energy facility. Also, this theory employed in various purposes as in [24]-[25]-[26].

2.3 Third Pillar: Clarifying relation between QoS's indicators as supportive factor in evaluation process.

In this study, we are exploiting new extension of soft set entailed in TSSs which introduced by Smarandache who highlighted it in [16].we utilized TSSs for first time in our interested scope for illustrating determined indicators in this study as nodes involved in tree. We described TSSs according to Smarandache as:

Let V be a universe of discourse, and T a non-empty subset of V, with P(T) the powerset of T.

Let Ind be a set of indicators (parameters, factors, etc.), Ind= $\{Ind_1, Ind_2, \dots, Ind_n\}$, for integer $n \geq 1$, where Ind_1, Ind_2, … , An are considered indicators of first level (since they have one-digit indexes).

Each indicator Ind i, $1 \leq i \leq n$, is formed by sub- indicators:

$Ind_1 = \{Ind_{1,1}, Ind_{1,2}, \dots\}$ $Ind_2 = \{ Ind_{2,1}, Ind_{2,2}, \dots \}$ $Ind_n = \{ Ind_{n,1}, Ind_{n,2}, \dots \}$

where the above $Ind_{i,j}$ are sub indicators (or indicators of second level) (since they have two-digit indexes). Again, each sub-attribute $Ind_{i,j}$ is formed by sub- indicators (or indicators of third level): $Ind_{i,j,k}$

And so on, as much refinement as needed into each application, up to sub-sub-…-sub- indicators (or indicators of m-level (or having m digits into the indexes): $Ind_{i1,i2,\dots,im}$

So, a graph-tree is formed, that we denote as Tree (Ind), whose root is Ind (considered of level zero), then nodes of level 1, level 2, up to level m.

We call the leaves of the graph-tree, all terminal nodes (nodes that have no descendants).

Then the TreeSoft Set is: F: P(Tree (Ind)) → P(T)

Tree (Ind) is the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and P(Tree(A)) is the powerset of the Tree (Ind).

All node sets of TSSs of level m are: Tree (Ind) = $\{ Ind_{i1} | i1= 1, 2, ... \}$

3. CSsEv: Cloud Services Evaluator Methodology

The evaluation process is conducted for evaluating candidates ClSPs after that selection process is starting to select optimal ClSP and worst one. These processes are implemented through the following steps.

Step 1: Determining nodes of TSSs

- Nodes of TSSs are determined where QoS'S indicators represents main nodes (Ind_n) in level 1 in form $\{Ind_1, Ind_2, \dots Ind_n\}$.
- Next nodes represents sub-indicators (sub_Ind) in next sub_levels which inherent of main level 1 $=\{sub\_Ind_{1i}, sub\_Ind_{2i}, .. sub\_Ind_{ni}\}$.
- Set of candidates of ClSPs as $\{ClSP_1, ClSP_2, \dots ClSP_n\}$ are recommended to contribute to evaluation and selection processes.

Step 2: Appreciating Indicators/sub-Indicators from main level1 to level in

- Decision Matrices are constructed based on Linguistic expert's for evaluating $ClSP_n$ over indicators (Ind_n) in level 1 $\{A_1, A_2 \ldots A_n\}$. Also, Decision Matrices are constructed based on Linguistic expert's for evaluating $ClSP_n$ over sub-indicators (sub_Ind_{ni}) in sub_level s $\{sub\_Ind_{1i}, sub\_Ind_{2i}, .. sub\_Ind_{ni}\}$.
- Linguistic expert's for evaluating $ClSP_n$ inspired from scale of single value Neutrosophic sets (SVNSs).

Step 2.1 Entropy technique begins to work in constructed decision matrices for $ClSP_n$ over indicators/ through following sub-steps:

Step 2.2: crisp matrices are generated through Eq.(1) in various decision matrices.

$$\boldsymbol{s}(\partial_{ij}) = \frac{(2+\mathcal{g}-q-\wp)}{3} \quad (1)$$

Where:

$\mathcal{g}, q, \wp$ refers to truth, false, and indeterminacy respectively.

Step 2.3: Crisp matrices are conjunction into an aggregated decision matrix.

$$Q_{ij} = \frac{(\sum_{j=1}^{N} \partial_{ij})}{F} \quad (2)$$

Where:

∂_{ij} refers to value of criterion in matrix, F refers to number of decision makers.

Step 2.4: aggregated decision matrix employed Eq.(3) to be normalized matrix

$$D_{ij}=\frac{Q_{ij}}{\sum_{j=1}^{n} Q_{ij}} \quad (3)$$

Where:

$\sum_{j=1}^{n} Q_{ij}$ represents sum of each criterion in aggregated matrix per column

Step 2.5: normalized matrix computes its entropy by Eq. (4).

$$e_{j=-h\sum_{i=1}^{n} D_{ij}} \ln D_{ij} \quad (4)$$

Where:

$$h = \frac{1}{\ln (ClSPs)} \quad (5)$$

ClSPs refers to number of alternatives.

Step 2.6: Compute weight vectors through deploying Eq.(6).

$$w_{j=} \frac{1 - e_j}{\sum_{j=1}^{n}(1 - e_j)} \quad (6)$$

Step 3: Ranking ClSPs and recommend optimal ClSP

COPRAS based on SVNSs as subjective technique of MCDM for ranking set of candidates of alternatives through the following steps.

Step 3.1: weighted decision matrix (WD) is generated as in Eq (7).

$$WD_{ij} = w_j * D_{ij} \quad (7)$$

Where:

w_j vector of entropy's weights

Step 3.2: Eq.s (8) ,(9) are deployed for calculating Sum of weighted decision matrix to

$S_{+i} = \sum_{j=1}^{n} WD_{+ij}$, for beneficial criteria (8)

$S_{-i} = \sum_{j=1}^{n} WD_{-ij}$, for nonbeneficial criteria (9)

Step 3.2: the relative importance of alternatives is calculated by Eq.(10).Also, quantity utility U_i for each alternative is calculated based on Eq. (11) to rank the alternatives.

$$Q_i = s_{+i} + \frac{s_{-min} \sum_{i=1}^{m} s_{-i}}{s_{-i} \sum_{i=1}^{m} (s_{-m}/s_{-i})} \quad (10)$$

Where:
I = 1, 2,…,m, and $s_{-m} = s_{-i}$ all criteria are beneficial.

$$U_i = \left[\frac{Q_i}{Q_{max}}\right] \times 100\% \quad (11)$$

Where:
the alternative with the highest U_i is the best one.

4. An Empirical Case Study: Validation of CSsEv framework

We validate the constructed framework of CSsEv through applying it on real case study. Hence, four ClSPs contributed to the evaluation process as candidates.

4.1 Obtaining Optimal Cloud over Indicators at Level 1 in Tree soft.

4.1.1 Tree soft set of indicators and sub-indicators has been designed.

1. At main level 1: there are three indicators of QoS which represent as primary nodes as {Ind_1, Ind_2, Ind_3} ={Security, Service quality, Management services} [27].
2. At level 2: there are six sub-indicators which represent as sub_nodes of primary nodes as {$sub\_Ind_{1.1}$, $sub\_Ind_{1.2}$ }of Ind_1={Confidentiality, Integrity};{$sub\_Ind_{2.1}$, $sub\_Ind_{2.2}$ }of Ind_2 ={Service stability, Reputation of vendor};};{$sub\_Ind_{3.1}$, $sub\_Ind_{3.2}$ }of Ind_3 ={Monitoring , Configuration}

4.1.2 Valuing indicators in TSSs :At Level 1

3. Three Neutrosophic decision matrices are constructed based on Linguistic for three experts. Utilizing SVNSs linguistic scale of [28] for evaluating four ClSPs over three indicators in level 1. Also, this scale is used for evaluating four ClSPs over six sub_indicators in level 2.
4. Three Neutrosophic decision n matrices become crisp matrices by Eq.(1).
5. These crisp matrices are aggregating into an aggregated decision matrix as listed in Table 1.
6. Table 2 represents normalized matrix based on Eq.(3).
7. Eq.(4) is employed for computing entropy for normalized matrix as in Table 3.
8. Final indicators' weights is obtained in Figure 1 through executing Eq.(5).

Table 1: an aggregated matrix of Indicators at level 1

	Ind_1	**Ind_2**	**Ind_3**
$ClSP_1$	0.477777778	0.222222222	0.538888889
$ClSP_2$	0.288888889	0.627777778	0.7
$ClSP_3$	0.5	0.633333333	0.538888889
$ClSP_4$	0.433333333	0.367777778	0.166666667

Table 2: Normalized matrix of Indicators at level 1

	$\mathbf{Ind_1}$	$\mathbf{Ind_2}$	$\mathbf{Ind_3}$
$\mathbf{ClSP_1}$	-0.356713893	-0.254485409	-0.35563586
$\mathbf{ClSP_2}$	-0.301182196	-0.366726066	-0.367794449
$\mathbf{ClSP_3}$	-0.35993395	-0.366956989	-0.35563586
$\mathbf{ClSP_4}$	-0.348419443	-0.321078433	-0.210577352
$\sum_{i=1}^{m} \mathbf{X_{ij}}$	-1.366249482	-1.309246898	-1.289643521
$-h\sum_{i=1}^{m} \mathbf{X_{ij}} \ln X_{ij}$	0.985065877	0.943967013	0.929832979

Table 3. Entropy of Normalized matrix of Indicators at level 1

	$\mathbf{Ind_1}$	$\mathbf{Ind_2}$	$\mathbf{Ind_3}$
$\mathbf{ClSP_1}$	0.281045752	0.120048019	0.277142857
$\mathbf{ClSP_2}$	0.169934641	0.339135654	0.36
$\mathbf{ClSP_3}$	0.294117647	0.342136855	0.277142857
$\mathbf{ClSP_4}$	0.254901961	0.198679472	0.085714286

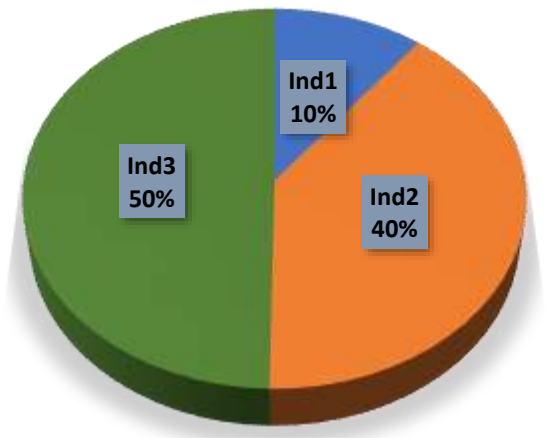

Figure 1: Weights of Indicators in Level 1

4.1.3 Ranking ClSPs and recommend optimal ClSP: At Level 1

1. Eq.(7) plays a critical role in normalized matrix to generate weighted decision matrix as in Table 4.
2. Considering all indicators are beneficial, we applied Eq.(8) to obtain sum weighted.

3. Quantity utility U_i for each alternative is calculated based on Eq. (11) to rank the alternatives and results illustrated in Figure 2.$ClSP_2$ is optimal cloud provider otherwise $ClSP_4$ is worst cloud provider.

Table 4: Weighted decision matrix of Indicators at level 1

	Ind$_1$	**Ind$_2$**	**Ind$_3$**
ClSP$_1$	0.024313606	0.028852563	0.186557868
ClSP $_2$	0.01470125	0.081508491	0.242332901
ClSP $_3$	0.025444471	0.082229805	0.186557868
ClSP $_4$	0.022051875	0.047750992	0.05769831

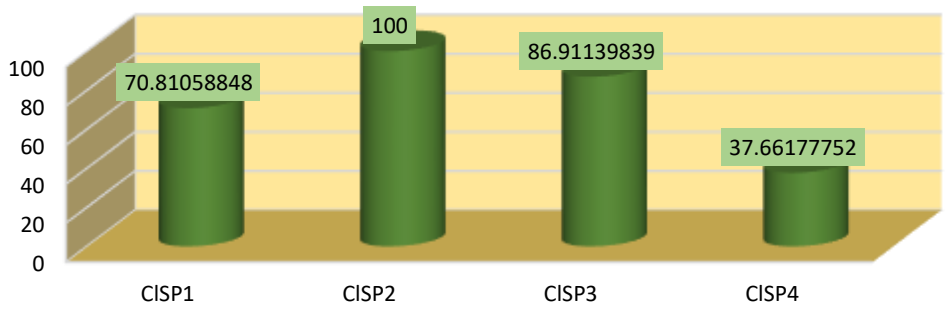

Figure 2: Ranking ClSPs over Indicators in Level 1

4.2 Obtaining Optimal Cloud over Indicators at Level 1 in Tree soft.

We repeat the previous three steps to evaluating and ranking ClSPs over sub_indicators at level 2.

4.2.1 Valuing Sub_indicators in TSSs : At Level 2

1. Three Neutrosophic decision matrices are constructed for {Ind $_{1.1}$, Ind $_{1.2}$}; {Ind $_{2.1}$, Ind $_{2.2}$}; {Ind $_{3,1}$,and Ind $_{3.2}$ }
2. These matrices are transformed into crisp matrices based on Eq.(1) and Eq.(2) aggragated each sub_indicator into aggregated matrix belongs to main node (indicators) at level 1.
3. Figure 3 indicated that sub_indicator 1.1 outperforms sub_indicator 1.2.
4. Figure 4 indicated that sub_indicator 2.1 outperforms sub_indicator 2.2.
5. Figure 5 indicated that sub_indicator 3.1 outperforms sub_indicator 3.2.

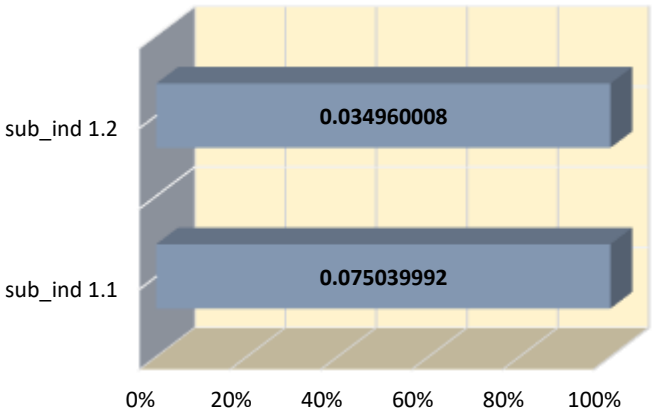

Figure 3: Final Weights of Sub Indicators 1.1 to 1.2 in Level 2

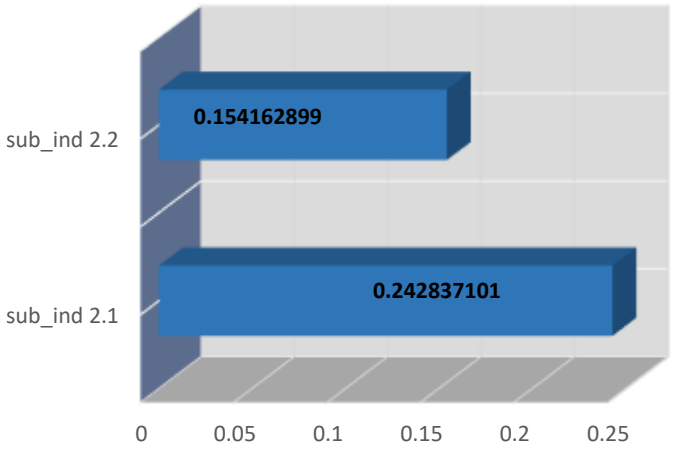

Figure 4: Final Weights of Sub Indicators 2.1 to 2.2 in Level 2

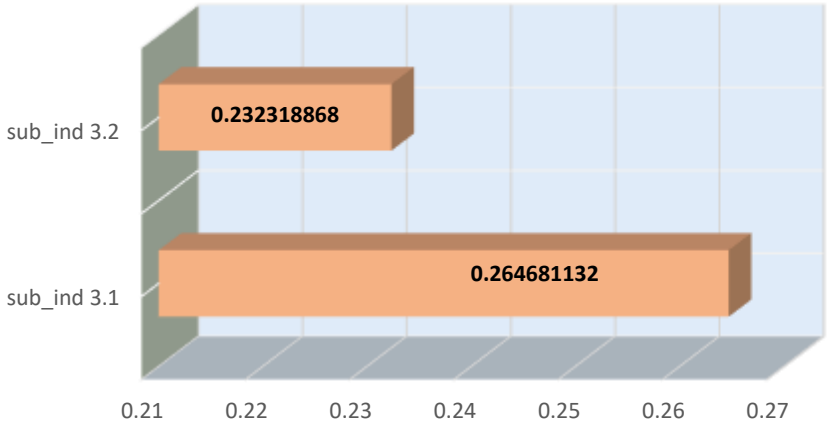

Figure 4: Final Weights of Sub Indicators 2.1 to 2.2 in Level 2

5. Conclusion

In the modern age, ClC has completely changed the way businesses operate. Selecting the best service provider for a given activity has grown challenging due to the growing number of cloud service providers (ClSPs) and the increased demand for cloud computing among consumers. This issue is considered catalyst for us to construct decision making framework. Herein, the constructed framework so-called CSsEv for evaluating set of candidates of ClSPs alternatives. By providing a linguistic evaluation for cloud services with incomplete or inaccurate knowledge, it assists cloud customers in identifying the optimal cloud service for their functional and non-functional requirements and lowers the likelihood of suffering significant losses over time. Thus, this framework employed various techniques for achieving its objective. Firstly, the influenced indicators and sub_indicators of QoS have been clarified into form of nodes of tree. Hence, TSSs applied for the first time in this field. Secondly, Neutrosophic theory in particular SVNSs is exploited and amalgamated with entropy to obtain weights for indicators and sub-indicators. These weights are leveraging for ranking ClSPs through employing COPRAS to achieve this objective. In this study we determined three indicators as primary nodes in initial level and six

sub_indicators as sub_nodes in level 2 which branched from primary nodes Finally, we applied CSsEv framework in real case study to ensure its authenticity. The results of application indicated that $ClSP_2$ is optimal cloud provider according to Quantity utility U_i in contrast $ClSP_4$ is worst cloud provider as in Figure 2.

6. Future Directions

In this field we can employ other extensions of soft set aside from TSSs as hyper soft set to utilize in this field for first time. Also, other extensions of Neutrosophic as Type-2 neutrosophic sets (T2NSs) or bipolar Neutrosophic to amalgamate with MCDM techniques for selecting right ClSP to fulfill customers' need.

References

[1] Tomar, A., Kumar, R. R., & Gupta, I. (2023). Decision making for cloud service selection: a novel and hybrid MCDM approach. *Cluster computing*, *26*(6), 3869–3887. DOI:10.1007/s10586-022-03793-y

[2] Cloud, H. (2011). The nist definition of cloud computing. *National institute of science and technology, special publication*, *800*(2011), 145.

[3] Kumar, A., Sah, B., Singh, A. R., Deng, Y., He, X., Kumar, P., & Bansal, R. C. (2017). A review of multi criteria decision making (MCDM) towards sustainable renewable energy development. *Renewable and sustainable energy reviews*, *69*, 596–609.

[4] Raut, R. D., Priyadarshinee, P., Gardas, B. B., & Jha, M. K. (2018). Analyzing the factors influencing cloud computing adoption using three stage hybrid SEM-ANN-ISM (SEANIS) approach. *Technological forecasting and social change*, *134*, 98–123.

[5] Büyüközkan, G., Uztürk, D., & Maden, A. (2023). Influential factor analysis for cloud computing technology service provider. *Technological forecasting and social change*, *192*(December 2022). DOI:10.1016/j.techfore.2023.122531

[6] Kumar Tiwari, R., & Kumar, R. (2022). A framework for prioritizing cloud services in neutrosophic environment. *Journal of king saud university - computer and information sciences*, *34*(6), 3151–3166. DOI:10.1016/j.jksuci.2020.05.009

[7] Ardagna, D., Casale, G., Ciavotta, M., Pérez, J. F., & Wang, W. (2014). Quality-of-service in cloud computing: modeling techniques and their applications. *Journal of internet services and applications*, *5*(1), 1–17.

[8] Saha, M., Panda, S. K., & Panigrahi, S. (2022). A Survey on Applications of Multi-Attribute Decision Making Algorithms in Cloud Computing. *ECS transactions*, *107*(1), 12887–12900. DOI:10.1149/10701.12887ecst

[9] Karabasevic, D., Popovic, G., Stanujkic, D., Maksimovic, M., & Sava, C. (2019). An approach for hotel type selection based on the single-valued intuitionistic fuzzy numbers. *Int. rev*, *1*, 7.

[10] Abdel-Monem, A., Mohamed, S. S., & Aziz, A. S. (2023). A Multi-Criteria Decision Making Methodology for Assessment Performance of Electrocoagulation System. *Multicriteria algorithms with applications*, *1*(1), 19–30.

[11] Kumar, R. R., Tomar, A., Shameem, M., & Alam, M. N. (2022). OPTCLOUD: An Optimal Cloud Service Selection Framework Using QoS Correlation Lens. *Computational intelligence and neuroscience*, *2022*. DOI:10.1155/2022/2019485

[12] Petrovic, I., & Kankaras, M. (2020). A hybridized IT2FS-DEMATEL-AHP-TOPSIS multicriteria decision making approach: Case study of selection and evaluation of criteria for determination of air traffic control radar position. *Decision making: applications in management and engineering*, *3*(1), 146–164.

[13] Smarandache, F., & Abdel-Basset, M. (2021). *Neutrosophic Operational Research: Methods and Applications*. , Neutrosophic Operational Research: Methods and Applications.

[14] Atanassov, K. T., & Atanassov, K. T. (1999). Interval valued intuitionistic fuzzy sets. *Intuitionistic fuzzy*

sets: theory and applications, 139–177.

[15] Smarandache, F. (1999). A unifying field in Logics: Neutrosophic Logic. In *Philosophy* (pp. 1–141). American Research Press.

[16] Smarandache, F. (2023). New Types of Soft Sets: HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set. *International journal of neutrosophic science*, *20*(4), 58–64. DOI:10.54216/IJNS.200404

[17] Garg, S. K., Versteeg, S., & Buyya, R. (2013). A framework for ranking of cloud computing services. *Future generation computer systems*, *29*(4), 1012–1023.

[18] Mousavi-Nasab, S. H., & Sotoudeh-Anvari, A. (2017). A comprehensive MCDM-based approach using TOPSIS, COPRAS and DEA as an auxiliary tool for material selection problems. *Materials & design*, *121*, 237–253.

[19] Jahan, A., Ismail, M. Y., Sapuan, S. M., & Mustapha, F. (2010). Material screening and choosing methods–a review. *Materials & design*, *31*(2), 696–705.

[20] Shameem, M., Kumar, R. R., Nadeem, M., & Khan, A. A. (2020). Taxonomical classification of barriers for scaling agile methods in global software development environment using fuzzy analytic hierarchy process. *Applied soft computing*, *90*, 106122.

[21] Mohamed, M. (2023). Toward Smart Logistics: Hybrization of Intelligence Techniques of Machine Learning and Multi-Criteria Decision-Making in Logistics 5.0. *Multicriteria algorithms with applications*, *1*(1), 42–57.

[22] Liu, F., Aiwu, G., Lukovac, V., & Vukic, M. (2018). A multicriteria model for the selection of the transport service provider: A single valued neutrosophic DEMATEL multicriteria model. *Decision making: applications in management and engineering*, *1*(2), 121–130.

[23] Karaşan, A., Boltürk, E., & Kahraman, C. (2019). A novel neutrosophic CODAS method: Selection among wind energy plant locations. *Journal of intelligent & fuzzy systems*, *36*(2), 1491–1504.

[24] Ismail, M. M., Ahmed, Z., Abdel-Gawad, A. F., & Mohamed, M. (2024). Toward Supply Chain 5.0: An Integrated Multi-Criteria Decision-Making Models for Sustainable and Resilience Enterprise. *Decision making: applications in management and engineering*, *7*(1), 160–186.

[25] Hussein, G. S., Zaied, A. N. H., & Mohamed, M. (2023). ADM: Appraiser Decision Model for Empowering Industry 5.0-Driven Manufacturers toward Sustainability and Optimization: A Case Study. *Neutrosophic systems with applications*, *11*, 22–30.

[26] Al-baker, S. F., & Mohamed, M. (2024). Exploring the Influences of Metaverse on Education Based on the Neutrosophic Appraiser Model, *23*(01), 134–145.

[27] Büyüközkan, G., Uztürk, D., & Maden, A. (2023). Influential factor analysis for cloud computing technology service provider. *Technological forecasting and social change*, *192*, 122531.

[28] Mohamed, M. (2023). Financial Risks Appraisal based on Dynamic Appraisal Framework. *Neutrosophic and information fusion*, *2*(1), 50–58. DOI:10.54216/nif.020104

Neutrosophic Treatment of the Modified Simplex Algorithm to find the Optimal Solution for Linear Models

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract

Science is the basis for managing the affairs of life and human activities, and living without knowledge is a form of wandering and a kind of loss. Using scientific methods helps us understand the foundations of choice, decision-making, and adopting the right solutions when solutions abound and options are numerous. Operational research is considered the best that scientific development has provided because its methods depend on the application of scientific methods in solving complex issues and the optimal use of available resources in various fields, private and governmental work in peace and war, in politics and economics, in planning and implementation, and in various aspects of life. Its basic essence is to use the data provided for the issue under study to build a mathematical model that is the optimal solution. It is the basis on which decision makers rely in managing institutions and companies, and when operations research methods meet with the neutrosophic teacher, we get ideal solutions that take into account all the circumstances and fluctuations that may occur in the work environment over time. One of the most important operations research methods is the linear programming method. Which prompted us to reformulate the linear models, the graphical method, and the simplex method, which are used to obtain the optimal solution for linear models using the concepts of neutrosophic science. In this research, and as a continuation of what we presented previously, we will reformulate the modified simplex algorithm that was presented to address the difficulty that we were facing when applying the direct simplex algorithm. It is the large number of calculations required to be performed in each step of the solution, which requires a lot of time and effort.

keywords: Linear programming; neutrosophic logic; neutrosophic simplex algorithm; modified neutrosophic simplex algorithm.

1. Introduction:

Operational research is the application of scientific methods to complex issues. It is the art of finding the best solutions and arriving at them through scientific methods so that nothing is left to chance or luck. With the computer, without which it would not have been possible to achieve solutions to many of these issues raised, the use of operations research methods helps us understand the foundations of selection and decision-making and reach agreement on the best ones in order to adopt the correct solutions when solutions abound and options are numerous [1,2,3,4,5], and even the studies presented complete the scientific and practical issues. It is preferable to use the concepts of neutrosophic science, the science that has proven its ability to provide appropriate solutions for all conditions that may occur in the case under study and helps decision makers to adopt ideal decisions that achieve the greatest profit and the lowest cost. Therefore, we have reformulated many research methods. Processes using neutrosophic concepts, see [6-18]. We continue to build neutrosophic processes research by reformulating the modified simplex algorithm and the synthetic algorithm used to find the optimal solution for linear models using neutrosophic concepts.

2. Discussion:

The neutrosophic linear model is given by the following general formula: 19]
The neutrosophic general form is given the linear mathematical model in the abbreviated form as follows:

$$\text{Max } or \text{ Min } f(x) = \sum_{j=1}^{n} (c_j \pm \varepsilon_j) x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i + \delta_i \;\; ; i = 1,2, \dots, m$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i + \delta_i \;\; ; i = 1,2, \dots, m$$

$$x_j \geq 0 \;\; ; j = 1,2, \dots, n$$

Where $Nc_j = c_j \pm \varepsilon_j$, $Nb_i = b_i \pm \delta_i$, a_{ij}, $j = 1,2, \dots, n$, $i = 1,2, \dots, m$ are constants having set or interval values according to the nature of the given problem, x_j are decision variables.

It is worthy to mention that the coefficients subscribed by the index N are of neutrosophic values.

The objective function coefficients $Nc_1, Nc_2, \dots, Nc_n$ have neutrosophic meaning are intervals of possible values:

That is, $Nc_j \in [\lambda_{j1}, \lambda_{j2}]$, where $\lambda_{j1}, \lambda_{j2}$ are the upper and the lower bounds of the objective variables x_j respectively, $j = 1,2, \dots, n$. Also, we have the values of the right-hand side of the inequality constraints $Nb_1, Nb_2, \dots, Nb_m$ are regarded as neutrosophic interval values:

$Nb_i \in [\mu_{i1}, \mu_{i2}]$, here, $\mu_{i1}, \mu_{i2}$are the upper and the lower bounds of the constraint

$i = 1,2, \dots, m$.

The optimal solution is obtained using the direct simplex method, which is summarized in three basic stages, as we presented: [20]

The first stage: converting the imposed model into the formal form

The second stage: converting the regular form into a basic form to obtain the non-negative basic solutions

The third stage: searching for the ideal solution required among the non-negative basic solutions.
In this research, we will reformulate the modified Simplex algorithm using the concepts of neutrosophic science. The algorithm was presented to address the difficulty that we faced when applying the direct Simplex algorithm, which is the large number of calculations required to be performed in each step of the solution, which requires a lot of time and effort. We explain the steps of the modified simplex algorithm through the following neutrosophic linear mathematical model:

$$Max\, Z = Nc_1 x_1 + Nc_2 x_2 + \cdots + Nc_n x_n$$

$$subject\ to \quad \begin{array}{l} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq Nb_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq Nb_2 \\ a_{31} x_1 + a_{32} x_2 + \cdots + a_{3n} x_n \leq Nb_3 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq Nb_m \end{array}$$

With the non-negativity conditions $x_1, x_2, \dots, x_n \geq 0$.

To find the optimal solution for this linear neutrosophic model using the modified simplex algorithm.

1- We write the neutrosophic linear model in standard form, we get the following model:

$$\max Z = c_{1N} x_1 + c_{2N} x_2 + \cdots + c_{nN} x_n + 0y_1 + 0y_2 + \cdots + 0y_m$$

$$subject\ to \quad \begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 = b_{1N} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + y_2 = b_{2N} \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n + y_3 = b_{3N} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + y_m = b_{mN} \end{matrix}$$

With the non-negativity conditions $x_1, x_2, \dots, x_n, y_1, y_2, y_3, \dots, y_m \geq 0$. The below tableau consists of the coefficient corresponding to the linear constraint variables and the coefficients of the objective function.
We convert the regular linear model to the basal form and place the coefficients in a short table whose first column includes the basic variables and whose top row includes the non- basic variables only.

- We define the pivot column, which is the column corresponding to the largest positive value in the objective function row because the objective function is a maximization function (but if the objective function is a minimization function, it is the column corresponding to the most negative values). Let this column be the column of the variable x_s.
- We define the pivot row; The pivot row is determined by the following indicator:

$$\theta = min\left[\frac{b_{iN}}{a_{is}}\right] = \frac{b_{tN}}{a_{ts}} > 0; \quad a_{is} > 0, b_{iN} > 0$$

Let this row be the line of the base variable y_t

- Then the fulcrum element is the element resulting from the intersection of the fulcrum column and the fulcrum row, i.e., We explain the second step in the following table:

Table No. 1: Anchor element table -

Non-Basic Variables / Basic Variables	x_1	x_2	…..	x_s	….	x_n	Nb_i
y_1	a_{11}	a_{12}	……	a_{1s}	…..	a_{1n}	Nb_1
y_2	a_{21}	a_{22}	…….	a_{1s}	….	a_{2n}	Nb_2
……..	…….	……..	…….	……	…..	……	…….
y_t	a_{t1}	a_{t2}	……	a_{ts}	……	a_{tn}	Nb_t
……	……	……	….	…..	…..	……	…….
y_m	a_{m1}	a_{m2}	…….	a_{1s}	….	a_{mn}	Nb_m
NZ	Nc_1	Nc_2	……	Nc_s	…..	Nc_n	$NZ - Nc_o$

We calculate the new elements corresponding to the pivot row and the pivot column according to the following steps:

1- We put opposite the pivot element $a_{ts}$the reciprocal of $\frac{1}{a_{ts}}$
2- We calculate the elements of the row corresponding to the pivot row (except the pivot row element) by dividing the elements of the pivot row by the anchor element a_{ts}
3- We calculate all the elements of the column opposite the fulcrum (except the fulcrum element) by dividing the elements of the fulcrum column by the fulcrum element a_{ts} and then multiplying them by)-1(
4- We calculate the other elements from the following relationships:

$$a'_{ij} = a_{ij} - a_{tj}\frac{a_{is}}{a_{ts}} = \frac{a_{ij}a_{ts} - a_{tj}a_{is}}{a_{ts}} \qquad (1)$$

$$Nb'_i = Nb_i - Nb_t\frac{a_{is}}{a_{ts}} = \frac{Nb_i a_{ts} - Nb_t a_{is}}{a_{ts}} \qquad (2)$$

$$Nc'_j = Nc_j - Nc_s\frac{a_{tj}}{a_{ts}} = \frac{Nc_j a_{ts} - Nc_s a_{tj}}{a_{ts}} \qquad (3)$$

We get the following table:

Table No. 2: The first stage in searching for the optimal solution

Non-Basic Variables / **Basic Variables**	x_1	x_2	…..	y_t	….	x_n	Nb_i'
y_1	a_{11}'	a_{12}'	……	$\frac{-a_{1s}}{a_{ts}}$	…..	a_{1n}'	Nb_1'
y_2	a_{21}'	a_{22}'	…….	$\frac{-a_{2s}}{a_{ts}}$	….	a_{21}'	Nb_2'
……..	…….	……..	…….	……	…..	……	…….
x_s	$\frac{a_{t1}}{a_{ts}}$	$\frac{a_{t2}}{a_{ts}}$	……	$\frac{1}{a_{ts}}$	……	$\frac{a_{tn}}{a_{ts}}$	$\frac{Nb_t}{a_{ts}}$
……	……	……	….	…..	…..	……	…….
y_m	a_{m1}'	a_{m1}'	…….	$\frac{-a_{ms}}{a_{ts}}$	….	a_{mn}'	Nb_m'
NZ	Nc_1'	Nc_2'	……	$\frac{-Nc_s}{a_{ts}}$	…..	Nc_n'	$NZ - Nc_0'$

We apply the stopping criterion of the Simplex algorithm to the objective function row in Table (2) below:
Since the objective function is of the maximize type, the objective function row in the table must not contain any positive value, (but if the objective function is of the minimize type, the objective function row in the new table must not contain any negative value), in the event that the Criterion: We return to step No. (3) and repeat the same steps until the stopping criterion is met and we obtain the desired ideal solution.
We explain the above through the following example: (This example was presented in research [20])

Example:

A company produces two types of products A, B using four raw materials F_1, F_2, F_3, F_4. The quantities needed from each of these materials to produce one unit of each of the two producers A, B, the available quantities of the raw materials, and the profit returned from one unit of both products are shown in the following table:

Table No. 3: Issue data

Products / Raw Materials	Required quantity per unit		Available quantities of the raw materials
	A	B	
F_1	2	3	[14,20]
F_2	2	1	[10,16]
F_3	0	3	[12,18]
F_4	3	0	[15,21]
Profit Returned per unit	[5,8]	[3,6]	

Required:

Finding the optimal production plan that makes the company's profit from the producers A, B as large as possible.

Symbolize the quantities produced from the product A with the symbol x_1, and from the product B with the symbol x_2 , the problem will be reformulated from the neutrosophic perspective as follow:

$$\max Z \in [5,8]\, x_1 + [3,6]\, x_2$$

$$subject\ to \begin{cases} 2x_1 + 3x_2 \leq [14,20] \\ 2x_1 + x_2 \leq [10,16] \\ 3x_2 \leq [12,18] \\ 3x_1 \leq [15,21] \end{cases}$$

$$x_1 \geq 0, x_2 \geq 0$$

We apply the modified simplex algorithm:

1- The standard form of the previous linear model is:

$$\max Z \in [5,8]\, x_1 + [3,6]\, x_2 + 0y_1 + 0y_2 + 0y_3 + 0y_4$$

$$subject\ to \begin{cases} 2x_1 + 3x_2 + y_1 = [14,20] \\ 2x_1 + x_2 + y_2 = [10,16] \\ 3x_2 + y_3 = [12,18] \\ 3x_1 + y_4 = [15,21] \end{cases}$$

$$x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0$$

2- We organize the previous information in the following modified simplex table:

Table No.4: Simplex table according to the modified Neutrosophic simplex algorithm

Non-basic var. / **Basic var.**	x_1	x_1	b_i
y_1	2	3	[14,20]
y_2	2	1	[10,16]
y_3	0	3	[12,18]
y_4	3	0	[15,21]
Objective fun.	[5,8]	[3,6]	$Z - 0$

We know $[a, b] \leq [c, d]$ if $a \leq c$ and $b \leq d$,Therefore.

It is clear that $\max([5,8],[3,6]) = [5,8]$ versus to the column of x_1 , meaning that the variable x_1 should be placed instead of one of the basic variables.

Now to demonstrate which basic variables should be ejected, the following calculation has been done:

$$\theta \in min\left[\frac{[14,20]}{2}, \frac{[10,16]}{2}, \frac{[15,21]}{3}\right] = \frac{[15,21]}{3} = [5,7]$$

The value of θ indicates that the row versus to the variable y_4, and the element positioned in the cross row/column is 3 which is the pivot element, divide the elements of the row versus to y_4 yields.

3- We calculate the elements of the new table using relationships (1), (2), (3), we obtain the following table

Table No.5: Table of the first step in searching for the optimal solution

Non-basic var. / **Basic var.**	x_1	x_2	b_i'
y_1	$\frac{-2}{3}$	**3**	[4,6]
y_2	$\frac{-2}{3}$	1	[0,4]
y_3	0	3	[12,18]
x_1	$\frac{1}{3}$	0	[5,7]
Object. Fun.	$\left[\frac{-8}{3}, \frac{-5}{3}\right]$	[3,6]	$Z - [25,56]$

We apply a stopping criterion in the algorithm. We find:
It still there is a non-negative value in the row of the objective function (i.e., [3,6])
This means that we have not yet reached the optimal solution, so we repeat the previous steps as follows:
which is versus to the x_2 column, this leads to the fact that the variable x_2 should be entered into the basic variables. Now, the question is: which basic variable should be ejected?
Track the following calculation to answer this question,
And here too because $[a,b] \leq [c,d]$ if $a \leq c$ and $b \leq d$,Therefore.

$$\theta \in \min\left[\frac{[4,6]}{3}, \frac{[0,4]}{1}, \frac{[12,18]}{3}\right] = \frac{[4,6]}{3} = [\frac{4}{3}, 2]$$

which is versus to the slack variable y_1, the pivot element equal 3, hence the row versus to y_1 should be divided by 3 , the required calculations yield the following tableau:

Table No. 6: Final solution table

Non-basic var. / basic var.	x_1	x_2	b_i'
x_2	$\frac{-2}{9}$	$\mathbf{\frac{1}{3}}$	$[\frac{4}{3}, 2]$
y_2	$\frac{-4}{9}$	$\frac{-1}{3}$	$[\frac{-4}{3}, 2]$
y_3	$\frac{2}{3}$	$\mathbf{-1}$	[8,12]
x_1	1	0	[5,7]
Object. Fun.	$[-6,-1]$	$[-2,-1]$	$Z - [29,68]$

We apply the algorithm stopping criterion. We find that the criterion has been met and thus we have reached the optimal solution.
The optimal solution for the linear model is:

$$x_1^* \in [5,7], x_2^* \in \left[\frac{4}{3}, 2\right], y_2^* \in \left[\frac{-4}{3}, 2\right], y_3^* \in [8,12], y_1^* = y_4^* = 0$$

The value of the objective function corresponds to:

$$\max Z \in [5,8].[5,7] + [3,6].\left[\frac{4}{3}, 2\right] = [25,56] + [4,12] = [29,68]$$

It is clear from the row of the objective function that all the elements are neutrosophic negative numbers, this means that we have reached to the optimal solution is:

$$x_1^* \in [5,7], x_2^* \in \left[\frac{4}{3}, 2\right], y_2^* \in \left[\frac{-4}{3}, 2\right], y_3^* \in [8,12], y_1^* = y_4^* = 0$$

Substitute the above optimal solution into the objective maximum function, yields:
This means that the company must produce quantity $x_1^* \in [5,7]$,of product A and quantity $x_2^* \in \left[\frac{4}{3}, 2\right]$of product B, thereby achieving a maximum profit of:

$$\max Z \in [5,8].[5,7] + [3,6].\left[\frac{4}{3}, 2\right] = [25,56] + [4,12] = [29,68]$$

It should be noted that the previous problem was solved in the research [20] using the direct simplex algorithm, and the solution tables are as follows:

Table No. 7: Simplex table according to the direct neutrosophic simplex algorithm

Non-basic var. / Basic var.	x_1	x_1	y_1	y_2	y_3	y_4	b_i
y_1	2	3	1	0	0	0	[14,20]
y_2	2	1	0	1	0	0	[10,16]
y_3	0	3	0	0	1	0	[12,18]
y_4	3	0	0	0	0	1	[15,21]
Objective fun.	[5,8]	[3,6]	0	0	0	0	$Z-0$

Table No. 8: Table of the first step in searching for the optimal solution

Non-basic var. / Basic var.	x_1	x_2	y_1	y_2	y_3	y_4	b_i
y_1	0	**3**	1	0	0	$\frac{-2}{3}$	[4,6]
y_2	0	1	0	1	0	$\frac{-2}{3}$	[0,4]
y_3	0	3	0	0	1	0	[12,18]
x_1	1	0	0	0	0	$\frac{1}{3}$	[5,7]
Object. Fun.	0	[3,6]	0	0	0	$[\frac{-8}{3}, \frac{-5}{3}]$	$Z-[25,56]$

Table No. 9: Final solution table

Non-basic var. / basic var.	x_1	x_2	y_1	y_2	y_3	y_4	b_i
x_2	0	1	$\frac{1}{3}$	0	0	$\frac{-2}{9}$	$[\frac{4}{3}, 2]$
y_2	0	0	$\frac{1}{3}$	1	0	$\frac{-4}{9}$	$[\frac{-4}{3}, 2]$
y_3	0	0	-1	0	1	$\frac{2}{3}$	[8,12]
x_1	1	0	0	0	0	$\frac{1}{3}$	[5,7]
Object. Fun.	0	0	[-2, -1]	0	0	$[\frac{-6}{9}, -1]$	$Z-[29,68]$

It is clear from the row of the objective function that all the elements are either zero or neutrosophic negative numbers, this means that we have reached to the optimal solution is:

$$x_1^* \in [5,7], x_2^* \in \left[\frac{4}{3}, 2\right], y_2^* \in \left[\frac{-4}{3}, 2\right], y_3^* \in [8,12], y_1^* = y_4^* = 0$$

Substitute the above optimal solution into the objective maximum function:

$$\max Z \in [5,8].[5,7] + [3,6].\left[\frac{4}{3}, 2\right] = [25,56] + [4,12] = [29,68]$$

3. Conclusion and results:

From the previous study, we note that we obtained the same optimal solution that was obtained when we used the direct simplex method, but with a much smaller number of calculations than the number we did in the direct simplex algorithm, as is clear by comparing the solution tables using the modified simplex method, Tables No.

(4)., No. (5), No. (6), with solution tables using the direct simplex method, Tables No. (7), No. (8), No. (9). Therefore, in order to shorten time and effort, we focus on the necessity of using the modified Simplex method to find the optimal solution for linear models. Especially when the model contains a large number of variables and constraints.

References

[1] Alali. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004. (Arabic version).
[2] Bugaha J.S, Mualla.W, Nayfeh.M, Murad. H, and other, Operations Research Translator into Arabic, The Arab Center for Arabization, Translation, Authoring and Publishing, Damascus,1998. (Arabic version).
[3] Al Hamid .M, Mathematical programming, Aleppo University, Syria, 2010. (Arabic version).
[4] Maissam Jdid, Operations Research, Faculty of Informatics Engineering, Al-Sham Private University Publications, 2021. (Arabic version).
[5] David.G. Luenbrgrr.YinyuYe, Linear and Nonlinear Programming, Springer Science + Business Media-2015.
[6] Florentin Smarandache, Maissam Jdid ,On Overview of Neutrosophic and Plithogenic Theories and Applications, Prospects for Applied Mathematics and Data Analysis (PAMDA), Vol .2, No.1, 2023.
[7] Smarandache, F., Introduction to Neutrosophic statistics, Sitech & Education Publishing, 2014.
[8] Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. American Research Press, Rehoboth, NM, 1999.
[9] Maissam Jdid, The Use of Neutrosophic linear Programming Method in the Field of Education, Handbook of Research on the Applications of Neutrosophic Sets Theory and Their Extensions in Education, Chapter 15, IGI-Global,2023
[10] Maissam Jdid, Studying Transport Models with the Shortest Time According to the Neutrosophic Logic, Journal Neutrosophic Sets and Systems, NSS Vol. 58, 2023.
[11] Broumi S. and Smarandache, F., Correlation coefficient of interval neutrosophic set, Appl. Mech. Mater., 436:511–517, 2013.
[12] Madeleine Al- Tahan, Some Results on Single Valued Neutrosophic (Weak) Polygroups, International Journal of Neutrosophic Science, Volume 2, Issue 1, PP: 38-46, 2020.
[13] Aslam, M., Khan, N. and AL-Marshadi, A. H. (2019). Design of Variable Sampling Plan for Pareto Distribution Using Neutrosophic Statistical Interval Method, Symmetry, 11 (1), 80.
[14] Victor Christianto, Robert N. Boyd, Florentin Smarandache, Three possible applications of Neutrosophic Logic in Fundamental and Applied Sciences, International Journal of Neutrosophic Science, Volume 1, Issue 2, PP: 90-95, 2020.
[15] Broumi, S.; Smarandache, F.; Talea, M.; Bakali, A. Operations on Interval Valued Neutrosophic Graphs; Infinite Study; Modern Science Publisher: New York, NY, USA, 2016
[16] Maissam Jdid, Said Broumi, Neutrosophic Rejection and Acceptance Method for the Generation of Random Variables, Neutrosophic Sets and Systems, NSS, Vol.56,2023
[17] and Quantum Optimization. Computers in Industry (Elsevier), 111, 121–139, 2019.
[18] Maissam Jdid, Florentin Smarandache, The Graphical Method for Finding the Optimal Solution for Neutrosophic linear Models and Taking Advantage of Non-Negativity Constraints to Find the Optimal Solution for Some Neutrosophic linear Models in Which the Number of Unknowns is More than Three, Journal Neutrosophic Sets and Systems, NSS Vol.58,2023
[19] Maissam Jdid, Huda E Khalid, Mysterious Neutrosophic Linear Models, International Journal of Neutrosophic Science, Vol.18, No. 2, 2022
[20] Maissam Jdid, AA Salama, Huda E Khalid, Neutrosophic Handling of the Simplex Direct Algorithm to Define the Optimal Solution in Linear Programming ,International Journal of Neutrosophic Science, Vol.18,No. 1, 2022

Neutrosophic Vision of the Expected Opportunity Loss Criterion (NEOL) Decision Making Under Risk

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract:
One of the major challenges facing decision-makers at the present time is obtaining complete information about the issue under study, due to the unstable conditions of the work environment that are beyond the control of decision-makers, which requires them to reach an optimal decision in light of these circumstances and fluctuations and to benefit from the data that is collected. Collected by specialists to determine the appropriate probability distribution corresponding to random cases of nature, here we are faced with the issue of making a decision in the event of risk because the probability distribution is a distribution linked to the data controlled by the conditions of the work environment, which entails a great risk. Decision makers bear the responsibility of choosing the optimal decision that reduces This risk is achieved and the greatest possible profit and the least possible loss are achieved. The issue of decision-making becomes more complex as the number of events increases, and we are in dire need of an ideal study of the issue that takes into account all the circumstances of the work environment. The concept of missed opportunity is very useful in analyzing the decision making under risk, after making the decision and the occurrence of events, the decision makers may regret and wish they had chosen actions different from those they chose at the beginning. To reduce the regret of the decision makers and minimize the expected lost opportunity, researchers in the field of classical operations research presented the criterion of the expected lost opportunity through which the decision can be determined. The ideal with the least percentage of regret. In this research, we present a neutrosophic vision of the expected opportunity loss criterion by taking the data of the issue under study. neutrosophic values are ranges whose lowest limit expresses profit in the worst conditions, and only the highest represents profit in the best conditions.

Key words: Operations research; decision-making theory; decision making under risk; neutrosophic science; neutrosophic decision-making theory; neutrosophic missed opportunity criterion.

Introduction: Decision-making theory depends on the data provided by specialists in collecting data on the issue under study and the type of this data: whether it is confirmed data, uncertain data, or

random data repeated according to a certain probability distribution law, through which the methods that must be followed to obtain the optimal decision are determined. The need to know the results of any decision before making it when the data is uncertain or the data is random and the decision maker does not know anything about the state that nature will take or even about the chances of any of it occurring, so the decision maker uses the primary information to obtain additional information that reduces the risk. In classical logic, a group of methods were used that helped the decision maker to make the ideal decision, and since this decision depends on specific classical values that do not take into account the changes that may occur in the work environment, and in light of the changes, fluctuations, and challenges that the decision maker faces in all fields, they increase day by day. Day after day, there is a need for a new study that relies on data that has a margin of freedom and takes into account all circumstances, from the best to the worst. Therefore, many researchers and those interested in studying operations research methods have presented many papers, including [9-1], which is considered a new vision, a neutrosophical vision for this. The methods are based on the concepts and foundations laid down by the founder of this logic, see [10]. Through the indeterminacy of the elements of the profit matrix and used in the decision-making process, an ideal neutrosophic decision is obtained. In this research, we will take the elements of the profit (or loss) matrix and the probabilities are neutrosophic values in the form of ranges, the lowest of which corresponds to the worst states of nature and the highest of which corresponds to the best states of nature, to formulate the expected opportunity loss criterion used to choose the optimal decision under risk. We will apply this criterion to the example that was presented in the research [9], where it was done. Using other criteria to make decisions under risk. By comparing the results provided by each criterion, decision makers can use the appropriate criterion for the issue under study.

Discussion:

The risk lies in the view of decision making when the movement of nature is random and subject to a probability distribution that may not be completely known, but rather a distribution assumed by experts, or by the decision maker himself, which reflects its effects on the decision itself. To reduce the risk, the probabilities are calculated or estimated. Through practical facts or a statistical study taken from previous experiments and studies, classical operations research methods presented a study of decision-making theory in its three cases: the case of confirmed data, uncertain data, and random data, where appropriate criteria were set for each case so that decision makers can make decisions that limit losses. But these decisions are appropriate for work conditions similar to the work conditions in which the data was collected, and any change may cause a large and unexpected loss. Therefore, in previous research [8] we presented a neutrosophical vision of some standards for decision-making in the case of uncertain data, and in another research [9]. We presented some criteria for decision-making under risk, as a complement to what we presented from the neutrosophical study of decision-making under risk. In this research, we present a neutrosophical vision for the expected opportunity loss criterion.

1- The classic general formulation of the decision problem:

The decision maker has alternatives $A(a_1, a_2, \dots, a_m)$where m is the number of alternatives available to the decision maker, and the states that nature can take in the future $\theta(\theta_1, \theta_2, \dots, \theta_n)$ where n is the number of states that nature can take at Its movement, and the amount of profit or loss that the decision maker will achieve is $X(a_i, \theta_j)$, or by short code X_{ij} .Then the profit matrix is given by the following table:

States of nature / Alternatives	θ_1	θ_2	...	θ_n
a_1	X_{11}	X_{12}	...	X_{1n}
a_2	X_{21}	X_{22}	...	X_{2n}
...	...	...	...	...
a_m	X_{m1}	X_{m2}	...	X_{mn}
$P(\theta_j)$	$P(\theta_1)$	$P(\theta_2)$	...	$P(\theta_n)$

Table No. (1) Classic general data for the decision-making issue (profit matrix)

2- The general neutrosophic formulation of the decision-making problem:

The decision maker has the alternatives $A(a_1, a_2, \dots, a_m)$where m is the number of alternatives available to the decision maker, and the states that nature can take in the future are $\theta(\theta_1, \theta_2, \dots, \theta_n)$where n is the number of states that nature can take when they move (they are independent of each other), We symbolize the amount of profit or loss that the decision maker will achieve $NX = X(a_i, \theta_j) \pm \varepsilon_{ij}$, or by short code NX_{ij} These are neutrosophic values, and ε_{ij} ,it is indeterminacy, it can be$\varepsilon_{ij} \in [\lambda_1, \lambda_2]$or $\varepsilon_{ij} \in \{\lambda_1, \lambda_2\}$. $^-0$

Also, the law of probability distribution to which the possible states of nature are subject, we take it as a neutrosophic number series or a neutrosophic mathematical function that corresponds to each state of nature with the probability of its occurrence:

$NP(\theta_j) = P(\theta_j) + \mu_j$. Where $^-0 \leq \sum_{j=1}^{n} NP(\theta_j) \leq 3^+$ and $0 \leq P(\theta_j) \leq 1$.

μ_j it is indeterminacy that can by $\mu_j \in [\delta_1, \delta_2]$ or $\mu_j \in \{\delta_1, \delta_2\}$. Based on the previous data, the goal is to choose the optimal alternative according to the available states of nature in order to obtain the greatest possible profit or the least possible loss. We organize the previous information in the following table:

States of nature / Alternatives	θ_1	θ_2	...	θ_n
a_1	$X_{11} \pm \varepsilon_{11}$	$X_{12} \pm \varepsilon_{12}$	...	$X_{1n} \pm \varepsilon_{1n}$
a_2	$X_{21} \pm \varepsilon_{21}$	$X_{22} \pm \varepsilon_{22}$	...	$X_{2n} \pm \varepsilon_{2n}$
...	...	...	...	...
a_m	$X_{m1} \pm \varepsilon_{m1}$	$X_{m2} \pm \varepsilon_{m2}$	...	$X_{mn} \pm \varepsilon_{mn}$
$P(\theta_j)$	$P(\theta_1)$	$P(\theta_2)$	...	$P(\theta_n)$

Table No. (2) Neutrosophic general data for the decision-making issue (profit matrix)

3- **In a previous study [9], we presented a neutrosophical vision of three criteria used to determine the optimal decision under risk:**

a. Neutrosophic aspiration level criterion.
b. Neutrosophic most likely criterion.
c. Neutrosophic largest expected values criterion.

We chose the appropriate alternative for the following question:

Example 1:

We have the following table of alternatives and states of nature:

States of nature / Alternatives	θ_1	θ_2	θ_3
a_1	[300,350]	[100,150]	[400,450]
a_2	$[-220,-170]$	[170,220]	[500,550]
a_3	$[-400,-350]$	[200,250]	[300,350]
a_4	[160,210]	[300,350]	[200,250]
$P(\theta_j)$	[0.3,0.45]	[0.1,0.25]	[0.6,0.75]

Table No. (3) Neutrosophic profit matrix table

It is required to determine the appropriate alternative using:

Neutrosophic aspiration level criterion:

According to the following data, the level of ambition of the decision maker:

The profit belongs to the range $M \in [300,350]$.

The loss belongs to the range $N \in [200,250]$.

The appropriate alternative that achieves the level of ambition of the decision maker in profit and loss is alternative a_1.

Neutrosophic most likely criterion:

From the table we notice that the most likely case is case θ_3, then the issue will lead to a decision in case of confirmation according to the following table:

Most likely case / Alternative	θ_3
a_1	[400,450]
a_2	[500,550]
a_3	[300,350]
a_4	[200,250]
$P(\theta_3)$	[0.6,0.75]

Table No. (4): Table of the most likely neutrosophic states

We choose the largest value in the condition column θ_3, which is $[500,550]$ corresponding to the alternative a_2, and a_2 is the appropriate alternative.

Largest expected values criterion:

States of nature / Alternatives	θ_1	θ_2	θ_3	$E(a_i)$
a_1	[300,350]	[100,150]	[400,450]	[340,532.5]
a_2	$[-220,-170]$	[170,220]	[500,550]	[251,391]
a_3	$[-400,-350]$	[200,250]	[300,350]	[80,167.5]
a_4	[160,210]	[300,350]	[200,250]	[198,369.5]
$P(\theta_j)$	[0.3,0.45]	[0.1,0.25]	[0.6,0.75]	

Table No. (5): Table of expected neutrosophic values

By comparing the elements of column $E(a_i)$, we notice that the largest expected values are $[340,532.5]$ corresponding to alternative a_1. Alternative a_1 is the appropriate alternative according to this criterion.

4- **In this research, we present a neutrosophical vision of another of the criteria used to choose the appropriate alternative decision making under risk:**

Expected opportunity loss criterion (EOL):

Based on the information contained in references [11-13], we find that the minimum expected lost opportunity criterion depends on choosing the decision that guarantees us the least regret, i.e., the decision with the lost opportunity, and it is calculated according to the following steps:

From the profit matrix:

we choose the largest profit value corresponding to each state of nature, θ_j, and let M_j be:

$$M_j = \underbrace{Max}_{i} X_{ij}$$

We form the Regret Matrix from the following relation:

$$X'_{ij} = M_j - X_{ij}$$

We obtain the following regret matrix:

States of nature / Alternatives	θ_1	θ_2	...	θ_n
a_1	X'_{11}	X'_{12}	...	X'_{1n}
a_2	X'_{21}	X'_{22}	...	X'_{2n}
...	...	...	...	...
a_m	X'_{m1}	X'_{m2}	...	X'_{mn}
$P(\theta_j)$	$P(\theta_1)$	$P(\theta_2)$	...	$P(\theta_n)$

Table No. (6) Classic regret matrix

c. We calculate the expected value corresponding to each alternative:

$$E(a_i) = \sum_{j=1}^{n} P(\theta_j) \cdot X'_{ij} \quad ; i = 1,2, \dots, m$$

We symbolize the appropriate alternative, through which we will determine the optimal decision, with the symbol E_K, and it is calculated from the following relation:

$$E_K = \underbrace{Min}_{i} [E(a_i)] \quad ; i = 1,2, \dots, m$$

Neutrosophic Vision of the Expected Opportunity Loss Criterion (NEOL):

Using the data in the general neutrosophic formulation of the risk decision problem we find: we choose the largest profit value corresponding to each state of nature, θ_j, and let NM_j be:

$$NM_j = \underbrace{Max}_{i} NX_{ij}$$

We form the Regret Matrix from the following relationship:

$$NX'_{ij} = NM_j - NX_{ij}$$

We obtain the following neutrosophic regret matrix:

States of nature / Alternatives	θ_1	θ_2	...	θ_n
a_1	NX'_{11}	NX'_{12}	...	NX'_{1n}
a_2	NX'_{21}	NX'_{22}	...	NX'_{2n}
...	...	...	...	...
a_m	NX'_{m1}	NX'_{m2}	...	NX'_{mn}
$NP(\theta_j)$	$NP(\theta_1)$	$NP(\theta_2)$	...	$NP(\theta_n)$

Table No. (7) Neutrosophic Regret Matrix

We calculate the expected neutrosophic value corresponding to each alternative in the regret matrix:

$$NE(a_i) = \sum_{j=1}^{n} NP(\theta_j) \, . \, NX'_{ij} \quad ; i = 1,2, \dots, m$$

We symbolize the optimal expected neutrosophic minimum value, through which we will determine the appropriate alternative, with the symbol NE_K, and it is calculated from the following relation:

$$NE_K = \underbrace{Min}_{i} [NE(a_i)] \quad ; i = 1,2, \dots, m$$

Example 2:

We apply the neutrosophic expected value criterion to the following data in Example No. (1):

States of nature / Alternatives	θ_1	θ_2	θ_3
a_1	[300,350]	[100,150]	[400,450]
a_2	$[-220,-170]$	[170,220]	[500,550]
a_3	$[-400,-350]$	[200,250]	[300,350]
a_4	[160,210]	[300,350]	[200,250]
$P(\theta_j)$	[0.3,0.45]	[0.1,0.25]	[0.6,0.75]

Table No. (8) Neutrosophic profit matrix table for the problem

What is required is to determine the appropriate alternative using the expected lost opportunity criterion:

We form the neutrosophic regret matrix using the following relation:

$$NX'_{ij} = NM_j - NX_{ij}$$

We obtain the following matrix:

States of nature / Alternatives	θ_1	θ_2	θ_3	$E(a_i)$
a_1	0	200	100	[80,125]
a_2	520	130	0	[169,266.5]
a_3	700	100	200	[340,490]
a_4	140	0	300	[222,288]
$P(\theta_j)$	[0.3,0.45]	[0.1,0.25]	[0.6,0.75]	

Table No. (9) Regret matrix table and expected value of the neutrosophic minimum

In our example, we find that the alternative a_1 corresponding to the minimum expected value of neutrosophic [80,125]is the appropriate alternative that achieves the lowest value of regret and achieves a profit whose expected value is [340,532.5].

Conclusion and results:

Through the previous study, we presented a neutrosophic vision of one of the important criteria used for decision-making under risk, which can be used in many life issues to reduce the regret resulting from making a decision on a specific issue by using neutrosophic value data, and to help decision makers in choosing an example decision in the case of risk, we present the following comparison is between the results of some of the criteria used to make a decision that suits all circumstances in the event of risk. We leave them to choose the criterion that can be relied upon and is appropriate for the issue under study through the following table.

Alternative and profit / criterion	Alternative	profit
Neutrosophic aspiration level criterion	a_1	[400,450]
Neutrosophic most likely criterion	a_2	[500,550]
Neutrosophic largest expected values criterion.	a_1	[340,532.5]
Neutrosophic expected opportunity loss criterion	a_1	[340,532.5]

Table No. (10): Comparison table

We note that alternative a_1 is the appropriate alternative according to three criteria. The criterion of the most likely neutrosophic state determines alternative a_2, best alternative.

References:

1. Mahmoud Ismail, Ahmed M. Ali, Ahmed Abdelhafeez, Ahmed Abdel-Rahim EI-Douh, Mahmoud Ibrahim and Ayman H. Abdel-aziem, **Neutrosophic Multi-Criteria Decision Making for Sustainable Procurement in Food Business**, Neutrosophic Sets and Systems, Vol. 57, 2023, pp. 283-293. DOI**: 10.5281/zenodo.8271435**
2. D. Nagarajan, V.M. Gobinath and Said Broumi, **Multicriteria Decision Making on 3D printers for economic manufacturing using Neutrosophic environment**, Neutrosophic Sets and Systems, Vol. 57, 2023, pp. 33-56. DOI**: 10.5281/zenodo.8271321**
3. R.K. Mohanty, B.K. Tripathy**, A New Approach to Neutrosophic Soft Sets and their Application in Decision Making,** Neutrosophic Sets and Systems, Vol. 60, 2023, pp. 159-174. DOI: **10.5281/zenodo.1022416**
4. Faisal Al-Sharqi, Ashraf Al-Quran, and Zahari Md. Rodzi, Multi-Attribute Group Decision-Making Based on Aggregation Operator and Score Function of Bipolar Neutrosophic Hypersoft Environment**,** Neutrosophic Sets and Systems, Vol. 61, 2023, pp. 465-492. DOI: **10.5281/zenodo.10428642**
5. Mountajab Alhasan , Malath F Alaswad, A Study of the Movement of a Neutrosophic Material Point in the Neutrosophic Plane by Using a Neutrosophic AH-Isometry, https://doi.org/10.54216/PAMDA.020101

6. Fatma Taher, Neutrosophic Multi-Criteria Decision-Making Methodology to Identify Key Barriers in Education, https://doi.org/10.54216/IJNS.220410
7. Antonios Paraskevas, Extended Event Calculus using Neutrosophic Logic: Method, Implementation, Analysis, Recent Progress and Future Directions, https://doi.org/10.61356/j.nswa.2024.116
8. Maissam Jdid, Basel Shahin , Fatima Al Suleiman ,Important Neutrosophic Rules for Decision-Making in the Case of Uncertain Data , https://doi.org/10.54216/IJNS.1803014
9. Maissam Jdid, Hla Hasan, The State of Risk and Optimum Decision According to Neutrosophic Rules ,https://doi.org/10.54216/IJNS.200107
10. Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications , https://doi.org/10.54216/PAMDA.020102
11. Bugaha J.S, Mualla.W, Nayfeh.M, Murad. H, Al-Awar. M. N-Operations Research Translator into Arabic, The Arab Center for Arabization, Translation, Authoring and Publishing, Damascus,1998. (Arabic version).
12. Rawad Almuarrawi, Maisam Jaded, Decision Making under Risk and Looking for the Optimal Strategies Based on the Available Information, Al-Baath University Journal -Series of Basic Sciences, Vol,38, No,20, PP,31-56,2016

A Study of Systems of Neutrosophic Linear Equations

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract

Operations research methods are among the modern scientific methods that have occupied a prominent place among the mathematical methods used in planning and managing various economic and military activities. They have been able to help specialists in developing ideal plans in terms of costs, production, storage, or investment of human energies. One of its most important methods is the method Linear programming, which was built based on the sets of linear equations that represent the constraints for any linear model. Based on the methods for solving the systems of linear equations, researchers were able to prepare algorithms for solving linear models, such as the direct Simplex algorithm and its modifications. After the emergence of neutrosophic science, we found that research methods had to be reformulated. Operations using the concepts of this science, and as a basis and foundation for neutrosophic linear programming. In this research, we will reformulate the systems of linear equations and some methods for solving them using the concepts of neutrosophic to be a basis for any study presented in the field of neutrosophic linear programming.

Keywords: Operations research; linear programming; systems of linear equations; neutrosophic science; systems of neutrosophic linear equations; methods for solving neutrosophic linear equations.

1. Introduction:

Through our endeavor to provide everything new and keep pace with scientific development and given the great importance of the linear programming method as one of the methods of operations research, we found it necessary to reformulate the systems of linear equations and some methods for solving them using the concepts of neutrosophic, the science in which research and studies that use its concepts topped the research list in most publishing houses. The reason for this is the accuracy of the results obtained in studies that use the concepts of this science, some research [1-13].

Operations research methods are among the modern scientific methods that have occupied a prominent place among the mathematical methods used in planning and managing various economic and military activities. They have been able to help specialists in developing ideal plans in terms of costs, production, storage, or investment of human energies. One of its most important methods is the method Linear programming, which relies on mathematical models to express any practical system from real life or a proposed idea. Mathematical models consist of an objective function and constraints. Classical operations research methods provided many algorithms that help to find the optimal solution for these models. The construction of these algorithms was based on sentences Linear equations. Based on the methods used to solve these sentences, such as the Jardin-Diver method and the Simplex method, the direct Simplex algorithm and its modifications were developed to solve linear models, and in view of the importance of the linear programming method and to keep pace with everything new in the field of scientific research and after the emergence of neutrosophic science, the science that caused a great revolution. In all fields of science, we found that it is necessary to reformulate the methods of operations research using the concepts of this science, the science that takes into account all the changes that may occur to the issue under study through the indeterminacy of the issue's data. Therefore, we present in this research a study of sentences of linear equations and some methods for solving them using the concepts of neutrosophic science to be a basis for any study presented in the field of neutrosophic linear programming .we will take the matrix of constants on the right side of these equations as neutrosophic values. Of the form $Nb_j = b_j + \delta_j$ where δ_j is indeterminacy and can take one of the forms$\delta_j \in \{\mu_{i1}, \mu_{i2}\}$ or $\delta_j \in [\mu_{i1}, \mu_{i2}]$, , also a_{ij} on the right side we will take it Neutrosophic values

i.e., $Na_{ij} = a_{ij} + \gamma_{ij}$ where $\gamma_{ij}$is indeterminacy and can take In one of the forms $\gamma_{ij} \in [\varphi_{ij1}, \varphi_{ij2}]$, or $\gamma_{ij} \in \{\varphi_{ij1}, \varphi_{ij2}\}$, which helps us obtain more accurate solution results that take into account all conditions.

2. Discussion:

Based on the study mentioned in the classical references on the systems of linear equations and methods for solving them in some references [14-17].

We present the following study according to the concepts of neutrosophic science:

The systems of linear equations in which the number of equations equals m and the number of variables in them equals n are given according to classical logic in the following general form:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

It is written in the following matrix form:

$$A.X = B$$

Where:

$$\mathrm{A} = \begin{bmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} \dots \mathrm{a}_{1\mathrm{n}} \\ \mathrm{a}_{21} & \mathrm{a}_{22} \dots \mathrm{a}_{2\mathrm{n}} \\ \dots\dots\dots\dots\dots\dots\dots \\ \mathrm{a}_{\mathrm{m1}} & \mathrm{a}_{\mathrm{m2}} \dots \mathrm{a}_{\mathrm{mn}} \end{bmatrix} \quad \mathrm{B} = \begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ .. \\ \mathrm{b}_\mathrm{m} \end{bmatrix} \qquad \mathrm{X} = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ .. \\ \mathrm{x}_\mathrm{n} \end{bmatrix}$$

Where a_{ij} and b_i are real numbers for all values of $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$

We distinguished three cases of the systems of linear equations:

The first case: The number of equations is equal to the number of variables, i.e., $m = n$

The second case: The number of equations is greater than the number of variables, i.e., $m > n$

Third case: The number of equations is less than the number of variables, i.e., $m < n$

Below we will present the systems of linear equations using the concepts of neutrosophic science, where we will take the real numbers a_{ij} and b_i as neutrosophic numbers, that is, of the form Nb_i and Na_{ij}, indefinite values. Perfectly determined, they can be any neighborhood of the real numbers a_{ij} and b_i, written in one of the forms. next:

$Na_{ij} = a_{ij} + \varepsilon_{ij}$ and $Nb_i = b_i + \mu_i$ where $\varepsilon_{ij} \in [\lambda_{1ij}, \lambda_{2ij}]$ or $\varepsilon_{ij} \in \{\lambda_{1ij}, \lambda_{2ij}\}$ or otherwise, then the systems of neutrosophic linear equations is written in the following form:

The systems of neutrosophic linear equations in which the number of equations equals m and the number of variables in them equals $n$are given in the following general form:

$$Na_{11}x_1 + Na_{12}x_2 + \cdots + Na_{1\mathrm{n}}x_n = Nb_1$$
$$Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2\mathrm{n}}x_n = Nb_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$Na_{\mathrm{mn}}x_1 + Na_{\mathrm{mn}}x_2 + \cdots + Na_{mn}x_m = Nb_m$$

In the following matrix form**:**

$$NA.\mathrm{X} = NB$$

Where:

$$NA = \begin{bmatrix} Na_{11} & Na_{12} \dots Na_{1n} \\ Na_{21} & Na_{22} \dots Na_{2n} \\ \dots\dots\dots\dots\dots\dots\dots \\ Na_{m1} & Na_{m2} \dots Na_{mn} \end{bmatrix} \quad NB = \begin{bmatrix} Nb_1 \\ Nb_2 \\ .. \\ Nb_m \end{bmatrix} \qquad \mathrm{X} = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ .. \\ \mathrm{x}_\mathrm{n} \end{bmatrix}$$

To find the general solution to the previous systems of equations, we examine them according to the three aforementioned cases:

The first case: The number of equations is equal to the number of variables, i.e., $m = n$.

We write the systems of equations as follows:

$$Na_{11}x_1 + Na_{12}x_2 + \cdots + Na_{1\mathrm{n}}x_n = Nb_1$$
$$Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2\mathrm{n}}x_n = Nb_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$Na_{n1}x_1 + Na_{n2}x_2 + \cdots + Na_{nn}x_n = Nb_n$$

In matrix form:

$$NA\,.X = NB$$

Where:

$$NA = \begin{bmatrix} Na_{11} & Na_{12} \dots Na_{1n} \\ Na_{21} & Na_{22} \dots Na_{2n} \\ \dots\dots\dots\dots\dots\dots \\ Na_{n1} & Na_{n2} \dots Na_{nn} \end{bmatrix} \quad NB = \begin{bmatrix} Nb_1 \\ Nb_2 \\ .. \\ Nb_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix}$$

The proverbs matrix is a square matrix whose determinant is $\Delta_N = |NA|$

Here we distinguish two cases:

1- $\Delta_N = 0$. This case results in two cases:

a. If $\Delta_N = 0$ and $\Delta_{N_{x_j}} \neq 0$ where $\Delta_{N_{x_j}}$ is the determinant resulting from the determinant of the matrix of proverbs Δ_N after replacing the column containing the proverbs of the unknown x_j with the column of constants (the values on the side The second of the equations) then the systems have no solution.

b. If $\Delta_N = 0$ and $\Delta_{N_{x_j}} = 0$, this means that the systems of equations are not linearly independent, meaning that some of them are linearly related to each other. To address this case, we delete one of the two linearly related equations, thus the number of equations decreases and becomes $m'$where $m' = m - 1$ and $m' < n$, which is identical to the second case that will be dealt with later.

c. If $\Delta_N \neq 0$, that is, the systems of equations are linearly independent and the systems have a single solution, which can be found in several ways. In this research, we study Gaussian- Jordan method, which is the basis for the direct simplex algorithm that we use to obtain the optimal solution for linear models.

Gaussian- Jordan method for solving systems of neutrosophic linear equations in which $m = n$:

To clarify the mathematical basis of this method, we write the equations in the following matrix form:

$$\begin{bmatrix} Na_{11} & Na_{12} \dots Na_{1n} \\ Na_{21} & Na_{22} \dots Na_{2n} \\ \dots\dots\dots\dots\dots\dots \\ Na_{n1} & Na_{n2} \dots Na_{nn} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} = \begin{bmatrix} Nb_1 \\ Nb_2 \\ .. \\ Nb_n \end{bmatrix} \quad (1)$$

Or in the following abbreviated form:

$$NA\,.X = NB \quad (2)$$

Since $\Delta_N = |NA| \neq 0$, this means that the matrix NA has an inverse that is NA^{-1}. We multiply both sides of equation (2) by $NA^{-1}$and we find:

$$NA^{-1}.(NA\,.X) = NA^{-1}.NB$$

Hence, we get:

$$I.\mathrm{X} = NB'$$

Which is written in the following detailed form:

$$\begin{bmatrix} 1 & 0 & 0 \dots 0 \\ 0 & 1 & 0 \dots 0 \\ \dots\dots\dots\dots\dots \\ 0 & 0 & 0 \dots 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ .. \\ Nb'_n \end{bmatrix} \quad (3)$$

This process is the basis of the Gaussian- Jordan method for solving a system of linear equations. In order to convert Figure (1) to Figure (2), we follow the following steps:

1- We express Figure (1) in the following table:

Table No. (1): Table of equations

Variables / Equations	x_1	x_2	....	x_n	NB
1	Na_{11}	Na_{12}	....	Na_{1n}	Nb_1
2	Na_{21}	Na_{22}	...	Na_{2n}	Nb_2
....	...	...	...	...	...
n	Na_{n1}	Na_{n2}	...	Na_{nn}	Nb_n

2- We convert the matrix NA to the unit matrix I by processing the lines of the table so that we make all non-diagonal elements in all its rows equal to zero and the diagonal elements equal to one. When we want to remove the variable x_s from the equation t, we follow these steps:

a- We divide all the elements of row t in which we want to make x_s equal to one by Na_{ts}, so x_s becomes equal to one and the other expressions change.

b- We set all elements of the column with x_s (except row t) equal to zero.

c- We calculate the rest of the elements of the new table from the following two relationships:

$$Na'_{ij} = \left(Na_{ij} - Na_{is}\frac{Na_{tj}}{Na_{ts}}\right) = \frac{Na_{ij}Na_{ts} - Na_{is}Na_{tj}}{Na_{ts}}$$
$$Nb'_{i} = \left(Nb_{i} - Na_{is}\frac{Nb_{t}}{Na_{ts}}\right) = \frac{Nb_{i}Na_{ts} - Na_{is}Nb_{t}}{Na_{ts}} \quad (4)$$

The element Na_{ts} is called the pivot element
From the previous processing we get the following table:

Table No. (2) Final solution table

Variables / Equations	Nx_1	Nx_2	....	Nx_n	NB'
1	1	0	....	0	Nb'_1
2	0	1	...	0	Nb'_2
....	...	...	...	...	...
n	0	0	...	1	Nb'_n

Through the table, the systems of linear equations are written in the following matrix form:

$$I.NX = NB'$$
$$\begin{bmatrix} 1 & 0 & 0 \dots 0 \\ 0 & 1 & 0 \dots 0 \\ \dots & \dots & \dots \dots \dots \\ 0 & 0 & 0 \dots 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ .. \\ Nb'_n \end{bmatrix} \Rightarrow$$
$$\begin{bmatrix} Nx_1 \\ Nx_2 \\ .. \\ Nx_n \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ .. \\ Nb'_n \end{bmatrix}$$
$$\Rightarrow Nx_1 = Nb'_1, Nx_2 = Nb'_2, \dots, Nx_n = Nb'_n$$

The second case:
The number of equations is greater than the number of variables, i.e., $m > n$*:*
In this case, we form a new system from the set of equations in which the number of equations is equal to the number of variables by excluding a number of equations of m − n. Then we solve the new systems as we did in the first case and replace the resulting solution in the equations that were excluded to ensure that they are satisfied*.*
The third case: The number of variables is greater than the number of equations, i.e., $m < n$*:*
In this case, we are faced with a set of equations of the following form*:*

$$\left.\begin{array}{c} Na_{11}x_1 + Na_{12}x_2 + \cdots + Na_{1\mathrm{m}}x_m + \cdots + Na_{1\mathrm{n}}x_n = Nb_1 \\ Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2\mathrm{m}}x_m + \cdots + Na_{2\mathrm{n}}x_n = Nb_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ Na_{m1}x_1 + Na_{m2}x_2 + \cdots + Na_{\mathrm{mm}}x_m + \cdots + Na_{mn}x_n = Nb_m \end{array}\right\| \quad (5)$$

Which is written in the following matrix form:

$$NA = \begin{bmatrix} Na_{11} & Na_{12} \dots Na_{1m} \dots \dots Na_{1n} \\ Na_{21} & Na_{22} \dots Na_{2m} \dots \dots Na_{1n} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ Na_{m1} & Na_{m2} \dots Na_{mm} \dots \dots Na_{mn} \end{bmatrix} \quad NB = \begin{bmatrix} Nb_1 \\ Nb_2 \\ .. \\ Nb_n \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_m \\ .. \\ .. \\ x_n \end{bmatrix}$$

In the following brief form:

$$NA_{(m.n)}.X_{(n.1)} = NB_{(m.1)} \quad (6)$$

1- We partition the matrix $NA_{(m.n)}$ into two matrices:
   a- A square matrix of rank $(m.m)$ and we denote it as $NC_{(m.m)}$.
   b- And a rectangular matrix of rank $(m.n-m)$ and we denote it $ND_{(m.n-m)}$
2- We partition the column matrix $X_{(n.1)}$ into two matrices $X'_{(m.1)}$ and $X''_{(n-m.1)}$.

Then the systems of equations (5) are written in the following matrix form:

$$\left[NC_{(m.m)}, ND_{(m.n-m)}\right] \cdot \begin{bmatrix} X'_{(m.1)} \\ X''_{(n-m.1)} \end{bmatrix} = NB_{(m.1)} \quad (7)$$
$$NC_{(m.m)}.X'_{(m.1)} + ND_{(m.n-m)}.X''_{(n-m.1)} = NB_{(m.1)}$$

From them we find that:

$$NC_{(m.m)}.X'_{(m.1)} = NB_{(m.1)} - ND_{(m.n-m)}.X''_{(n-m.1)} \quad (8)$$

Assuming that $|NC| \neq 0$, we multiply both sides in relation (8) by NC^{-1} and we find:

$$NC^{-1}.NC.X' = NC^{-1}.(NB - ND.X^{"})$$
$$I.X' = NC^{-1}.NB - NC^{-1}.ND.X^{"}) \quad (9)$$

Assuming that $NC^{-1}.NB = NB'$ and $NC^{-1}.ND = ND'_{(m.n-m)}$we find that:

$$\begin{bmatrix} 1 & 0 & 0 \dots 0 \\ 0 & 1 & 0 \dots 0 \\ \dots & \dots & \dots \dots \dots \\ 0 & 0 & 0 \dots 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_m \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ .. \\ Nb'_m \end{bmatrix} - \begin{matrix} Nd'_{11} & Nd'_{12} \dots Nd'_{1(n-m)} \\ Nd'_{21} & Nd'_{22} \dots Nd'_{1(n-m)} \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ \lfloor Nd'_{m1} & Nd'_{m2} \dots Nd'_{m(n-m)} \rfloor \end{matrix} . \begin{bmatrix} x_{m+1} \\ x_{m+2} \\ \dots \\ x_n \end{bmatrix} \quad (10)$$

It is transformed into a set of linear equations as follows:

$$Nx_1 = Nb'_1 - (Nd'_{11}x_{m+1} + Nd'_{12}x_{m+2} + \dots + Nd'_{1(n-m)}x_n)$$
$$Nx_2 = Nb'_2 - (Nd'_{21}x_{m+1} + Nd'_{22}x_{m+2} + \dots + Nd'_{2(n-m)}x_n)$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$Nx_m = Nb'_m - (Nd'_{m1}x_{m+1} + Nd'_{m2}x_{m+2} + \dots + Nd'_{m(n-m)}x_n)$$

This means that we were able to calculate m in terms of $(n-m)$in terms of $(n-m)$, $x_{m+1}, x_{m+2}, \dots, x_n$,we note that the values of the variables $x_1, x_2, \dots, x_m$, it relates to the values taken by the variables $x_{m+1}, x_{m+2}, \dots, x_n$, or in other words, what we give to the variables $x_{m+1}, x_{m+2}, \dots, x_n$, and that for every sentence of Values such as $\beta_{m+1}, \beta_{m+2}, \dots, \beta_n$for these variables we get a set of values for the variables $x_1, x_2, \dots, x_m$ is:

$$Nx_1 = Nb'_1 - (Nd'_{11}\beta_{m+1} + Nd'_{12}\beta_{m+2} + \dots + Nd'_{1(n-m)}\beta_n)$$
$$Nx_2 = Nb'_2 - (Nd'_{21}\beta_{m+1} + Nd'_{22}\beta_{m+2} + \dots + Nd'_{2(n-m)}\beta_n)$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$Nx_m = Nb'_m - (Nd'_{m1}\beta_{m+1} + Nd'_{m2}\beta_{m+2} + \dots + Nd'_{m(n-m)}\beta_n)$$

Thus, we obtain a solution that includes all the variables of sentence (5)

The solution is arranged as follows:

$$(\beta_1, \beta_2, \dots, \beta_m, \beta_{m+1}, \beta_{m+2}, \dots, \beta_n)$$

But since the variables $x_{m+1}, x_{m+2}, \dots, x_n$can take an infinite number of qualitative values (even if they are restricted by certain conditions), we obtain an infinite number of corresponding values for the variables $x_1, x_2, \dots, x_m$.

Therefore, if $|NC| \neq 0$, then the set of equations (5) has an infinite number of acceptable solutions of the form:

$$(x_1, x_2, \dots, x_m, x_{m+1}, x_{m+2}, \dots, x_n)$$

Thus, we obtain a solution that includes all variables of the sentence, which is the ordered solution:

$$(\beta_1, \beta_2, \dots, \beta_m, \beta_{m+1}, \beta_{m+2}, \dots, \beta_n)$$

3. Basic solutions

Since sentence (5) has an infinite number of acceptable solutions, we will try to limit ourselves to a limited number of them by setting the variables $x_{m+1}, x_{m+2}, \dots, x_n$ equal to zero. Then sentence (9) takes the following form:

$$\begin{bmatrix} 1 & 0 & 0 \dots 0 \\ 0 & 1 & 0 \dots 0 \\ \dots & \dots & \dots \dots \dots \\ 0 & 0 & 0 \dots 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_m \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ .. \\ Nb'_m \end{bmatrix} \quad (11)$$

From it we get:

$$x_1 = Nb'_1 \ , x_2 = Nb'_2 \ , \dots, x_m = Nb'_m$$

So, the complete solution is:

$$(Nb'_1, Nb'_2, \dots, Nb'_m, 0,0, \dots, 0)$$

We call this solution the basic solution because it is attributed to the rule with single normal vectors in the space R^m as follows:

$$e_1 = \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \\ \lfloor .. \\ 0 \rfloor \end{matrix} \quad e_2 = \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \\ \lfloor .. \\ 0 \rfloor \end{matrix} \quad \dots \quad \dots \quad e_m = \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ \lfloor .. \\ m \rfloor \end{matrix}$$

The set of vectors $e_1, e_2, \dots, e_m$ form a rule because they are linearly independent, and the vector NB' can be expressed in terms of it using the factorials $x_1, x_2, \dots, x_m$ as follows:

$$NB' = e_1x_1 + e_2x_2 + \dots + e_mx_m$$

We call the variables$x_1, x_2, \dots, x_m$, basic variables and we call other variables $x_{m+1}, x_{m+2}, \dots, x_n$ free or non-basic variables because they take qualitative values.

The process of choosing the variables $x_1, x_2, \dots, x_m$ to be basic variables is a random process, as we can form other basic solutions, knowing that the possibilities available to obtain basic solutions are:

$$C_n^m = \frac{n!}{m!\,(n-m)!}$$

It is a finite number of infinite acceptable solutions.

Example 1:

Find the joint solution of the following two linear equations:

$$2x_1 + 7x_2 + 3x_3 + 2x_4 = [2,5]$$
$$3x_1 + 9x_2 + 4x_3 + x_4 = [3,7]$$
$$x_1 + 5x_2 + 3x_3 + 4x_4 = [4,8]$$

In the set of equations, the number of variables is $n = 4$ and the number of equations is $m = 3$. Therefore, the number of basic variables is equal to 3 and the number of non-basic free variables is $n - m = 1$. The number of possible solutions is calculated from the relationship:

$$C_n^m = \frac{n!}{m!\,(n-m)!}$$

i.e.,

$$C_4^3 = \frac{4!}{3!\,(4-3)!} = 4$$

Write as follows:

$$(x_1, x_2, x_3, 0)\ ,(x_1, x_2, 0, x_4)\ ,(x_1, 0, x_3, x_4), (0, x_2, x_3, x_4)$$

To obtain these solutions, we write the systems of equations in the following form:

$$2x_1 + 7x_2 + 3x_3 = [2,5] - 2x_4$$
$$3x_1 + 9x_2 + 4x_3 + x_4 = [3,7] - x_4$$
$$x_1 + 5x_2 + 3x_3 = [4,8] - 4x_4$$

The previous sentence is written in the following matrix form:

$$\begin{bmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} [2,5] \\ [3,7] \\ [4,8] \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} . [x_4] \quad (*)$$

$$C = \begin{bmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{bmatrix} \quad X' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad NB = \begin{bmatrix} [2,5] \\ [3,7] \\ [4,8] \end{bmatrix} \quad D = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \quad X^{"} = [x_4]$$

We calculate the determinant $|C|$.

We find:

$$|C| = \begin{vmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{vmatrix} = -3 \neq 0$$

To find the solutions, we find the reciprocal of the matrix, $C = \begin{bmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{bmatrix}$

We find:

$$C^{-1} = \begin{bmatrix} \frac{-7}{3} & 2 & \frac{-1}{3} \\ \frac{5}{3} & -1 & \frac{-1}{3} \\ -2 & 1 & 1 \end{bmatrix}$$

We compensate in the relationship:

$$NC^{-1}.NC.X' = NC^{-1}.(NB - ND.X^{"})$$

We get*:*

$$\begin{bmatrix} \frac{-7}{3} & 2 & \frac{-1}{3} \\ \frac{5}{3} & -1 & \frac{-1}{3} \\ -2 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{bmatrix} = \begin{bmatrix} \frac{-7}{3} & 2 & \frac{-1}{3} \\ \frac{5}{3} & -1 & \frac{-1}{3} \\ -2 & 1 & 1 \end{bmatrix} . \left(\begin{bmatrix} [2,5] \\ [3,7] \\ [4,8] \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} . [x_4] \right)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Nx_1 \\ Nx_2 \\ Nx_3 \end{bmatrix} = \begin{bmatrix} \frac{-7}{3} & 2 & \frac{-1}{3} \\ \frac{5}{3} & -1 & \frac{-1}{3} \\ -2 & 1 & 1 \end{bmatrix} . \begin{bmatrix} [2,5] \\ [3,7] \\ [4,8] \end{bmatrix} - \begin{bmatrix} \frac{-7}{3} & 2 & \frac{-1}{3} \\ \frac{5}{3} & -1 & \frac{-1}{3} \\ -2 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} . [x_4]$$

It is transformed into the following systems of equations:

$$Nx_1 = \begin{bmatrix} \left[0,\frac{-1}{3}\right] \\ -\left[1,\frac{4}{3}\right] \\ [3,5] \end{bmatrix} - \begin{bmatrix} \frac{-16}{3} \\ 1 \\ 1 \end{bmatrix}.[x_4]$$

Setting the free variable x_4 equal to zero we get:

$$Nx_1 = \begin{bmatrix} \left[0,\frac{-1}{3}\right] \\ -\left[1,\frac{4}{3}\right] \\ [3,5] \end{bmatrix}$$

i.e.,

$$x_1 = \left[0,\frac{-1}{3}\right], x_2 = -\left[1,\frac{4}{3}\right], x_3 = [3,5]$$

Thus, we obtain the first neutrosophic basic solution, which is:

$$(x_1, x_2, x_3, 0) = \left(\left[0,\frac{-1}{3}\right], -\left[1,\frac{4}{3}\right], [3,5], 0\right)$$

In the same way we obtain other basic solutions.

4. Dissolved basic solutions:

The base solution is a degenerate and invalid solution if in the final result we obtain a value of zero for the variables that we chose as the base.

Gaussian- Jordan method for solving a set of linear equations in which $m < n$:

Based on the previous mathematical principles, the basic steps of the Gaussian- Jordan method are as follows:

1- We write the systems of equations (5) in the following matrix form:

$$I.X' + NC^{-1}.D.X'' = NC^{-1}.NB = NB'$$

$$[I\ , NC^{-1}.ND].\begin{bmatrix} X' \\ X'' \end{bmatrix} = NB' \qquad (12)$$

Which is written in the following detailed form:

$$\begin{bmatrix} 1 & 0 & 0 \dots 0 & Nd'_{11} & Nd'_{12} \dots Nd'_{1(n-m)} \\ 0 & 1 & 0 \dots 0 & Nd'_{21} & Nd'_{22} \dots Nd'_{1(n-m)} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots 1 & Nd'_{m1} & Nd'_{m2} \dots Nd'_{m(n-m)} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ .. \\ Nb'_m \end{bmatrix} \qquad (13)$$

The transition from Figure (5) to Figure (12) is done by following the same steps that we mentioned in the previous paragraph, but this method does not give us a basic solution unless we set all the free variables equal to zero. If we do that, we only get the first solution, and to get all the solutions, we do the following steps:

a. We organize the following table:

Table No. (3) The first table for the Gaussian- Jordan method

Variables / Equations	x_1	x_2	....	x_m	x_{m+1}	x_{m+2}	....	x_n	NB
1	a_{11}	a_{12}	....	a_{1m}	a_{1m+1}	a_{1m+2}	....	a_{1n}	Nb'_1
2	a_{21}	a_{22}	...	a_{2m}	a_{2m+1}	a_{2m+2}	....	a_{2n}	Nb'_2
....	...	...	...	...	...	...	...	...	...
m	a_{m1}	a_{m2}	...	a_{mm}	a_{mm+1}	a_{mm+2}	...	a_{mn}	Nb'_m

b. We find the identity matrix $I_{m\times m}$ by processing the rows of the previous table in the same way that was explained in the previous paragraph. This is done by specifying the variables that will be entered into the base and let them be $x_1, x_2, \dots, x_m$. As a result of this processing, we get the following table:

Table No. (4): Table of the first basic solution

Variables / Equations	x_1	x_2	....	x_m	x_{m+1}	x_{m+2}	....	x_n	NB'
1	1	0	....	0	Nd'_{11}	Nd'_{12}	....	Nd'_{1n-m}	Nb'_1
2	0	1	...	0	Nd'_{21}	Nd'_{22}	....	Nd'_{2n-m}	Nb'_2
....	...	...	...	...	...	...	...	...	...
m	0	0	...	1	Nd'_{m1}	Nd'_{22}	...	Nd'_{mn-m}	Nb'_2

c. Setting all the free variables in Table (4) equal to zero, we obtain the following first basic solution:

$$(Nb'_1, Nb'_2, \dots, Nb'_m, 0,0, \dots, 0)$$

d. To obtain a second basic solution, we replace one of the basic variables, say x_m, with one of the non-basic variables x_{m+1}, by selecting the appropriate pivot element, and here it is Nq'_{m1}. We work to delete x_{m+1} from all equations except equation m .We set the coefficient of this variable in this equation equal to one. We perform the appropriate calculations through the two relations (4). We solve the following second basic solution:

$$(Nb'_1, Nb'_2, \dots, Nb'_{m-1}, 0, Nb'_{m+1}, 0, \dots, 0)$$

To obtain other basic solutions, we repeat what was stated in step (d).

Non-negative basic solutions:

If all or some of the variables are conditioned to be non-negative, then some of the basic solutions are unacceptable because they violate the condition. In such a case, we have to look for positive basic solutions from among the basic solutions.

Due to the difficulty of applying the method mentioned in the example, especially in the case that includes a large number of variables, the Gaussian- Jordan method was developed so that positive solutions are directly obtained.

The new method was called the simplex method, which is carried out according to the following steps:

The simplex method for finding non-negative basic solutions to a system of linear equations in which $m < n$:

In the systems of equations (5):

1- We make all elements of the constant's column NB on the second side of the equations non-negative, by multiplying the equation whose second side is negative by (-1)

2- We put the coefficients of the new systems in a table.

3- We form a rule consisting of m variables by choosing the variable that we want to enter into the rule, for example, x_s, and then we calculate the index.

$$\theta = Min\left[\frac{Nb_i}{Na_{is}}\right] = \frac{Nb_t}{Na_{ts}} > 0; \quad Na_{is} > 0, Nb_i > 0$$

We call the element Na_{ts} the pivot element, we delete the variable x_s from all equations according to the Gaussian- Jordan method, except for the equation t, in which its coefficient is equal to one. We repeat the previous step until we form a base consisting of m variables.

4- Setting the non-basic variables equal to zero we obtain the following non-negative basic solution:

$$(Nb'_1, Nb'_2, \dots, Nb'_{m-1}, 0, Nb'_{m+1}, 0, \dots, 0)$$

5- To obtain other new non-negative basic solutions, we choose one of the variables to be a basic variable, then we determine the pivot element and repeat the work we did for the previously mentioned variable x_s. We obtain a new non-negative basic solution, and so we continue working until we obtain all non-negative basic solutions.

We explain the above through the following example:

Example 2:

$$x_1 - 3x_4 + 2x_5 = -[1,3]$$
$$x_2 + 2x_4 - 3x_5 = [2,8]$$

We multiply the first equation by (-1) until the condition $Nb_i > 0$ is met, we obtain the following new systems:

$$-x_1 - 3x_3 - 2x_5 = [1,3]$$
$$x_2 + 2x_4 - 3x_5 = [2,8]$$

The stopping criterion is when we do not find a free column that we did not use for switching that contains a positive element (at least one), meaning that all the elements of the free columns that were not used during the swap are negative values.

In the systems of equations, the number of variables is $n = 5$ and the number of equations is $m = 2$ Therefore, the number of basic variables is equal to 2 and the number of non-basic free variables is $n - m = 3$. The number of possible solutions is calculated from the relationship:

$$C_n^m = \frac{n!}{m!\,(n-m)!}$$

i.e.,

$$C_5^2 = \frac{5!}{2!\,(5-2)!} = 10$$

Write as follows:

$$(x_1, x_2, 0,0,0)\ , (x_1, 0, x_3, 0,0), (x_3, 0,0, x_4, 0), (x_1, 0,0,0, x_5), (0, x_2, x_3, 0,0)$$
$$(0, x_2, 0, x_4, 0), (0, x_2, 0,0, x_5), (0,0, x_3, x_4, 0), (0,0, x_3, 0, x_5), (0,0,0, x_4, x_5)$$

To obtain these solutions, we organize the following table:

Table No. (5): The first table for the simplex method

Variables / Equations	x_1	x_2	x_3	x_4	x_5	NB
1	-1	0	0	3	-2	$[1,3]$
2	0	1	0	2	-3	$[2,8]$

To find a basic solution to the set of equations, we choose a variable, for example x_4, to be a basic variable, and to determine the appropriate anchor element, we calculate the index:

$$\theta = Min\left[\frac{Nb_i}{Na_{is}}\right] = Min\left[\frac{[1,3]}{3}, \frac{[2,8]}{2}\right] = \frac{[1,3]}{3}$$

That is, the fulcrum is $a_{14} = 3$. By performing the necessary calculations to delete the variable x_4 from the two equations, we obtain the following table:

Table No. (6) The second table for the simplex method

Variables / Equations	x_1	x_2	x_3	x_4	x_5	NB'
x_4	$-\frac{1}{3}$	0	0	1	$-\frac{2}{3}$	$\left[\frac{1}{3}, 1\right]$
2	$\frac{2}{3}$	1	0	0	$\frac{5}{3}$	$\left[\frac{4}{3}, 6\right]$

We choose another variable to be a basic variable. We note that the variable x_2 is ready to be a basic variable, and thus we get the following table:

Table No. (7): Final solution table

Variables / Equations	x_1	x_2	x_3	x_4	x_5	NB'
x_4	$-\frac{1}{3}$	0	0	1	$-\frac{2}{3}$	$\left[\frac{1}{3}, 1\right]$
x_2	$\frac{2}{3}$	1	0	0	$\frac{5}{3}$	$\left[\frac{4}{3}, 6\right]$

Thus, we obtain a base consisting of the variables x_2 , x_4. We set the free variables equal to zero, and we obtain the following non-negative neutrosophic basic solution:

$$\left(0, \left[\frac{4}{3}, 6\right], 0, \left[\frac{1}{3}, 1\right], 0\right)$$

To obtain other solutions, we repeat what we did to determine the previous solution.

5. Conclusion and results

As a basis for neutrosophic linear programming, we presented in this research a study of the sets of neutrosophic linear equations, and the Gaussian- Jordan method, which is considered the mathematical basis for the simplex method used to find positive basic solutions, which can be used when there are restrictions on some or all of the variables to be positive values, which in turn was the basis for the method. Direct simplex used to find the optimal solution for linear models. Through the examples that we presented on the systems of neutrosophic equations, we arrived at basic neutrosophic solutions that express unspecified values. Such sentences can be used in cases where the data provided to the systems that operate according to these systems of equations is subject to change. Here we can benefit from the margin of freedom offered by neutrosophic values.

References

[1] Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Applied Mathematics and Data Analysis, Vo .2, No .1, 2023

[2] Mountajab Alhasan, Malath F Alaswad. A Study of the Movement of a Neutrosophic Material Point in the Neutrosophic Plane by Using a Neutrosophic AH-Isometry, Vol 2, No 1, 2023

[3] Wang, J., & Zhang, H. (2017). Generalized Neutrosophic Sets and Their Application in Multi-criteria Decision-making. Springer.

[4] Ye, J. (2019). Neutrosophic logic and its applications in information fusion. Information Fusion, 48, 32-40.

[5] Maissam Jdid, Math 1, Faculty of Informatics Engineering, Al-Sham Private University Publications,2020. (Arabic version)

[6] Maissam Jdid, Operations Research, Faculty of Informatics Engineering, Al-Sham Private University Publications,2021. (Arabic version)
[7] Alali. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004. (Arabic version)
[8] Bugaha J.S, Mualla.W, and others -Operations Research Translator into Arabic, The Arab Center for Arabization, Translation, Authoring and Publishing, Damascus,1998. (Arabic version).

Neutrosophic Treatment of Duality Linear Models and the Binary Simplex Algorithm

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract
One of the most important theories in linear programming is the dualistic theory and its basic idea is that for every linear model has dual linear model, so that solving the original linear model gives a solution to the dual model. Therefore, when we solving the linear programming model, we actually obtain solutions for two linear models. In this research, we present a study of the models. The neutrosophic dual and the binary simplex algorithm, which works to find the optimal solution for the original and dual models at the same time. The importance of this algorithm is evident in that it is relied upon in several operations research topics, such as integer programming algorithms, some nonlinear programming algorithms, and sensitivity analysis in linear programming...

Keywords: Neutrosophic Science; Neutrosophic Linear Models; Dual Models; Neutrosophic dual Linear Models; binary Simplex Algorithm.

1. Introduction:

Most companies and institutions rely on studies provided by experts and researchers using operations research methods in order to ensure a safe work environment away from danger and danger and to give decision makers a margin of freedom, after the studies and research presented using the concepts of neutrosophic science in most fields of science have proven their ability to provide more accurate results. From the results that we were obtaining using classical data-driven studies, we reformulated many operations research topics using neutrosophic concepts [1-20], and to complement what we have done, we present in this research a study of dual linear models and the binary simplex algorithm that gives us a solution for the original and dual models at the same time so that we can provide a clear study that helps decision makers in companies and institutions develop plans and programs through which the greatest profit and lowest cost are achieved.

Discussion:

In our practical life, we encounter many problems that are formulated in the form of linear mathematical models consisting of an objective function and a set of constraints in the form of equations or inequalities. The linear model is written in many formulas that are distinguished by the type of the objective function and the form of the constraints. Many references have been provided that studies of linear and dual linear models are based on classical data, and the appropriate algorithms for solving each of them and how to find the dual model for any model [21-25]. Since classical data results in optimal solutions, which are classical values, restricted values appropriate to the conditions in which the data were collected, and for any change in these conditions, these solutions are inappropriate and may cause the facility a large unexpected loss, so we found it appropriate to reformulate these studies using concepts neutrosophic science is done by taking data with neutrosophic values suitable for all conditions. In previous research, we presented neutrosophic linear models, the direct neutrosophic simplex algorithm, and the graphical method for finding the optimal solution for linear models,]18,19,20[. As a

continuation of what we have done previously, we present in this research a study whose purpose is to reformulate the binary simplex algorithm to find the optimal solution for the original model and the dual model at the same time.

We present this study in two stages:

The first stage:

Finding the neutrosophic dual models using the binary table.

When we want to find the model associated with any linear model, we start by placing this model in the symmetrical form, so we mention the symmetrical form.

The symmetrical form of the neutrosophic linear model: [19]

We say of a linear model that it is in the symmetrical form if all variables are constrained to be non-negative and if all constraints are given in the form of inequalities (and the inequalities of the maximization model constraints must be in the form (≤ less than or equal to) while the inequalities of the minimization model constraints must be in the form (≥ is greater than or equal to), then the linear model is written in the neutrosophic symmetric form in one of two cases:

First case:

$$NZ = \sum_{j=1}^{n} (c_j \pm \varepsilon_j) x_j \longrightarrow Max$$

Constraints:

$$\sum_{j=1}^{n} Na_{ij} x_j \leq b_i \pm \delta_i \quad ; \quad i = 1,2, \dots, m$$

$$x_j \geq 0$$

Second case:

$$NL = \sum_{j=1}^{n} (c_j \pm \varepsilon_j) x_j \longrightarrow Min$$

Constraints:

$$\sum_{j=1}^{n} Na_{ij} x_j \geq b_i \pm \delta_i \quad ; \quad i = 1,2, \dots, m$$

$$x_j \geq 0$$

In both cases, we have $x_j \geq 0$, which are the decision variables, unknown values that we obtain after solving the linear model.

$Nc_j = c_j \pm \varepsilon_j$ and $Nb_i = b_i \pm \delta_i$ and $Na_{ij} = a_{ij} \pm \mu_{ij}$ where (j=1,2,…,n , i=1,2,…,m) are the data of the problem under study, and they are neutrosophic values, indefinite values that enjoy a margin of freedom and are taken according to The nature of the situation represented by the linear model

First method:

Constructing neutrosophic dual linear models using tables:

If the original model is given in detailed form, we put it in the symmetrical form as stated in the previous paragraph, then we draw a binary table for the original and dual models according to the following steps:

1- The coefficients of the objective function in the original model are the constants column in the dual model, and the constants column in the original model are the coefficients of the objective function in the dual model.
2- We invert the signs of the inequalities of the constraints (if they were in the original model of type ≤ they become in the dual model of type ≥)
3- We change the objective from maximizing in the original model to minimizing in the dual model
4- We place each constraint (row) in the original model corresponding to a column in the dual model, meaning there is one variable for each constraint in the original model
5- The variables in the original model and the dual model satisfy the non-negativity constraints.

We explain the above through the following two cases:

The first case: The original model is symmetrical and of the maximization type

Find

$$NZ = Nc_1x_1 + Nc_2x_2 + \cdots + Nc_nx_n \longrightarrow Max$$

Constants:

$$Na_{11}x_1 + Na_{12}x_2 + \cdots + \mathrm{Na}_{1n}x_n \leq Nb_1$$

$$Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2n}x_n \leq Nb_2$$

…… … … … … … … … … … … … … … … …

$$Na_{m1}x_1 + Na_{m2}x_2 + \cdots + Na_{mn}x_n \leq Nb_m$$

$$x_1, x_2, \ldots, x_n \geq 0$$

The binary table for the original model and the accompanying model is as follows:

Table No. (1) Objective follower of the maximization type

		Original model			
		objective function / constants	$Nc_1x_1 + Nc_2x_2 + \cdots + Nc_nx_n$		Max / Constants column
Dual vibrable	y_1	1	$Na_{11}x_1 + Na_{12}x_2 + \cdots + Na_{1n}x_n$	≤	Nb_1
	y_2	2	$Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2n}x_n$	≤	Nb_2
	...	...	… … … … … … … … … … … … … … …	≤	...
	y_m	m	$Na_{m1}x_1 + Na_{m2}x_2 + \cdots + Na_{mn}x_n$	≤	Nb_m
		Non-negativity constraints	$x_1, x_2, \ldots, x_n$	≥	0
		Dual model			
		Objective function / constraints	$Nb_1y_1 + Nb_2y_2 + \cdots + Nb_iy_m$		Min / Constants column
		1	$Na_{11}y_1 + Na_{21}y_2 + \cdots + Na_{m1}y_m$	≥	Nc_1
		2	$Na_{12}y_1 + Na_{22}y_2 + \cdots + Na_{m2}y_m$	≥	Nc_2
		...	… … … … … … … … … … … … … … …	≥	...
		n	$Na_{1n}y_1 + Na_{2n}y_2 + \cdots + Na_{mn}y_m$	≥	Nc_n

Non-negativity constraints	$y_1, y_2, \dots, y_m$	≥	0

The second case: The original model is symmetrical and of the reduction type:
Find

$$NL = Nc_1x_1 + Nc_2x_2 + \cdots + Nc_nx_n \longrightarrow Min$$

Constants:

$$Na_{11}x_1 + Na_{12}x_2 + \cdots + \mathrm{N}a_{1n}x_n \geq Nb_1$$

$$Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2n}x_n \geq Nb_2$$

$$\dots\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$Na_{m1}x_1 + Na_{m2}x_2 + \cdots + Na_{mn}x_n \geq Nb_m$$

$$x_1, x_2, \dots, x_n \geq 0$$

The binary table for the original model and the accompanying model is as follows:

Table No. (2) objective follower in the original model of the reduce type

		Original model			
		objective function / constants	$Nc_1x_1 + Nc_2x_2 + \cdots + Nc_nx_n$		Min / Constants column
Dual vibrable	y_1	1	$Na_{11}x_1 + Na_{12}x_2 + \cdots + Na_{1n}x_n$	≥	Nb_1
	y_2	2	$Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2n}x_n$	≥	Nb_2
	...	...	... ...	≥	...
	y_m	m	$Na_{m1}x_1 + Na_{m2}x_2 + \cdots + Na_{mn}x_n$	≥	Nb_m
		Non-negativity constraints	$x_1, x_2, \dots, x_n$	≥	0
		Dual model			
		objective function / constants	$Nb_1y_1 + Nb_2y_2 + \cdots + Nb_iy_m$		Max / Constants column
		1	$Na_{11}y_1 + Na_{21}y_2 + \cdots + Na_{m1}y_m$	≤	Nc_1
		2	$Na_{12}y_1 + Na_{22}y_2 + \cdots + Na_{m2}y_m$	≤	Nc_2
		...	... ...	≤	...
		n	$Na_{1n}y_1 + Na_{2n}y_2 + \cdots + Na_{mn}y_m$	≤	Nc_n
		Non-negativity constraints	$y_1, y_2, \dots, y_m$	≥	0

The second stage: formulation of the binary neutrosophic algorithm.

The binary simplex algorithm (for the original and dual models) neutrosophic. Through this algorithm, we can find the two ideal solutions for both the original and dual models at the same time. Before starting the binary simplex algorithm, we must mention the modified simplex algorithm that we will use within the steps of the binary algorithm.

Modified simplex algorithm:

In the modified Simplex algorithm, after converting the regular linear model to the basic form, we place the coefficients in a short table whose first column includes the basic variables and whose top row includes the non-basic variables only. We define the pivot column, which is the column corresponding to the largest positive value in the objective function row if the objective function is a maximization function (but if the objective function is a minimization function, it is the column corresponding to the most negative values). Let this column be the column of the variable x_s. We define the pivot row. The pivot row is determined. Through the following indicator:

$$N\theta = min\left[\frac{Nb_i}{Na_{is}}\right] = \frac{Nb_t}{Na_{ts}} > 0; \quad Na_{is} > 0, Nb_i > 0$$

Let this line be the line of the base variable y_t, then the anchor element is the element resulting from the intersection of the anchor column and the anchor line, i.e., Na_{ts}. Then we calculate the new elements corresponding to the anchor line and the anchor column as follows:

1- We put opposite the pivot element Na_{ts} the reciprocal of $\frac{1}{Na_{ts}}$
2- We calculate the elements of the row corresponding to the pivot row (except the pivot element) by dividing the elements of the pivot row by the pivot element Na_{ts}
3- We calculate all the elements of the column opposite the pivot (except the pivot element) by dividing the elements of the pivot column by the pivot element Na_{ts} and then multiplying them by)-1(
4- We calculate the other elements from the following relationships:

$$Nb_i' = Nb_i - Nb_t\frac{Na_{is}}{Na_{ts}} = \frac{Nb_iNa_{ts} - Nb_tNa_{is}}{Na_{ts}}$$
$$Na_{ij}' = Na_{ij} - Na_{tj}\frac{Na_{is}}{Na_{ts}} = \frac{Na_{ij}Na_{ts} - Na_{tj}Na_{is}}{Na_{ts}}$$
$$Nc_j' = Nc_j - Nc_s\frac{Na_{tj}}{Na_{ts}} = \frac{Nc_jNa_{ts} - Nc_sNa_{tj}}{Na_{ts}}$$

5- We apply the stopping criterion of the direct Simplex algorithm on the objective function row. If the objective function is of the maximize type, the objective function row in the table must not contain any positive value. (But if the objective function is of the minimization type, the objective function row in the new table must not be contains any negative value), if the criterion is not met, we repeat the same steps until the stopping criterion is met and we obtain the desired ideal solution.

Steps of the binary simplex algorithm:

a. We write the two models in basic form by adding or subtracting additional variables or using synthetic variables and isolating the non-restricting variables.

Basal form of the original model:

Find

$$NZ = Nc_1x_1 + Nc_2x_2 + \cdots + Nc_nx_n + 0u_1 + 0u_2 + \cdots + 0u_m \longrightarrow Max$$

Constans:

$$Na_{11}x_1 + Na_{12}x_2 + \cdots + Na_{1n}x_n + u_1 = Nb_1$$
$$Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2n}x_n + u_2 = Nb_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$Na_{m1}x_1 + Na_{m2}x_2 + \cdots + Na_{mn}x_n + u_m = Nb_m$$
$$x_j \geq 0 \ ; \mathrm{j} = 1,2,\ldots,\mathrm{n}$$
$$u_i \geq 0 \ ; \mathrm{i} = 1,2,\ldots,\mathrm{m}$$

Here we do not require that $Nb_i \geq 0$.

Basic form of the dual model:

Find

$$NL = Nb_1y_1 + Nb_2y_2 + \cdots + Nb_iy_m + 0v_1 + 0v_2 + \cdots + 0v_n \longrightarrow Min$$

Constans:

$$Na_{11}y_1 + Na_{21}y_2 + \cdots + Na_{m1}y_m - v_1 = Nc_1$$
$$Na_{12}y_1 + Na_{22}y_2 + \cdots + Na_{m2}y_m - v_2 = Nc_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$Na_{1n}y_1 + Na_{2n}y_2 + \cdots + Na_{mn}y_m - v_n = Nc_n$$
$$y_i \geq 0 \ ; i = 1,2, \ldots, m$$
$$v_j \geq 0 \ ; \mathrm{j} = 1,2, \ldots, \mathrm{n}$$

Here we do not require that $Nc_j \geq 0$

The two models have the same coefficients, and the matrix of instances of the dual model is the transpose of the matrix of instances of the original model. We write the two models in the following binary table:

Table No. (3) Standard format for the original and companion models

		Original model			
		objective function / constants	$Nc_1x_1 + Nc_2x_2 + \cdots + Nc_nx_n + 0u_1 + 0u_2 + \cdots + 0u_m$		Max / Constants column
Dual vibrable	y_1	1	$Na_{11}x_1 + Na_{12}x_2 + \cdots + Na_{1n}x_n + u_1$	=	Nb_1
	y_2	2	$Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2n}x_n + u_2$	=	Nb_2
	...	...	... ...	=	...
	y_m	m	$Na_{m1}x_1 + Na_{m2}x_2 + \cdots + Na_{mn}x_n + u_m$	=	Nb_m
		Non-negativity constraints	$x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m$	≥	0
		Dual model			
		objective function / constants	$Nb_1y_1 + Nb_2y_2 + \cdots + Nb_iy_m + 0v_1 + 0v_2 + \cdots + 0v_n$		Min / Constants column
		1	$Na_{11}y_1 + Na_{21}y_2 + \cdots + Na_{m1}y_m - v_1$	=	Nc_1
		2	$Na_{12}y_1 + Na_{22}y_2 + \cdots + Na_{m2}y_m - v_2$	=	Nc_2
		...	... ...	=	...
		n	$Na_{1n}y_1 + Na_{2n}y_2 + \cdots + Na_{mn}y_m - v_n$	=	Nc_n
		Non-negativity constraints	$y_1, y_2, \ldots, y_m, v_1, v_2, \ldots, v_n$	≥	0

b. We place the variables and coefficients of the original model in the modified simplex table, and we place the variables of the dual model outside the table as follows:

Table No. (4): The binary table for the original and dual models according to the modified Simplex algorithm

			Basic variables with a (-) sign in the dual model				
			$-v_1$	$-v_2$	...	$-v_n$	
		Non-basic vibrable / basic vibrable	x_1	x_2	...	x_n	**Follow the objective of the dual model** NB_i
No n-	y_1	u_1	Na_{11}	Na_{12}	...	Na_{1n}	Nb_1

	y_2	u_2	Na_{21}	Na_{22}	…	Na_{2n}	Nb_2
	…	…	…	…	…	…	…
	y_m	u_m	Na_{m1}	Na_{m2}	…	Na_{mn}	Nb_m
		objective of the original model	Nc_1	Nc_2	…	Nc_n	$L-0\rightarrow Min$ / $Z-0\rightarrow Max$

Binary simplex algorithm for the original and dual models:

From the modified simplex algorithm of the original model, we obtain the optimal solution of the original model when all the elements are in the last row (the objective function row of the original model) $Nc_j \leq 0$; $\mathrm{j} = 1,2,\dots,\mathrm{n}$ and at the same time all the elements are in the column The last (associated objective function column) $Nb_i \geq 0$; $i = 1,2,\dots,m$ and we get the optimal solution for the dual model when all elements in the last column (associated objective function column) are $Nb_i \geq 0$; $i = 1,2,\dots,m$ and at the same time the last row (the objective function row of the original model) $Nc_j \leq 0$; $\mathrm{j} = 1,2,\dots,\mathrm{n}$ (because it will correspond to $Nc_j = -v_j$) which are the two conditions Same for both models. Therefore, when searching for the optimal solution for both models together, we must work to make all elements $Nb_i \geq 0$; $i = 1,2,\dots,m$ and to make all elements $Nc_j \leq 0$; $\mathrm{j} = 1,2,\dots,\mathrm{n}$ to achieve this, we rely on one of the two models, put its variables and coefficients in a table, and place the dual model in an external frame of that table. In general, we find that the necessity of placing the two models in a short table does not allow us to get rid of the negative constants on the right side, and therefore the general case of the previous binary table can it must include negative constants $Nb_i < 0$, and the elements of the last row can include positive elements $Nc_j > 0$, so when searching for the optimal solution for the two models, we must work to address these elements based on one of the two models.

Depending on the original model, we do this in two stages:

The first stage:

We make the constant Nb_i non-negative, which corresponds to obtaining a non-negative basic solution for the original model.

The second stage:

We make all elements of the objective function row non-positive (in the case of the objective function, maximization), and this corresponds to obtaining the optimal solution required for the original model.

Based on the dual model, we do this in two stages:

The first stage:

We must make the elements of the dual model objective function column $Nb_i \geq 0$; $\mathrm{i} = 1,2,\dots,\mathrm{m}$,The last row is non-negative

The second stage:

We must make the free constants for the dual model $-Nc_j$ non-positive, and this corresponds to obtaining the optimal solution for the dual model. We explain the above through the following example:

Find the optimal solution for both the following neutrosophic linear model and its dual using the binary algorithm

Example:

Find:

$$[5,8]\ x_1 + [3,6]\ x_2 \longrightarrow Max$$

Constans:

$$2x_1 + 3x_2 \leq [14,20]$$

$$2x_1 + x_2 \leq [10,16]$$

$$3x_2 \leq [12,18]$$
$$3x_1 \leq [15,21]$$

$$x_1 \geq 0, x_2 \geq 0$$

We form the binary table of the model and the dual model:

Table No. (5) The original model and its dual mode

		Original model			
		objective function constants	$[5,8]\ x_1 + [3,6]\ x_2$		Max Constants column
Dual vibrable	y_1	1	$2x_1 + 3x_2$	$\leq$	[14,20]
	y_2	2	$2x_1 + x_2$	$\leq$	[10,16]
	y_3	3	$3x_2$	$\leq$	[12,18]
	y_4	4	$3x_1$	$\leq$	[15,21]
		Non-negativity constraints	x_1, x_2	$\geq$	0
		Dual model			
		objective function constants	$[14,20]y_1 + [10,16]y_2 + [12,18]y_3 + [15,21]y_4$		Min Constants column
		1	$2y_1 + 2y_2 + 3y_4$	$\geq$	[5,8]
		2	$3y_1 + y_2 + 3y_4$	$\geq$	[3,6]
		Non-negativity constraints	y_1, y_2, y_3, y_4	$\geq$	0

In the following table, we wrote the two models in standard form:

		Original model			
		objective function constants	$[5,8]\ x_1 + [3,6]\ x_2 + 0u_1 + 0u_2 + 0u_3 + 0u_4$		Max Constants column
Dual vibrable	y_1	1	$2x_1 + 3x_2 + u_1$	$=$	[14,20]
	y_2	2	$2x_1 + x_2 + u_2$	$=$	[10,16]
	y_3	3	$3x_2 + u_3$	$=$	[12,18]
	y_4	4	$3x_1 + u_4$	$=$	[15,21]
		Non-negativity constraints	$x_1, x_2, u_1, u_2, u_3, u_4$	$\geq$	0
		Dual model			
		objective function constants	$[14,20]y_1 + [10,16]y_2 + [12,18]y_3 + [15,21]y_4 + 0v_1 + 0v_2$		Min Constants column
		1	$2y_1 + 2y_2 + 3y_4 - v_1$	$=$	[5,8]
		2	$3y_1 + y_2 + 3y_4 - v_2$	$=$	[3,6]

Non-negativity constraints	$y_1, y_2, y_3, y_4, v_1, v_2$	≥	0

Table No. (6) Standard format for the original model and the dual model

We notice from the table that the standard form of the original model includes a ready-made base of additional variables u_1, u_2, u_3, u_4, but for the dual model there is no ready-made base. Therefore, we multiply the two restrictions by (-1) and we obtain the basic form of the dual model.

The following table shows the basic form of the original and dual models:

Table No. (7): The basic shape of the original model and the dual model

		Original model		
	objective function constants	$[5,8]\ x_1 + [3,6]\ x_2 + 0u_1 + 0u_2 + 0u_3 + 0u_4$		Max Constants column
Dual vibrable y_1	1	$2x_1 + 3x_2 + u_1$	=	[14,20]
Dual vibrable y_2	2	$2x_1 + x_2 + u_2$	=	[10,16]
Dual vibrable y_3	3	$3x_2 + u_3$	=	[12,18]
Dual vibrable y_4	4	$3x_1 + u_4$	=	[15,21]
	Non-negativity constraints	$x_1, x_2, u_1, u_2, u_3, u_4$	≥	0
		Dual model		
	objective function constants	$[14,20]y_1 + [10,16]y_2 + [12,18]y_3 + [15,21]y_4 + 0v_1 + 0v_2$		Min Constants column
	1	$-2y_1 - 2y_2 - 3y_4 + v_1$	=	$-[5,8]$
	2	$-3y_1 - y_2 - 3y_4 + v_2$	=	$-[3,6]$
	Non-negativity constraints	$y_1, y_2, y_3, y_4, v_1, v_2$	≥	0

We put the two models in the modified Simplex algorithm table and we get the following table:

Table No. (8): The binary table for the original and dual models according to the modified Simplex algorithm

		According to the original model		
		$-v_1$	$-v_2$	
	Non-basic vibrable / basic vibrable	x_1	x_2	objective function NB_i
Non-basic vibrable y_1	u_1	2	3	[14,20]
Non-basic vibrable y_2	u_2	2	1	[10,16]
Non-basic vibrable y_3	u_3	0	3	[12,18]
Non-basic vibrable y_4	u_4	3	0	[15,21]
	objective function Nc_i	[5,8]	[3,6]	$L - 0$ / $Z - 0$

			According to the dual model				
			u_1	u_2	u_3	u_4	
		Non-basic vibrable / basic vibrable	y_1	y_2	y_3	y_4	objective function Original model Nc_i
Non-basic	x_1	v_1	-2	-2	0	-3	$-[5,8]$
	x_2	v_2	-3	-1	-3	0	$-[3,6]$
		objective function Dual model NB_i	$[14,20]$	$[10,16]$	$[12,18]$	$[15,21]$	$Z-0$ / $L-0$

The first stage:

1- For the original model:

Since the values in the constant's column are all positive, we study the values in the objective function row and determine the largest positive value. We find:

$$\max([5,8],[3,6]) = [5,8]$$

It is an expression of the variable x_1. This means that it will enter the base. To determine the element that will exit from the base, we calculate the index $N\theta$, where:

$$N\theta \in min\left[\frac{[14,20]}{2}, \frac{[10,16]}{2}, \frac{[15,21]}{3}\right] = \frac{[15,21]}{3} = [5,7]$$

We find that the pivot column is the column of the non-base variable x_1, meaning that the variable x_1 will enter the base instead of the variable u_4, and the pivot element is the element resulting from the intersection of the pivot row and the pivot column, which is (3)

We perform the switching between variables using a modified simplex algorithm.

2- For the dual model:

We study the elements of the objective function row. We notice that all the values are positive. Therefore, we study the elements of the constant's column. We find that they are all negative values. We choose the most negative of them, which is $(-[5,8])$ which is the row of the base variable v_1, so its row is the pivot row. To determine the pivot column and the pivot element, we calculate the index $N\theta'$where:

$$N\theta' \in Max\left[\frac{[14,20]}{-2}, \frac{[10,16]}{-2}, \frac{[15,21]}{-3}\right] = \frac{[15,21]}{-3}$$

So, the column of the non-base variable u_4 is the pivot column, meaning that the variable u_4 will enter the base instead of the variable v_1, and the pivot element is the element resulting from the intersection of the pivot row and the pivot column, which is (-3). We perform the switching between the variables using the modified simplex algorithm, from (1) and (2) We get the following double table:

Table No. (9): The binary table for the first stage, the solution according to the original and dual models

	According to the original model		
	$-y_4$	$-v_2$	
Non-basic vibrable / basic vibrable	u_4	x_2	objective function Original model NB_i

Non-basic vibrable	y_1	u_1	$\frac{-2}{3}$	3	[4,6]
	y_2	u_2	$\frac{-2}{3}$	1	[0,4]
	y_3	u_3	0	3	[12,18]
	v_1	x_1	$\frac{1}{3}$	0	[5,7]
		objective function Original model Nc_i	$\left[\frac{-8}{3},\frac{-5}{3}\right]$	[3,6]	$L-[25,56]$ $Z-[25,56]$

			According to the dual model				
			u_1	u_2	u_3	x_1	
		Non-basic vibrable / basic vibrable	y_1	y_2	y_3	v_1	objective function Original model Nc_i
Non-basic vibrable	u_4	y_4	$\frac{2}{3}$	$\frac{2}{3}$	0	$\frac{-1}{3}$	$\left[\frac{8}{3},\frac{5}{3}\right]$
	x_2	v_2	-3	-1	-3	0	$-[3,6]$
		objective function Original model Nc_i	[4,6]	[0,4]	[12,18]	[5,7]	$Z-[25,56]$ $L-[25,56]$

The second phase:

We apply the stopping criterion of the algorithm

For the original model:

Since the values in the constant's column are all positive, we study the values in the objective function row. We notice that there is a positive value, which is [3,6], meaning that we have not yet reached the optimal solution. Therefore, we specify the pivot column, which is the column of the variable x_2 corresponding to the only positive value in the objective function row. [3,6] In order to determine the pivot row and the pivot element, we calculate the index $N\theta$, where:

$$N\theta \in \min\left[\frac{[4,6]}{3},\frac{[0,4]}{1},\frac{[12,18]}{3}\right]=\frac{[4,6]}{3}$$

It corresponds to the base element u_1, so its row is the pivot row and the pivot element is $(\mathbf{3})$. We swap between the variables using the modified simplex algorithm.

For the dual model:

We study the elements of the objective function row. We notice that all the values are positive. Therefore, we study the elements of the constant's column. We find that there is a single negative value, which is $(-[3,6])$, which is the line of the base variable v_2, so its row is the pivot row. To determine the pivot column and the pivot element, we calculate the index $N\theta'$ where:

$$N\theta' \in Max\left[\frac{[4,6]}{-3},\frac{[0,4]}{-1},\frac{[12,18]}{-3}\right]=\frac{[4,6]}{-3}$$

So, the column of the non-base variable y_1 is the pivot column, meaning that the variable y_1 will enter the base instead of the variable v_2, and the pivot element is the element resulting from the intersection of the pivot row and

the pivot column, which is (-3). We perform the switching between the variables using the modified simplex algorithm, from (1) and (2) We get the following double table:

Table No. (10): The binary algorithm table for the second stage

			According to the original model		
			$-y_4$	$-y_1$	
		Non-basic vibrable / **basic vibrable**	u_4	u_1	objective function Dual model NB_i
Non-basic vibrable	v_2	x_2	$\frac{-2}{9}$	$\frac{1}{3}$	$\left[\frac{4}{3},2\right]$
	y_2	u_2	$\frac{-4}{9}$	$\frac{-1}{3}$	$\left[\frac{4}{3},2\right]$
	y_3	u_3	$\frac{2}{3}$	-1	$[8,12]$
	v_1	x_1	1	0	$[5,7]$
		objective function Original model Nc_i	$[-6,-1]$	$[-2,-1]$	$L-[29,68]$ / $Z-[29,68]$

			According to the dual model				
			x_2	u_2	u_3	x_1	
		Non-basic vibrable / **basic vibrable**	v_2	y_2	y_3	v_1	objective function Original model Nc_i
Non-basic vibrable	u_4	y_4	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{-2}{3}$	-1	$[-6,-1]$
	u_1	y_1	$\frac{-1}{3}$	$\frac{1}{3}$	1	0	$[-2,-1]$
		objective function Dual model NB_i	$\left[\frac{4}{3},2\right]$	$\left[\frac{4}{3},2\right]$	$[8,12]$	$[5,7]$	$Z-[29,68]$ / $L-[29,68]$

We apply the stopping criterion of the algorithm:

1- For the original model, we study the elements of the objective function row until the criterion for stopping the algorithm is met, which is the absence of any positive element
2- For the dual model, we also study the elements of the constants column until the criterion for stopping the algorithm is met, which is the absence of any negative element
3- We find that the criterion has been met and thus we have reached the optimal solution

The optimal solution of the original model is:

$$x_2^* \in \left[\frac{4}{3},2\right] \ , u_2^* \in \left[\frac{4}{3},2\right] \ , u_3^* \in [8,12] \ , x_1^* \in [5,7] \ , u_1^* = u_4^* = 0$$

The value of the objective function corresponds to:

$$Z^* = max\, Z \in [29,68]$$

The optimal solution of the dual model is:

$$y_1^* \in [1,2]\ , y_4^* \in [1,6]\ , v_2^* = y_2^* = y_3^* = v_1^* = 0$$

The value of the objective function corresponds to:

$$L^* = MinL \in [29,166]$$

We note that:

$$Z^* = Max\, Z \in [29,68] \leq L^* = MinL \in [29,166]$$

This solution is acceptable according to the following theory:

If $(x_1, x_2, \dots, x_n)$is an acceptable solution to the original model of type Max and $(y_1, y, \dots, y_m)$was an acceptable solution for the dual model of type Min, so the value of the objective function of the original model does not exceed the value of the objective function of the dual model for these two solutions, that is, it is

$$\sum_{j=1}^{n} Nc_j x_j \leq \sum_{i=1}^{m} Nb_i y_i$$

This is for all acceptable solutions for both models (including the optimal solution)

5. Conclusion and Results:

From the previous study, we arrived at a solution for the original and dual models at the same time, which are neutrosophic values from which we know the minimum and maximum profit that we can obtain, because the interpretation of the optimal solution for the original model is that it gives us the best production plan that makes the value of that production as large as possible, within Available capabilities. As for the optimal solution for the dual model, it gives us the best values for the prices of raw materials, which, if used without waste, will also give us the best production plan, and the result is the maximum profit.

References

[1] Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Applied Mathematics and Data Analysis, Vo .2, No .1, 2023

[2] Maissam Jdid- Hla Hasan, The state of Risk and Optimum Decision According to Neutrosophic Rules, International Journal of Neutrosophic Science (IJNS), Vol. 20, No.1,2023

[3] Mohammed Alshikho, Maissam Jdid, Said Broumi, Artificial Intelligence and Neutrosophic Machine learning in the Diagnosis and Detection of COVID 19, Journal Prospects for Applied Mathematics and Data Analysis, Vol 01, No,02 USA,2023

[4] Maissam Jdid, Neutrosophic Nonlinear Models, Journal Prospects for Applied Mathematics and Data Analysis, Vo .2, No .1, 2023

[5] Maissam Jdid, Neutrosophic Mathematical Model of Product Mixture Problem Using Binary Integer Mutant, Journal of Neutrosophic and Fuzzy Systems (JNFS), Vo .6, No .2, 2023

[6] Maissam Jdid, The Use of Neutrosophic linear Programming Method in the Field of Education, Handbook of Research on the Applications of Neutrosophic Sets Theory and Their Extensions in Education, Chapter 15, IGI-Global,2023

[7] Maissam Jdid, Florentin Smarandache, Said Broumi, Inspection Assignment Form for Product Quality Control, Journal of Neutrosophic Systems with Applications, Vol. 1, 2023

[8] Maissam Jdid, Said Broumi, Neutrosophic Rejection and Acceptance Method for the Generation of Random Variables, Neutrosophic Sets and Systems, NSS, Vol.56,2023
[9] Maissam Jdid, Florentin Smarandache, The Use of Neutrosophic Methods of Operation Research in the Management of Corporate Work, Journal of Neutrosophic Systems with Applications, Vol. 3, 2023
[10] Maissam Jdid, Florentin Smarandache, Lagrange Multipliers and Neutrosophic Nonlinear Programming Problems Constrained by Equality Constraints Journal of Neutrosophic Systems with Applications, Vol. 6, 2023
[11] Maissam Jdid, Florentin Smarandache, Optimal Neutrosophic Assignment and the Hungarian Method, Journal Neutrosophic Sets and Systems, NSS Vol.57,2023
[12] Maissam Jdid, Florentin Smarandache, Graphical Method for Solving Neutrosophical Nonlinear Programming Models, Journal of Neutrosophic Systems with Applications, Vol. 9, 2023
[13] Maissam Jdid, Florentin Smarandache, Optimal Agricultural Land Use: An Efficient Neutrosophic Linear Programming Method, Journal of Neutrosophic Systems with Applications, Vol. 10, 2023
[14] Maissam Jdid, Nada A Nabeeh Generating Random Variables that follow the Beta Distribution Using the Neutrosophic Acceptance-Rejection Method *1, Journal Neutrosophic Sets and Systems, NSS Vol.58,2023
[15] Maissam Jdid, Studying Transport Models with the Shortest Time According to the Neutrosophic Logic, Journal Neutrosophic Sets and Systems, NSS Vol.58,2023
[16] Florentin Smarandache, Maissam Jdid, NEUTROSOPHIC TRANSPORT AND ASSIGNMENT ISSUES, Publisher: Global Knowledge's: 978_1 _59973_769_0
[17] Maissam Jdid, NEUTROSOPHIC TRANSPORT AND ASSIGNMENT ISSUES, Publisher: Global Knowledge's: 978_1 _59973_770_6, (Arabic version).
[18] Maissam Jdid, AA Salama, Huda E Khalid ,Neutrosophic Handling of the Simplex Direct Algorithm to Define the Optimal Solution in Linear Programming ,International Journal of Neutrosophic Science, Vol.18,No. 1, 2022
[19] Maissam Jdid, Huda E Khalid, Mysterious Neutrosophic Linear Models, International Journal of Neutrosophic Science, Vol.18, No. 2, 2022
[20] Maissam Jdid, Florentin Smarandache, The Graphical Method for Finding the Optimal Solution for Neutrosophic linear Models and Taking Advantage of Non-Negativity Constraints to Find the Optimal Solution for Some Neutrosophic linear Models in Which the Number of Unknowns is More than Three, Journal Neutrosophic Sets and Systems, NSS Vol.58,2023
[21] Alali. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004. (Arabic version).
[22] Al Hamid. Mohammed Dabbas, Mathematical programming, Aleppo University, Syria, 2010. (Arabic version).
[23] Linear and Nonlinear Programming-DavidG. Luenbrgrr.YinyuYe- Springer Science + Business Media-2015.
[24] Maissam Jdid, Bi-Linear Model of Natural Resource Use Published in Tver State University, 2002
[25] Maissam Jdid Operations Research, Faculty of Informatics Engineering, Al-Sham Private University Publications, 2021.

Finding a Basic Feasible Solution for Neutrosophic Linear Programming Models: Case Studies, Analysis, and Improvements

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract: Since the inception of operations research, linear programming has received the attention of researchers in this field due to the many areas of its use. The focus was on the methods used to find the optimal solution for linear models. The direct simplex method, with its three basic stages, begins by writing the linear model in standard form and then finding a basic solution that is improved according to the simplex steps until We get the optimal solution, but we encounter many linear models that do not give us a basic solution after we put it in a standard form, and here we need to solve a rule through which we reach the optimal solution. For these models, researchers and scholars in the field of operations research introduced the simplex method with an artificial basis, which helped to Find the optimal solution for linear models, given the importance of this method and as a complement to the previous research we presented using the concepts of neutrosophic science. In this research, we will reformulate the simplex algorithm with an artificial basis using concepts of neutrosophic science.

Keywords: Linear Programming; Simplex Method; Neutrosophic Science; Simplex Neutrosophic Method; Artificial Variable.

1. Introduction

The great scientific development that our contemporary world has witnessed has led to the emergence of what is called operations research. This name refers to the group of scientific methods used in analyzing problems and searching for optimal solutions. Operations research is considered one of the modern sciences whose applications have achieved wide success in various fields of life. One of the methods of operations research is the linear programming method that allows us to model, analyze, and solve a wide range of issues that have resulted from the great scientific development that our contemporary world is witnessing [1]. In all previous studies, we have reached the optimal solution for solvable models, and this solution has a specific value resulting from specific data provided by the study. The field studies that were conducted are linked to the conditions that existed, but the reality of the situation indicates that the conditions surrounding the work environment are not fixed and the future cannot be predicted. These specific values for profits and available resources are subject to instantaneous change. Out of interest in keeping pace with scientific development, we have in this research reformulated one of the most important methods used to find the optimal solution for linear models: the simplex method with an artificial basis using the concepts of neutrosophic science, the science that has proven its ability to provide the best solution in many fields

of science. Therefore, researchers' interest has focused on providing studies and research in various fields according to the concepts of this science [2-6].

The purpose of solving linear models is to choose the optimal solution from the set of acceptable solutions. This is done based on a base solution that is improved using the direct simplex algorithm that was presented according to the concepts of neutrosophic science in the research [7]. It consists of three basic stages:

(i). The stage of converting the imposed model into an equivalent systematic form [8].

(ii). The stage of converting the regular form into a basic form to obtain the non-negative basic solutions.

(iii). The stage of searching for the ideal solution requires from among the non-negative basic solutions [7].

Therefore, the process of searching for the optimal solution does not begin until after obtaining a base solution, but in many linear models we face great difficulty in obtaining the base solution, so the simplex method with an artificial base was proposed, where a base is formed consisting of a set of artificial variables that are not negativity is added to constraints that do not contain a basic variable, thus obtaining the basic solution. Then we improve it using the direct simplex algorithm until we obtain the optimal solution. In this research, we will reformulate the simplex algorithm with an artificial basis to find the optimal solution for linear models for which it is difficult to obtain a basic solution, using the concepts of neutrosophic.

2. Problem statement

Find the optimal solution for the following neutrosophic linear model:

$$Max\ Z = NC_1x_1 + NC_2x_2 + \cdots + NC_nx_n + NC_0$$

Constraints:

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + \varepsilon_1 = Nb_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + \varepsilon_2 = Nb_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n + \varepsilon_3 = Nb_3 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + \varepsilon_m = Nb_m \\ x_1, x_2, \dots, x_n \geq 0 \end{gathered}$$

where $NC_j = C_j \pm \varepsilon_j$, $Nb_i = b_i \pm \delta_i$, a_{ij}, $j = 1,2,\dots,n$, $i = 1,2,\dots,m$ are constants having set or interval values according to the nature of the given problem, x_j are decision variables. It is worth mentioning that the coefficients subscribed by the index N are of neutrosophic values. The objective function coefficients $NC_1, NC_2, \dots, NC_n$ have neutrosophic meaning are intervals of possible values: That is, $Nc_j \in [\lambda_{j1}, \lambda_{j2}]$, where $\lambda_{j1}, \lambda_{j2}$ are the upper and the lower bounds of the objective variables x_j respectively, $j = 1,2,\dots,n$. Also, we have the values of the right-hand side of the inequality constraints $Nb_1, Nb_2, \dots, Nb_m$ are regarded as neutrosophic interval values:
$Nb_i \in [\mu_{i1}, \mu_{i2}]$, here, μ_{i1}, μ_{i2} are the upper and the lower bounds of the constraint $i = 1,2,\dots,m$.

In the previous model, we note that the number of variables is n and the number of constraints is m, and this model is in the standard form.
We move to the second stage, which is to find a basic solution. Here we use the simplex algorithm with an artificial base, which is represented by the following:

(i). From the standard form, we form an artificial base form by adding to the left side of each of the constraint equations a non-negative artificial variable ε_i. Thus, we form a base consisting of the non-negative variables $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$.

(ii). Since the artificial variables are introduced into constraints that were originally linear equations, these variables must take the value of zero so that the linear constraints are not affected.

(iii). Therefore, we must move all of them from the base until they become non-base variables, and to be able to make this transition, we use the direct simplex algorithm.

(iv). We introduce these variables into the objective function with the likes of M (where M is a sufficiently large positive number that is at least greater than any $|Nc_j|$) and preceded by a minus sign (because the objective function is a maximization function) so that we do not transfer them back to the base variables again.

(v). We obtain the following basic form of the neutrosophic linear model:

$$Max\, Z = NC_1x_1 + NC_2x_2 + \cdots + NC_nx_n - M\varepsilon_1 - M\,\varepsilon_2 - \cdots - M\,\varepsilon_m + NC_0$$

Constraints:

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + \varepsilon_1 = Nb_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + \varepsilon_2 = Nb_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n + \varepsilon_3 = Nb_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + \varepsilon_m = Nb_m \end{gathered}$$

$$x_j \geq 0\ , \varepsilon_i > 0\ , Nb_i > 0\ ; j = 1,2, \ldots, n\ and\ i = 1,2, \ldots, m$$

(vi). After obtaining the basic solution, we use the direct simplex algorithm to improve this solution to reach the optimal solution. Therefore, we arrange the previous information in Table 1.

Table 1. General data of the model.

Variables / Basic	x_1	x_2	….	x_n	ε_1	ε_2	….	ε_m	b_i
ε_1	a_{11}	a_{12}	….	a_{1n}	1	0	….	0	b_1
ε_2	a_{21}	a_{22}	…	a_{2n}	0	1	0	0	b_2
….	…	…	…	…	…	…	…	…	…
ε_m	a_{m1}	a_{m2}	…	a_{mn}	0	0	…	1	b_m
objective function	NC_1	NC_2	….	NC_n	$-M$	$-M$	…	$-M$	$Z - NC_0$

We get rid of the artificial variables. Here we study the constants b_i corresponding to the artificial variables and choose the largest of them, let it be b_t corresponding to the variable ε_t and we consider its row to be the pivot row. Then we determine the pivot element in it by dividing the elements of the objective function row (elements NC_j) by the elements of the ε_t row and then we take the smallest positive ratio θ where:

$$\theta = \underset{j}{\text{Min}}\left[\frac{NC_j}{a_{tj}} > 0\right] = \frac{NC_s}{a_{ts}}$$

where $a_{tj} > 0$, then the pivot element is a_{ts}, and we exchange the variables x_s and ε_t, According to the direct neutrosophic Simplex algorithm instructions, see [7]. We repeat step (vi) until we get rid of all the artificial variables and obtain a normal base consisting of the basic variables.

After getting rid of the artificial variables, we return to working according to the direct neutrosophic simplex algorithm.

3. Examples

3.1 Example: All constraints are of type equals

Find the ideal solution for the following linear model:

$$Max\ Z = -12x_1 + [6,9]x_2 + 3x_3$$

Constraints:

$$8x_1 - x_2 + 4x_3 = [4,6]$$
$$6x_1 - 3x_2 + 3x_3 = [-12,-9]$$
$$x_1, x_2, x_3 \geq 0$$

Solution:

1. We convert the model to the standard form, multiply the second equation by (-1) and we obtain the following model:
   Find a rule solution for the following neutrosophic linear model:
   $$Max\ Z = -12x_1 + [6,9]x_2 + 3x_3$$
   Constraints:
   $$8x_1 - x_2 + 4x_3 = [4,6]$$
   $$-6x_1 + 3x_2 - 3x_3 = [9,12]$$
   $$x_1, x_2, x_3 \geq 0$$

2. We add the artificial variables and enter them into the objective function with a capital letter M preceded by a minus sign. Here we take $M = 15$.
   Find a rule solution for the following neutrosophic linear model:
   $$Max\ Z = -12x_1 + [6,9]x_2 + 3x_3 - 15\varepsilon_1 - 15\varepsilon_2$$
   Constraints:
   $$8x_1 - x_2 + 4x_3 = [4,6]$$
   $$-6x_1 + 3x_2 - 3x_3 = [9,12]$$
   $$x_1, x_2, x_3, \varepsilon_1, \varepsilon_2 \geq 0$$

   We arrange the previous information in Table 2.

Table 2. Artificial base.

Variables / Basic	x_1	x_2	x_3	ε_1	ε_2	b_i
ε_1	8	-1	4	1	0	[4,6]
ε_2	-6	3	-3	0	1	[9,12]
objective function	-12	[6,9]	3	-15	-15	$Z-0$

Since the rule is artificial, we study the constants $b_i$and find that the largest of them belong to the group [9,12] corresponding to the variable ε_2. Therefore, we divide the objective function row by the positive elements in the ε_2 row and calculate the index θ, and we find that:

$$\theta = \min_j \left[\frac{[6,9]}{3}\right] = \frac{[6,9]}{3}$$

Thus, the pivot element is (3) corresponding to x_2. Therefore, we replace x_2 with ε_2, then the variable x_2 becomes a base variable and ε_2 comes out of the base. We perform the necessary calculations and obtain Table 3.

Table 3. The first change table in the base.

Variables / Basic	x_1	x_2	x_3	ε_1	ε_2	b_i
ε_1	6	0	3	1	1/3	$[7, 10]$
x_2	-2	1	-1	0	1/3	$[3, 4]$
objective function	[0,6]	0	[9,12]	-15	$[-18, -17]$	$Z - [18, 36]$

The artificial variable ε_1 is still present in the base, so we perform another substitution, adopting the pivot line as the line opposite it. To determine the pivot column, we calculate the index θ, we find:

$$\theta = \min_j \left[\frac{[0,6]}{6}, \frac{[9,12]}{3}\right] \in \frac{[0,6]}{6}$$

Thus, the pivot element is (6) corresponding to x_1, so we move x_1 to the base instead of ε_1, so we get the following Table 4.

Table 4. The second change in the base.

Variables / Basic	x_1	x_2	x_3	ε_1	ε_2	b_i
x_1	1	0	$\frac{1}{2}$	$\frac{3}{6}$	$\frac{1}{18}$	$\left[\frac{7}{6}, \frac{10}{6}\right]$
x_2	0	1	0	$\frac{1}{3}$	$\frac{4}{9}$	$\left[\frac{16}{3}, \frac{22}{3}\right]$
objective function	0	0	9	$[-18, -15]$	$\left[-18, \frac{-50}{3}\right]$	$Z - [18,46]$

From the previous table, we note that the base variables x_1, x_2, and thus we have obtained an initial solution for the linear model, which gives us the following rule solution:

$$\left(x_1 \in \left[\frac{7}{6}, \frac{10}{6}\right], x_2 \in \left[\frac{16}{3}, \frac{22}{3}\right], x_3 = 0, \varepsilon_1 = 0, \varepsilon_2 = 0\right)$$

But it is clear from the table that this solution is not the ideal solution because, in the target function line, there is a positive value corresponding to the variable x_3. Therefore, we apply the direct simplex algorithm to improve the basic solution. We obtain the ideal solution from Table 5.

Table 5. The optimal solution for the model.

Variables / Basic	x_1	x_2	x_3	ε_1	ε_2	b_i
x_3	2	0	1	1	$\frac{1}{9}$	$\left[\frac{7}{3},\frac{10}{3}\right]$
x_2	0	1	0	$\frac{1}{3}$	$\frac{4}{9}$	$\left[\frac{16}{3},\frac{22}{3}\right]$
objective function	-18	0	0	$[-27,-24]$	$\left[-19,\frac{-53}{3}\right]$	$Z-[39,76]$

The optimal solution for the linear model:

$$x_1 = 0, x_2 \in \left[\frac{16}{3},\frac{22}{3}\right], x_3 \in \left[\frac{7}{3},\frac{10}{3}\right], \varepsilon_1 = 0, \varepsilon_2 = 0$$

In this solution, the goal function takes its greatest value, which is:

$$Z \in [39,76]$$

The solution can be verified by substituting the constraints and the objective function statement, we note that the values in the ideal solution of the previous linear model are neutrosophic values.

3.2 Example: Constraints are mixed

Find the ideal solution for the following linear model:

$$Min\, Z = -3x_1 + [8,10]x_2 + [0,6]x_3$$

Constraints:

$$x_1 - 2x_2 + x_3 \leq [3,7]$$
$$-4x_1 + x_2 + 2x_3 \geq [9,6]$$
$$2x_1 - x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

Converting this model to standard form the problem becomes:

Find the ideal solution for the following linear model:

$$Min\, Z = -3x_1 + [8,10]x_2 + [0,6]x_3 + 0y_1 + 0y_2$$

Constraints:

$$x_1 - 2x_2 + x_3 + y_1 = [3,7]$$
$$-4x_1 + x_2 + 2x_3 - y_2 = [9,6]$$
$$2x_1 - x_3 = 1$$
$$x_1, x_2, x_3, y_1, y_2 \geq 0$$

The variable y_1 in the first constraint is a basic variable, and since there are no other basic variables, we add artificial variables to the second and third restrictions and enter them into the objective function in sufficiently positive times because the model is a minimization model, and thus we obtain the following basic form:

Find the ideal solution for the following linear model:

$$Min\ Z = -3x_1 + [8,10]x_2 + [0,6]x_3 + 0y_1 + 0y_2 + 12\varepsilon_1 + 12\varepsilon_2$$

Constraints:

$$x_1 - 2x_2 + x_3 + y_1 = [3,7]$$
$$-4x_1 + x_2 + 2x_3 - y_2 + \varepsilon_1 = [9,6]$$
$$2x_1 - x_3 + \varepsilon_2 = 1$$
$$x_1, x_2, x_3, y_1, y_2, \varepsilon_1, \varepsilon_2 \geq 0$$

We follow the same steps mentioned in Example 1 to remove the artificial variables from the base and insert the basic variables. After obtaining the base solution, we use the direct simplex method to find the optimal solution.

Important Notes:

- If the row ε_i does not include a positive element and $b_t > 0$, this indicates a conflict of constraints and the problem is unsolvable.
- If we cannot find a positive ratio $\frac{NC_j}{a_{tj}}$, we calculate the largest negative ratio θ' where:

$$\theta' = Max\left[\frac{NC_j}{a_{tj}} < 0\right] = \frac{NC_s}{a_{ts}}$$

where $a_{tj} > 0$, so a_{ts} is the pivot element and it is a positive element.

4. Conclusions

In this study, we presented one of the important methods for finding the optimal solution for neutrosophic linear models, which is the synthetic simplex method that we resort to when we are unable to find a rule solution. We found that the optimal solution that we obtained is neutrosophic values, indeterminate values, perfectly defined, belonging to a field that represents its minimum. The smallest value that the objective function can take and the highest alone represent the highest value of the objective function, which is proportional to the conditions surrounding the system's operating environment, which can be represented by the linear model.

References

1. Jdid, M., & Smarandache, F. (2023). Neutrosophic Static Model without Deficit and Variable Prices. Neutrosophic Sets and Systems, 60: 124-132.
2. Smarandache, F., & Jdid, M. (2023). An Overview of Neutrosophic and Plithogenic Theories and Applications. Prospects for Applied Mathematics and Data Analysis, 2(1), 19-26. https://doi.org/10.54216/PAMDA.020102.
3. Smarandache, F., & Jdid, M. (2023). An Overview of Neutrosophic and Plithogenic Theories and Applications.
4. Jdid, M., & Smarandache, F. (2023). An Efficient Optimal Solution Method for Neutrosophic Transport Models: Analysis, Improvements, and Examples. Neutrosophic Systems With Applications, 12, 56–67. https://doi.org/10.61356/j.nswa.2023.111.
5. Salamah, A., Alaswad, M. F., & Dallah, R. (2023). A Study of a Neutrosophic Differential Equation by Using the One-Dimensional Geometric AH-Isometry. Journal Prospects for Applied Mathematics and Data Analysis, 1(1).
6. Al-Odhari, A. (2023). Some Algebraic Structure of Neutrosophic Matrices. Prospects for Applied Mathematics and Data Analysis (PAMDA), 1(02), 37-44. https://doi.org/10.54216/PAMDA.010204.
7. Jdid, M., Salama, A. A., & Khalid, H. E. (2022). Neutrosophic Handling of the Simplex Direct Algorithm to Define the Optimal Solution in Linear Programming. International Journal of Neutrosophic Science (IJNS), 18(1). https://doi.org/10.54216/IJNS.180104.
8. Jdid, M., & Khalid, H. E. (2022). Mysterious Neutrosophic linear models. International Journal of Neutrosophic Science, 18(2), 243-253. https://doi.org/10.54216/IJNS.180207.

Multi-Criteria Group Decision-Making with ELECTRE-III Method for Selection of Female Spouse in Pythagorean Neutrosophic Environment

Rajesh Kumar Saini,[1] **Ashik Ahirwar,**[2] **Florentin Smarandache,**[3] **Mukesh Kushwaha**[4]

[1,2,4] Department of Mathematical Sciences and Computer Applications, Bundelkhand University, Jhansi, Uttar Pradesh, India. Emails: prof.rksaini@bujhansi.ac.in, bu.gashik@bujhansi.ac.in, bu.gmukesh@bujhansi.ac.in.

[3] University of New Mexico, Mathematics, Physics and Natural Sciences Division 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu.

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Abstract

Selecting a female partner in the 21st century is more challenging as compared to the 20th century. The globalized world, changing societal norms, and technology have increased the pool of potential partners, making it harder to find a compatible match. The prevalence of social media and online dating has also changed how people interact, leading to new challenges in establishing genuine connections. This paper presents a novel Multiple-Criteria Group Decision Making (MCGDM) model for selecting a female spouse using the ELECTRE-III method in a Pythagorean neutrosophic environment. The proposed model takes into account multiple criteria and addresses the challenges of handling conflicting and uncertain information. The Pythagorean neutrosophic set theory (PNST) is used to handle the vagueness and uncertainty of the decision-making process. The proposed model is validated using a case study of selecting a female spouse based on education, family background, beauty, and skill capability. The case study results demonstrate the proposed model's effectiveness in addressing the complexity and uncertainty of the decision-making process and providing a useful tool for decision-makers in selecting a spouse.

Keywords: Female Spouse, Multi-Criteria Group Decision-Making, Pythagorean Neutrosophic Sets, ELECTRE-III Method.

1 | Introduction

The process of selecting a spouse [1, 2] is a complex and challenging decision that individual's face in their lives. It is not only a personal decision but also has social, cultural, and economic implications [3]. Therefore, the importance of making an informed decision is paramount. With advancements in technology and the availability of information, individuals have access to a wide range of options for selecting a spouse [4, 5]. However, the abundance of choices often leads to confusion and indecision, making it difficult to make the right choice. To assist individuals in making the best decision, Multi-Criteria Decision Making [MCDM] methods [6, 7] have been used extensively in recent years. These methods help individuals to evaluate the alternatives based on multiple criteria and select the best one. The existing MCDM models, however, do not take into account the uncertainties present in the decision-making process. These uncertainties arise due to

the imprecise nature of the criteria, the preferences of the decision-maker, and the available information. However, to choose the appropriate alternative is very difficult because of vague information in some cases.

To overwhelmed such circumstances, Zadeh established the notion of fuzzy sets [FSs] [8] to solve those problems which comprise uncertainty and vagueness. As a result of the observation that some situations cannot be resolved by fuzzy sets, Turksen [9-11] proposed the concept of interval-valued fuzzy sets [IVFS]. In some circumstances, we must consider membership unbiased as well as non-membership values for the appropriate representation of an object under ambiguous and indefinite circumstances that neither FSs nor IVFSs could manage. Atanassov [12] proposed the idea of intuitionistic fuzzy sets [IFSs] to get around these issues. Atanassov's theory only deals with incomplete data when both membership and non-membership values are taken into account, but intuitionistic fuzzy set theory is unable to cope with contradictory and imprecise data. To deal with such incompatible and imprecise data Smarandache [13] expanded on Atanassov's [12] work with IFSs and introduced Neutrosophic sets [NSs] as a potent tool for dealing with ambiguous, insufficient, and inconsistent information that is encountered in real-world issues. NSs are comparable to FSs and IFSs. So, using NSs directly for ELECTRE-III is not always easy. Wang et al. [14] in created a subclass of NSs called as single-valued Neutrosophic sets [SVNSs] to be used with the NSs. The author of suggested a geometric explanation based on NSs. In the Neutrosophic context, Gulfam et al. [15] created new methodologies and a new distance formula for SVNSs. the idea of a single-valued Neutrosophic soft expert set that was first out in by fusing soft expert sets with SVNSs. To address this gap, this research paper proposes a new MCGDM model for the selection of a female spouse using the ELECTRE-III method in a Pythagorean neutrosophic environment. Pythagorean neutrosophic sets [PNSs] are an extension of NSs, which provide a more comprehensive and flexible framework for dealing with uncertainties. The proposed model incorporates the concept of PNSs to handle the uncertainties present in the decision-making process.

The proposed model is designed to consider the preferences of the decision-maker, the imprecise nature of the criteria, and the uncertainties present in the decision-making process. It is expected to provide a more comprehensive and accurate assessment of the alternatives, thereby assisting the decision-maker in making the best possible decision. To demonstrate the effectiveness of the proposed model, it is applied to a real-life case study. The case study involves selecting a female spouse based on multiple criteria, such as education, income, family background, and personal qualities. The results of the case study are analysed and compared with the existing MCGDM models to show the superiority of the proposed model.

The introduction should briefly place the study in a broad context and highlight why it is important. It should define the purpose of the work and its significance. The current state of the research field should be reviewed carefully and key publications cited. Please highlight controversial and diverging hypotheses when necessary. Finally, briefly mention the main aim of the work and highlight the principal conclusions. As far as possible, please keep the introduction comprehensible to scientists outside your particular field of research. References should be numbered in order of appearance and indicated by a numeral or numerals in square brackets, e.g., [1] or [2,3], or [4–6]. See the end of the document for further details on references.

1.1 | Motivation of the Research Work

The process of selecting a spouse [16, 17] is a complex and important decision that has a significant impact on an individual's personal and social life. The existing MCDM models, however, do not take into account the uncertainties present in the decision-making process. These uncertainties arise due to the imprecise nature of the criteria, the preferences of the decision-maker, and the available information. Therefore, there is a need for a new MCGDM model that can address these uncertainties and assist individuals in making better-informed decisions when selecting a spouse.

1.2 | Objective of the Investigation

The objective of this research paper is to propose a new MCGDM model for the selection of a female spouse using the ELECTRE-III method in a Pythagorean neutrosophic environment.(see [18-22]). The proposed

model incorporates the concept of PNSs to handle the uncertainties present in the decision-making process. The ELECTRE-III method is used to rank the alternatives based on the preferences of the DM. The proposed model is expected to provide a more comprehensive and accurate assessment of the alternatives, thereby assisting the decision-maker in making the best possible decision.

1.3 | Related Work of the Study

Several studies have been conducted in the field of MCDM to assist individuals in selecting a spouse. The Analytic Hierarchy Process [AHP] and the Technique for Order of Preference by Similarity to Ideal Solution [TOPSIS] ([11], [23], [24]) are two widely used methods for selecting a spouse. However, these methods do not take into account the uncertainties present in the decision-making process.

PNSs are an extension of NS, which provide a more comprehensive and flexible framework for dealing with uncertainties. Several studies have been conducted on the application of PNSs in MCDM. However, the use of PNSs in the selection of a spouse has not been explored.

The ELECTRE-III method is a widely used MCDM method that takes into account the preferences of the decision-maker and the criteria weights. Several studies have been conducted on the application of the ELECTRE-III method in various fields. However, its application in the selection of a spouse using PNSs has not been explored.

Therefore, this research paper proposes a new MCGDM model that combines the use of PNSs and the ELECTRE-III method to address the uncertainties present in the decision-making process when selecting a spouse. The proposed model is expected to provide a more comprehensive and accurate assessment of the alternatives, thereby assisting the DM in making the best possible decision.

2 | Preliminaries

In this section, some essentials introductory concept of NSs, SVNSs, PNSs are briefly presented which will enable the conversation in following sections.

Definition 2.1. [25]: Let Z be a non-empty set. A NS A on Z, comprising $\alpha_A(z)$ as membership, $\beta_A(z)$ as indeterminacy and $\gamma_A(z)$ as non-membership functions, defined as

$$A = \{\langle z, \alpha_A(z), \beta_A(z), \gamma_A(z)\rangle : z \in Z\},$$

where $\alpha_A(z), \beta_A(z), \gamma_A(z) \in \left]^{-}0, 1^{+}\right[$ such that $^{-}0 \le \alpha_A(z) + \beta_A(z) + \gamma_A(z) \le 3^{+}$ for all $z \in Z$.

Definition 2.2. [15]: A SVNS A on a non-empty universal set Z is defined as

$$A = \{\langle z, \alpha_A(z), \beta_A(z), \gamma_A(z)\rangle : z \in Z\},$$

where $\alpha_A(z), \beta_A(z), \gamma_A(z) \in [0,1]$ such that $0 \le \alpha_A(z) + \beta_A(z) + \gamma_A(z) \le 3$ for all $z \in Z$. And $\alpha_A(z)$ as membership, $\beta_A(z)$ as indeterminacy and $\gamma_A(z)$ as non-membership functions.

Definition 2.3. [26]: Let Z be a discourse. A PNS A on Z is defined as

$$A = \{\langle z, \alpha_A(z), \beta_A(z), \gamma_A(z)\rangle : z \in Z\}$$

where $\alpha_A(z), \beta_A(z), \gamma_A(z) \in [0,1]$ such that $0 \le (\alpha_A(z))^2 + (\beta_A(z))^2 + (\gamma_A(z))^2 \le 2$ for all $z \in Z$. $\alpha_A(z)$ as membership, $\beta_A(z)$ as indeterminacy and $\gamma_A(z)$ as non-membership. Here truth $(\alpha_A(z))$ and falsity $(\gamma_A(z))$ are dependent components and indeterminacy $(\beta_A(z))$ is an independent component. The triplet

$A=\left(\alpha_A(z),\beta_A(z),\gamma_A(z)\right)$ is called a Pythagorean neutrosophic number [PNN]. For suitability, we denote a PNN $A=\left(\alpha_A(z),\beta_A(z),\gamma_A(z)\right)$ as $A=\left(\alpha_A,\beta_A,\gamma_A\right)$, throughout in this article.

Definition 2.4. ([20],[21],[27]): Let three PNN $A=\left(\alpha_A,\beta_A,\gamma_A\right)$, $A_1=\left(\alpha_{A_1},\beta_{A_1},\gamma_{A_1}\right)$ and $A_2=\left(\alpha_{A_2},\beta_{A_2},\gamma_{A_2}\right)$ then the elementary mathematical operations over these PNNs are defined as:

(i). Complement: $A^c=\left(\gamma_A,\beta_A,\alpha_A\right)$

(ii). Union: $A_1\bigcup A_2=\left(\max\left\{\alpha_{A_1},\alpha_{A_2}\right\},\min\left\{\beta_{A_1},\beta_{A_2}\right\},\min\left\{\gamma_{A_1},\gamma_{A_2}\right\}\right)$

(iii). Intersection: $A_1\bigcap A_2=\left(\min\left\{\alpha_{A_1},\alpha_{A_2}\right\},\max\left\{\beta_{A_1},\beta_{A_2}\right\},\max\left\{\gamma_{A_1},\gamma_{A_2}\right\}\right)$

(iv). Addition: $A_1\oplus A_2=\left(\sqrt{\alpha_{A_1}^2+\alpha_{A_2}^2-\alpha_{A_1}^2\cdot\alpha_{A_2}^2},\beta_{A_1}\cdot\beta_{A_2},\gamma_{A_1}\cdot\gamma_{A_2}\right)$ (1)

(v). Multiplication: $A_1\otimes A_2=\left(\alpha_{A_1}\cdot\alpha_{A_2},\sqrt{\beta_{A_1}^2+\beta_{A_2}^2-\beta_{A_1}^2\cdot\beta_{A_2}^2},\sqrt{\gamma_{A_1}^2+\gamma_{A_2}^2-\gamma_{A_1}^2\cdot\gamma_{A_2}^2}\right)$ (2)

(vi). Scalar Multiplication: $r.A=\left(\sqrt{1-\left(1-\alpha_A^2\right)^r},\left(\beta_A\right)^r,\left(\gamma_A\right)^r\right);\ r>0.$ (3)

(vii). Exponentiation: $A^r=\left(\left(\alpha_A\right)^r,\sqrt{1-\left(1-\beta_A^2\right)^r},\sqrt{1-\left(1-\gamma_A^2\right)^r}\right);r>0.$ (4)

Definition 2.5. [De-Neutrosophication]

(i). Score Function: $s\left(A\right)=\alpha_A^2-\beta_A^2-\gamma_A^2$, provided $-1\le s(A)\le 1$ (5)

(ii). Accuracy Function: $a\left(A\right)=\alpha_A^2+\beta_A^2+\gamma_A^2$, provided $0\le a(A)\le 2$ (6)

(iii). Normalized Euclidean Distance:

$$d\left(A_1,A_2\right)=\sqrt{\left(\alpha_{A_1}{}^2-\alpha_{A_2}{}^2\right)^2+\left(\beta_{A_1}{}^2-\beta_{A_2}{}^2\right)^2+\left(\gamma_{A_1}{}^2-\gamma_{A_2}{}^2\right)^2} \quad (7)$$

Definition 2.6. [Comparison]

(i). If $s(A_1)>s(A_2)$, then $A_1\succ A_2$ (A_1 is superior to A_2)

(ii). If $s(A_1)=s(A_2)$, then

a. If $a(A_1)>a(A_2)$, then $A_1\succ A_2$ (A_1 is superior to A_2)

b. If $a(A_1)=a(A_2)$, then $A_1\ \square\ A_2$ (A_1 is equivalent to A_2)

Definition 2.7. [Aggregation]: Let some PNN are $A_i=\left(\alpha_{A_i},\beta_{A_i},\gamma_{A_i}\right)$ and their respective weights are $w=\left(w_1,w_2,w_3,...,w_n\right)$ such that $\sum_{i=1}^n w_i=1$. then

(i). Pythagorean Neutrosophic Weighted Aggregation [PNWA] Operator

$$\text{PNWA}_w\left(A_1,A_2,A_3,...,A_n\right)=w_1A_1\oplus w_2A_2\oplus w_3A_3\oplus...\oplus w_nA_n$$

$$=\left(\sqrt{1-\prod_{i=1}^n\left(1-\alpha_{A_i}^2\right)^{w_i}},\prod_{i=1}^n\left(\beta_{A_i}\right)^{w_i},\prod_{i=1}^n\left(\gamma_{A_i}\right)^{w_i}\right) \quad (8)$$

(ii). Pythagorean Neutrosophic Weighted Geometric [PNWG] Operator

$$\mathrm{PNWG}_w\left(A_1, A_2, A_3, ..., A_n\right) = w_1A_1 \otimes w_2A_2 \otimes w_3A_3 \otimes ... \otimes w_nA_n$$

$$= \left(\prod_{i=1}^{n} \left(\alpha_{A_i}\right)^{w_i}, \sqrt{1 - \prod_{i=1}^{n} \left(1 - \beta_{A_i}^2\right)^{w_i}}, \sqrt{1 - \prod_{i=1}^{n} \left(1 - \gamma_{A_i}^2\right)^{w_i}} \right) \tag{9}$$

3 | Two-Phase Pythagorean Neutrosophic ELECTRE III Method (Algorithm)

In this section, the PNSs and the conventional ELECTRE III technique [7] are combined to form the two-phase Pythagorean neutrosophic ELECTRE-III (PN-ELECTRE-III) group decision support system. In Pythagorean neutrosophic environment, let for a multi-criteria group decision making [MCGDM] problem, $Y = \left\{y_1, y_2, y_3, ..., y_a, ..., y_f\right\}$ be a set of accessible f alternatives and $C = \left\{c_1, c_2, c_3, ..., c_b, ..., c_g\right\}$ be a set of g criteria assigning to individually alternative. A group $V = \left\{v_1, v_2, v_3, ...v_t, ...v_h\right\}$ of h decision-maker [DM] allocates the feasibility data of alternative $y_a \in Y$ with respect to criterion $c_b \in C$ as; $c_b(y_a)$. As all the criteria can have their own and unequal importance considering objective of the MCGDM problem thus criteria weight vector will be denoted by $w = \left\{w_1, w_2, w_3, ..., w_b, ..., w_g\right\}$ such that $\sum_{b=1}^{g} w_b = 1.$ Likewise, the importance expert weight vector will be $\ell = \left\{\ell_1, \ell_2, \ell_3, ..., \ell_t, ..., \ell_h\right\}$ such that $\sum_{t=1}^{h} \ell_t = 1.$ Let the subscript set of criterions i.e. $\tau = \left\{1, 2, 3, ..., g\right\}.$

3.1 | Phase I: (Pythagorean Neutrosophic Evaluation Phase)

Step-1. For the evaluation of, feasibility ratings of alternatives, importance weights of criteria, important weights of experts, and establishment of threshold functions, we first build certain linguistic variables/terms in the form of PNN.

Step-2. Methodically measuring each alternative y_a w.r.t. each criterion c_b. Expert v_t delivers his/her assessment data in the form of Pythagorean neutrosophic decision matrix [PNDM] $P^{(t)} = [P_{ab}^{(t)}]_{f \times g}$, as in table 1, where $P_{ab}^{(t)} = \left(\alpha_{P_a}^{(t)}(c_b), \beta_{P_a}^{(t)}(c_b), \gamma_{P_a}^{(t)}(c_b)\right)$ is the PNN assigned by the DM v_t.

Table 1. PNDM by DM v_t

$P^{(t)}$	c_1	c_2	$\cdots$	c_g
y_1	$\left(\alpha_{P_1}^{(t)}(c_1), \beta_{P_1}^{(t)}(c_1), \gamma_{P_1}^{(t)}(c_1)\right)$	$\left(\alpha_{P_1}^{(t)}(c_2), \beta_{P_1}^{(t)}(c_2), \gamma_{P_1}^{(t)}(c_2)\right)$	$\cdots$	$\left(\alpha_{P_1}^{(t)}(c_g), \beta_{P_1}^{(t)}(c_g), \gamma_{P_1}^{(t)}(c_g)\right)$
y_2	$\left(\alpha_{P_2}^{(t)}(c_1), \beta_{P_2}^{(t)}(c_1), \gamma_{P_2}^{(t)}(c_1)\right)$	$\left(\alpha_{P_2}^{(t)}(c_2), \beta_{P_2}^{(t)}(c_2), \gamma_{P_2}^{(t)}(c_2)\right)$	$\cdots$	$\left(\alpha_{P_2}^{(t)}(c_g), \beta_{P_2}^{(t)}(c_g), \gamma_{P_2}^{(t)}(c_g)\right)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
y_f	$\left(\alpha_{P_f}^{(t)}(c_1), \beta_{P_f}^{(t)}(c_1), \gamma_{P_f}^{(t)}(c_1)\right)$	$\left(\alpha_{P_f}^{(t)}(c_2), \beta_{P_f}^{(t)}(c_2), \gamma_{P_f}^{(t)}(c_2)\right)$	$\cdots$	$\left(\alpha_{P_f}^{(t)}(c_g), \beta_{P_f}^{(t)}(c_g), \gamma_{P_f}^{(t)}(c_g)\right)$

Step-3. Weight of t^{th} expert can be strong minded by the subsequent equation

$$\ell_t = \alpha_t + \gamma_t \left(\frac{\alpha_t}{\alpha_t + \beta_t} \right) \Bigg/ \sum_{t=1}^{h} \left(\alpha_t + \gamma_t \left(\frac{\alpha_t}{\alpha_t + \beta_t} \right) \right) \tag{10}$$

where the ℓ_t satisfy the normalized condition $\sum_{t=1}^{h} \ell_t = 1.$

Step-4. The separate opinions of DM or experts essential to be combined into a collective opinion to construct aggregated Pythagorean neutrosophic decision matrix (PNDM) $P' = \left(P'_{ab}\right)_{f\times g}$ by using PNWA operator, see equation**Error! Reference source not found.**,

$$P'_{ab} = PNWA_{\ell}\left(P_{ab}^{(1)}, P_{ab}^{(2)}, P_{ab}^{(3)}, ..., P_{ab}^{(h)}\right) = \ell_1 P_{ab}^{(1)} \oplus \ell_2 P_{ab}^{(2)} \oplus \ell_3 P_{ab}^{(3)} \oplus ... \oplus \ell_h P_{ab}^{(h)}$$
$$= \left(\sqrt{1-\prod_{t=1}^{h}\left(1-\left(\alpha_{ab}^{(t)}\right)^2\right)^{\ell_t}}, \prod_{t=1}^{h}\left(\beta_{ab}^{(t)}\right)^{\ell_t}, \prod_{t=1}^{h}\left(\gamma_{ab}^{(t)}\right)^{\ell_t}\right) \tag{11}$$

where $P'_{ab} = \left(\alpha_{P'_a}(c_b), \beta_{P'_a}(c_b), \gamma_{P'_a}(c_b)\right)$ shown in table 2 as follows:

Table 2. Aggregated Pythagorean Neutrosophic Decision Matrix $\left(\text{A - PNDM}\right)$

P'	c_1	c_2	$\cdots$	c_g
y_1	$\left(\alpha_{P'_1}(c_1), \beta_{P'_1}(c_1), \gamma_{P'_1}(c_1)\right)$	$\left(\alpha_{P'_1}(c_2), \beta_{P'_1}(c_2), \gamma_{P'_1}(c_2)\right)$	$\cdots$	$\left(\alpha_{P'_1}(c_g), \beta_{P'_1}(c_g), \gamma_{P'_1}(c_g)\right)$
y_2	$\left(\alpha_{P'_2}(c_1), \beta_{P'_2}(c_1), \gamma_{P'_2}(c_1)\right)$	$\left(\alpha_{P'_2}(c_2), \beta_{P'_2}(c_2), \gamma_{P'_2}(c_2)\right)$	$\cdots$	$\left(\alpha_{P'_2}(c_g), \beta_{P'_2}(c_g), \gamma_{P'_2}(c_g)\right)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
y_f	$\left(\alpha_{P'_f}(c_1), \beta_{P'_f}(c_1), \gamma_{P'_f}(c_1)\right)$	$\left(\alpha_{P'_f}(c_2), \beta_{P'_f}(c_2), \gamma_{P'_f}(c_2)\right)$	$\cdots$	$\left(\alpha_{P'_f}(c_g), \beta_{P'_f}(c_g), \gamma_{P'_f}(c_g)\right)$

Step-5. Let C_B and C_N represent the corresponding collections of criteria that are of the benefit-type and cost-types. The combined PNDM can be transformed into the normalized aggregated PNDM, $P = \left(P_{ab}\right)_{f\times g}$ which displays the evaluation information of each alternative w.r.t. each benefit or cost criterion, in standard form, for supplementary calculations, table 3 establishes how the matrix P is erected. You can define the PNN for P_{ab} as trails:

$$P_{ab} = \left(\alpha_{P_a}(c_b), \beta_{P_a}(c_b), \gamma_{P_a}(c_b)\right) = \begin{cases} P'_{ab} = \left(\alpha_{P'_a}(c_b), \beta_{P'_a}(c_b), \gamma_{P'_a}(c_b)\right), & \text{if } c_b \in Y_B \\ \left(P'_{ab}\right)^C = \left(\gamma_{P'_a}(c_b), \beta_{P'_a}(c_b), \alpha_{P'_a}(c_b)\right), & \text{if } c_b \in Y_N \end{cases} \tag{12}$$

Table 3. Normalized Aggregated Pythagorean Neutrosophic Decision Matrix $\left(\text{NA-PNDM}\right)$

P	c_1	c_2	$\cdots$	c_g
y_1	$\left(\alpha_{P_1}(c_1), \beta_{P_1}(c_1), \gamma_{P_1}(c_1)\right)$	$\left(\alpha_{P_1}(c_2), \beta_{P_1}(c_2), \gamma_{P_1}(c_2)\right)$	$\cdots$	$\left(\alpha_{P_1}(c_g), \beta_{P_1}(c_g), \gamma_{P_1}(c_g)\right)$
y_2	$\left(\alpha_{P_2}(c_1), \beta_{P_2}(c_1), \gamma_{P_2}(c_1)\right)$	$\left(\alpha_{P_2}(c_2), \beta_{P_2}(c_2), \gamma_{P_2}(c_2)\right)$	$\cdots$	$\left(\alpha_{P_2}(c_g), \beta_{P_2}(c_g), \gamma_{P_2}(c_g)\right)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
y_f	$\left(\alpha_{P_f}(c_1), \beta_{P_f}(c_1), \gamma_{P_f}(c_1)\right)$	$\left(\alpha_{P_f}(c_2), \beta_{P_f}(c_2), \gamma_{P_f}(c_2)\right)$	$\cdots$	$\left(\alpha_{P_f}(c_g), \beta_{P_f}(c_g), \gamma_{P_f}(c_g)\right)$

Step-6. Not all criteria might have likewise important. Let $w_b^{(t)} = \left(\alpha_w^{(t)}(c_b), \beta_w^{(t)}(c_b), \gamma_w^{(t)}(c_b)\right)$ denote the PNN that the expert v_t assigned for the comparative weight of criterion c_b. By aggregating the opinions of experts on c_b, determine the PN weight $w_b = \left(\alpha_w(c_b), \beta_w(c_b), \gamma_w(c_b)\right)$ as follows:

$$w_b = PNWA_{\ell}\left(w_b^{(1)}, w_b^{(2)}, w_b^{(3)}, ..., w_b^{(h)}\right) = \ell_1 w_b^{(1)} \oplus \ell_2 w_b^{(2)} \oplus \ell_3 w_b^{(3)} \oplus ... \oplus \ell_h w_b^{(h)}$$
$$= \left(\sqrt{1-\prod_{t=1}^{h}\left(1-\left(\alpha_b^{(t)}\right)^2\right)^{\ell_t}}, \prod_{t=1}^{h}\left(\beta_b^{(t)}\right)^{\ell_t}, \prod_{t=1}^{h}\left(\gamma_b^{(t)}\right)^{\ell_t}\right) \tag{13}$$

Thus, the following criteria weight row matrix can be obtained.

$$w_b = \begin{pmatrix} \left(\alpha_w(c_1), \beta_w(c_1), \gamma_w(c_1)\right) \\ \left(\alpha_w(c_2), \beta_w(c_2), \gamma_w(c_2)\right) \\ \cdots \\ \cdots \\ \left(\alpha_w(c_g), \beta_w(c_g), \gamma_w(c_g)\right) \end{pmatrix}^T \tag{14}$$

Step-7. The weighted normalized aggregated PNDM, $P^* = \left(P^*_{ab}\right)_{f\times g}$ is formed as exposed in table 4 in the order of integrating the information from the normalized aggregated PNDM and the criteria weight matrix. $P^*_{ab} = \left(\alpha_{P^*_a}(c_b), \beta_{P^*_a}(c_b), \gamma_{P^*_a}(c_b)\right)$ may be generated by using the multiplication operator, which is specified in equation **Error! Reference source not found.**, i.e.

$$P^*_{ab} = \left(\alpha_{P_a}(c_b), \beta_{P_a}(c_b), \gamma_{P_a}(c_b)\right) \otimes \left(\alpha_w(c_b), \beta_w(c_b), \gamma_w(c_b)\right)$$

$$= \left(\alpha_{P_a}(c_b)\cdot\alpha_w(c_b), \sqrt{\beta^2_{P_a}(c_b) + \beta^2_w(c_b) - \beta^2_{P_a}(c_b)\cdot\beta^2_w(c_b)}, \sqrt{\gamma^2_{P_a}(c_b) + \gamma^2_w(c_b) - \gamma^2_{P_a}(c_b)\cdot\gamma^2_w(c_b)}\right) \tag{15}$$

Table 4. Weighted Normalized Aggregated PNDM (WNA-PNDM)

P^*	c_1	c_2	$\cdots$	c_g
y_1	$\left(\alpha_{P^*_1}(c_1), \beta_{P^*_1}(c_1), \gamma_{P^*_1}(c_1)\right)$	$\left(\alpha_{P^*_1}(c_2), \beta_{P^*_1}(c_2), \gamma_{P^*_1}(c_2)\right)$	$\cdots$	$\left(\alpha_{P^*_1}(c_g), \beta_{P^*_1}(c_g), \gamma_{P^*_1}(c_g)\right)$
y_2	$\left(\alpha_{P^*_2}(c_1), \beta_{P^*_2}(c_1), \gamma_{P^*_2}(c_1)\right)$	$\left(\alpha_{P^*_2}(c_2), \beta_{P^*_2}(c_2), \gamma_{P^*_2}(c_2)\right)$	$\cdots$	$\left(\alpha_{P^*_2}(c_g), \beta_{P^*_2}(c_g), \gamma_{P^*_2}(c_g)\right)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
y_f	$\left(\alpha_{P^*_f}(c_1), \beta_{P^*_f}(c_1), \gamma_{P^*_f}(c_1)\right)$	$\left(\alpha_{P^*_f}(c_2), \beta_{P^*_f}(c_2), \gamma_{P^*_f}(c_2)\right)$	$\cdots$	$\left(\alpha_{P^*_f}(c_g), \beta_{P^*_f}(c_g), \gamma_{P^*_f}(c_g)\right)$

3.2 | Phase II: (Pythagorean Neutrosophic Ranking Phase (PN-ELECTRE III))

The first phase of the Pythagorean decision support system gathers the PN calculation information for individually alternative, and the second step uses the PFELECTRE III approach, which uses the combined evaluations to yield the whole ranking of alternatives.

3.2.1 | Module I: Evolving Outranking Relations

The first phase of the Pythagorean decision support system gathers the PN calculation information for individually alternative, and the second step uses the PFELECTRE III approach, which uses the combined evaluations to yield the whole ranking of alternatives.

Step-8. In order to reveal concordance and discordance indices, gauge the degree of credibility, and rate the alternatives, the ELECTRE-III model's ([28, 29]) evaluation approach includes developing an indifference threshold function, preference threshold function, and veto threshold function. Let $q(c_b)$ be the indifference threshold function and $p(c_b)$ be the preference thresholds function for corresponding criteria c_b. Let if for any two given alternatives $y_1, y_2 \in Y$, $c(y_1) \geq c(y_2)$, then,

$$\begin{aligned} &c(y_1) \succ c(y_2) + p(c(y_2)) \Leftrightarrow y_1 P y_2 \\ &c(y_2) + q(c(y_2)) \prec c(y_1) \prec c(y_2) + p(c(y_2)) \Leftrightarrow y_1 Q y_2 \\ &c(y_2) \prec c(y_1) \prec c(y_2) + q(c(y_2)) \Leftrightarrow y_1 I y_2 \end{aligned} \tag{16}$$

where $c(y)$ is the criterion score value of the alternative y, and P signifies strong preference, Q weak preference, I indifference.

Step-9. For each pair of alternatives, the comprehensive concordance index $\varsigma(y_1, y_2)$ is calculated,

$$\varsigma(y_1, y_2) = \sum_{b=1}^{g} w_b \varsigma_b(y_1, y_2), \tag{17}$$

where represent the weight of b^{th} criteria and $\varsigma_b(y_1, y_2)$ represent the partial concordance indices over the criteria c_b

$$\varsigma_b(y_1, y_2) = \begin{cases} 0, & \text{if} \quad c_b(y_2) - c_b(y_1) \succ p(c_b) \\ 1, & \text{if} \quad c_b(y_2) - c_b(y_1) \le q(c_b) \\ \dfrac{p(c_b) - \left(c_b(y_2) - c_b(y_1)\right)}{p(c_b) - q(c_b)}, & otherwise \end{cases} \tag{18}$$

Thus $0 \le \varsigma_b(y_1, y_2) \le 1$.

Table 5 shows partial concordance indices $\varsigma_b = \left[\varsigma_b(y_i, y_j)\right]_{f \times f} \left(i, j = 1,2,3,..., f, i \ne j\right)$ and $\left(b = 1,2,3,..., g\right)$ over each criterion $c_b \in C$ can be obtained using equation.

Table 5. Partial Concordance indices over each criteria

ς_b	y_1	y_2	$\cdots$	y_{f-1}	y_f
y_1	–	$\varsigma_b(y_1, y_2)$	$\cdots$	$\varsigma_b(y_1, y_{f-1})$	$\varsigma_b(y_1, y_f)$
y_2	$\varsigma_b(y_2, y_1)$	–	$\cdots$	$\varsigma_b(y_2, y_{f-1})$	$\varsigma_b(y_2, y_f)$
$\vdots$	$\vdots$	$\vdots$	–	$\vdots$	$\vdots$
y_{f-1}	$\varsigma_b(y_{f-1}, y_1)$	$\varsigma_b(y_{f-1}, y_2)$	$\cdots$	–	$\varsigma_b(y_{f-1}, y_f)$
y_f	$\varsigma_b(y_f, y_1)$	$\varsigma_b(y_f, y_2)$	$\cdots$	$\varsigma_b(y_f, y_{f-1})$	–

And after that for each pair of alternatives, table 6 shows the comprehensive concordance index $\varsigma = \left\lfloor \varsigma_{ij} \right\rfloor_{f \times f}$ $\left(i, j = 1,2,3,..., f, i \ne j\right)$ is calculated using equation.

Table 6. Comprehensive Concordance index

ς	y_1	y_2	$\cdots$	y_{f-1}	y_f
y_1	–	ς_{12}	$\cdots$	$\varsigma_{1(f-1)}$	ς_{1f}
y_2	ς_{21}	–	$\cdots$	$\varsigma_{2(f-1)}$	ς_{2f}
$\vdots$	$\vdots$	$\vdots$	–	$\vdots$	$\vdots$
y_{f-1}	$\varsigma_{(f-1)1}$	$\varsigma_{(f-1)2}$	$\cdots$	–	$\varsigma_{(f-1)f}$
y_f	ς_{f1}	ς_{f2}	$\cdots$	$\varsigma_{f(f-1)}$	–

Step-10. Calculating the Discordance Index of the Assertion $y_1 S y_2$

For each criterion, the discordance index $\xi_b(y_1, y_2)$ is calculated.

$$\xi_b(y_1, y_2) = \begin{cases} 0, & \text{if} \quad c_b(y_2) - c_b(y_1) \le p(c_b) \\ 1, & \text{if} \quad c_b(y_2) - c_b(y_1) \succ v(c_b) \\ \dfrac{\left(c_b(y_2) - c_b(y_1)\right) - p(c_b)}{v(c_b) - p(c_b)}, & otherwise \end{cases} \tag{19}$$

Thus $0 \le \xi_b(y_1, y_2) \le 1$.

For each criterion, table 7 Shows the discordance index $\xi_b = \left[\xi_b(y_i, y_j)\right]_{f \times f} \left(i, j = 1,2,3,..., f, i \ne j\right)$ and $\left(b = 1,2,3,..., g\right)$ is calculated using equation.

Table 7. Discordance indices over each criteria

ξ_b	y_1	y_2	$\cdots$	y_{f-1}	y_f
y_1	–	$\xi_b(y_1, y_2)$	$\cdots$	$\xi_b(y_1, y_{f-1})$	$\xi_b(y_1, y_f)$
y_2	$\xi_b(y_2, y_1)$	–	$\cdots$	$\xi_b(y_2, y_{f-1})$	$\xi_b(y_2, y_f)$
$\vdots$	$\vdots$	$\vdots$	–	$\vdots$	$\vdots$
y_{f-1}	$\xi_b(y_{f-1}, y_1)$	$\xi_b(y_{f-1}, y_2)$	$\cdots$	–	$\xi_b(y_{f-1}, y_f)$
y_f	$\xi_b(y_f, y_1)$	$\xi_b(y_f, y_2)$	$\cdots$	$\xi_b(y_f, y_{f-1})$	–

Step-11. Disclosure of Credibility Index

The credibility index, denoted by the notation $\pi(y_1, y_2)$, is used to determine the degree of outranking relation $y_1 S y_2$ is defined as:

$$\pi(y_1, y_2) = \begin{cases} \varsigma(y_1, y_2), & \text{if} \quad \xi_b(y_1, y_2) \le \varsigma(y_1, y_2) \quad \forall b \in \tau \\ \varsigma(y_1, y_2) \times \prod_{b \in \tau'} \dfrac{1 - \xi_b(y_1, y_2)}{1 - \varsigma(y_1, y_2)}, & otherwise \end{cases} \tag{20}$$

where $\tau' = \{b \in \tau : \xi_b(y_1, y_2) > \varsigma(y_1, y_2)\}$.

The credibility index, denoted by the notation $\pi = \lfloor \pi_{ij} \rfloor_{f \times f}, (i, j = 1,2,3,..., f, i \neq j)$, is used to determine the degree of outranking relation $y_1 S y_2$ can be determine using equation which is expressed in Table 8.

Table 8. Credibility index

π	y_1	y_2	$\cdots$	y_{f-1}	y_f
y_1	–	π_{12}	$\cdots$	$\pi_{1(f-1)}$	π_{1f}
y_2	π_{21}	–	$\cdots$	$\pi_{2(f-1)}$	π_{2f}
$\vdots$	$\vdots$	$\vdots$	–	$\vdots$	$\vdots$
y_{f-1}	$\pi_{(f-1)1}$	$\pi_{(f-1)2}$	$\cdots$	–	$\pi_{(f-1)f}$
y_f	π_{f1}	π_{f2}	$\cdots$	$\pi_{f(f-1)}$	–

3.2.2 | Module II: The Exploitation of Outranking Relations (The Exploitation of Outranking Relations)

The ELECTRE III [30, 31] standard ranking method uses a structured algorithm with two intermediate ranking methods, one of which is descending distillation, where the alternatives are ordered from best to worst, and the other of which is based on ascending order from the worst to best option (ascending distillation). Li and Wang, however, propose a novel ranking method based on the derivation of three concepts: the concordance credibility degree, the discordance credibility degree, and the net credibility degree.

(i). For each alternative, the concordance credibility degree defined as:

$$\rho^+(y_a) = \sum_{y_b \in Y} \pi(y_a, y_b), \ \forall\, y_a \in Y \tag{21}$$

The concordance credibility degree represents outranked y_a.

(ii). For each alternative, the discordance credibility degree defined as:

$$\rho^-(y_a) = \sum_{y_b \in Y} \pi(y_b, y_a), \quad \forall y_a \in Y \tag{22}$$

The discordance credibility degree represents outranked y_b.

(iii). For each alternative, the net credibility degree defined as:

$$\rho(y_a) = \rho^{+}(y_a) - \rho^{-}(y_a), \quad \forall y_a \in Y \tag{23}$$

with the value of net credibility degree $\rho(y_a)$, alternative y_a becomes more alluring. Thus, the potential solutions may be ordered in decreasing order according to their level of net believability.

4 | Real-Life Case Study: Best Female Spouse Selection for Male

The male spouse selection problem is a hypothetical case study that presents a situation where a man is faced with the task of selecting a spouse. In this scenario, the man is presented with a list of potential partners, each with their own set of characteristics and qualities. The man must evaluate each candidate based on these factors and ultimately choose the one that he believes is the best match for him. To solve this problem, the man may use a variety of criteria to evaluate each candidate. These criteria may include things like personality, intelligence, physical attractiveness, shared interests or values, and communication skills. The man may also consider factors such as cultural or religious background, family connections, and social status. Similar to the female spouse selection problem, the selection process will be influenced by a range of personal preferences and individual priorities. The man may have a clear idea of what he is looking for in a partner, or he may be more open to considering a wider range of options. He may also seek input from friends or family members, or consult with a professional matchmaker or dating coach.

Again, in any case, the selection process is likely to involve a considerable amount of self-reflection and careful consideration. The man will need to weigh the pros and cons of each potential partner and ultimately make a decision that he feels confident and comfortable with. While there is no one "right" way to approach this problem, the key is to stay true to oneself and to prioritize those qualities and characteristics that are most important to the individual. Ultimately, the goal is to find a partner with whom one can build a strong, healthy, and fulfilling relationship.

4.1 | Available Alternatives

Let we have three Alternative $Y = \{y_1, y_2, y_3\}$ to a wife or female spouse selection for male.

4.2 | Best Criteria for Consideration

There are several favorable/ Benefit-type/ Positive (+) and unfavorable/ Cost-type/ Negative (-) criteria that can make a female spouse suitable/unsuitable for a man. Here are some of the top criteria $C = \{c_1, c_2, c_3, c_4\}$

- **Trustworthiness (c_1 +):** A spouse who is trustworthy is someone who can be relied upon to keep their promises, maintain confidentiality, and be faithful.
- **Respectful (c_2 +):** A respectful spouse is someone who values and acknowledges their partner's thoughts, feelings, and opinions.
- **Inarticulate (c_3 -):** Good communication skills are essential in any relationship, and a wife who can communicate effectively can avoid misunderstandings and resolve conflicts more easily.
- **Supportive (c_4 +):** A supportive spouse is someone who stands by their partner through thick and thin, offers encouragement, and helps them achieve their goals.

4.3 | Sequential Process

In the phases that follow, the entire PN-ELECTRE III process is utilized to determine the ideal wife or female spouse for male candidate.

4.3.1 | Phase I: Pythagorean Neutrosophic Assessment Phase

Step-1: Configuration of Linguistic variables- As shown in table 9, the relevance of weight degree to criteria and experts are given using the decision support system of the recommended approach in the form of linguistic terms/variable that are provided by PNNs. Table 10 shows the performance ratings using linguistic phrases and variables.

Table 9: LVfor importance weighted rating of criteria and expert.

Linguistic Variable	Code	PNNs
Very Good	VG	(0.90, 0.32, 0.20)
Good	*G*	(0.70, 0.23, 0.35)
Medium	M	(0.50, 0.55, 0.50)
Bad	B	(0.35, 0.23, 0.70)
Very Bad	VB	(0.20, 0.32, 0.90)

Table 10: LV for Favorable Rating of Alternatives and Threshold functions.

Linguistic Variable	**Code**	**PNNs**
Extremely Favourable	EF	(0.95, 0.15, 0.11)
Very Favourable	VF	(0.72, 0.32, 0.25)
Medium Favourable	MF	(0.62, 0.42, 0.32)
Medium Unfavourable	MU	(0.32, 0.42, 0.62)
Very Unfavourable	VU	(0.25, 0.32, 0.72)
Extremely Unfavourable	EU	(0.11, 0.15, 0.95)

Step-2: Calculation of the weights of DMs - Table 11 lists the importance rankings that the male candidate granted to each of the DMs v_t. Employing equation it is possible to govern each expert's individual weight.

Table 11. Assigning and Calculation of weights of DMs

DMs	LV	PNNs	Weights (ℓ_t)
v_1	VG	(0.90, 0.32, 0.20)	0.3811
v_2	G	(0.70, 0.23, 0.35)	0.3505
v_3	M	(0.50, 0.55, 0.50)	0.2685

Step-3: Assigning of the DMs Opinions- The LV used in table 11 to designate the distinct viewpoints of each DMs on the decision-making board with regard to individually alternative and all taken-into-account criteria. The Pythagorean Neutrosophic Decision Matrices [PNDMs] $P^{(1)}, P^{(2)}$ and $P^{(3)}$ that highlight the exclusive opinions of the DMs v_1, v_2 and v_3, correspondingly, are exposed in tables 12, 13 and 14.

Table 11: Opinion of DMs in LV on Favourability Rating of alternatives w.r.t. Criterion

		c_1	c_2	c_3	c_4
v_1	y_1	EF	VF	MU	VF
	y_2	VU	EF	MU	MF
	y_3	VU	MF	VF	EU
v_2	y_1	VF	MU	MU	EF
	y_2	VU	VF	MF	VF
	y_3	EU	MU	VF	EF
v_3	y_1	MU	MF	VF	VU
	y_2	MF	EF	VU	EU
	y_3	VF	MF	MU	EU

Table 12. Opinion of DMsin PNNs

$P^{(1)}$	c_1	c_2	c_3	c_4
y_1	(0.95,0.15,0.11)	(0.72,0.32,0.25)	(0.32,0.42,0.62)	(0.72,0.32,0.25)
y_2	(0.25,0.32,0.72)	(0.95,0.15,0.11)	(0.32,0.42,0.62)	(0.62,0.42,0.32)
y_3	(0.25,0.32,0.72)	(0.62,0.42,0.32)	(0.72,0.32,0.25)	(0.11,0.15,0.95)

Table 13. Opinion of DMsin PNNs

$P^{(2)}$	c_1	c_2	c_3	c_4
y_1	(0.72,0.32,0.25)	(0.32,0.42,0.62)	(0.32,0.42,0.62)	(0.95,0.15,0.11)
y_2	(0.25,0.32,0.72)	(0.72,0.32,0.25)	(0.62,0.42,0.32)	(0.72,0.32,0.25)
y_3	(0.11,0.15,0.95)	(0.32,0.42,0.62)	(0.72,0.32,0.25)	(0.95,0.15,0.11)

Table 14. Opinion of DMsin PNNs

$P^{(3)}$	c_1	c_2	c_3	c_4
y_1	(0.32,0.42,0.62)	(0.62,0.42,0.32)	(0.72,0.32,0.25)	(0.25,0.32,0.72)
y_2	(0.62,0.42,0.32)	(0.95,0.15,0.11)	(0.25,0.32,0.72)	(0.11,0.15,0.95)
y_3	(0.72,0.32,0.25)	(0.62,0.42,0.32)	(0.32,0.42,0.62)	(0.11,0.15,0.95)

Step-4: Aggregation of the DMs Opinion- The unique opinions of each DMs are pooled based on the normalized weights assigned by the PNWA operator, see equation **Error! Reference source not found.** and the DMs. The combined Pythagorean neutrosophic decision matrix $P' = \left(P'_{ab}\right)_{f\times g}$ is shown in table 15.

Table 15: Aggregated Pythagorean neutrosophic decision matrix (A-PNDM)

P'	c_1	c_2
y_1	(0.8309,0.2579,0.2333)	(0.6002,0.3786,0.3672)
y_2	(0.4033,0.3442,0.5791)	(0.9107,0.1956,0.1466)
y_3	(0.4489,0.2453,0.5973)	(0.5454,0.4200,0.4034)
P'	c_3	c_4
y_1	(0.4905,0.3904,0.4858)	(0.8191,0.2453,0.2490)
y_2	(0.4520,0.3904,0.5118)	(0.5989,0.2896,0.3930)
y_3	(0.6564,0.3442,0.3190)	(0.7492,0.1500,0.4462)

Step-5: Normalization of Aggregated PNDM- Let C_B and C_N denote the corresponding groups of criteria that are of the benefit-type (Positive) criteria $C_B = \{c_1, c_2, c_4\}$, Cost type (Negative) criteria $C_N = \{c_3\}$.. The

aggregated PNDM, $P' = \left(P'_{ab}\right)_{f\times g}$, can be converted into the normalized aggregated PNDM, $P = \left(P_{ab}\right)_{7\times 8}$ using equation, which demonstrate the evaluation data of each alternative w.r.t. each benefit or cost criterion, in standard form, for additional calculations. Table 16 demonstrates how the matrix P is built. You can define the PNN for P_{ab}:

Table 16 : *Normalized Aggregated PNDM* $\left(NA-PNDM\right)$

P	c_1	c_2
y_1	$(0.8309, 0.2579, 0.2333)$	$(0.6002, 0.3786, 0.3672)$
y_2	$(0.4033, 0.3442, 0.5791)$	$(0.9107, 0.1956, 0.1466)$
y_3	$(0.4489, 0.2453, 0.5973)$	$(0.5454, 0.4200, 0.4034)$
P	c_3	c_4
y_1	$(0.4858, 0.3904, 0.4905)$	$(0.8191, 0.2453, 0.2490)$
y_2	$(0.5118, 0.3904, 0.4520)$	$(0.5989, 0.2896, 0.3930)$
y_3	$(0.3190, 0.3442, 0.6564)$	$(0.7492, 0.1500, 0.4462)$

Step-6: Formation of weight matrix of criteria- The DMs panel's linguistic labels for each criterion, PN-weights, and normalised weights of the criteria are shown in table 17 and table 18.

Table 17. Linguistic variable to unfold the importance of criteria

$Criteria \rightarrow$ $Expert \downarrow$	c_1	c_2	c_3	c_4
v_1	VG	M	VB	B
v_2	M	G	VG	VB
v_3	G	VG	B	G

Table 18. PN - Weights of Criteria

$Criteria \rightarrow$ $Expert \downarrow$	c_1	c_2	c_3	c_4
v_1	$(0.90, 0.32, 0.20)$	$(0.50, 0.55, 0.50)$	$(0.20, 0.32, 0.90)$	$(0.35, 0.23, 0.70)$
v_2	$(0.50, 0.55, 0.50)$	$(0.70, 0.23, 0.35)$	$(0.90, 0.32, 0.20)$	$(0.20, 0.32, 0.90)$
v_3	$(0.70, 0.23, 0.35)$	$(0.90, 0.32, 0.20)$	$(0.35, 0.23, 0.70)$	$(0.70, 0.23, 0.35)$

Normalized weights of criteria using equation **Error! Reference source not found.** and equation

$$W_{\{c_1,c_2,c_3,c_4\}} = \begin{pmatrix} (0.7741, 0.3540, 0.3204) \\ (0.7395, 0.3503, 0.3450) \\ (0.6847, 0.2928, 0.4966) \\ (0.4661, 0.2582, 0.6346) \end{pmatrix}^T \quad \text{and} \quad \begin{matrix} w(c_1) = 0.2565 \\ w(c_2) = 0.2513 \\ w(c_3) = 0.2665 \\ w(c_4) = 0.2257 \end{matrix}$$

Step-7: Creation of weighted normalized aggregation PNDM- The weighted normalized aggregated PNDM, $P^* = \left(P^*{}_{ab}\right)_{f\times g}$ is created using equation **Error! Reference source not found.** as shown in table 19.

Table 19. Weighted Normalized Aggregated PNDM $\left(\text{WNA - PNDM}\right)$

P^*	c_1	c_2
y_1	$(0.6432, 0.4284, 0.3892)$	$(0.4438, 0.4985, 0.4877)$
y_2	$(0.3122, 0.4785, 0.6353)$	$(0.6735, 0.3953, 0.3714)$
y_3	$(0.3475, 0.4218, 0.6502)$	$(0.4033, 0.5267, 0.5122)$

$P*$	c_3	c_4
y_1	(0.3326, 0.4744, 0.6541)	(0.3818, 0.3505, 0.6631)
y_2	(0.3504, 0.4744, 0.6329)	(0.2791, 0.3807, 0.7035)
y_3	(0.2184, 0.4405, 0.7558)	(0.3492, 0.2961, 0.7222)

Step-8: Calculation of Score degrees w.r.t. WNA-PNDM- Table 20 comprises the computed score values of consistent PNNs in the weighted normalized aggregated PNDM using equation

Table 20. The score degree of WNA - PNDM

$s(P)$	c_1	c_2	c_3	c_4
y_1	0.0787	–0.2894	–0.5423	–0.4168
y_2	–0.5351	0.1594	–0.5028	–0.5619
y_3	–0.4799	–0.3771	–0.7176	–0.4873

4.3.2 | Phase II: Pythagorean Neutrosophic Ranking Phase (PN-ELECTRE III)

4.3.2.1 | Module-I: The Construction of Outranking Relations

Step-9. Establishment of Threshold Functions- For each criteria χ_b, here are preference threshold values $p(\chi_b)$ and indifference threshold values $q(\chi_b)$ and Veto threshold values $v(\chi_b)$ as shown in Table 21.

Table 20 Assigning of PNNs to threshold functions and its score values

	c_1	c_2	c_3	c_4
$q(c_j)$	EU	VU	MF	VU
	(0.11, 0.15, 0.95)	(0.25, 0.32, 0.72)	(0.62, 0.42, 0.32)	(0.25, 0.32, 0.72)
$s\left(q(c_j)\right)$	–0.9129	–0.5583	0.1056	–0.5583
$p(c_j)$	MU	MF	VF	MU
	(0.32, 0.42, 0.62)	(0.62, 0.42, 0.32)	(0.72, 0.32, 0.25)	(0.32, 0.42, 0.62)
$s\left(p(c_j)\right)$	–0.4584	0.1056	0.3535	–0.4584
$v(c_j)$	VF	EF	EF	VF
	(0.72, 0.32, 0.25)	(0.95, 0.15, 0.11)	(0.95, 0.15, 0.11)	(0.72, 0.32, 0.25)
$s\left(v(c_j)\right)$	0.3535	0.8679	0.8679	0.3535

Step-10. Calculation of difference in the score degrees- The differences in the score values/degrees of the feasibility of every pair of alternatives are represented by Table 21. i.e., the if part of equation

Table 21. The differences in the score values / degrees

c_1			c_2		
–	–0.6138	–0.5586	–	0.4488	–0.0877
0.6138	–	0.0552	–0.4488	–	–0.5365
0.5586	–0.0552	–	0.0877	0.5365	–
c_3			c_4		
–	0.0395	–0.1753	–	–0.1451	–0.0705
–0.0395	–	–0.2148	0.1451	–	0.0746
0.1753	0.2148	–	0.0705	–0.0746	–

Step-11: Calculation of Partial Concordance Indicines and Concordance Matrix-

The partial concordance Matrices w.r.t. each criterion is given in the Table 22, calculated using equation

Table 22. The partial concordance Matrices

c			c_2		
–	0.3419	0.2205	–	0.0000	0.2912
0.0000	–	0.0000	0.8351	–	0.9672
0.0000	0.0000	–	0.027	0.0000	–
c_3			c_4		
–	1.0000	1.0000	–	0.0000	0.0000
1.0000	–	1.0000	0.0000	–	0.0000
0.7188	0.5595	–	0.0000	0.0000	–

Comprehensive concordance matrix is given by Table 23, employing by equation

Table 23. Comprehensive concordance matrix

c		
0	0.3542	0.3962
0.4764	0	0.5096
0.1983	0.1491	0

Step-12: Calculation of Discordance Matrix- Discordance matrices w.r.t. each criteria shown in Table 24, calculated using equation

Table 24. Discordance matrices

d_1			d_2		
–	0.0000	0.0000	–	0.4502	0.0000
1.0000	–	0.6326	0.0000	–	0.0000
1.0000	0.4966	–	0.0000	0.5653	–
d_3			d_4		
–	0.0000	0.0000	–	0.3859	0.4778
0.0000	–	0.0000	0.7433	–	0.6565
0.0000	0.0000	–	0.6514	0.4727	–

Step-13: Calculation of Credibility Index- Table 25 shows the credibility index of alternative, calculated using equation.

Table 25. credibility index of alternative

Credibility		
–	0.2867	0.3427
0.0000	–	0.2674
0.0000	0.0279	–

4.3.2.2 | Module II: The Exploitation of Outranking Relations

Step-14: Calculation of Credibility Index- The ranking of the alternatives have been given in the table 26, employing by equations and

Table 26. Ranking of Alternatives

Alternatives	μ^{+}	μ^{-}	μ	Ranking
y_1	0.6294	0	0.6294	1
y_2	0.2674	0.3146	–0.0472	2
y_3	0.0279	0.6101	–0.5822	3

From the Table 26, we find $y_1 \succ y_2 \succ y_3$. Thus, first is the best alternative to select as wife/female spouse for a male candidate.

5 | Conclusions

Debuting a new age that brings innovative challenges in the desire to forge genuine and lasting connections, the emergence of social media and the spread of online dating platforms have also changed the fundamental fabric of human contact. Our dedication to solving this modern problem is attested to by this study article. In a Pythagorean neutrosophic framework, we have pioneered a novel fusion of MCGDM and the ELECTRE-III technique, through which we have developed a special model that is well-tuned to address the particular issues of mate choosing in the twenty-first century. Our concept is based on the idea that choosing a spouse is a multifaceted process that takes a number of factors into account. It provides a methodological framework that is organized and methodical and well-suited to manage the sometimes contradictory and ambiguous information that usually pervades this crucial decision-making process. The use of PNST is a cornerstone of our strategy. This notion acts as a mental compass that steers us clear of the hazy seas of ambiguity and doubt, which frequently cloud our judgment when it comes to choosing a mate. This method equips decision-makers with a strong toolset to handle the complexity that come with choosing a life partner since it is based on rigorous mathematical and computational underpinnings.

In conclusion, the necessity for sophisticated decision-making tools becomes more and more obvious as the dynamics of interpersonal interactions continue to change in response to the globalized, technologically advanced world of the 21st century. In order to face this problem, our study has developed a potent MCGDM framework that is solidly based on Pythagorean neutrosophic theory. We hope that our contribution will serve as a guiding light for people and DM as they set out to create lasting and happy relationships, giving them confidence and clarity in their quest for love and companionship in the modern day.

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Author Contributaion

R K Saini: Conceptualized and designed the study, mathematical formulation of the case study, conducted data analysis, and provided supervision throughout the research process.

Ashik Ahirwar: Contributed to data analysis, conducted data calculations, manuscript writing, and provided technical assistance at various stages of the project.

Florentin Smarandache: Provided critical revision of the manuscript and offered supervision, ensuring the integrity and quality of the research.

Mukesh Kushwaha: Contributed to data collection, conduction of data analysis, and provided some writing assistance at various stages.

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Data Availability

The availability and access to data utilized in our study have been meticulously sourced and referenced from reputable online platforms. We have diligently collected pertinent information from various reliable sources and ensure the integrity and reproducibility of our research findings, facilitating further exploration and analysis in this study.

Conflicts of Interest

It is declared that the authors have been personally and actively involved in substantial work leading to the paper and will take public responsibility for its content. No violation of the Ethics involved in this paper.

Ethical Approval

We affirm that our research article does not involve human subjects; therefore, ethical approval and informed consent did not apply to this study. All data utilized in our research were obtained from publicly available sources or were simulated/generated for analysis. We ensure that all data handling procedures strictly adhere to ethical standards and legal regulations regarding data usage and privacy. Our commitment to transparency and integrity underscores our dedication to upholding the highest ethical standards in scientific research.

References

[1] J. Erola, J. Harkonen, and J. Dronkers, "More Careful or Less Marriageable? Parental Divorce, Spouse Selection and Entry into Marriage," Social Forces, vol. 90, no. 4, pp. 1323–1345, Jun. 2012, doi: 10.1093/sf/sos073.

[2] E. Voland and C. Engel, "Female Choice in Humans: A Conditional Mate Selection Strategy of the Krummhörn Women (Germany, 1720-1874)," Ethology, vol. 84, no. 2, pp. 144–154, Apr. 2010, doi: 10.1111/j.1439-0310.1990.tb00791.x.

[3] M. Zimmer, "Assortative mating and ethnicity in the low wage population: an examination of spouses' earnings," Appl Econ Lett, vol. 3, no. 5, pp. 311–315, May 1996, doi: 10.1080/135048596356429.

[4] K. Yagura, "Does Labour Migration Offer Opportunities for Meeting Prospective Spouses? The Case of Migrant Workers in Cambodia," Popul Space Place, vol. 18, no. 3, pp. 277–294, May 2012, doi: 10.1002/psp.683.

[5] I. Aslaksen, T. Wennemo, and R. Aaberge, "'Birds of a Feather Flock Together': The Impact of Choice of Spouse on Family Labor Income Inequality," Labour, vol. 19, no. 3, pp. 491–515, Sep. 2005, doi: 10.1111/j.1467-9914.2005.00311.x.

[6] S. Ge, "Women's College Decisions: How Much Does Marriage Matter?," J Labor Econ, vol. 29, no. 4, pp. 773–818, Oct. 2011, doi: 10.1086/660774.

[7] M. Akram, F. Ilyas, and A. N. Al-Kenani, "Two-Phase Group Decision-Aiding System using ELECTRE III Method in Pythagorean Fuzzy Environment," Arab J Sci Eng, vol. 46, no. 4, pp. 3549–3566, Apr. 2021, doi: 10.1007/s13369-020-05003-6.

[8] L. A. Zadeh, "Fuzzy sets," Inform. and Control, vol. 8, pp. 338–353, 1965.

[9] I. B. Türkşen, "Interval-valued fuzzy sets and compensatory AND," Fuzzy Sets Syst, vol. 51, no. 2, pp. 295–307, 1992, doi: DOI.

[10] I. B. Türkşen, "Interval-valued strict preference with Zadeh triples," Fuzzy Sets Syst, vol. 7, no. 2, pp. 183–195, 1996.

[11] B. Ashtiani, F. Haghighirad, A. Makui, and G. ali Montazer, "Extension of fuzzy TOPSIS method based on interval-valued fuzzy sets," Appl Soft Comput, vol. 9, no. 2, 457–461, 2009.

[12] K. Atanassov, "Intuitionistic Fuzzy Sets," Fuzzy Sets Syst, vol. 20, pp. 87–96, 1986.

[13] F. Smarandache, A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic. American Research Press, 1999.

[14] J. Peng, J. Wang, J. Wang, H. Zhang, and X. Chen, "Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems," Int J Syst Sci, vol. 46, no. 14, pp. 2575–2590, 2015.

[15] J. Ye, "Single-valued neutrosophic cross-entropy for multicriteria decision-making problems," Appl Math Model, vol. 38, no. 3, pp. 1170–1175, 2014.

[16] R. Hagiwara, "Which do you Choose, Marriage or Career?: Econometric Analysis Using JPSC Panel Data," International Journal of Economic Policy Studies, vol. 7, no. 1, pp. 114–134, Jan. 2012, doi: 10.1007/BF03405740.

[17] H. Chun and I. Lee, "Why do married men earn more: productivity or marriage selection?" Econ Inq, vol. 39, no. 2, pp. 307–319, Apr. 2001, doi: 10.1111/j.1465-7295.2001.tb00068.x.
[18] H.-F. Li and J.-J. Wang, "An improved ranking method for ELECTRE III," in International Conference on Wireless Communications, Networking and Mobile Computing, 2007, 6659–62.
[19] T. Kaya and C. Kahraman, "ELECTRE III method for selection among wastewater treatment technologies," Expert Syst Appl, vol. 37, no. 9, pp. 6582–6589, 2010.
[20] J. Rajan and M. Krishnaswamy, "Similarity Measures of Pythagorean Neutrosophic Sets with Dependent Neutrosophic Components Between T and F." Journal of New Th. 33 (2020) 85-94.
[21] R. Jansi, K. Mohana, and F. Smarandache, "Correlation Measure for Pythagorean Neutrosophic Sets with T and F as Dependent Neutrosophic Components Neutrosophic Sets and Systems." Dec. 2019.
[22] S. A. Abbas, M. K. Khan, N. Ali, K. Zaman, and G. Li, "Using Pythagorean Neutrosophic programming approach to evaluate and select the best supplier for a manufacturing company," J Ambient Intell Humaniz Comput, vol. 11, pp. 4211–4224, 2020.
[23] A. Elhassouny; Florentin Smarandache "Neutrosophic-simplified-TOPSIS Multi-Criteria Decision-Making using combined Simplified-TOPSIS method and Neutrosophics." 2016 IEEE International Conference on Fuzzy Systems, 24-29 July 2016.
[24] M. Akram, W. A. Dudek, and F. Ilyas, "Group decision making based on pythagorean fuzzy TOPSIS method," Int. J. Intell. Syst., vol. 34, no. 7, pp. 1455–1475, 2019.
[25] F. Smarandache, "Neutrosophic Set–a generalization of the intuitionistic fuzzy set," International Journal of Pure and Applied Mathematics, vol. 24, no. 3, pp. 287–297, 1998.
[26] X. Wang, H. Liu, J. Zhang, F. Ma, and H. Zhang, "Applied Pythagorean Neutrosophic programming approach to rank different photovoltaic power plant locations in China," J Clean Prod, vol. 312, p. 127733, 2021.
[27] F. Smarandache and S. Broumi, "New Operations on Pythagorean Neutrosophic Sets," Complex & Intelligent Systems, vol. 5, no. 2, pp. 127–142, 2019.
[28] S. S. Hashemi, S. H. R. Hajiagha, E. K. Zavadskas, and H. A. Mahdiraji, "Multicriteria group decision making with ELECTRE III method based on interval-valued intuitionistic fuzzy information," Appl Math Model, vol. 40, no. 2, pp. 1554–1564, Apr. 2016.
[29] A. Guitouni and J.-M. Martel, "Applying the ELECTRE III method to select the best location for a landfill site in Quebec, Canada," Waste Management & Res, vol. 16, no. 6, 539–549, 1998.
[30] S. S. Hashemi, R. Hajiagha, E. K. Zavadskas, and H. A. Mahdiraji, "Multi criteria group decision making with ELECTRE III method based on interval-valued intuitionistic fuzzy information," Appl Math Model, vol. 40, pp. 1554–1564, 2016,
[31] T. Gao, S. Na, X. Dang, and Y. Zhang, "Study of the Competitiveness of Quanzhou Port on the Belt and Road in China Based on a Fuzzy-AHP and ELECTRE III Model," Sustainability, vol. 10, no. 4, p. 1253, Apr. 2018.

Double Indeterminacy — Neutrosophic Study of an Approximation Techniques Used to Find Random Variables

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract
The main interest in statistical analysis is to generate a series of random variables that follow the probability distribution in which the system under study operates. In almost all simulation tests, we need to generate random variables that follow a distribution, a distribution that adequately describes and represents the physical process involved in the experiment at That point. During the experiment, it may be necessary to simulate a real and perform the process of generating a random variable from a distribution many times depending on the complexity of the model to be simulated in order to obtain more accurate simulation results. In previous research, we presented a neutrosophical study of the process of generating random numbers and some techniques that can be used to convert these random numbers into variables. Randomness follows the probability distributions according to which the system to be simulated operates. These techniques were specific to probability distributions defined by a probability density function that is easy to deal with in terms of finding the cumulative distribution function and the inverse function of the cumulative distribution function or by calculating the values of this function at a certain value, and in reality, we encounter Many systems operate according to these distributions, which requires techniques other than the techniques presented. Therefore, in this research we will present a neutrosophical study of the approximation technique for generating random variables that follow probability distributions known as a complex probability density function. We will apply this study to find random variables that follow the distribution. Standard natural
Keywords: Generating neutrosophic random numbers; generating neutrosophic random variables; approximation technique for generating neutrosophic random variables; standard normal distribution.

1. Introduction:

In light of the great development witnessed by our contemporary world, the complexity of systems, and the large material and non-material losses that can result from operating any system without prior study, there must be a scientific method that enables us to know the results that we can obtain when operating systems and helps us reduce these losses. The simulation process was the modern tool through which we could predict the results that we could obtain through the operation of these systems over time. It became the modern-day tool that helps us study many systems that could not have been studied or predict the results that we could obtain. Through the operation of these systems over time, the simulation process depends on generating a series of random numbers subject to a regular probability distribution over the field [0, 1], then converting these random numbers into random variables that follow the probability distribution in which the system to be simulated operates. Since the process Simulation depends on the data that is collected and is subject to change and is not fixed over the length of time during which the system operates. Therefore, we must prepare a study that takes into account all the conditions and changes that the operating environment of the system to be simulated may undergo. In other words, a study based on the concepts of neutrosophic logic. Logic, which studies ideas and concepts that are neither true nor false, but in between, which means (neutrality, indeterminacy, ambiguity, contradiction, etc.), is expressed in three dimensions: truth T, neutrality I, and error F, each of which has degrees. This new vision of concepts has prompted many researchers and scholars to reformulate many scientific concepts according to the concepts of this logic [1-12]. As a complementary study to these studies and given the importance of the science of operations research, we have presented and reformulated many methods of operations research using the concepts of this logic [12-15], in order to obtain more accurate simulation results, we presented a neutrosophical study of the process of generating random numbers and some techniques that can be used to transform these random numbers into random variables that follow the probability distributions according to which the system to be simulated operates [16-23], and these

techniques were special With probability distributions known as a probability density function, it is easy to deal with in terms of finding the cumulative distribution function and its inverse function. Also, calculating the value of this function at a certain value is easy. It should be noted here that there are many systems that operate according to probability distributions known as a complex probability density function that requires the process of generating random variables depends on special techniques. In this research, we present a neutrosophical study of the approximation technique for generating random variables that follow probability distributions defined by a complex probability density function. As an example, we will apply this technique to generate random variables that follow the standard normal distribution.

2. Discussion:

The process of neutrosophic simulation depends on generating a series of random numbers that follow a uniform distribution over the field [0,1], then converting these random numbers into neutrosophic random numbers using the information contained in the research [16], to generate random variables that follow the probability distribution according to which the system operates. What is to be simulated? If the probability density function of the distribution is easy, we use the appropriate method from among the methods presented in research [16-23]. It should be noted here that there are many systems that operate according to probability distributions defined by a complex probability density function that requires the process of generating random variables. Tracing it to special techniques, we present in this research a neutrosophical study of the approximation technique for generating random variables that follow probability distributions defined by a complex probability density function,

2.1. The basic idea of approximation techniques: [24,25]

Approximation techniques are related to the inverse transformation technique, and are used when the probability distribution is known as a complex probability density function. In approximation methods, we approximate one of three different quantities:

- Follow the probability density of the distribution to be simulated.
- Follow the cumulative distribution of the distribution to be simulated.
- The inverse transformation of the cumulative distribution function of the distribution to be simulated.

Approximation methods are used when the cumulative distribution function cannot be obtained from the given probability density function, or when it is impossible to obtain the opposite transformation even if $F(x)$ exists. When this is true, an approximation must be made to the probability density function to facilitate the derivation of the function. Cumulative distribution, or an approximation must be made to $F(x)$ directly. Once the cumulative distribution function of the approximated probability density function is obtained, the inverse transformation can be applied and the required random variables can be generated.

2.2. In the classic study, this technique was applied to generate random variables that follow the standard normal distribution, and the following relationship was obtained:

$$z = \frac{1}{k} ln\left[\frac{1+R_1}{1-R_1}\right] \quad (1)$$

To generate random variables that follow the general normal distribution $N(\mu, \sigma^2)$, we use the following transformation:

$$x = \sigma z + \mu \Longrightarrow z = \frac{x-\mu}{\sigma}$$

Substituting into relationship (1) we get the following relation:

$$\Longrightarrow x = \frac{\sigma}{k} ln\left[\frac{1+R_1}{1-R_1}\right] + \mu \quad (2)$$

Example 1:

Using the approximation technique, find a random variable that follows a normal distribution defined by the parameters $\mu = 15$ and $\sigma = 2$.

the solution:

Using the mean square method, we generate a random number that follows a uniform distribution in the range $[0,1]$.

The mean square method is given by the following relation:

$$R_{i+1} = Mid[R_i^2]\ ; i = 0,1,2,3, \dots \qquad (3)$$

Where Mid symbolizes the middle four ranks of R_i^2, and R_i is chosen, i.e., a fractional random number consisting of four ranks (called a seed) that does not contain a zero in any of its four ranks, and let $R_0 = 0.1234$, we get: $R_1 = 0.5227$

Applying relation (3) we get:

$$x_1 = \frac{\sigma}{k} ln\left[\frac{1+R_1}{1-R_1}\right] + \mu = 2.\sqrt{\frac{\pi}{8}}\ .ln\left[\frac{1+0.5227}{1-0.5227}\right] + 15$$

$$x_1 = 16.4541$$

2.3. Neutrosophic study to generate random variables that follow the standard normal distribution using the approximation technique:

Based on the study mentioned in references [24,25] for the quadrature technique for generating random variables that follow the standard normal distribution, and the study mentioned in reference [2] to define neutrosophic probability distributions, the study mentioned in reference [16] for generating neutrosophic random numbers, and the study mentioned in the two references. [17,18] The neutrosophic study of the inverse transformation method. We can present the following neutrosophic formulation of the approximation technique used to generate random variables that follow the standard normal distribution, and then we use the appropriate transformation to formulate the appropriate relationship to generate random variables that follow the general normal distribution:

To generate random variables that follow the standard normal distribution defined by the following probability density function:

$$\boldsymbol{f(z) = \frac{1}{\sqrt{2\pi}}\ e^{-\frac{z^2}{2}} \quad -\infty \le z \le +\infty}$$

- One of the most important approximations of the quantity $e^{-\frac{z^2}{2}}$ is the Kahn approximation, which is given by the following relation:

$$e^{-\frac{z^2}{2}} \approx \frac{2e^{-kz}}{(1+e^{-kz})^2} \qquad ; z > 0$$

in order to

$$k = \sqrt{\frac{8}{\pi}}$$

- We calculate the standard cumulative distribution function from the following approximate relation:

$$F(z) = \int_0^z \frac{1}{\sqrt{2\pi}} \cdot \frac{2e^{-kt}}{(1+e^{-kt})^2} dt = \left[\frac{2}{1+e^{-kx}}\right] - 1$$

- We generate a series of random numbers that follow a uniform distribution over the domain $[0,1]$ and let it be: [24,25]

$$R_1, R_2, \dots, R_n$$

- We convert these random numbers into neutrosophic random numbers [16] and we get:

$$R_{1N}, R_{2N}, \dots, R_{nN}$$

- We use the inverse transformation [17,18] as follows:

$$F(z) = \left[\frac{2}{1+e^{-kz}}\right] - 1 = R_{iN} \Longrightarrow \frac{2}{1+e^{-kz}} = 1 + R_{iN}$$
$$\Longrightarrow 1 + e^{-kz} = \frac{2}{1+R_{iN}} \Longrightarrow e^{-kz} = \frac{2}{1+R_{iN}} - 1$$

$$\Rightarrow e^{-kz} = \frac{1 + R_{iN}}{1 - R_{iN}}$$

$$z_{iN} = \frac{1}{k} ln\left[\frac{1 + R_{iN}}{1 - R_{iN}}\right] \quad (4)$$

When we want to generate neutrosophic random variables that follow the general neutrosophic normal distribution $N(\mu_N, \sigma_N)$, we use the following transformation:

$$\Rightarrow x_{iN} = \sigma_N z_{iN} + \mu_N$$

We get the following relation:

$$\Rightarrow x_{iN} = \frac{\sigma_N}{k} ln\left[\frac{1 + R_{iN}}{1 - R_{iN}}\right] + \mu_N \quad (5)$$

- Neutrosophic random variables are obtained according to the following conditions:

The first case: Neutrosophic random numbers and probability distribution in the classical form

The second case: classical random numbers and the probability distribution is in the neutrosophic form.

The third case: neutrosophic random numbers and probability distribution in neutrosophic form.

Example 2:

Using the approximation technique, find a random variable that follows the normal distribution defined by the parameters $\mu_N \in [15,17]$ and $\sigma_N \in \{2,3,4\}$

the solution:

From the data, we notice that the probability distribution is in the neutrosophic form, and here we can apply either the second case or the third case.

For this, we follow the following steps:

- Using the mean square method, we generate a random number that follows a uniform distribution in the range $[0,1]$.
- The mean square method is given by the following relation:

$$R_{i+1} = Mid[R_i^2]\ ; i = 0,1,2,3, \ldots$$

Where Mid symbolizes the middle four levels of R_i^2, and R_i is chosen, i.e., a fractional random number composed of four places (called a seed) that does not contain a zero in any of its four places, and let $R_0 = 0.1234$, we get $R_1 = 0.5227$

- We transform this random number into a neutrosophic random number and here we distinguish three forms of the field $[0,1]$ with a margin of indeterminacy, in the three forms we have $\varepsilon \in [0, n]$ and $0 < n < 1$

The first form: $[0 + \varepsilon, 1]$ indeterminacy at the minimum of the field. We substitute in the relation:

$$R_{iN} = \frac{R_i - \varepsilon}{1 - \varepsilon}$$

The second form: $[0,1 + \varepsilon]$ The indeterminacy at the upper limit of the range, we substitute into the relation:

$$R_{iN} = \frac{R_i}{1 + \varepsilon}$$

The third form: $[0 + \varepsilon, 1 + \varepsilon]$ The indeterminacy in the upper and lower limits of the field, we substitute in the relation:

$$R_{iN} = R_i - \varepsilon$$

We take $\varepsilon \in [0,0.02]$ then we obtain the appropriate neutrosophic random numbers for each of the three forms as follows:

From the first form we get the following neutrosophic random number:

$$R_{1N} = \frac{R_1 - \varepsilon}{1 - \varepsilon} = \frac{0.5227 - [0,0.02]}{1 - [0,0.02]} = \frac{[0.5027,0.5227]}{[1,0.98]} \in [0.5027,0.5334]$$

From the second figure we get the following two neutrosophic random numbers:

$$R_{1N} = \frac{R_1}{1+\varepsilon} = \frac{0.5227}{1+[0,0.02]} = \frac{0.5227}{[1,1.02]} \in [0.5125,0.5227]$$

From the third figure we get the following two neutrosophic random numbers:

$$R_{1N} = R_1 - \varepsilon = 0.5227 - [0,0.02] \in [0.5027,0.5227]$$

To generate neutrosophic random variables that follow the normal distribution defined by $\mu_N \in [15,17]$ and $\sigma_N \in \{2,3,4\}$, we note that the probability distribution is given in the neutrosophic form, meaning that we obtain random variables that follow it from the second case or from the third case.

According to the second case:

the random numbers are classical and the probability distribution is in the neutrosophic form:

The data of the problem becomes:

That is, $R_1 = 0.5227$, and the distribution is defined by $\mu_N \in [15,17]$ and $\sigma_N \in \{2,3,4\}$.

Substituting into relation (5) we get:

$$x_{1N} = \frac{\sigma_N}{k} ln\left[\frac{1+R_1}{1-R_1}\right] + \mu_N$$

$$x_{1N} = \{2,3,4\}.\sqrt{\frac{\pi}{8}} ln\left[\frac{1+\ 0.5227}{1-\ 0.5227}\right] + [15,17]$$

For $\sigma = 2$:

$$x_{1N} = 2.\sqrt{\frac{\pi}{8}}.ln\left[\frac{1.5227}{0.4773}\right] + [15,17]$$

$$\left[\frac{1.5227}{0.4773}\right] = 3.1902$$

$$\ln(3.1902) = 1.1601$$

$$\sqrt{\frac{\pi}{8}} = 0.3927$$

$$2.\sqrt{\frac{\pi}{8}} = 0.7854$$

$$2.\sqrt{\frac{\pi}{8}}.ln\left[\frac{1.5227}{0.4773}\right] = 0.7854 * 1.1601 = 0.9111$$

$$x_{1N} = 2.\sqrt{\frac{\pi}{8}}.ln\left[\frac{1.5227}{0.4773}\right] + [15,17]$$

$$x_{1N} = 0.9111 + [15,17] \in [15.9111,17.9111]$$

$$x_{1N} \in [15.9111,17.9111]$$

For $\sigma = 3$:

$$x_{1N} = 1.1781 + [15,17] \in [16.1781,18.1781]$$

$$x_{1N} \in [16.1781,18.1781]$$

For $\sigma = 4$:

$$x_{1N} = 1.5708 + [15,17] \in [16.5708,18.5708]$$

$$x_{1N} \in [16.5708,18.5708]$$

Thus, for any $R_1 = 0.5227$, $\mu_N \in [15,17]$ and $\sigma_N \in \{2,3,4\}$ then:

$$x_{1N} \in \{[15.9111,17.9111], [16.1781,18.1781], [16.5708,18.5708]\}$$

We note that we have obtained a set consisting of three elements that are fields, and this means the multiple choices that the random variable that follows the normal distribution can take, which provides us with highly accurate simulation results due to the double indeterminacy that we obtained.

According to the third case, the random numbers are neutrosophic, i.e.:

From the first form we have: $R_{1N} \in [0.5027, 0.5334]$

The data of the problem becomes:

Using the neutrosophic random number $R_{1N} \in [0.5027,0.5334]$ which follows a uniform distribution in the range $[0+\varepsilon, 1]$ where $\varepsilon \in [0,0.02]$, find a random variable that follows the normal distribution defined by $\mu_N \in [15,17]$ and $\sigma_N \in \{2,3,4\}$

We substitute into the following relation:

$$x_{1N} = \frac{\sigma_N}{k} ln\left[\frac{1+R_1}{1-R_1}\right] + \mu_N$$

We get the neutrosophic random variable:

$$x_{1N} \in \{[16.3859,18.4887], [17.0788,19.2329], [17.7718,19.9773]\}$$

From the second figure we have: $\boldsymbol{R_{1N} \in [0.5125, 0.5227]}$

The data of the problem becomes:

Using the neutrosophic random number $R_{1N} \in [0.5125,0.5227]$ which follows a uniform distribution in the range $[0,1+\varepsilon]$ where $\varepsilon \in [0,0.02]$, find a random variable that follows the normal distribution defined by $\mu_N \in [15,17]$ and $\sigma_N \in \{2,3,4\}$

In the same way as before, we obtain the neutrosophic random variable:

$$x_{1N} \in \{[16.3859,18.4540], [17.0788,19.1809], [17.7718,19.9079]\}$$

From the third form we have: $\boldsymbol{R_{1N} \in [0.5027, 0.5227]}$

The data of the problem becomes:

Using the neutrosophic random number $R_{1N} \in [0.5027,0.5227]$ which follows a uniform distribution in the domain $[0+\varepsilon, 1+\varepsilon]$ where $\varepsilon \in [0,0.02]$, find a random variable that follows the normal distribution defined by $\mu_N \in [15,17]$ and $\sigma_N \in \{2,3,4\}$

In the same way as before, we obtain the neutrosophic random variable:

$$x_{1N} \in \{[16.3859,18.4540], [17.0788,19.7334], [17.7718,19.9079]\}$$

Comparison between the classical study and the neutrosophic study:

1- In the classical study, example (1), the data are specific values

The random number $R_1 = 0.5227$The normal distribution is defined by the parameters $\mu = 15$ and $\sigma = 2$.

We obtained a specific value for the required random variable:

$$x_1 = 16.4541$$

2- In the neutrosophic study:

The second case:

the random number $R_1 = 0.5227$. The normal distribution is defined by $\mu_N \in [15,17]$ and $\sigma_N \in \{2,3,4\}$we obtain a nitrosophic random variable (double indeterminacy):

$$x_{1N} \in \{[15.9111,17.9111], [16.1781,18.1781], [16.5708,18.5708]\}$$

The third case:

the neutrosophic random number normal distribution defined by $\mu_N \in [15,17]$ and $\sigma_N \in \{2,3,4\}$, according to the three forms of the neutrosophic random number, we obtain a neutrosophic random variable (double indeterminacy) which is:

According to the first figure:

$$x_{1N} \in \{[16.3859,18.4887], [17.0788,19.2329], [17.7718,19.9773]\}$$

According to the second figure:

$$x_{1N} \in \{[16.3859,18.4540], [17.0788,19.1809], [17.7718,19.9079]\}$$

According to the third figure:

$$x_{1N} \in \{[16.3859,18.4540], [17.0788,19.7334], [17.7718,19.9079]\}$$

3. Conclusion and Results:

From the above, we note that to obtain random variables that follow the normal distribution using the approximation technique, we use this technique with respect to the standard normal distribution, and then we use the transformation relationship defined between the normal distribution and the standard normal distribution to obtain the relationship through which we obtain random variables that follow the general normal distribution, and here When we take the data in the studied system as classical values, we get a single value for the random variable, and this value is appropriate and gives appropriate simulation results for working conditions that are completely similar to the conditions in which the data was collected, and any change may cause an unexpected loss. However, when we take the data as neutrosophic values, the random variable belongs the minimum range corresponds to the worst working conditions that the system to be simulated can experience, and the highest range corresponds to the best working conditions that the system to be simulated can experience. Therefore, simulating systems using the concepts of neutrosophic logic provides us with more accurate results than the results provided by the classic study.

References

[1] Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Doi :https://doi.org/10.54216/PAMDA.020102

[2] F. Smarandache. Introduction to Neutrosophic statistics, Sitech & Education Publishing, 2014.

[3] Faisal Al-Sharqi, Ashraf Al-Quran, and Zahari Md. Rodzi , Multi-Attribute Group Decision-Making Based on Aggregation Operator and Score Function of Bipolar Neutrosophic Hypersoft Environment, Neutrosophic Sets and Systems, Vol. 61, 2023, pp. 465-492. DOI: 10.5281/zenodo.10428642

[4] Mountajab Alhasan, Malath F Alaswad. A Study of the Movement of a Neutrosophic Material Point in the Neutrosophic Plane by Using a Neutrosophic AH-Isometry, Journal Prospects for Applied Mathematics and Data Analysis, Doi: https://doi.org/10.54216/PAMDA.020101,Vol 2, No 1, 2023.

[5] Mona Gharib, Florentin Smarandache, Mona Mohamed, CSsEv: Modelling QoS Metrics in Tree Soft Toward Cloud Services Evaluator based on Uncertainty Environment, Doi :https://doi.org/10.54216/IJNS.230204, Vol,23, No,2

[6] Said Broumi, S. krishna Prabha, Fermatean Neutrosophic Matrices and Their Basic Operations, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 572-595. DOI: 10.5281/zenodo.8404536

[7] Mohamed Abdel-Basset, Uncertainty-infused Representation Learning Using Neutrosophic-based Transformer Network, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 644-657. DOI: 10.5281/zenodo.10028758

[8] Florentin Smarandache, Introduction to Neutrosophic Genetics Doi: https://doi.org/10.54216/IJNS.130103 , Vol 13, No, 1.

[9] Maissam Jdid, Hla Hasan, The State of Risk and Optimum Decision According to Neutrosophic Rules, **Doi :**https://doi.org/10.54216/IJNS.200107, Volume 20, Issue 1

[10] S. Loganayaki, N. Rajeswari, K. Kirupa, S. Broumi**,** Optimization of triangular neutrosophic based economic order quantity model under preservation technology and power demand with shortages, Vol. 64 (2024): Neutrosophic Sets and Systems

[11] M. Anandhkumar, H. Prathab, S. M. Chithra, A. S. Prakaash, A. Bobin, **Secondary K-Range Symmetric Neutrosophic Fuzzy Matrices**, Volume 23, Issue 4, **Doi :**https://doi.org/10.54216/IJNS.230402

[12] Mohammed Alshikho, Maissam Jdid, Said Broumi, A Study of a Support Vector Machine Algorithm with an Orthogonal Legendre Kernel According to Neutrosophic logic and Inverse Lagrangian Interpolation, **Doi :** https://doi.org/10.54216/JNFS.050105

[13] Florentin Smarandache, Maissam Jdid, Research on the topics of neutrosophic operations research, https://fs.unm.edu/NeutrosophicOperationsResearch.pdf , Biblio Publishing, Volume (1),2023

[14] Maissam Jdid, Neutrosophic linear models and algorithms to find their optimal solution, Biblio Publishing, ISBN, 978_1 _59973_778_2, USA,2023 https://fs.unm.edu/NeutrosophicLinearModels-ar.pdf

[15] Maissam Jdid, NEUTROSOPHIC TRANSPORT AND ASSIGNMENT ISSUES, Publisher: Global Knowledge's, ISBN, 978_1 _59973_770_6, USA,2023 **https://fs.unm.edu/NeutrosophicOptimization-ar.pdf**,

[16] Maissam Jdid, Rafif Alhabib and A. A. Salama, Fundamentals of Neutrosophical Simulation for Generating Random Numbers Associated with Uniform Probability Distribution, Neutrosophic Sets and Systems, Vol. 49, 2022, pp. 92-102 _DOI: 10.5281/zenodo.6426375

[17] Maissam Jdid, A. Salama, Using the Inverse Transformation Method to Generate Random Variables that follow the Neutrosophic Uniform Probability Distribution, **Doi:** https://doi.org/10.54216/JNFS.060202

[18] Maissam Jdid, Rafif Alhabib and A. A. Salama**,** The Basics of Neutrosophic Simulation for Converting Random Numbers Associated with a Uniform Probability Distribution into Random Variables Follow an Exponential Distribution**,** Neutrosophic Sets and Systems, Vol. 53, 2023, pp. 358-366. DOI: 10.5281/zenodo.7536049

[19] Maissam Jdid, Said Broumi**,** Neutrosophical Rejection and Acceptance Method for the Generation of Random Variables**,** Neutrosophic Sets and Systems, Vol. 56, 2023, pp. 153-166. DOI: 10.5281/zenodo.8194749

[20] Maissam Jdid and Nada A. Nabeeh, Generating Random Variables that follow the Beta Distribution Using the Neutrosophic Acceptance-Rejection Method, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 139-148. DOI: 10.5281/zenodo.8404445
[21] Maissam Jdid, Florentin Smarandache, and Khalifa Al Shaqsi, Generating Neutrosophic Random Variables Based Gamma Distribution, Plithogenic Logic and Computation Vol. 1 (2024), 16-24, DOI: https://doi.org/10.61356/j.plc.2024.18760, https://sciencesforce.com/index.php/plc/article/view/87
[22] Maissam Jdid, Florentin Smarandache, Generating Neutrosophic Random Variables Following the Poisson Distribution Using the Composition Method (The Mixed Method of Inverse Transformation Method and Rejection Method), Neutrosophic Sets and Systems, Vol. 64, 2024, pp. 132-140. DOI: 10.5281/zenodo.10703089
[23] Maissam Jdid, Florentin Smarandache, The Box and Muller Technique for Generating Neutrosophic Random Variables Follow a Normal Distribution, Volume 23, Issue 4,**Doi:**https://doi.org/10.54216/IJNS.230406
[24] Alali. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004.
[25] Bukajh JS -. Mualla, W... and others-Operations Research Book translated into Arabic - The Arab Center for Arabization, Translation, Authoring and Publishing -Damascus ,1998.

Optimal Agricultural Land Use: An Efficient Neutrosophic Linear Programming Method

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract: The increase in the size of the problems facing humans, their overlap, the division of labor, the multiplicity of departments, as well as the diversity of products and commodities, led to the complexity of business and the emergence of many administrative and production problems. It was necessary to search for appropriate methods to confront these problems. The science of operations research, with its diverse methods, provided the optimal solutions. It addresses many problems and helps in making scientific and thoughtful decisions to carry out the work in the best way within the available capabilities. Operations research is one of the modern applied sciences that uses the scientific method as a basis and method in research and study, and its basic essence is to build a model that helps management in making decisions related to difficult administrative problems. For example, the military field, financial aspects, industry, in construction for building bridges and huge projects to evaluate the time taken for each project and reduce this time, financial markets and stocks and forecasting economic conditions, in hospital management and controlling the process of nutrition and medicines within the available capabilities, in agriculture Agricultural marketing and many other problems that have been addressed using classical operations research methods. We know that the agricultural sector is one of the important sectors in every country, and the agricultural production process is regulated by those responsible for securing the needs of citizens. Also, those responsible for the agricultural sector are responsible for rationalizing the agricultural process so that the surplus is saved. Due to the difficult circumstances that the country may be going through, in this research, we will reformulate the general model for the optimal distribution of agricultural lands using the concepts of neutrosophic science.

Keywords: Operations Research; Neutrosophic Science; Neutrosophic Linear Programming; Optimal Agricultural Land Use Model.

1. Introduction

Securing the needs of citizens is necessary and one of the major responsibilities that falls on officials in the state. This matter requires a scientific study of the reality of the state's situation and optimal exploitation of the available capabilities, that is, organizing work in all sectors of the state in a way that guarantees citizens a stable life in all circumstances. This matter prompted scientists and researchers are prepare scientific studies that help decision-makers make ideal decisions to manage the work of these sectors. The classical linear programming method was one of the most widely used methods [1-3], and it was relied upon even though the solutions it provided were appropriate solutions for conditions similar to those in which, data is collected about the case under study. Any change in this data will affect the optimal solution and thus the decisions of decision-makers, which

requires us to search for a new scientific method that provides us with optimal solutions suitable for all circumstances and takes into account all changes that may occur. In the work environment, we find that using the concepts of neutrosophic science, the science that takes into account the changes that can occur in the work environment through the indeterminacy of neutrosophic values. So, we have to reformulate many practical issues using the concepts of this science, which can be viewed from what was presented by the American scientist Florentin Smarandache, the founder of this science, and many researchers in various scientific fields [4-19]. Given the importance of the linear programming method, we presented in previous research the neutrosophic linear models [20]. In another research, we presented one of the most important methods used to find the optimal solution for the models. Linear, which is the simplex neutrosophic method [21]. Among the uses of neutrosophic linear programming, we presented a study on its use in the field of education [22]. As a continuation of our previous work, we present in this research a study whose purpose is to reformulate the model of optimal use of agricultural land using the concepts of neutrosophic. This will help decision-makers obtain an optimal solution that secures the needs of citizens for agricultural crops in all circumstances that the country may go through.

The structure of this paper is organized as follows. In Section 2, we briefly examine the concept used for solving the problem. In Section 3, we apply the introduced approach to a case study and discuss the obtained results. In Section 4, we discuss the conclusions of the paper in Section 3.

2. Materials and methods

The most important stage in linear programming is the stage of creating a linear programming model, and we mean expressing realistic relationships with assumed mathematical relationships based on the study and analysis of reality. In order to formulate a linear programming model, the following basic elements must be present:

- Determine the goal in a quantitative manner
  It is expressed by the objective function, which is the function for which the maximum or minimum value is required. It must be possible to express the goal quantitatively, such as if the goal is to achieve the greatest possible profit or secure the smallest possible cost.

- Determine the constraints
  The constraints on the available resources must be specific, that is, the resources must be measurable, and expressed in a mathematical formula in the form of inequalities or equals.

- Determine the goal in a quantitative manner
  This element indicates that the problem should have more than one solution so that linear programming can be applied because if the problem had one solution, there would be no need to use linear programming because its benefit is focused on helping to choose the best solution from among the multiple solutions [1].

3. Results and discussion

3.1 Problem definition

We will apply the above in Section 2, to the model of optimal use of agricultural land, using the concepts of neutrosophic science. We will take data that is affected by the surrounding conditions, and neutrosophic values.

- Text of the issue:

Let us assume that we have n agricultural areas (plain or cultivated), the area of each of which is equal to $A_1, A_2, \cdots\cdots, A_n$, we want to plant it with m types of agricultural crops to secure the community's requirements for it. Knowing that we need of crop i the amount b_i , if the average

productivity of one area in plain j of crop i is equal to Na_{ij}tons/ha. Where $j = 1,2, \dots \dots, n$ and $i = 1,2, \dots \dots, m$, and the profit returned from one unit of crop i equal to Np_i, Where Np_i is a neutrosophic value, an undefined non-specific value that designates a perfect and can be any neighbor of the value a_{ij}, also Np_iwhich can be any neighbor of p_i.

- Required:

Determine the amount of area needed to be cultivated with each crop and in all regions to achieve the greatest possible profit and meet the needs of society.

- Formulation of the mathematical model:

We symbolize by x_{ij} the amount of area in area j that must be cultivated with crop, and we place the data for the problem in Table 1.

Table 1. Issue data.

Regions / *Crops*	1	2	……	n	*Order amount* b_i	*profit amount* Np_i
1	Na_{11} x_{11}	Na_{12} x_{12}	……	Na_{1n} x_{1n}	b_1	Np_1
2	Na_{21} x_{21}	Na_{22} x_{22}	……	Na_{2n} x_{2n}	b_2	Np_2
……	……	……	……	……	……	……
m	Na_{m1} x_{m1}	Na_{m2} x_{m2}	……	Na_{mn} x_{mn}	b_m	Np_m
Available space a_i	a_1	a_2	……	a_n		

Then we find that the conditions imposed on the variables x_{ij} are:

1. Space restrictions

   The total area allocated to various crops in area j must be equal to a_j, that is, it must be:

$$x_{11} + x_{12} + \cdots\cdots + x_{m1} = a_1$$
$$x_{12} + x_{22} + \cdots\cdots + x_{m2} = a_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_{1n} + x_{2n} + \cdots\cdots + x_{mn} = a_n$$

2. Conditions for meeting community requirements

   The total production of crop i in all regions must not be less than the amount b_i, that is, it must be:

$$Na_{11}x_{11} + Na_{12}x_{12} + \cdots\cdots + Na_{1n}x_{1n} \geq b_1$$
$$Na_{21}x_{21} + Na_{22}x_{22} + \cdots\cdots + Na_{2n}x_{2n} \geq b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Na_{m1}x_{m1} + Na_{m2}x_{m2} + \cdots\cdots + Na_{mn}x_{mn} \geq b_m$$

3. Find the objective function
   - We note that the profit resulting from the production of crop i only and from all regions is equal to the product of the profit times the quantity and i.e.:

$$Np_i(Na_{i1}x_{i1} + Na_{i2}x_{i2} + \cdots\cdots + Na_{in}x_{in})$$

Thus, we find that the objective function, which expresses the total profit resulting from all crops, is equal to:

$$Z = Np_1\left(\sum_{j=1}^{n} Na_{1j}\ x_{1j}\right) + Np_2\left(\sum_{j=1}^{n} Na_{2j}\ x_{2j}\right) + \cdots\cdots + Np_m\left(\sum_{j=1}^{n} Na_{mj}\ x_{mj}\right) \to Max$$

- From the above, we get the following mathematical model:

Find the maximum value of

$$Z = Np_1\left(\sum_{j=1}^{n} Na_{1j}\ x_{1j}\right) + Np_2\left(\sum_{j=1}^{n} Na_{2j}\ x_{2j}\right) + \cdots\cdots + Np_m\left(\sum_{j=1}^{n} Na_{mj}\ x_{mj}\right) \to Max$$

Within restrictions:

$$x_{11} + x_{12} + \cdots\cdots + x_{m1} = a_1$$
$$x_{12} + x_{22} + \cdots\cdots + x_{m2} = a_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_{1n} + x_{2n} + \cdots\cdots + x_{mn} = a_n$$

$$a_{11}x_{11} + a_{12}x_{12} + \cdots\cdots + a_{1n}x_{1n} \geq b_1$$
$$a_{21}x_{21} + a_{22}x_{22} + \cdots\cdots + a_{2n}x_{2n} \geq b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_{m1} + a_{m2}x_{m2} + \cdots\cdots + a_{mn}x_{mn} \geq b_m$$

$$x_{ij} \geq 0 \ \ ; i = 1,2,\cdots\cdots, m\ , j = 1,2,\cdots\cdots, n$$

3.2 Example

Let us assume that we want to exploit four agricultural areas A_1, A_2, A_3, A_4, and the area of each of them, respectively, is 60,150,20,10, by planting them with the following crops: wheat, barley, cotton, tobacco, and beet, from which we need the following: $800, 200,600,1000,2500$ Let us assume that the regions' productivity of these crops and their prices are given in Table 2.

Table 2. Example data.

Regions / Crops	A_1	A_2	A_3	A_4	The order	Price per ton
wheat	{4,6}	*4*	3	*6*	*2500*	{1400,1600}
barley	7	*5*	*4*	{3,5}	*1000*	{900,1100}
cotton	*4*	{9,11 }	*8*	*5*	*600*	{4500,6000}
tobacco	6	{2,4}	*0*	*0*	*200*	{4000,5000}
beet	*3*	{10,14}	*10*	6	*800*	{400,700}
Space	*60*	*150*	*20*	*10*		

- Required:

Formulate the mathematical model for this issue so that the production value is as large as possible. To formulate the mathematical model, we extract the following linear conditions:

- Space restrictions

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 60$$
$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} = 150$$
$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 20$$
$$x_{14} + x_{24} + x_{34} + x_{44} + x_{54} = 10$$

- Order restrictions

$$\{4,6\}x_{11} + 4x_{12} + 3x_{13} + 6x_{14} \geq 2500$$
$$7x_{21} + 5x_{22} + 4x_{23} + \{3,5\}x_{24} \geq 1000$$
$$4x_{31} + \{9,11\}x_{32} + 8x_{33} + 5x_{34} \geq 600$$
$$6x_{41} + \{2,4\}x_{42} + 0x_{43} + 0x_{44} \geq 200$$
$$3x_{51} + \{10,14\}x_{52} + 10x_{53} + 6x_{54} \geq 800$$

- Non-Negative restrictions

$$x_{ij} \geq 0 \ ; i = 1,2,3,4,5 \quad \text{and} \quad j = 1,2,3,4$$

- Objective function that express the value of production is:

$$Z = \{1400,1600\}(\{4,6\}x_{11} + 4x_{12} + 3x_{13} + 6x_{14}) + \{900,1100\}(7x_{21} + 5x_{22} + 4x_{23} + \{3,5\}x_{24}) + \{4500,6000\}(4x_{31} + \{9,11\}x_{32} + 8x_{33} + 5x_{34}) + \{4000,5000\}(6x_{41} + \{2,4\}x_{42} + 0x_{43} + 0x_{44}) + \{400,700\}(3x_{51} + \{10,14\}x_{52} + 10x_{53} + 6x_{54}) \rightarrow \text{Max}$$

- Mathematical model:
  - Find the maximum value of

$$Z = \{1400,1600\}(\{4,6\}x_{11} + 4x_{12} + 3x_{13} + 6x_{14}) + \{900,1100\}(7x_{21} + 5x_{22} + 4x_{23} + \{3,5\}x_{24}) + \{4500,6000\}(4x_{31} + \{9,11\}x_{32} + 8x_{33} + 5x_{34}) + \{4000,5000\}(6x_{41} + \{2,4\}x_{42} + 0x_{43} + 0x_{44}) + \{400,700\}(3x_{51} + \{10,14\}x_{52} + 10x_{53} + 6x_{54}) \rightarrow \text{Max}$$

  - Within restrictions:

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 60$$
$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} = 150$$
$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 20$$
$$x_{14} + x_{24} + x_{34} + x_{44} + x_{54} = 10$$
$$\{4,6\}x_{11} + 4x_{12} + 3x_{13} + 6x_{14} \geq 2500$$
$$7x_{21} + 5x_{22} + 4x_{23} + \{3,5\}x_{24} \geq 1000$$
$$4x_{31} + \{9,11\}x_{32} + 8x_{33} + 5x_{34} \geq 600$$
$$6x_{41} + \{2,4\}x_{42} + 0x_{43} + 0x_{44} \geq 200$$
$$3x_{51} + \{10,14\}x_{52} + 10x_{53} + 6x_{54} \geq 800$$
$$x_{ij} \geq 0 \ ; i = 1,2,3,4,5 \quad \text{and} \quad j = 1,2,3,4$$

  - It is a linear model; we use the simplex method to obtain an optimal solution.

4. Conclusions

In this study, the authors presented a new formulation of the model for optimal use of agricultural land using the concepts of neutrosophic science, where we took data that are affected by

the surrounding conditions. Neutrosophic values take into account fluctuations in ambient conditions, from natural factors that can affect crop yields to price fluctuations that can affect profit. We obtained a linear neutrosophic mathematical model that can be solved using the simplex neutrosophic method that was presented in previous research. Then, the optimal solution is the values of the variables that express the areas that can be allocated in each of the regions for each crop.

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Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Alali. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004. (Arabic version).
2. Linear and Nonlinear Programming-DavidG. Luenbrgrr. YinyuYe- Springer Science + Business Media-2015.
3. Maissam Jdid Operations Research, Faculty of Informatics Engineering, Al-Sham Private University Publications, 2021
4. Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Applied Mathematics and Data Analysis, Vo .2,No .1, 2023
5. Maissam jdid ,Important Neutrosophic Economic Indicators of the Static Model of Inventory Management without Deficit, Journal of Neutrosophic and Fuzzy Systems (JNFS),Vo .5,No .1, 2023
6. Mohammed Alshikho, Maissam Jdid, Said Broumi, Artificial Intelligence and Neutrosophic Machine learning in the Diagnosis and Detection of COVID 19, Journal Prospects for Applied Mathematics and Data Analysis ,Vol 01, No,02 USA,2023
7. Mohammed Alshikho, Maissam Jdid, Said Broumi ,A Study of a Support Vector Machine Algorithm with an Orthogonal Legendre Kernel According to Neutrosophic logic and Inverse Lagrangian Interpolation, , Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .5,No .1, 2023
8. Maissam Jdid, Khalifa Alshaqsi, Optimal Value of the Service Rate in the Unlimited Model $M\backslash M\backslash 1$, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .6,No .1, 2023
9. Maissam Jdid, Florentin Smarandache, The Use of Neutrosophic Methods of Operation Research in the Management of Corporate Work, , Journal of Neutrosophic Systems with Applications, Vol. 3, 2023
10. Maissam Jdid, Florentin Smarandache , Said Broumi, Inspection Assignment Form for Product Quality Control, Journal of Neutrosophic Systems with Applications, Vol. 1, 2023
11. Maissam Jdid, Neutrosophic Mathematical Model of Product Mixture Problem Using Binary Integer Mutant, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .6,No .2, 2023
12. Maissam Jdid, Neutrosophic Nonlinear Models, Journal Prospects for Applied Mathematics and Data Analysis, Vo .2,No .1, 2023
13. Maissam Jdid , Florentin Smarandache, Lagrange Multipliers and Neutrosophic Nonlinear Programming Problems Constrained by Equality Constraints, Journal of Neutrosophic Systems with Applications, Vol. 6, 2023

14. Maissam Jdid, Said Broumi, Neutrosophic Rejection and Acceptance Method for the Generation of Random Variables, Neutrosophic Sets and Systems, NSS, Vol.56, 2023.
15. Maissam Jdid, Florentin Smarandache, Optimal Neutrosophic Assignment and the Hungarian Method, Neutrosophic Sets and Systems, NSS, Vol.57, 2023.
16. Sudeep Dey, Gautam Chandra Ray, Separation Axioms in Neutrosophic Topological Spaces, Neutrosophic Systems with Applications, vol.2, (2023): pp. 38–54. https://doi.org/10.61356/j.nswa.2023.9.
17. Ahmed Sleem, Nehal Mostafa, Ibrahim Elhenawy, Neutrosophic CRITIC MCDM Methodology for Ranking Factors and Needs of Customers in Product's Target Demographic in Virtual Reality Metaverse, Neutrosophic Systems with Applications, vol.2, (2023): pp. 55–65. https://doi.org/10.61356/j.nswa.2023.10.
18. Ibrahim M. Hezam, An Intelligent Decision Support Model for Optimal Selection of Machine Tool under Uncertainty: Recent Trends, Neutrosophic Systems with Applications, vol.3, (2023): pp. 35–44. https://doi.org/10.61356/j.nswa.2023.12.
19. Maissam jdid- Hla Hasan, The state of Risk and Optimum Decision According to Neutrosophic Rules, International Journal of Neutrosophic Science (IJNS), Vol. 20, No. 1, 2023
20. Maissam Jdid, Huda E Khalid, Mysterious Neutrosophic Linear Models, International Journal of Neutrosophic Science, Vol.18, No. 2, 2022
21. Maissam Jdid, AA Salama, Huda E Khalid, Neutrosophic Handling of the Simplex Direct Algorithm to Define the Optimal Solution in Linear Programming, International Journal of Neutrosophic Science, Vol.18, No. 1, 2022
22. Maissam Jdid, The Use of Neutrosophic linear Programming Method in the Field of Education, Handbook of Research on the Applications of Neutrosophic Sets Theory and Their Extensions in Education, Chapter 15, IGI-Global, 2023.

Decision Making in the Case of Confirmed Data Neutrosophic Linear Models to Choose the Advertising Medium

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract

In light of the great development witnessed by our contemporary world, it has become necessary to focus on scientific methods and use the quantitative method to reach more accurate decisions, appropriate to the surrounding circumstances and factors. The process of decision-making and choosing the optimal alternative depends on the type and quality of data that describes the issue for which the decision is to be made. Regarding it, in this chapter we present a study of the issue of determining the ideal advertising medium to display a company's products. This issue is considered one of the issues of decision-making in the case of confirmed data, so we build the appropriate mathematical model and through the optimal solution to it we can make the ideal decision through which the company achieves its goal from the campaign. Informative, we will divide this study into two parts. In the first section, we will develop a general formula for this issue, and the data will be classical values. We will obtain a linear mathematical model. In the second section, we will formulate the issue from the perspective of neutrosophic science, meaning we will take the data as neutrosophic values, obtaining a linear neutrosophic model.

Keywords: Decision making in the case of confirmed data; Linear models; Neutrosophic science; Neutrosophic linear models; The issue of choosing the advertising medium

1. Introduction

In the world of rapid changes, there has become an urgent need to make a rational decision based on quantitative methods that limit the level of risk, especially if the decisions are critical and the decision issues are huge and complex. Decision making theory is one of the important theories provided by operations research. The scientific basis for this theory depends on the data of the case under study. This data forms the basic description of the problem, from which we can determine the type of solution for the case. In decision theory, we have three types of data: certain data, uncertain data, and random data that repeat according to a certain probability distribution law. In the case of confirmed data, we can formulate the problem in the form of a mathematical model with the optimal solution that enables us to choose the appropriate alternative. Classic studies of decision theory provided appropriate rules for choosing the optimal decision for each type of data. After the emergence of neutrosophic logic, many studies were presented in most fields using the concepts of this logic. This is evidenced by what was presented by (Smarandache et al., 2023), and as for the theory of decision-making, (Jdid et al., 2022) presented a neutrosophic study of the theory of decision-making in the case of uncertain data, and (Jdid et al.,2023) also presented a neutrosophic study of the theory of decision-making. In case of risk. In this chapter, we present a neutrosophical study of the issue of choosing the appropriate advertising medium, which is a question of making a decision (the case of confirmed data), meaning that the decision maker can determine the results of each of the available alternatives with certainty. Using the information provided by (Bukajh et al., 1998), we provide a general formulation of this issue. The optimal solution to the mathematical model represented in it helps decision makers choose the appropriate decision that achieves the company's goal from the advertising campaign. Using the information provided by (Jdid et al., 2022), we will reformulate the problem using the concepts of neutrosophic logic and thus obtain a linear neutrosophic model. We use the simple neutrosophic algorithm presented by (Jdid et al., 2022) or the modified simple neutrosophic algorithm presented by (Jdid et al., 2024) to obtain the ideal neutrosophic solution that enables us to make the appropriate decision that achieves the company's goal from the advertising campaign.

2. Discussion

1- Mathematical background:

The science of operations research, with its various methods, has provided optimal solutions to many issues and helped make decisions based on scientific foundations. Its basic essence is to build a model and then use appropriate algorithms to obtain the optimal solution to it, through which the ideal and appropriate decision for the nature of the issue under study is determined. The most important methods of operations research are the linear programming method because we find that many real-world problems can be represented by a linear model and also because there are many algorithms that help us obtain the optimal solution. Given this importance, we have reformulated the linear models and the most important algorithms used to find the optimal solution using the concepts of neutrosophic logic. In this chapter, we present a neutrosophical study of one of the decision-making issues in the case of confirmed data, which can be formulated through a linear model whose optimal solution gives us the appropriate decision, which is the issue of choosing the advertising medium.

1-1- A model for choosing the advertising medium. See (Bukajh et al., 1998).

The text of the issue as stated in the reference:

An advertising company wants to plan a media campaign using three different means: television, radio, and magazines. This campaign aims to reach the largest possible number of potential customers. The results of the market study give the following table:

Table 1: Issue data

advertising medium Information	TV		Radio	Magazines
	During the day	During the evening		
Cost of advertising unit	40000$	75000$	30000$	15000$
The number of customers that can be reached for the advertising unit	400000	900000	500000	200000
The number of female customers that can be reached by the advertising unit	300000	400000	200000	100000

Assuming that the company does not want to spend more than 800000$ on advertising, and that it wants the following:

a. The number of views from women should reach at least two million.
b. The cost of advertising on TV is set at 500000$.
c. To purchase at least three advertising units on TV during the day and two units in the evening.
d. The number of advertising units on radio and magazines should range between 5 and 10 units for each.

We assume that the number of advertising units purchased on TV during the day and in the evening, on the radio, and on magazines, respectively, is: x_1, x_2, x_3, x_4 so the number of potential customers estimated in thousands is:

$$400x_1 + 900x_2 + 500x_3 + 200x_4$$

Since the company does not want to spend more than 800000$ on advertising, we get the following constraint:

$$40000x_1 + 75000x_2 + 30000x_3 + 15000x_4 \leq 800000$$

The restriction on the number of female views that can be achieved through the media campaign is:

$$300000x_1 + 400000x_2 + 200000x_3 + 100000x_4 \geq 2000000$$

The cost restriction for TV advertising is:

$$40000x_1 + 75000x_2 \leq 500000$$

The restriction number of units in the TV:

$$x_1 \geq 3$$

$$x_2 \geq 2$$

The restriction on the number of advertising units on radio is:

$$5 \leq x_3 \leq 10$$

The restriction on the number of advertising units on magazines is:

$5 \leq x_4 \leq 10$

Mathematical model:
Find:

$$MaxZ = 400000x_1 + 900000x_2 + 500000x_3 + 200000x_4$$

Constraints:

$$40000x_1 + 75000x_2 + 30000x_3 + 15000x_4 \leq 800000$$

$$300000x_1 + 400000x_2 + 200000x_3 + 100000x_4 \geq 2000000$$

$$40000x_1 + 75000x_2 \leq 500000$$

$$x_1 \geq 3$$

$$x_2 \geq 2$$

$$x_3 \geq 5$$

$$x_3 \leq 10$$

$$x_4 \geq 5$$

$$x_4 \leq 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

1-2- A study of neutrosophic linear models, see (Jdid et al., 2022).

Neutrosophic linear models are linear models that include in their composition a set of neutrosophic variables, which can undergo change as a result of the work environment, and the presence of neutrosophic values guarantees that the facility that operates according to these models has a safe workflow, in which the variables in the objective function are neutrosophic values, i.e. $c_j \pm \varepsilon_j$ Where ε_j is the indeterminacy and it can be $\varepsilon_j \in [\lambda_1, \lambda_2]$or $\varepsilon_j \in \{\lambda_1, \lambda_2\}$ where $j = 1,2, \dots, n$
Also, the values that express the available possibilities are neutrosophic values. This means that $b_i \pm \delta_i$ where δ_i is indeterminacy and it is possible to be
$\delta_i \in [\mu_1, \mu_2]$or $\delta_i \in \{\mu_1, \mu_2\}$ where $i = 1,2, \dots, m$, and therefore the neutrosophic linear model is written in the following general form:
The general neutrosophic formula for the linear mathematical model is given in abbreviated form as follows:

$$NZ = \sum_{j=1}^{n} (c_j \pm \varepsilon_j) x_j \longrightarrow Max \; or \; Min$$

Constraints:

$$\sum_{j=1}^{n} Na_{ij} x_j \begin{pmatrix} \geq \\ \leq \\ = \end{pmatrix} b_i \pm \delta_i \quad ; \; i = 1,2, \dots, m$$

$$x_j \geq 0$$

Where $c_j + \varepsilon_j$, $b_i \pm \delta_i$, a_{ij}, $j = 1,2, \dots, n$, $i = 1,2, \dots, m$ are constants having set or interval values according to the nature of the given problem, x_j are decision variables.

Future research direction:

In this chapter, we will develop a general formulation of one of the important issues that we encounter in our daily lives, which is the issue of choosing the appropriate advertising medium through which we achieve access to the largest possible number of customers to display a company's products. This is in two cases:

1-3- Model data are classical values

Formulation of the issue:

A company wants to establish an advertising campaign to display its products using $\boldsymbol{n}$ media means. This campaign aims to reach the largest possible number of customers. The team responsible for collecting information about each of the means provided the following data:

Table 2: Data for the classic problem

advertising medium Information	A_1	A_2	...	A_n
Cost of advertising unit	c_1	c_2	...	c_n
The number of customers that can be reached for the advertising unit	b_1	b_2	...	b_n
The number of female customers that can be reached by the advertising unit	w_1	w_2	...	w_n
The number of male customers that can be reached for the advertising unit	m_1	m_2	...	m_n

Assuming that the company does not want to spend more than $d\$$ on advertising, and that it wants the following:

a. The number of female views must reach at least S.
b. The number of male views should reach at most K.
c. The cost of advertising through the advertising medium A_r should not exceed the amount $d_r\$$.
d. The number of advertising units in the advertising medium A_t should range between E and F unit.

Assuming that the number of advertising units purchased in each advertising method $A_1, A_2, \dots, A_n$ is $x_1, x_2, \dots, x_n$, then the number of potential customers is:

$$b_1x_1 + b_2x_2 + \cdots + b_nx_n$$

Since the company does not want to spend more than d dollars on advertising, we get the following constraint:

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \leq d$$

The restriction on the number of female views that can be achieved through the media campaign is:

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n \geq S$$

The restriction on the number of male views that can be achieved through the media campaign is:

$$m_1x_1 + m_2x_2 + \cdots + m_nx_n \leq K$$

The cost entry for advertising using the advertising medium A_r is:

$$c_rx_r \leq d_r$$

The restriction on the number of advertising units in the advertising medium A_t is:

$$E \leq x_t \leq F$$

Mathematical model:

Find:

$$MaxZ = b_1x_1 + b_2x_2 + \cdots + b_nx_n$$

Constraints:

$$c_1x_1 + c_2x_2 + \cdots c_nx_n \leq d$$

$$w_1x_1 + w_2x_2 + \cdots w_nx_n \geq S$$

$$m_1x_1 + m_2x_2 + \cdots m_nx_n \leq K$$

$$c_rx_r \leq d_r$$

$$E \leq x_t \leq F$$

$$x_1, x_2, \dots, x_n \geq 0$$

Example 1:

We rephrase the example in reference (Bukajh et al., 1998) as follows:

A company that produces women's bags and shoes wants to establish a media campaign to display its products using three media means: TV, radio, and road signs. This campaign aims to reach the largest possible number of customers. The team responsible for collecting information about each of the three means provided the following data:

Table 3: Classic example data

advertising medium Information	TV		Radio	Road signs
	During the day	During the evening		
Cost of advertising unit	40000$	75000$	30000$	15000$
The number of customers that can be reached for the advertising unit	400000	900000	500000	200000
The number of female customers that can be reached by the advertising unit	300000	400000	200000	100000

Assuming that the company does not want to spend more than 800000$ on advertising, and that it wants the following:

a. The number of views from women should reach at least two million.
b. The cost of advertising on TV is set at 500000$.
c. To purchase at least three advertising units on TV during the day and two units in the evening.
d. The number of advertising units on radio and road billboards should range between 5 and 10 units for each.

We assume that the number of advertising units purchased on TV during the day and in the evening, on the radio, and on road signs, respectively, is: x_1, x_2, x_3, x_4 so the number of potential customers estimated in thousands is:

$$400x_1 + 900x_2 + 500x_3 + 200x_4$$

Since the company does not want to spend more than 800000$ on advertising, we get the following constraint:

$$40000x_1 + 75000x_2 + 30000x_3 + 15000x_4 \leq 800000$$

The restriction on the number of female views that can be achieved through the media campaign is:

$$300000x_1 + 400000x_2 + 200000x_3 + 100000x_4 \geq 2000000$$

The cost restriction for TV advertising is:

$$40000x_1 + 75000x_2 \leq 500000$$

The restriction number of units in the TV:

$$x_1 \geq 3$$

$$x_2 \geq 2$$

The restriction on the number of advertising units on radio is:

$$5 \leq x_3 \leq 10$$

The restriction on the number of advertising units on road signs is:

$5 \leq x_4 \leq 10$

Mathematical model:

Find:

$$MaxZ = 400000x_1 + 900000x_2 + 500000x_3 + 200000x_4$$

Constraints:

$$40000x_1 + 75000x_2 + 30000x_3 + 15000x_4 \leq 800000$$

$$300000x_1 + 400000x_2 + 200000x_3 + 100000x_4 \geq 2000000$$
$$40000x_1 + 75000x_2 \leq 500000$$
$$x_1 \geq 3$$
$$x_2 \geq 2$$
$$x_3 \geq 5$$
$$x_3 \leq 10$$
$$x_4 \geq 5$$
$$x_4 \leq 10$$
$$x_1, x_2, x_3, x_4 \geq 0$$

The model we obtained is a linear model. We use the direct simplex algorithm or one of its modifications to find the optimal solution through which the company achieves the goal of the media campaign.

1-4- Model data are neutrosophic values:

Based on the study presented in the paper (Jdid et al., 2022) of the linear neutrosophic models, we can develop the following general formula for the neutrosophic media selection model.

Formulation of the issue:
A company wants to establish an advertising campaign to display its products using n media means. This campaign aims to reach the largest possible number of customers. The team responsible for collecting information about each of the means provided the following data:

Table 4: Issue data Neutrosophic values

advertising medium Information	A_1	A_2	...	A_n
Cost of advertising unit	$C_{1N} = c_1 \pm \varepsilon_1$	$C_{2N} = c_2 \pm \varepsilon_2$	...	$C_{nN} = c_n \pm \varepsilon_n$
The number of customers that can be reached for the advertising unit	$B_{1N} = b_1 \pm \delta_1$	$B_{2N} = b_2 \pm \delta_2$	...	$B_{nN} = b_n \pm \delta_n$
The number of female customers that can be reached by the advertising unit	$W_{1N} = w_1 \pm \gamma_1$	$W_{2N} = w_2 \pm \gamma_2$	...	$W_{nN} = w_n \pm \gamma_n$
The number of male customers that can be reached for the advertising unit	$M_{1N} = m_1 \pm \beta_1$	$M_{2N} = m_2 \pm \beta_2$	...	$M_{nN} = m_n \pm \beta_n$

Where $\varepsilon_j, \delta_j, \gamma_j, \beta_j$ is the indeterminacy of the cost of the advertising unit, the number of possible customers, the number of female customers, and the number of male customers. It may be a field or a group and expresses the increase or decrease to which the data of the issue may be exposed.
Assuming that the company does not want to spend more than d\$ on advertising, and that it wants the following:

a. The number of female views must reach at least S.
b. The number of male views should reach at most K.
c. The cost of advertising through the advertising medium A_r should not exceed the amount d_r\$.
d. The number of advertising units in the advertising medium $A_t$should range between E and F unit.

Assuming that the number of advertising units purchased in each advertising method $A_1, A_2, \dots, A_n$ is $x_1, x_2, \dots, x_n$, then the number of potential customers is:

$$B_{1N}x_1 + B_{2N}x_2 + \cdots + B_{nN}x_n$$

Since the company does not want to spend more than d dollars on advertising, we get the following constraint:

$$C_{1N}x_1 + C_{2N}x_2 + \cdots + C_{nN}x_n \leq d$$

The restriction on the number of female views that can be achieved through the media campaign:

$$W_{1N}x_1 + W_{2N}x_2 + \cdots + W_{nN}x_n \geq S$$

The restriction on the number of male views that can be achieved through the media campaign is:

$$M_{1N}x_1 + M_{2N}x_2 + \cdots + M_{nN}x_n \leq K$$

The cost entry for advertising using the advertising medium A_r is:

$$C_{rN}x_r \leq d_r$$

The restriction on the number of advertising units in the advertising medium A_t is:

$$E \leq x_t \leq F$$

Mathematical model:

Find:

$$MaxZ_N = B_{1N}x_1 + B_{2N}x_2 + \cdots + B_{nN}x_n$$

Constraints:

$$C_{1N}x_1 + C_{2N}x_2 + \cdots + C_{nN}x_n \leq d$$

$$W_{1N}x_1 + W_{2N}x_2 + \cdots + W_{nN}x_n \geq S$$

$$M_{1N}x_1 + M_{2N}x_2 + \cdots + M_{nN}x_n \leq K$$

$$C_{rN}x_r \leq d_r$$

$$E \leq x_t \leq F$$

$$x_1, x_2, \ldots, x_n \geq 0$$

Example 2:

Neutrosophic formulation of the example in reference **(**Bukajh et al., 1998) using the information in reference (Jdid et al., 2022):

A company that produces women's bags and shoes wants to establish a media campaign to display its products using three media means: TV, radio, and road signs. This campaign aims to reach the largest possible number of customers. The team responsible for collecting information about each of the three means provided the following data:

Table 5: Example data Neutrosophic values

advertising medium Information	TV		Radio	Road signs
	During the day	During the evening		
Cost of advertising unit	$40000 + \varepsilon_1\$$	$75000 + \varepsilon_2\$$	$30000 + \varepsilon_3\$$	$15000 + \varepsilon_4\$$
The number of customers that can be reached for the advertising unit	400000	900000	500000	200000
The number of female customers that can be reached by the advertising unit	300000	400000	200000	100000

Assuming that the company does not want to spend more than $800000\$$ on advertising, and that it wants the following:

a. The number of views from women should reach at least two million
b. The cost of advertising on TV is set at $500000\$$
c. To purchase at least three advertising units on TV during the day and two units in the evening.
d. The number of advertising units on radio and road billboards should range between 5 and 10 units for each.

We assume that the number of advertising units purchased on TV during the day and in the evening, on the radio, and on road signs, respectively, is: x_1, x_2, x_3, x_4 so the number of potential customers estimated in thousands is:

$$400x_1 + 900x_2 + 500x_3 + 200x_4$$

Since the company does not want to spend more than $800000\$$ on advertising, we get the following constraint:

$$(40000 \pm \varepsilon_1)x_1 + (75000 \pm \varepsilon_2)x_2 + (30000 \pm \varepsilon_3)x_3 + (15000 \pm \varepsilon_4)x_4 \leq 800000$$

The restriction on the number of female views that can be achieved through the media campaign is:

$$300000x_1 + 400000x_2 + 200000x_3 + 100000x_4 \geq 2000000$$

The cost restriction for TV advertising is:

$$(40000 + \varepsilon_1)x_1 + (75000 + \varepsilon_2)x_2 \leq 500000$$

The restriction number of units in the TV:

$$x_1 \geq 3$$

$$x_2 \geq 2$$

The restriction on the number of advertising units on radio is:

$$5 \leq x_3 \leq 10$$

The restriction on the number of advertising units on road signs is:

$5 \leq x_4 \leq 10$

Taking $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = [0,5000]$,we get the following form:

Mathematical model:
Find:

$$MaxZ = 400000x_1 + 900000x_2 + 500000x_3 + 200000x_4$$

Constraints:

$$[40000,45000]x_1 + [75000,80000]x_2 + [30000,35000]x_3 + [15,20000]x_4 \leq 800000$$

$$300000x_1 + 400000x_2 + 200000x_3 + 100000x_4 \geq 2000000$$

$$(40000 + \varepsilon_1)x_1 + (75000 + \varepsilon_2)x_2 \leq 500000$$

$$x_1 \geq 3$$

$$x_2 \geq 2$$

$$x_3 \geq 5$$

$$x_3 \leq 10$$

$$x_4 \geq 5$$

$$x_4 \leq 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

The model we obtained is a linear neutrosophic model. We use the direct simplex algorithm (see (Jdid et al.2022), or the modified neutrosophic simplex algorithm (see (Jdid et al.2024), to find the optimal solution through which the company achieves the goal of the media campaign.

3. Conclusion and results

Through the previous study, we presented a general formulation of an important issue that we encounter in daily life and which many companies need to market their products. We presented the issue using classical data and obtained a linear model through which the company achieves the optimal goal of the advertising campaign. In order for the study to be more realistic and appropriate for all circumstances, we by formulating the issue from the perspective of neutrosophic science, we obtained a neutrosophic linear model whose optimal solution is neutrosophic values that enjoy a margin of freedom, take into account all circumstances, and achieve the company's goal of the advertising campaign.

References

[1] Florentin Smarandache , Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Prospects for Applied Mathematics and Data Analysis, Volume 2 , Issue 1, PP: 19-26 , 2023 ,Doi :https://doi.org/10.54216/PAMDA.020102

[2] Maissam Jdid , Basel Shahin , Fatima Al Suleiman , Important Neutrosophic Rules for Decision-Making in the Case of Uncertain Data , International Journal of Neutrosophic Science ,Volume 18 , Issue 3, PP: 166-176 , 2022, Doi :https://doi.org/10.54216/IJNS.1803014

[3] Maissam Jdid, Hla Hasan, The State of Risk and Optimum Decision According to Neutrosophic Rules, International Journal of Neutrosophic Science, Volume 20 , Issue 1, PP: 77-85 , 2023,Doi :https://doi.org/10.54216/IJNS.200107

[4] Bukajh J.S -. Mualla, W... and others - Operations Research Book translated into Arabic - The Arab Center for Arabization, Translation, Authoring and Publishing -Damascus -1998.

[5] Maissam Jdid, Huda E. Khalid, Mysterious Neutrosophic Linear Models, International Journal of Neutrosophic Science, Volume 18 , Issue 2, PP: 243-253 , 2022,Doi :https://doi.org/10.54216/IJNS.180207

[6] Maissam Jdid, A. A. Salama, Huda E. Khalid, Neutrosophic Handling of the Simplex Direct Algorithm to Define the Optimal Solution in Linear Programming, International Journal of Neutrosophic Science, Volume 18 , Issue 1, PP: 30-41 , 2022,Doi :https://doi.org/10.54216/IJNS.180104

[7] Maissam Jdid, Florentin Smarandache, Neutrosophic Treatment of the Modified Simplex Algorithm to find the Optimal Solution for Linear Models, International Journal of Neutrosophic Science, Volume 23 , Issue 1, PP: 117-124 , 2024 ,Doi :https://doi.org/10.54216/IJNS.230110

Generating Neutrosophic Random Variables Following the Poisson Distribution Using the Composition Method (The Mixed Method of Inverse Transformation Method and Rejection Method)

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract:
Simulation is a numerical technique used to perform tests on a numerical computer, and involves logical and mathematical relationships interacting with each other to describe the behavior and structure of a complex system in the real world over a period of time. Analysis using simulation is a "natural" and logical extension of the mathematical analytical models inherent in operations research, because most operations research methods depend on building mathematical models that closely approximate the real-world environment and we obtain the optimal solution for them using algorithms appropriate to the type of these models. The importance of the simulation process comes In all branches of science, there are many systems that cannot be studied directly, due to the great difficulty that we may encounter when studying, and the high cost, in addition to the fact that some systems cannot be studied directly. The simulation process depends on generating a series of numbers. Randomness subject to a uniform probability distribution over the domain [0,1] , then converting these numbers into random variables subject to the law of probability distribution by which the system to be simulated works, using known transformation methods. In previous research, we presented a neutrosophical vision of the reverse transformation method and the method of rejection and acceptance. Which are used to transform random numbers into random variables that follow probability distributions such as: uniform distribution, exponential distribution, beta distribution..., In this research, we present a neutrosophical vision of the Composition method (the mixed method of inverse transformation method and rejection method)**,** used to generate random variables that follow... To some Poisson distribution, the aim is to obtain neutrosophic random variables that we use when simulating systems that operate according to this distribution in order to obtain more accurate simulation results.

Introduction:

To keep pace with the great scientific development that our contemporary world is witnessing, it was necessary to reformulate operations research methods according to the

basic concepts of neutrosophic logic, because the margin of freedom enjoyed by neutrosophic values gives more accurate results, which has prompted many researchers to prepare many researches in various fields of science. Especially in the field of mathematics and its applications [1-19], when performing the simulation process for any system according to classical logic, we begin by generating random numbers that follow a regular probability distribution over the domain $[0,1]$ using one of the known methods, and then we convert these random numbers into variables. Randomness follows the probability distribution in which the system to be simulated operates. The simulation process we conduct produces specific results that do not take into account changes that may occur in the system's operating environment. To obtain more accurate results, we have presented, in previous research, a neutrosophical vision of the following topics:

In the paper [20] we generated neutrosophic random numbers that follow a uniform neutrosophic distribution over the domain $[0,1]$.In research [21] we used the inverse transformation method to convert neutrosophic random numbers into neutrosophic random variables that follow a uniform distribution over the domain $[0,1]$.In research [22], we used the inverse transformation method to convert random numbers into random variables that follow the neutrosophic exponential distribution. In research [23] we used the rejection method to transform random numbers into random variables that follow the probability distribution according to which the system to be simulated operates. In the research [24] using the rejection method to generate random numbers that follow the beta distribution.

In this research, we present a neutrosophical study of transforming random numbers into random variables that follow the Poisson distribution using the composition method (The mixed method of inverse transformation method and rejection method), a distribution that has many uses in practical life. Such as inventory control, queueing theory, quality control, traffic flow, and many other fields of management science.

Discussion:

Classic Composition method: [25-26]

The Composition method is based on the inverse transformation method and the rejection and acceptance method is special for generating random variables that follow complex probability distributions.

Using the conditional distribution of the variable x, we assume that $f(x)$ is the law of the probability distribution to be simulated, and that $g(x|y)$ is the conditional distribution of If y belongs to the cumulative distribution $H(y)$ and $P(x,y)$ is the joint distribution of (x,y), then:

$$P(x,y) = h(y)g(x|y)$$

Thus, we find:

$$f(x) = \int_{-\infty}^{+\infty} P(x,y)dy = \int_{-\infty}^{+\infty} h(y)g(x|y)dy$$

When the time periods between possible events are distributed exponentially, the number of events that occur in one period of time has a Poisson distribution given by the following probability density function:

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad ; x = 0,1,2, \dots, \infty$$

Where λ is the number of expected occurrences in one time, this indicates that the time period between events is exponentially distributed with an average of $\frac{1}{\lambda}$. Using the relationship between the exponential distribution and the Poisson distribution, we can generate random variables that follow the Poisson distribution.

The neutrosophic vision of the method of installation:

The mixed method is based on the neutrosophic countertransference method [21-22] and the neutrosophic rejection and acceptance method [23].

Using the conditional distribution of the variable x, we assume that $f_N(x)$ is the law of the probability distribution to be simulated, and that $g_N(x|y)$ is the conditional distribution of If y belongs to the cumulative distribution $H_N(y)$ and $P_N(x,y)$ is the joint distribution of (x,y), then:

$$P_N(x,y) = h_N(y) g_N(x|y)$$

Thus, we find:

$$f_N(x) = \int_{-\infty}^{+\infty} P_N(x,y)dy = \int_{-\infty}^{+\infty} h_N(y) g_N(x|y)dy$$

When the time periods between possible events are distributed exponentially, the number of events that occur in one period of time has a neutrosophic Poisson distribution given by the following probability density function:

Where λ_N is a neutrosophic value from reference [27]. We find that what is meant by neutrosophic data are completely indeterminate values written in the following standard formula $N = a + bI$ where $a \; and \; b$ are real or complex coefficients, a represents the specified part and bI the indeterminate part (indeterminacy). For the number N, it could be $[\lambda_1 , \lambda_2]$ or $\{\lambda_1 , \lambda_1\}$ or...otherwise it is any set close to the real value a, expressing the number of expected occurrences in One time, this indicates that the time period between events is exponentially distributed with an average of $\frac{1}{\lambda_N}$. Using the relationship between the neutrosophic exponential distribution and the neutrosophic Poisson distribution, we can generate neutrosophic random variables that follow the Poisson distribution.

Here we distinguish three cases:

First case: the random numbers are neutrosophic and the probability distribution is classical.

Second case: classical random numbers and neutrosophic probability distribution.

Third case: neutrosophic random numbers and neutrosophic probability distribution.

We start with the first case; the random numbers are neutrosophic and the probability distribution is classical:

In this case the probability density function of the Poisson distribution takes the following form:

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad ; x = 0,1,2, \dots, \infty$$

Where λ is the number of expected occurrences in one time, this indicates that the time period between events is exponentially distributed with an average of $\frac{1}{\lambda}$, since the random numbers must be neutrosophic to obtain them, we follow the following steps:

a. Using the mean square method given by the following relation:

$$R_{i+1} = Mid[R_i^2] \; ; i = 0,1,2,3, - - \qquad (1)$$

Where Mid symbolizes the middle four ranks of R_i^2, and R_i is chosen, any fractional random number composed of four places (called a seed) and does not contain a zero in any of its four places, [25-26], we generate a series of random numbers that follow the distribution regular over the domain $[0,1]$, we get the following series:

$$R_1, R_2, R_3 \dots R_m, \dots \qquad (2)$$

b. Using the study given in reference [20] we convert these random numbers into neutrosophic random numbers and here we distinguish three forms of the field [0,1] with margin of indeterminacy, in the three forms we have $\varepsilon \in [0, n]$ and $0 < n < 1$

The first form: $[0 + \varepsilon, 1]$ indeterminacy at the minimum of the field we find:

$$R_{Ni} \in \left[R_i, \frac{R_i - n}{1 - n}\right] \; ; \quad 0 < n < 1$$

It is calculated from the following relation:

$$R_{Ni} = \frac{R_i - \varepsilon}{1 - \varepsilon} \quad ; \quad \varepsilon \in [0, n]$$

The second form: $[0, 1 + \varepsilon]$ indeterminacy at the upper limit of the field we find:

$$R_{Ni} \in \left[R_i, \frac{R_i}{1 + n}\right] \; ; \quad 0 < n < 1$$

It is calculated from the following relation:

$$R_{Ni} = \frac{R_i}{1 + \varepsilon} \quad ; \quad \varepsilon \in [0, n]$$

The third form: $[0 + \varepsilon, 1 + \varepsilon]$ Indeterminacy at the upper and lower limits of the field we find:

$$R_{Ni} \in [R_i, R_i - n] \quad ; \quad 0 < n < 1$$

It is calculated from the following relation:

$$R_{Ni} = R_i - \varepsilon \ ; \quad \varepsilon \in [0, n]$$

From each of the previous forms we get the following series of neutrosophic random numbers:

$$R_{N1}, R_{\mathrm{N}2}, R_{\mathrm{N}3} \dots R_{Nm}, \dots \quad (3)$$

c. Using the study mentioned in References [21-22], we convert these neutrosophic random numbers into neutrosophic random numbers that follow the exponential distribution defined by the following relation:

$$h(y) = \lambda . e^{-\lambda y}$$

Using the relation:

$$y_{Ni} = -\frac{1}{\lambda} ln R_{Ni}$$

We obtain the series of neutrosophic random numbers that follow the following exponential distribution:

$$y_{N1}, y_{N2}, y_{N3} \dots y_{Nm}, \dots \quad (4)$$

We apply the accept-reject method given in reference [23]:

We take the cumulative sum of these numbers if the following inequality:

$$\sum_{i=1}^{x} y_{Ni} \leq 1 \pm \varepsilon \leq \sum_{i=1}^{x+1} y_{Ni+1}$$

Then we consider the number x to be subject to the Poisson distribution, where x is the number of random numbers y_{Ni} that are subject to the exponential distribution.

$h(y) = \lambda . e^{-\lambda y}$, the sum of which we took and the number did not exceed $1 \pm \varepsilon$, but if we added another number $y_{\mathrm{N}i+1}$ the sum became greater than $1 \pm \varepsilon$. However, if the inequality is not met, we return to Step (a), we repeat the work until we obtain the required number of random numbers that follow the Poisson distribution.

The second case: classical random numbers and Poisson-Neutrosophic distribution.

The probability density function of the neutrosophic Poisson distribution is defined by the following relationship: [28]

$$f_N(x) = \frac{\lambda_N^x e^{-\lambda_N}}{x!} \quad ; x = 0,1,2, \dots, \infty$$

Where λ_N is a neutrosophic value that expresses the number of expected occurrences in one time, this indicates that the time period between events is exponentially distributed with an average of $\frac{1}{\lambda_N}$.

a. Since classical random numbers follow a uniform distribution over the interval [0,1], we take the sequence of random numbers from relation (2).

b. Using the inverse transformation method, we convert these random numbers into neutrosophic random numbers that follow the neutrosophic exponential distribution defined by the following relation:

$$h_N(y) = \lambda_N . e^{-\lambda_N y}$$

c. Using the conversion relation:

$$y'_{Ni} = -\frac{\ln R_i}{\lambda_N}$$

We obtain the series of neutrosophic random numbers that follow the following exponential distribution:

$$y'_{N1}, y'_{N2}, y'_{N3}, \dots, y'_{Nm}, \dots \quad (4)$$

We take the cumulative sum of these numbers if the following inequality:

$$\sum_{i=1}^{x} y'_{Ni} \leq 1 \pm \varepsilon \leq \sum_{i=1}^{x+1} y'_{Ni+1}$$

Then we consider the number x to be subject to the Poisson distribution, where x is the number of random numbers y'_{Ni} which is subject to the exponential distribution.

$h(y) = \lambda . e^{-\lambda y}$ whose sum we took and the number did not exceed $1 \pm \varepsilon$, but if we added another number y'_{Ni+1} the sum became greater than $1 \pm \varepsilon$, but if the inequality is not satisfied we return to step (a), repeating the work until we obtain the required number of random numbers that follow the Poisson distribution.

The third case: neutrosophic random numbers and neutrosophic probability distribution.

From the study in the first case, we obtain the series of neutrosophic random numbers as in (3).

Poisson Neutrosophic distribution, i.e., the probability density function is defined as it is in the second case. To convert neutrosophic random numbers $R_{N1}, R_{\mathrm{N2}}, R_{\mathrm{N3}} \dots R_{Nm}, \dots$, into random numbers that follow the exponential distribution, we use the following relation:

$$y''_{Ni} = -\frac{\ln R_{Ni}}{\lambda_N}$$

We obtain the series of neutrosophic random numbers that follow the following exponential distribution:

$$y''_{N1}, y''_{N2}, y''_{N3}, \dots, y''_{Nm}, \dots \quad (5)$$

We take the cumulative sum of these numbers if the following inequality:

$$\sum_{i=1}^{x} y''_{Ni} \leq 1 \pm \varepsilon \leq \sum_{i=1}^{x+1} y''_{Ni+1}$$

achieved, then we consider the number x to be subject to the Poisson distribution, where x is the number of random numbers $y_{Ni}^{"}$ that obey the neutrosophic exponential distribution $h_N(y) = \lambda_N . e^{-\lambda_N y}$. The sum of which we took and the number did not exceed $1 \pm \varepsilon$, but if we added another number $y_{Ni+1}^{"}$, the sum became greater than $1 \pm \varepsilon$. However, if the inequality is not met, we return to step (a), and we repeat the work until we obtain the number the required random numbers follow a Poisson distribution.

Conclusion and results:

In this research, we presented a neutrosophic vision of the composition method used to generate random numbers that follow complex probability distributions from simple distributions. Random numbers that follow them can be generated using the neutrosophic inverse transformation or the neutrosophic rejection and acceptance method, using the relations provided by students and researchers in the field of mathematical statistics that link the probability distributions. Complex with simple probability distributions, such as the following relation between the Poisson distribution and the exponential distribution: When the time periods between possible events are exponentially distributed, the number of events that occur in one period of time has a Poisson distribution, which is relied upon to generate neutrosophic random numbers that follow the distribution Poisson, which has many uses in practical life, Such as inventory control, queueing theory, quality control, traffic flow, and many other fields of management science.

References:

1- Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, **Doi :**https://doi.org/10.54216/PAMDA.020102.

2- Florentin Smarandache, Maissam Jdid, Research on the topics of neutrosophic operations research, Volume (1), https://fs.unm.edu/NeutrosophicOperationsResearch.pdf, Biblio Publishing, 2023.

3- Abdulrahman Astambli, Mohamed Bisher Zeina and Yasin Karmouta, On Some Estimation Methods of Neutrosophic Continuous Probability Distributions Using One-Dimensional AH-Isometry, Neutrosophic Sets and Systems, Vol. 53, 2023, pp. 641-652. DOI: 10.5281/zenodo.7536101

4- Yaser Ahmad Alhasan and Raja Abdullah Abdulfatah, Division of refined neutrosophic numbers Neutrosophic Sets and Systems, Vol. 60, 2023, pp. 1-5. DOI: 10.5281/zenodo.10224078

5- Maissam Jdid, NEUTROSOPHIC TRANSPORT AND ASSIGNMENT ISSUES, Publisher: Global Knowledge's, ISBN, 978_1 _59973_770_6, (Arabic version). USA,2023.

6- Marwah Yahya Mustafa , Zakariya Yahya Algamal, Neutrosophic inverse power Lindley distribution: A modeling and application for bladder cancer patients, **Doi :**https://doi.org/10.54216/IJNS.210218

7- Renee Miriam M., Nivetha Martin , Aleeswari A. , Said Broumi, Rework Warehouse Inventory Model for Product Distribution with Quality Conservation in Neutrosophic Environment, **Doi :**https://doi.org/10.54216/IJNS.210215
8- Maissam Jdid, Neutrosophic linear models and algorithms to find their optimal solution, Biblio Publishing, ISBN, 978_1 _59973_778_2, (Arabic version). USA,2023.
9- Maissam Jdid, Florentin Smarandache, A Study of Systems of Neutrosophic Linear Equations, https://doi.org/10.54216/IJNS.230202.
10- Maissam Jdid, Florentin Smarandache, Neutrosophic Treatment of the Modified Simplex Algorithm to find the Optimal Solution for Linear Models https://doi.org/10.54216/IJNS.230110.
11- Maissam Jdid, Basel Shahin , Fatima Al Suleiman ,Important Neutrosophic Rules for Decision-Making in the Case of Uncertain Data , https://doi.org/10.54216/IJNS.1803014
12- Maissam Jdid, Hla Hasan, The State of Risk and Optimum Decision According to Neutrosophic Rules ,https://doi.org/10.54216/IJNS.200107
13- Zahraa A. Khalaf, Fatimah M. Mohammed, Weakly Generalized M-Closed and Strongly M-Generalized Closed Sets in Fuzzy Neutrosophic Topological Spaces, **Doi :**https://doi.org/10.54216/IJNS.210201
14- Hamiyet Merkepci , Ahmed Hatip, Algorithms for Computing Pythagoras Triples and 4-Tiples in Some Neutrosophic Commutative Rings, **Doi :**https://doi.org/10.54216/IJNS.200310
15- Najla M. Alnaqbi, Samira A. Alnuaimi , M. Elhoseny, A Neutrosophic AHP Analysis for Using Video Conferences in Smart Learning: A Systematic Review, **Doi :**https://doi.org/10.54216/IJNS.200307
16- Noura Metawa , Rhada Boujlil , Maha Metawea, Multi-Valued Neutrosophic Sets for Forecasting Cryptocurrency Volatility, **Doi:**https://doi.org/10.54216/IJNS.200306
17- Faisal Al-Sharqi , Abd Ghafur Ahmad , Ashraf Al-Quran, Mapping on Interval Complex Neutrosophic Soft Sets, **Doi :**https://doi.org/10.54216/IJNS.190406
18- Nahia Mourad, Assessment of Structural Cracks in Aircraft Using a Decision-Making Approach Based on Enhanced Entropy and Single-Valued Neutrosophic Sets, **Doi :**https://doi.org/10.54216/IJNS.190404
19- Maissam Jdid, Studying Transport Models with the Shortest Time According to the Neutrosophic Logic, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 631-638. DOI: 10.5281/zenodo.8404545
20- Maissam Jdid, Rafif Alhabib and A. A. Salama, Fundamentals of Neutrosophical Simulation for Generating Random Numbers Associated with Uniform Probability Distribution, Neutrosophic Sets and Systems, Vol. 49, 2022, pp. 92-102. DOI: 10.5281/zenodo.6426375
21- Maissam Jdid, A. Salama, Using the Inverse Transformation Method to Generate Random Variables that follow the Neutrosophic Uniform Probability Distribution, **Doi :**https://doi.org/10.54216/JNFS.060202

22- Maissam Jdid, Rafif Alhabib and A. A. Salama, The Basics of Neutrosophic Simulation for Converting Random Numbers Associated with a Uniform Probability Distribution into Random Variables Follow an Exponential Distribution, Neutrosophic Sets and Systems, Vol. 53, 2023, pp. 358-366. DOI: 10.5281/zenodo.7536049

23- Maissam Jdid, Said Broumi, Neutrosophical Rejection and Acceptance Method for the Generation of Random Variables, Neutrosophic Sets and Systems, Vol. 56, 2023, pp. 153-166. DOI: 10.5281/zenodo.8194749

24- Maissam Jdid and Nada A. Nabeeh, Generating Random Variables that follow the Beta Distribution Using the Neutrosophic Acceptance-Rejection Method, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 139-148. DOI: 10.5281/zenodo.8404445

25- Bugaha J.S, Mualla.W, and others -Operations Research Translator into Arabic, The Arab Center for Arabization, Translation, Authoring and Publishing, Damascus,1998. (Arabic version).

26- Alali. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004. (Arabic version).

27- Smarandache, F, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico, Gallup, NM 87301, USA,2002.

New Modification of Ranked Set Sampling for Estimating Population Mean: Neutrosophic Median Ranked Set Sampling with an Application to Demographic Data

Anamika Kumari,[1] **Rajesh Singh,**[1] **Florentin Smarandache**[2]

[1] Department of Statistics, Banaras Hindu University, Varanasi, Uttar Pradesh 221005, India

[2] Department of Mathematics, University of New Mexico, Gallup, USA

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Abstract

The study addressed the limitations of classical statistical methods when dealing with ambiguous data, emphasizing the importance of adopting neutrosophic statistics as a more effective alternative. Classical methods falter in managing uncertainty inherent in such data, necessitating a shift towards methodologies like neutrosophic statistics. To address this gap, the research introduced a novel sampling approach called "neutrosophic median ranked set sampling" and incorporated neutrosophic estimators tailored for estimating the population mean in the presence of ambiguity. This modification aims to address the inherent challenges associated with estimating the population mean when dealing with neutrosophic data. The methods employed involved modifying traditional ranked set sampling techniques to accommodate neutrosophic data characteristics. Additionally, neutrosophic estimators were developed to leverage auxiliary information within the framework of median-ranked set sampling, enhancing the accuracy of population mean estimation under uncertain conditions. The methods employed involved modifying traditional ranked set sampling techniques to accommodate neutrosophic data characteristics. Bias and mean squared error equations for the suggested estimators were provided, offering insights into their theoretical underpinnings. To illustrate the effectiveness and practical applications of the proposed methodology and estimators, a numer-ical demonstration and simulation study have been conducted using the R programming language. The key results highlighted the superior performance of the proposed estimators compared to existing alternatives, as demonstrated through comprehen-sive evaluations based on mean squared error and percentage relative efficiency criteria. The conclusions drawn underscored the effectiveness of the neutrosophic median ranked set sampling approach and suggested estimators in estimating the pop-ulation mean under conditions of uncertainty, particularly when utilizing neutrosophic auxiliary information and validated real-life applicability. The methodology and estimators presented in the study were shown to yield interval-based results, pro-viding a more realistic representation of uncertainty associated with population parameters. This interval estimation, coupled with minimum mean squared error considerations, enhanced the efficacy of the estimators in determining population mean values. The novelty of the work lies in its introduction of a tailored sampling approach and estimators designed specifically for neutrosophic data, filling a significant gap in the literature. By extending classical statistics to accommodate ambiguity, the study offers a substantial advancement in statistical methodology, particularly in domains where precise data is scarce and uncertainty is prevalent. Furthermore, the empirical validation through numerical demonstrations and simulation studies using the R programming language adds robustness to the proposed methodology and contributes to its practical applicability.

Keywords Neutrosophic statistics · Study variable · Auxiliary variable · Bias · Mean squared error · Percentage relative efficiency

1 Introduction

Sampling is a crucial practice for a variety of reasons, such as cost and time constraints. In sampling theory, the goal of estimation procedures is to enhance the effectiveness of estimators for population parameters while minimizing sampling errors. To achieve this, auxiliary information is utilized to improve estimator efficiency, and this information can be incorporated at various stages of the process. When highly correlated auxiliary information is not readily available, it can be gathered from previous surveys. Estimation techniques like ratio, product, and regression are commonly employed in this context.

For instance, [25] introduced a modified ratio estimator that incorporates the coefficient of variation of auxiliary information. References [3, 9, 14, 16, 19, 21] have also proposed population parameter estimation methods using auxiliary information. However, our focus here is on ranked set sampling.

Efforts in sampling continually strive for improvements in estimator efficiency, cost-effectiveness, simplicity, and time savings. Ranked set sampling (RSS) offers a superior alternative to simple random sampling (SRS) in various fields, including medicine, agriculture, earth sciences, statistics, and mathematics, especially when measurements are cumbersome, time-consuming, or expensive. The RSS technique was initially described for population mean estimation by [13], and the mathematical theory behind RSS was provided by [35]. Dell and Cultter [7] demonstrated that under both perfect and imperfect ranking scenarios, the mean estimate in RSS remains unbiased.

Numerous researchers, such as [2, 5, 6, 8, 10, 12, 17, 20, 33] have contributed to the field of RSS. For recent work, one can prefer [4, 23, 36, 37].

Only when the data have a symmetric distribution, RSS provide an efficient estimate for the estimation of population parameters; otherwise, the results will be less favorable. To acquire the most accurate parameter estimates, various RSS sampling methods are available. These include double RSS, median ranked set sampling (MRSS), quantile RSS, extreme RSS, a combination of median and extreme RSS, except extreme RSS, and more. For asymmetric one can go for MRSS. In this context, our focus will be on MRSS. [15] originally introduced MRSS, and subsequently, [18] presented a ratio estimator under MRSS. Al-Omari [1] provided ratio estimation under both SRS and MRSS, while [11] contributed a difference-cum-ratio and exponential-type estimator under MRSS.

Classical RSS deals solely with precise data, assuming no uncertainty exists. However, in practice, data can be uncertain and imprecise, containing sets or intervals. To address such situations, fuzzy logic is a valuable tool that handles data with imprecision. When analyzing fuzzy data, fuzzy statistics are employed, with ambiguous, uncertain, unclear, or imprecise characteristics. Yet, they do not account for the degree of indeterminacy inherent in the data. Neutrosophic statistics, an extension of fuzzy logic, offer a way to measure both indeterminacy and determinate aspects of uncertain or imprecise data. When dealing with data that contains indeterminacy, neutrosophic statistics are employed. Neutrosophic statistics expand upon classical statistics and encompass fuzzy and intuitionistic statistics. Neutrosophy is applicable when observations in a population or sample lack precision, are indeterminate, or are vague. Some examples of neutrosophic data include district-wise water level measurements with intervals, variations in machinery part sizes due to measurement errors, and day-wise temperature measurements resulting in interval-type data.

The concept of neutrosophy was initially introduced by [26–32], and extensive literature on neutrosophic sets, logic, and statistics can be found in his works. In the realm of sampling theory, [34] recently addressed the estimation of population parameters under a neutrosophic environment, introducing neutrosophic ratio-type estimators for population means under SRS. One can also prefer [22, 24] for neutrosophic estimators. However, there has been little focus on neutrosophic RSS (NRSS) for estimating population parameters. References [36, 37] proposed NRSS and a generalized estimator for the computation of the population mean.

Efficiency improvements in sampling framework and estimators are a constant objective in sampling. In this context, we propose an enhanced novel sampling framework neutrosophic median ranked set sampling (NMRSS) and improved NMRSS estimators for population mean estimation, with a particular emphasis on minimizing MSE and enhancing precision.

There is not much emphasis on the problem of estimation when dealing with indeterminate, vague, imprecise, set-type, or interval-type data that is characteristic of neutrosophic data, which is more prevalent in real-world scenarios than crisp data. It is a well-proven theory through a literature review that RSS is an improvement over SRS and MRSS is most efficient when data do not follow symmetric distribution. There is no work in the estimation of parameters using the MRSS technique when there is the presence of errors, unclear, inaccurate, indeterminate, vague, imprecise, set-type, or interval-type data. Hence, our research is novel and significant in the field of sampling where we came with neutrosophic data and SRS and RSS are not efficient sampling techniques to work with. Our work may be more desirable than other work in several real situations like agriculture sciences, mathematical sciences, biological sciences, poultry, business, economics, commerce, social sciences, etc.

The efficacy of the innovative sampling framework, neutrosophic median ranked set Sampling (NMRSS), alongside enhanced NMRSS estimators for population mean estima-

tion, was validated through empirical analysis using both real and simulated population data. Results unequivocally demonstrated the superiority of the recommended estimators over existing ones, showcasing lower MSEs and higher PREs, with t_{p2} emerging as the optimal estimator. As sample sizes and correlation coefficients increased, the MSE decreased and PRE increased for the recommended estimator, indicating its robustness and efficiency. NMRSS estimators surpassed their counterparts NSRS, with RSS proving to be a superior replacement for SRS. Similarly, under NRSS, MRSS outperformed RSS, particularly when data exhibited non-symmetric distributions. Comparative analysis revealed that MSEs and PREs of suggested estimators obtained through classical MRSS fell between those obtained using NMRSS, suggesting the latter's superiority. The study underscored the inadequacy of classical RSS in handling vague or indeterminate data, emphasizing NMRSS as a superior method for estimating uncertain or interval data.

It can be computationally burdensome to analyze neutrosophic data, particularly when processing large datasets with advanced models or algorithms. It can be difficult to interpret analysis based on neutrosophic data, particularly when dealing with contradicting or ambiguous results. Compared to other approaches or even just SRS, RSS implementation can be more complicated. Before sampling, the population must be carefully ranked and sorted, which can take time and require specialized knowledge.

Our study is designed as follows: Sect. 1 presents an introduction, and Sect. 2 outlines motivation, needs, and research gaps. Section 3 outlines the modified and novel sampling method, neutrosophic median ranked set sampling method. Section 4 presents existing NMRSS estimators. Section 5 presents proposed NMRSS estimators, while Sect. 6 presents an application with the help of empirical study using demographic data, and Sect. 7 offers a simulation study using artificial data for different sets of values. Section 8 is dedicated to a discussion, and Sect. 9 covers concluding remarks and future research directions.

2 Motivation, Need and Research Gap

This article's main objective is to introduce a novel approach known as the "neutrosophic median ranked set sampling" for dealing with interval-type or neutrosophic data. Our study focuses on sampling theory, marking the first instance of proposing an MRSS technique tailored to neutrosophic data, along with the development of NMRSS estimators for population mean estimation. This marks a significant step in expanding the field of sampling theory by comparing these estimators with existing neutrosophic methods such as ratio, product, and regression estimators. MRSS is considered a superior alternative to SRS and RSS when data does not follow symmetric distribution, making it an attractive avenue for further exploration in the context of NMRSS.

Several factors drive our exploration of NMRSS and its associated estimators for population parameter estimation. A primary motivation is to introduce MRSS and MRSS estimators in a neutrosophic setting. Previous research in survey sampling has predominantly focused on clear, well-defined data, while classical sampling methods yield precise results, albeit with potential risks of inaccuracies, overestimations, or underestimations. However, classical methods fall short when handling set-type or undetermined data, characteristic of neutrosophic data, which is more prevalent in real-world scenarios than crisp data. Thus, there is a growing need for additional neutrosophic statistical techniques. Traditional statistical approaches are ill-suited to compute accurate estimates of unknown parameters when dealing with indeterminate, vague, imprecise, or interval-type data. Neutrosophic statistics serve as a suitable replacement for classical statistics in such scenarios.

The necessity to close the gap between classical and neutrosophic statistics, and inspired by [36, 37] work, our work introduces a new sampling technique NMRSS, and enhanced NMRSS estimators for population mean estimation. Despite thorough research in the field, we found no prior studies in survey sampling that addressed the estimation of population means in the presence of auxiliary variables using neutrosophic data under MRSS. This research represents a significant step toward filling this gap and contributes to the evolving domain of neutrosophic statistics.

It has been well established by multiple authors that RSS is a more suitable option than SRS when dealing with cumbersome, expensive, or time-consuming measurements and MRSS is a better option than RSS when data does not follow the symmetric distribution. The challenges associated with measurements in a neutrosophic context exacerbate these problems. Therefore, our research introduces an NMRSS method to enhance the accuracy of the population mean estimators in this unique context.

3 Sampling Methodology

Numerous methods can be used to display the neutrosophic observations, and the neutrosophic numbers may include an unknown interval $[a, b]$. We are describing neutrosophic values as $Z_{rssN} \equiv Z_{rssL} + Z_{rssU} I_{rssN}$ with $I_{rssN} \in [I_{rssL}, I_{rssU}]$, N is here to represent the neutrosophic number. Hence, our neutrosophic observations will lie in an interval $Z_{rssN} \in [a, b]$, where 'a' and 'b' denote the neutrosophic data's lower and upper values.

In MRSS, a small subset of randomly chosen population units are measured after they have been ranked solely based on observation or experience. In the context of MRSS,

m independent sets of random samples, each consisting of m units, are drawn from the overall population. Every unit within a set has an equal probability of selection. The members of each random set are then arranged in order based on the characteristics of the auxiliary variable. Then, if the sample size m is even, identify $(\frac{m}{2})$th smallest ranked unit from the first $(\frac{m}{2})$ sets, and select $(\frac{m+2}{2})$th smallest ranked unit from the other $(\frac{m}{2})$ sets. In case when the sample size m is odd, identify $(\frac{m+1}{2})$th smallest ranked unit from all sets for actual quantification. Throughout the process, $rm\ (=n)$ units have been acquired as this cycle is replicated r times.

The method of NMRSS consists of selecting $m_N \in [m_L, m_U]$ bivariate random samples of size $m_N \in [m_L, m_U]$ from a population of size N, and then ranking inside each sample concerning for auxiliary variable $X_N \in [X_L, X_U]$ associated with $Y_N \in [Y_L, Y_U]$. The book by [32] entitled "Introduction to Neutrosophic Statistics" will be the basis for the ranking of the neutrosophic number. To show the process of ranking, we are utilizing here two sets as $X_{1N} \in [X_{1L}, X_{1U}]$ and $X_{2N} \in [X_{2L}, X_{2U}]$, also their midpoints are as $X_{1\text{mid}N} = [X_{1L} + X_{1U}]/2$ and $X_{2\text{mid}N} = [X_{2L} + X_{2U}]/2$. The ordering of neutrosophic numbers $m_N \in [m_L, m_U]$ can be done as $X_{1N} \in [X_{1L}, X_{1U}]$ will be less than $X_{2N} \in [X_{2L}, X_{2U}]$ if $X_{1\text{mid}N} \leq X_{2\text{mid}N}$, also if both are same that is $X_{1\text{mid}N} = X_{2\text{mid}N}$ then we will compare or see by $X_{1L} \leq X_{2L}$. Further, if again $X_{1L} = X_{2L}$ then this implies $X_{1U} = X_{2U}$ and hence $X_{1N} \in [X_{1L}, X_{1U}] = X_{2N} \in [X_{2L}, X_{2U}]$, so the neutrosophic number ranking will be carried out in this manner.

In the whole NMRSS structure if $m_N \in [m_L, m_U]$ is even, we count $(\frac{m_N}{2})th$ smallest rank unit of the first $(\frac{m_N}{2})$ data of set size $m_N \in [m_L, m_U]$, for the first $(\frac{m_N}{2})$ measurement units in the entire NMRSS structure, and then we scrap the remaining units. In a similar manner, we count the $(\frac{m_N+2}{2})th$ smallest observation from the remaining $(\frac{m_N}{2})$ data set for the remaining $(\frac{m_N}{2})$ measurement units and discard the remaining observations. If $m_N \in [m_L, m_U]$ is odd, we select $(\frac{m_N+1}{2})th$ smallest rank unit from all $m_N \in [m_L, m_U]$ sets of size $m_N \in [m_L, m_U]$. This process counts the total $m_N \in [m_L, m_U]$ neutrosophic bivariate units. After r cycles of these steps, the total $n_N = m_N r \in [n_L, n_U]$ bivariate NMRSS units are obtained.

Here, we have set sample NMRSSo for an odd set size and NMRSSe for an even set size. For jth cycle MRSSo is denoted by $(X_{1(\frac{m_N+1}{2})j}, Y_{1[\frac{m_N+1}{2}]j})$, $(X_{2(\frac{m_N+1}{2})j}, Y_{2[\frac{m_N+1}{2}]j})$, ..., $(X_{m_N(\frac{m_N+1}{2})j}, Y_{m_N[\frac{m_N+1}{2}]j})$; $(j = 1, 2, ..., r)$. Let the sample means of X_N and Y_N are, respectively, as

$$\overline{x}_{(o)N} = \frac{1}{rm_N} \sum_{j=1}^{r} \sum_{i=1}^{m_N} x_{i\left(\frac{m_N+1}{2}\right)j} \tag{1}$$

$$\overline{y}_{(o)N} = \frac{1}{rm_N} \sum_{j=1}^{r} \sum_{i=1}^{m_N} y_{i\left[\frac{m_N+1}{2}\right]j} \tag{2}$$

and their corresponding variances are given as

$$\begin{aligned} \text{Var}(\overline{x}_{(o)N}) &= \text{Var}\left(\frac{1}{rm_N} \sum_{j=1}^{r} \sum_{i=1}^{m_N} x_{i\left(\frac{m_N+1}{2}\right)j}\right) \\ &= \frac{1}{rm_N} \sigma^2_{x\left(\frac{m_N+1}{2}\right)} \end{aligned} \tag{3}$$

$$\begin{aligned} \text{Var}(\overline{y}_{(o)N}) &= \text{Var}\left(\frac{1}{rm_N} \sum_{j=1}^{r} \sum_{i=1}^{m_N} y_{i\left[\frac{m_N+1}{2}\right]j}\right) \\ &= \frac{1}{rm_N} \sigma^2_{y\left[\frac{m_N+1}{2}\right]}. \end{aligned} \tag{4}$$

Similarly, for jth cycle NMRSSe is denoted by $(X_{1(\frac{m}{2})j}, Y_{1[\frac{m}{2}]j})$, $(X_{2(\frac{m}{2})j}, Y_{2[\frac{m}{2}]j})$, ..., $(X_{\frac{m}{2}(\frac{m}{2})j}, Y_{\frac{m}{2}[\frac{m}{2}]j})$, $(X_{\frac{m}{2}+1(\frac{m+2}{2})j}, Y_{\frac{m}{2}+1[\frac{m+2}{2}]j})$, $(X_{\frac{m}{2}+2(\frac{m+2}{2})j}, Y_{\frac{m}{2}+2[\frac{m+2}{2}]j})$, ..., $(X_{m(\frac{m+2}{2})j}, Y_{m[\frac{m+2}{2}]j})$. Again, let the sample means of X_N and Y_N are, respectively, as

$$\overline{x}_{(e)N} = \frac{1}{rm_N} \sum_{j=1}^{r} \left(\sum_{i=1}^{\frac{m_N}{2}} x_{i\left(\frac{m_N}{2}\right)} + \sum_{i=\frac{m_N}{2}+1}^{m_N} x_{i\left(\frac{m_N+2}{2}\right)j} \right) \tag{5}$$

$$\overline{y}_{(e)N} = \frac{1}{rm_N} \sum_{j=1}^{r} \left(\sum_{i=1}^{\frac{m_N}{2}} y_{i\left[\frac{m_N}{2}\right]} + \sum_{i=\frac{m_N}{2}+1}^{m_N} y_{i\left[\frac{m_N+2}{2}\right]j} \right) \tag{6}$$

and their corresponding variances are given as

$$\begin{aligned} &\text{Var}(\overline{x}_{(e)N}) \\ &= \text{Var}\left(\frac{1}{rm_N} \sum_{j=1}^{r} \left(\sum_{i=1}^{\frac{m_N}{2}} x_{i\left(\frac{m_N}{2}\right)} + \sum_{i=\frac{m_N}{2}+1}^{m_N} x_{i\left(\frac{m_N+2}{2}\right)j} \right)\right) \\ &= \frac{1}{2rm_N} \left(\sigma^2_{x\left(\frac{m_N}{2}\right)} + \sigma^2_{x\left(\frac{m_N+2}{2}\right)} \right) \end{aligned} \tag{7}$$

$$\begin{aligned} &\text{Var}(\overline{y}_{(e)N}) \\ &= \text{Var}\left(\frac{1}{rm_N} \sum_{j=1}^{r} \left(\sum_{i=1}^{\frac{m_N}{2}} y_{i\left[\frac{m_N}{2}\right]} + \sum_{i=\frac{m_N}{2}+1}^{m_N} y_{i\left[\frac{m_N+2}{2}\right]j} \right)\right) \\ &= \frac{1}{2rm_N} \left(\sigma^2_{y\left[\frac{m_N}{2}\right]} + \sigma^2_{y\left(\frac{[m_N+2]}{2}\right)} \right) \end{aligned} \tag{8}$$

Consider a neutrosophic random sample of size $n_N \in [n_L, n_U]$ using NMRSS, which is acquired from a finite

population of N units $(U_1, U_2, ..., U_N)$. The neutrosophic study and auxiliary variable are $Y_N \in [Y_L, Y_U]$ and $X_N \in [X_L, X_U]$. Let $\overline{y}_{[n]N} \in [\overline{y}_{[n]L}, \overline{y}_{[n]U}]$ and $\overline{x}_{(n)N} \in [\overline{x}_{(n)L}, \overline{x}_{(n)U}]$ be the sample means of the neutrosophic study and auxiliary variables respectively, and also, let $\overline{Y}_N \in [\overline{Y}_L, \overline{Y}_U]$ and $\overline{X}_N \in [\overline{X}_L, \overline{X}_U]$ be the population means of the neutrosophic study and auxiliary variables, respectively. The correlation coefficient between both neutrosophic study and auxiliary variables is $\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}]$.

Let the neutrosophic mean error terms are $\epsilon_{0N} \in [\epsilon_{0L}, \epsilon_{0U}]$ and $\epsilon_{1N} \in [\epsilon_{1L}, \epsilon_{1U}]$. To obtain the bias and MSE of the estimators, we write

$\overline{y}_{[n]N} = \overline{Y}_N(1+\epsilon_{0N}), \overline{x}_{(n)N} = \overline{X}_N(1+\epsilon_{1N})$

$E(\epsilon_{0N}^2) = \frac{\mathrm{Var}(\overline{y}_{[n]N})}{\overline{Y}_N^2} = V_{yrN}$

$E(\epsilon_{1N}^2) = \frac{\mathrm{Var}(\overline{x}_{(n)N})}{\overline{X}_N^2} = V_{xrN}$

$E(\epsilon_{0N}, \epsilon_{0N}) = \frac{\mathrm{Cov}(\overline{y}_{[n]N}, \overline{x}_{(n)N})}{\overline{Y}_N \overline{X}_N} = V_{yxrN}$

$e_{0N}^2 \in [e_{0L}^2, e_{0U}^2]$; $e_{1N}^2 \in [e_{1L}^2, e_{1U}^2]$; $e_{0N}e_{1N} \in [e_{0L}e_{1L}, e_{0U}e_{0U}]$;

$V_{yrN} \in [V_{yrL}, V_{yrU}]$; $V_{xrN} \in [V_{xrL}, V_{xrU}]$; $V_{yxrN} \in [V_{yxrL}, V_{yxrU}]$.

4 Existing Estimators

The usual unbiased estimator for the population mean $\overline{Y}$ using NMRSS technique is given by

$$\overline{y}_{[n]N} = \frac{1}{n_N}\sum_{i=1}^{n_N} y_{[i]N}. \tag{9}$$

The variance of the estimator $\overline{y}_{[n]N}$ is given by

$$\mathrm{var}(\overline{y}_{[n]N}) = \overline{Y}_N^2 V_{yrN}. \tag{10}$$

The ratio estimator under NMRSS for the population mean $\overline{Y}$

$$\overline{y}_{rN} = \overline{y}_{[n]N}\left(\frac{\overline{X}_N}{\overline{x}_{(n)N}}\right). \tag{11}$$

The MSE of the estimator $\overline{y}_{rN}$ is given by

$$\mathrm{MSE}(\overline{y}_{rN}) = \overline{Y}_N^2(V_{yrN} + V_{xrN} - 2V_{yxrN}). \tag{12}$$

The product estimator under NMRSS for the population mean $\overline{Y}$

$$\overline{y}_{pN} = \overline{y}_{[n]N}\left(\frac{\overline{x}_{(n)N}}{\overline{X}_N}\right). \tag{13}$$

The MSE of the estimator $\overline{y}_{pN}$ is given by

$$\mathrm{MSE}(\overline{y}_{rN}) = \overline{Y}_N^2(V_{yrN} + V_{xrN} + 2V_{yxrN}). \tag{14}$$

The regression estimator under NMRSS for the population mean $\overline{Y}$

$$\overline{y}_{\mathrm{reg}N} = \overline{y}_{[n]N} + \beta(\overline{X}_N - \overline{x}_{(n)N}). \tag{15}$$

The MSE of the estimator $\overline{y}_{\mathrm{reg}N}$ is given by

$$\mathrm{MSE}(\overline{y}_{\mathrm{reg}N}) = \overline{Y}_N^2\left(V_{yrN} - \frac{{V_{yxrN}}^2}{V_{xrN}}\right). \tag{16}$$

The exponential ratio-type estimator under NMRSS for the population mean $\overline{Y}$ is represented as

$$\overline{y}_{\exp rN} = \overline{y}_{[n]N}\exp\left(\frac{\overline{X}_N - \overline{x}_{(n)N}}{\overline{X}_N + \overline{x}_{(n)N}}\right). \tag{17}$$

The MSE of the estimator $\overline{y}_{\exp rN}$ is given by

$$\mathrm{MSE}(\overline{y}_{\exp N}) = \overline{Y}^2\left(V_{yrN} + \frac{V_{yrN}}{4} - V_{yxrN}\right). \tag{18}$$

The exponential product type estimator under NMRSS for the population mean $\overline{Y}$ is represented as

$$\overline{y}_{\exp pN} = \overline{y}_{[n]N}\exp\left(\frac{\overline{x}_{(n)N} - \overline{X}_N}{\overline{x}_{(n)N} + \overline{X}_N}\right). \tag{19}$$

The MSE of the estimator $\overline{y}_{\exp pN}$ is given by

$$\mathrm{MSE}(\overline{y}_{\exp pN}) = \overline{Y}^2\left(V_{yrN} + \frac{V_{yrN}}{4} + V_{yxrN}\right). \tag{20}$$

5 Proposed Estimators

No single estimator is universally effective in all situations. Hence, it is preferable to employ estimators that yield minimal MSE and high precision. The objective of this section is to formulate estimators that demonstrate effective performance across a broader range of circumstances. In pursuit of this goal, we have embraced the [14] estimator within the framework of NMRSS and introduced two novel estimators for the mean of a finite population under NMRSS, utilizing auxiliary variables.

$$(1)\, P_{1N} = \overline{y}_{[n]N}(g_{1N}+1) + g_{2N}\log\left(\frac{\overline{x}_{(n)N}}{\overline{X}_N}\right), \tag{21}$$

where the constants g_{1N} and g_{2N} ensure that the MSE of the estimators is minimized.
When we express the estimator P_{1N} from Eq. (21) into terms of ϵ's, we obtain

$$P_{1N} = \overline{Y}_N (1+\epsilon_{01}) \\ (g_{1N}+1) + g_{2N} \log\left(\frac{\overline{X}_N (1+\epsilon_{1N})}{\overline{X}_N}\right). \quad (22)$$

Taking expectations by focusing on first-order approximation, we obtain MSE,

$$\text{MSE}(P_{1N}) = \overline{Y}_N{}^2 V_{yrN} + g_{1N}^2 A_{1N} + g_{2N}^2 B_{1N} \\ +2g_{1N}C_{1N} + 2g_{2N}D_{1N} + 2g_{1N}g_{2N}E_{1N}, \quad (23)$$

where

$$A_{1N} = \overline{Y}_N{}^2(1+V_{yrN}) \\ B_{1N} = V_{xrN} \\ C_{1N} = \overline{Y}_N{}^2 V_{yrN} \\ D_{1N} = \overline{Y}\, V_{yxrN} \\ E_{1N} = \overline{Y}\left(V_{yxrN} - \frac{1}{2}V_{xrN}\right).$$

To find out the minimum MSE for P_{1N}, we partially differentiate equation (23) w.r.t. g_{1N} & g_{2N} and equating to zero, we obtain

$$g_{1N}{}^* = \frac{B_{1N}C_{1N} - D_{1N}E_{1N}}{E_{1N}^2 - A_{1N}B_{1N}} \quad (24)$$

$$g_{2N}{}^* = \frac{A_{1N}D_{1N} - C_{1N}E_{1N}}{E_{1N}^2 - A_{1N}B_{1N}}. \quad (25)$$

By putting the optimum value of g_{1N} & g_{2N} in the Eq. (23), we can determine the minimum MSE of P_{1N} as

$$\text{MinMSE} \\ = C_{1N} + \frac{B_{1N}C_{1N}^2 + A_{1N}D_{1N}^2 - 2C_{1N}D_{1N}E_{1N}}{E_{1N}^2 - A_{1N}B_{1N}} \quad (26)$$

$$(2)\, P_{2N} = g_{3N}\overline{y}_{[n]N} + g_{4N}\exp\left(\frac{\overline{X}_N - \overline{x}_{(n)N}}{\overline{X}_N + \overline{x}_{(n)N}}\right) \\ \left(1 + \log\frac{\overline{x}_{(n)N}}{\overline{X}_N}\right). \quad (27)$$

When we express the estimator P_{2N} from Eq. (27) into terms of ϵ's, we obtain

$$P_{2N} = g_{3N}\overline{Y}_N(1+\epsilon_{0N}) + g_{4N}\exp\left(\frac{-\epsilon_{1N}}{2+\epsilon_{1N}}\right) \\ \times(1+\log(1+\epsilon_{1N})\ ) \quad (28)$$

$$P_{2N} - \overline{Y}_N = (g_{3N}-1)\overline{Y}_N + g_{3N}\overline{Y}_N\epsilon_{0N} \\ +g_{4N}\left(1 + \frac{\epsilon_{1N}}{2} - \frac{5\epsilon_{1N}{}^2}{8}\right) \quad (29)$$

$$\text{Bias}(P_{2N}) = \overline{Y}_N(g_{3N}-1) + g_{4N}\left[1 - \frac{5}{8}V_{xrN}\right]. \quad (30)$$

Case 1: If sum of weights is fixed $(g_{3N}+g_{4N}=1)$
The MSE of the estimator P_{2N} is shown as

$$\text{MSE}(P_{2N}) = \overline{Y}_N^2\left[V_{yrN} + g_{4N}^2 V_{xrN} - 2g_{4N}V_{yxrN}\right]. \quad (31)$$

To find out the minimum value of MSE for P_{2N}, we partially differentiate equation (31) w.r.t. g_{4N}, and equating to zero we get

$$g_{4N}{}^* = \frac{V_{yxrN}}{V_{xrN}}. \quad (32)$$

By putting the optimum value of g_{4N} in Eq. (31), we can determine the minimum MSE of P_{2N} as

$$\text{MinMSE} = \overline{Y}_N{}^2\left(V_{yrN} - \frac{V_{yxrN}{}^2}{V_{xrN}}\right). \quad (33)$$

Case 2: If the sum of weights is adjustable $(g_{3N} + g_{4N} \neq 1)$

$$P_{2N} - \overline{Y}_N = (g_{3N}-1)\overline{Y}_N + g_{3N}\overline{Y}_N\epsilon_{0N} + g_{4N} \\ \left(1 + \frac{\epsilon_{1N}}{2} - \frac{5\epsilon_1{}^2}{8}\right). \quad (34)$$

Squaring on both sides we get

$$(P_{2N} - \overline{Y}_N)^2 \\ = \overline{Y}_N{}^2 + \overline{Y}_N{}^2 g_{3N}^2(1+\epsilon_{01}^2) + g_{4N}^2\left(1-\epsilon_{1N}^2\right) \\ -2g_{3N}\overline{Y}_N{}^2 - 2g_{4N}\overline{Y}_N\left(1 - \frac{5\epsilon_{1N}^2}{8}\right) \\ +2g_{3N}g_{4N}\overline{Y}_N\left(1 - \frac{5\epsilon_{1N}{}^2}{8} + \frac{\epsilon_{0N}\epsilon_{1N}}{2}\right). \quad (35)$$

Taking expectations by focusing on first-order approximation, we obtain MSE,

$$\text{MSE}(P_{2N}) = \overline{Y}_N{}^2 + g_{3N}^2 A_{2N} + g_{4N}^2 B_{2N} - 2g_{3N}C_{2N} \\ -2g_{4N}D_{2N} + 2g_{3N}g_{4N}E_{2N}, \quad (36)$$

where

$$A_{2N} = \overline{Y}_N{}^2(1+V_{yrN})$$
$$B_{2N} = 1-V_{xrN}$$
$$C_{2N} = \overline{Y}_N{}^2$$
$$D_{2N} = \overline{Y}_N\left(1-\frac{5}{8}V_{xrN}\right)$$
$$E_{2N} = \overline{Y}_N\left(1-\frac{5}{8}V_{xrN}+\frac{1}{2}V_{yxrN}\right).$$

To find out the minimum MSE for P_{2N}, we partially differentiate equation (36) w.r.t. g_{3N} & g_{4N} and equating to zero we obtain

$$g_{3N}{}^* = \frac{B_{2N}C_{2N}-D_{2N}E_{2N}}{A_{2N}B_{2N}-E_{2N}^2} \tag{37}$$

$$g_{4N}{}^* = \frac{A_{2N}D_{2N}-C_{2N}E_{2N}}{A_{2N}B_{2N}-E_{2N}^2}. \tag{38}$$

By putting the optimum value of g_{3N} & g_{4N} in the Eq. (36), we can determine the minimum MSE of P_{2N} as

$$\text{MinMSE} = C_{2N}+\frac{B_{2N}C_{2N}^2+A_{2N}D_{2N}^2-2C_{2N}D_{2N}E_{2N}}{E_{2N}^2-A_{2N}B_{2N}}. \tag{39}$$

$$(3)\,P_{3N} = g_{5N}\overline{y}_{[n]N}+\ g_{6N}\left(\frac{\overline{X}_N}{\overline{x}_{(n)N}}\right)\exp\left(\frac{\overline{X}_N-\overline{x}_{(n)N}}{\overline{X}_N+\overline{x}_{(n)N}}\right). \tag{40}$$

When we express the estimator P_{3N} from Eq. (40) into terms of $\epsilon's$, we obtain

$$P_{3N} = g_{5N}\overline{Y}_N(1+\epsilon_{0N})+g_{6N} \times(1+\epsilon_{1N})^{-1}\ \exp\left(\frac{-\epsilon_1}{2+\epsilon_{1N}}\right) \tag{41}$$

$$P_{3N}-\overline{Y}_N = (g_{5N}-1)\overline{Y}_N+g_{5N}\overline{Y}_N\epsilon_{01} +g_{6N}\left(1-\frac{3\epsilon_{1N}}{2}+\frac{15\epsilon_{1N}{}^2}{8}\right) \tag{42}$$

$$\text{Bias}(P_{3N}) = \overline{Y}_N(g_{5N}-1)+g_{6N}\left[1+\frac{15}{8}V_{xrN}\right]. \tag{43}$$

Case 1: If sum of weights is fixed ($g_{5N}+g_{6N}=1$)
The MSE of the estimator P_{3N} is shown as

$$\text{MSE}(P_{3N}) = \overline{Y}_N{}^2\left[V_{yrN}+g_{6N}^2V_{xrN}-2g_{6N}V_{yxrN}\right]. \tag{44}$$

To find out the minimum value of MSE for P_{3N}, we partially differentiate equation (44) w.r.t. g_{6N} and equating to zero, we get

$$g_{6N}{}^* = \frac{V_{yxrN}}{V_{xrN}}. \tag{45}$$

By putting the optimum value of g_{6N} in the Eq. (44), we can determine the minimum MSE of P_{3N} as

$$\text{MinMSE} = \overline{Y}_N{}^2\left(V_{yrN}-\frac{V_{yxrN}^2}{V_{xrN}}\right). \tag{46}$$

Case 2: If the sum of weights is adjustable ($g_{5N}+g_{6N}\neq 1$)

$$P_{3N}-\overline{Y}_N = (g_{5N}-1)\overline{Y}_N+g_{5N}\overline{Y}\epsilon_{0N} +g_{6N}\left(1-\frac{3\epsilon_{1N}}{2}+\frac{15\epsilon_{1N}{}^2}{8}\right). \tag{47}$$

Squaring on both sides we get

$$\begin{aligned}(P_{3N}-\overline{Y}_N)^2 &= \overline{Y}_N{}^2+\overline{Y}_N{}^2g_{5N}{}^2(1+\epsilon_{0N}^2)+g_{6N}{}^2\left(1+6\epsilon_{1N}^2\right)\\ &\quad -2g_{5N}\overline{Y}_N{}^2-2g_{6N}\overline{Y}_N\left(1+\frac{15\epsilon_{1N}{}^2}{8}\right)\\ &\quad +2g_{5N}g_{6N}\overline{Y}_N\left(1+\frac{15\epsilon_{1N}{}^2}{8}-\frac{3\epsilon_{0N}\epsilon_{1N}}{2}\right).\end{aligned} \tag{48}$$

Taking expectations by focusing on first-order approximation, we obtain MSE,

$$\text{MSE}\,(P_{3N}) = \overline{Y}_N{}^2+g_{5N}^2A_{3N}+g_{6N}^2B_{3N}-2g_{5N}C_{3N} -2g_{6N}D_{3N}+2g_{5N}g_{6N}E_{3N}, \tag{49}$$

where

$$A_{3N} = \overline{Y}_N{}^2(1+V_{yrN})$$
$$B_{3N} = 1+6V_{xrN}$$
$$C_{3N} = \overline{Y}_N{}^2$$
$$D_{3N} = \overline{Y}_N\left(1+\frac{15}{8}V_{xrN}\right)$$
$$E_{3N} = \overline{Y}_N\left(1+\frac{15}{8}V_{xrN}-\frac{3}{2}V_{yxrN}\right).$$

To find out the minimum MSE for P_{3N}, we partially differentiate equation (49) w.r.t. g_{5N} & g_{6N} and equating to zero, we get

$$g_{5N}{}^* = \frac{B_{3N}C_{3N}-D_{3N}E_{3N}}{A_{3N}B_{3N}-E_{3N}^2} \tag{50}$$

$$g_{6N}{}^{*} = \frac{A_{3N}D_{3N} - C_{3N}E_{3N}}{A_{3N}B_{3N} - E_{3N}^2}. \tag{51}$$

By putting the optimum value of g_{5N} & g_{6N} in the Eq. (49), we can determine the minimum MSE of P_{3N} as

$$\text{MinMSE} = C_{3N} + \frac{B_{3N}C_{3N}^2 + A_{3N}D_{3N}^2 - 2C_{3N}D_{3N}E_{3N}}{E_{3N}^2 - A_{3N}B_{3N}} \tag{52}$$

where $P_{iN} \in [P_{iL}, P_{iU}]; i = 1, 2, 3, A_{iN} \in [A_{iL}, A_{iU}]; i = 1, 2, 3, B_{iN} \in [B_{iL}, B_{iU}]; i = 1, 2, 3$

$C_{iN} \in [C_{iL}, C_{iU}]; i = 1, 2, 3, D_{iN} \in [D_{iL}, D_{iU}]; i = 1, 2, 3, E_{iN} \in [E_{iL}, E_{iU}]; i = 1, 2, 3.$

6 Numerical Illustrations

In this section, we compare the performance of the suggested estimators to the other existing estimators considered in this paper. We have taken real-life natural growth rate data from the sample registration system (2020). The data mentioned in the sample registration system (2020) have four neutrosophic variables for every state, but in our research, we use birth rate vs natural growth rate. Here, birth rate is the neutrosophic auxiliary variable $X_N \in [X_L, X_U]$ and natural growth rate is a neutrosophic study variable $Y_N \in [Y_L, Y_U]$.

To obtain the relative performance of proposed estimators under NMRSS, we have taken two cases of odd and even set sizes. Further, in odd case we have drawn total $n_N = m_N r = 12$ samples with set size $m_N = [3, 3]$ and replication $r = 4$ and for even case we have drawn total $n_N = m_N r = 16$ samples with set size $m_N = [4, 4]$ with replication $r = 4$ from the given population of size 36 by utilizing the method of NMRSS. The NMRSS sample for the study and auxiliary variables are drawn simultaneously using the NMRSS technique outlined in Sect. 3. The following is a definition of the percent relative efficiency (PRE) formula:

$$\text{PRE (Estimators)} = \frac{\text{MSE}(\overline{y}_{[n]N})}{\text{MSE(estimator)}} \times 100. \tag{53}$$

7 Simulation Studies

We verify the effectiveness of the suggested estimator through simulation studies with other existing estimators like the conventional, ratio, regression estimator, etc. This is done via the following steps

1. It is well known that a neutrosophic normal distribution (NND) will be followed by neutrosophic random variables (NRV), i.e.$(X_N, Y_N) \sim NN[(\mu_{xN}, \sigma_{xN}^2), (\mu_{yN}, \sigma_{yN}^2)]$, $X_N \in [X_L, X_U]$, $Y_N \in [Y_L, Y_U]$, $\mu_{xN} \in [(\mu_{xL}, \mu_{xU})]$, $\mu_{yN} \in [(\mu_{yL}, \mu_{yU})]$, $\sigma_{xN}^2 \in [\sigma_{xL}^2, \sigma_{xU}^2]$, $\sigma_{yN}^2 \in [\sigma_{yL}^2, \sigma_{yU}^2]$.

4-variate random observations with a size of $N = 1000$ have been produced from a 4-variate normal distribution with mean $(\mu_{xL}, \mu_{yL}, \mu_{xU}, \mu_{yU}) = (50, 50, 60, 60)$ and covariance matrix

$$\begin{bmatrix} \sigma_{xL}^2 & \rho_{xyL}\sigma_{xL}\sigma_{yL} & 0 & 0 \\ \rho_{xyL}\sigma_{xL}\sigma_{yL} & \sigma_{yL}^2 & 0 & 0 \\ 0 & 0 & \sigma_{xU}^2 & \rho_{xyU}\sigma_{xU}\sigma_{yU} \\ 0 & 0 & \rho_{xyU}\sigma_{xU}\sigma_{yU} & \sigma_{yU}^2 \end{bmatrix},$$

where we have $\sigma_{xL}^2 = 100\,\sigma_{yL}^2 = 100$, $\sigma_{xU}^2 = 121\,\sigma_{yU}^2 = 121$.

2. For this $N = 1000$ simulated population, the parameters were computed.

3. A sample of size n with $m_N = 3$ for odd case and $m_N = 4$ for even case with $r = 4$ has been selected from this simulated population.

4. To find the MSE of each estimator under study, use the sample data.

5. To get MSEs, the entire step 3–4 process was repeated 10,000 times. The MSE of each estimator of the population mean is the average of the 10,000 values that were obtained.

6. The formula has been used to find each estimator's PRE with regard to $\overline{y}_{[n]N}$.

8 Discussion

The study established mathematical expressions for the very first NMRSS estimators, approximating up to the first order. Subsequently, to examine the properties of the proposed NMRSS estimators, numerical illustrations and simulation studies were conducted. The former used real-world natural growth rate data, while the latter involved an artificial neutrosophic dataset with varying correlation coefficients and sample sizes. The results were encapsulated in Tables 2, 3, 4, and 5, showcasing MSEs and PREs for both existing and proposed NMRSS estimators for odd and even cases. The PREs of the NMRSS estimators were calculated over estimators under NSRS and NRSS, and these outcomes are shown in Tables 6 and 7.

In Tables 2 and 3, the MSEs of the existing and suggested estimators are given along with PRE for odd and even cases respectively. The superiority of the suggested NMRSS estimators over the existing ones is displayed in Tables 2 and 3 in the bolded text. We also see that the MSE and PRE of the recommended estimator are lesser and higher than those of the other existing estimators. The tables make this clear that recommended estimators outperformed existing ones, offering lower MSEs and higher PREs, and t_{p2} being the best estimator.

Table 1 The data of natural growth rate as per SRS 2020

State	BRl	BRu	NGRl	NGRu	State	BRl	BRu	NGRl	NGRu
Andhra Pradesh	15	16	9	10.1	Uttar Pradesh	22.1	26.1	19.3	16.7
Assam	14.3	21.9	8.9	15.5	Uttarakhand	15.6	17	10.5	10.3
Bihar	21	26.2	15.7	20.7	West Bengal	11.2	16.1	10.8	5.4
Chhattisgarh	17.3	23.4	11	15	Arunachal Pradesh	15	17.8	11.8	10.6
NCT of Delhi	14.1	15.5	10.6	11.4	Goa	11.7	12.4	6.9	5.3
Gujrat	17.1	21.1	12	15.1	Himachal Pradesh	10	15.7	8.7	5.6
Haryana	17.7	21.2	12.3	14.7	Manipur	12.8	13.5	9.5	8
Jammu & Kashmir	11.1	16.1	7	11.3	Meghalaya	12.9	25.1	19.6	8.5
Jharkhand	17.6	23.4	13.1	17.9	Mizoram	11.7	16.8	13	7.1
Karnataka	15	17.5	10.2	10.5	Nagaland	11.8	12.9	9	8.4
Kerala	13.1	13.3	6.1	6.3	Sikkim	14	18.2	14.5	9.7
Madhya Pradesh	18.8	26	13.1	19.2	Tripura	10.7	13.4	8	4.2
Maharashtra	14.6	15.3	9.1	10.1	Andaman & Nicobar	10	11.5	5.4	4.7
Odisha	13.1	18.7	6.6	11.2	Chandigarh	12.8	18.1	14	9
Punjab	13.6	14.9	6.6	7.9	Dadar Nagar Haveli	18	21.4	18.1	13.3
Rajasthan	20.8	24.4	15.7	18.6	Ladakh	10.8	15.2	10	6.5
Tamil Nadu	13.6	14	6.8	8.5	Lakshadweep	13.1	19.9	12.7	8.1
Telangana	15.9	16.8	9.6	11.7	Puducherry	13.1	13.1	7	5.6

Table 2 The MSE and PRE of the estimators for odd case

Estimators	MSE	PRE
$\overline{y}_{[n]N}$	[0.351545, 1.209508]	[100, 100]
$\overline{y}_{rN}$	[0.110387, 0.200354]	[318, 603]
$\overline{y}_{pN}$	[0.849801, 3.365292]	[35, 41]
$\overline{y}_{\text{reg}N}$	[0.085726, 0.117522]	[410, 1029]
$\overline{y}_{\exp rN}$	[0.198829, 0.561602]	[176, 215]
$\overline{y}_{\exp pN}$	[0.568536, 2.144071]	[56, 61]
t_{p1}	**[0.085315, 0.115937]**	**[412, 1043]**
t_{p2}	**[0.013749, 0.028534]**	**[2556, 4238]**
t_{p3}	**[0.020461, 0.029224]**	**[1718, 4138]**

Table 3 The MSE and PRE of the estimators for even case

Estimators	MSE	PRE
$\overline{y}_{[n]N}$	[0.402361, 0.760615]	[100, 100]
$\overline{y}_{rN}$	[0.13007, 0.203868]	[309, 373]
$\overline{y}_{pN}$	[0.918725, 1.843483]	[41, 43]
$\overline{y}_{\text{reg}N}$	[0.083821, 0.121898]	[480, 623]
$\overline{y}_{\exp rN}$	[0.235706, 0.416477]	[170, 182]
$\overline{y}_{\exp pN}$	[0.630034, 1.236284]	[61, 63]
t_{p1}	**[0.083418, 0.121109]**	**[482, 628]**
t_{p2}	**[0.010757, 0.018951]**	**[3740, 4013]**
t_{p3}	**[0.017894, 0.027317]**	**[2248, 2784]**

Similarly, in Tables 4 and 5, the MSEs of the recommended and existing estimators are given along with PRE through a simulation study based on artificial neutrosophic data for different values of the correlation coefficient and different sample sizes. Like Tables 2 and 3, the superiority of the suggested NMRSS estimators over the existing ones is displayed in Tables 4 and 5. We also see that the MSEs and PREs of the recommended estimators are lesser and higher, respectively than those of other existing estimators. Hence, Tables 4 and 5 mirrored these findings, with the proposed estimators continuing to outshine existing ones, demonstrating lower MSEs and higher PREs in the simulation study too.

Tables 4 and 5 show that with the increase in sample sizes and correlation coefficients, the MSE decreases, and the PRE increases for the recommended estimator. Therefore, under NMRSS, the suggested estimators exhibit sensitivity similar to that of classical ranked set sampling.

Table 6 features PRE values of the proposed NMRSS estimators compared to NSRS counterparts. We observe from Table 6 that all PRE values exceeded 100 indicating that all the NMRSS estimators are superior to corresponding estimators under NSRS, as RSS is the best replacement for SRS.

Table 7 features PRE values of the proposed NMRSS estimators over NRSS counterparts. We see from Table 7 that all PRE values exceeded 100, indicating that all the NMRSS estimators are superior to corresponding estimators under NRSS, as MRSS is the best replacement for RSS when data do not follow the symmetric distribution.

Table 4 MSEs and PREs of the recommended and existing estimators under NMRSS for odd case

Estimators	$\rho = 0.9$		$\rho = 0.8$	
	MSE	PRE	MSE	PRE
$\overline{y}_{[n]N}$	[25.8558, 66.8489]	[100, 100]	[29.1643, 63.5223]	[100, 100]
$\overline{y}_{rN}$	[9.4335, 13.6782]	[274, 489]	[17.2001, 26.2012]	[170, 242]
$\overline{y}_{pN}$	[82.3835, 248.6207]	[27, 31]	[83.7130, 234.4552]	[27, 35]
$\overline{y}_{\text{reg}N}$	[8.3641, 11.7963]	[309, 567]	[14.4049, 20.4974]	[202, 310]
$\overline{y}_{\text{expr}N}$	[12.6315, 24.1884]	[205, 276]	[17.8592, 28.1602]	[163, 226]
$\overline{y}_{\text{exp}pN}$	[49.1065, 141.6597]	[47, 53]	[51.1156, 132.2873]	[48, 57]
t_{p1}	[8.3489, 11.7476]	[310, 569]	[14.3792, 20.4354]	[203, 311]
t_{p2}	[3.2519, 7.7395]	[795, 864]	[4.2796, 12.1591]	[522, 681]
t_{p3}	[2.9685, 4.3086]	[871,1552]	[5.4381, 8.2076]	[536, 774]
Estimators	$\rho = 0.7$		$\rho = 0.6$	
	MSE	PRE	MSE	PRE
$\overline{y}_{[n]N}$	[33.8702, 68.0986]	[100, 100]	[35.8566, 67.8329]	[100, 100]
$\overline{y}_{rN}$	[26.6266, 41.3841]	[127, 165]	[31.8830, 52.6924]	[112, 129]
$\overline{y}_{pN}$	[81.3494, 223.5612]	[30, 42]	[82.8448, 211.6999]	[32, 43]
$\overline{y}_{\text{reg}N}$	[22.0309, 32.2804]	[154, 211]	[25.1446, 38.9294]	[143, 174]
$\overline{y}_{\text{expr}N}$	[25.2190, 38.6479]	[134, 176]	[28.4930, 44.1718]	[126, 154]
$\overline{y}_{\text{exp}pN}$	[52.5804, 129.7364]	[52, 64]	[53.9739, 123.6756]	[55, 66]
t_{p1}	[21.9881, 32.1936]	[154, 212]	[25.0939, 38.8304]	[143, 175]
t_{p2}	[4.3620, 13.3893]	[509, 776]	[4.7355, 14.1430]	[480, 757]
t_{p3}	[8.2888, 13.2007]	[409, 516]	[9.9709, 16.7169]	[360, 406]

Table 5 MSEs and PREs of the recommended and existing estimator under NMRSS for even case

Estimators	$\rho = 0.9$		$\rho = 0.8$	
	MSE	PRE	MSE	PRE
$\overline{y}_{[n]N}$	[16.6856, 49.0243]	[100, 100]	[19.6011, 46.4910]	[100, 100]
$\overline{y}_{rN}$	[6.8564, 9.9145]	[243, 494]	[12.4269, 19.1244]	[158, 243]
$\overline{y}_{pN}$	[50.6041, 182.1930]	[27, 33]	[52.2026, 171.3741]	[27, 38]
$\overline{y}_{\text{reg}N}$	[6.2676, 8.8094]	[266, 556]	[10.8760, 15.4188]	[180, 302]
$\overline{y}_{\text{expr}N}$	[8.7599, 17.7120]	[190, 277]	[12.8356, 20.6181]	[153, 225]
$\overline{y}_{\text{exp}pN}$	[30.6337, 103.8513]	[47, 54]	[32.7235, 96.7430]	[48, 60]
t_{p1}	[6.2610, 8.7838]	[267, 558]	[10.8639, 15.3859]	[180, 302]
t_{p2}	[2.1239, 5.8077]	[786, 844]	[2.6848, 9.1361]	[509, 730]
t_{p3}	[2.2015, 3.2341]	[758,1516]	[3.9952, 6.1923]	[491, 751]
Estimators	$\rho = 0.7$		$\rho = 0.6$	
	MSE	PRE	MSE	PRE
$\overline{y}_{[n]N}$	[23.5002, 49.9260]	[100, 100]	[25.0812, 49.5001]	[100, 100]
$\overline{y}_{rN}$	[18.9892, 30.1761]	[124, 165]	[22.6217, 38.5168]	[111, 129]
$\overline{y}_{pN}$	[52.1644, 163.5474]	[31, 45]	[53.1778, 154.5742]	[32, 47]
$\overline{y}_{\text{reg}N}$	[16.4401, 24.2585]	[143, 206]	[18.8473, 29.1725]	[133, 170]
$\overline{y}_{\text{expr}N}$	[18.2256, 28.3171]	[129, 176]	[20.6468, 32.2471]	[121, 154]
$\overline{y}_{\text{exp}pN}$	[34.8132, 95.0028]	[53, 68]	[35.9249, 90.2758]	[55, 70]
t_{p1}	[16.4191, 24.2121]	[143, 206]	[18.8220, 29.1197]	[133, 170]
t_{p2}	[2.7098, 10.0417]	[497, 867]	[2.9217, 10.6288]	[466, 858]
t_{p3}	[5.8993, 9.9317]	[398, 503]	[7.0985, 12.5925]	[353, 393]

Table 6 PREs of the NMRSS estimators over estimators under NSRS

$n = 12$ Estimators	$\rho = 0.9$ PRE	$\rho = 0.8$ PRE	$\rho = 0.7$ PRE	$\rho = 0.6$ PRE
$\overline{y}_{[n]N}$	[119, 209]	[122, 187]	[118, 160]	[119, 152]
$\overline{y}_{rN}$	[118, 118]	[122, 130]	[117, 125]	[123, 140]
$\overline{y}_{pN}$	[121, 250]	[120, 233]	[121, 226]	[120, 210]
$\overline{y}_{\text{reg}N}$	[116, 119]	[125, 126]	[117, 118]	[123, 127]
$\overline{y}_{\text{expr}N}$	[117, 151]	[124, 139]	[116, 119]	[120, 126]
$\overline{y}_{\text{exppN}}$	[120, 237]	[121, 218]	[120, 201]	[120, 186]
t_{p1}	[115, 119]	[125, 125]	[117, 118]	[123, 127]
t_{p2}	[124, 203]	[120, 230]	[124, 264]	[125, 256]
t_{p3}	[119, 120]	[124, 132]	[118, 130]	[124, 144]
$n = 16$ Estimators	$\rho = 0.9$ PRE	$\rho = 0.8$ PRE	$\rho = 0.7$ PRE	$\rho = 0.6$ PRE
$\overline{y}_{[n]N}$	[122, 240]	[125, 205]	[121, 171]	[122, 161]
$\overline{y}_{rN}$	[122, 122]	[125, 135]	[121, 132]	[126, 148]
$\overline{y}_{pN}$	[124, 300]	[123, 277]	[124, 260]	[123, 242]
$\overline{y}_{\text{reg}N}$	[119, 122]	[128, 129]	[120, 121]	[126, 130]
$\overline{y}_{\text{expr}N}$	[119, 162]	[128, 143]	[118, 124]	[123, 130]
$\overline{y}_{\text{exppN}}$	[123, 281]	[124, 251]	[123, 224]	[123, 207]
t_{p1}	[119, 122]	[127, 129]	[120, 121]	[126, 130]
t_{p2}	[128, 240]	[123, 283]	[128, 323]	[129, 318]
t_{p3}	[123, 125]	[128, 139]	[121, 141]	[128, 156]

Table 7 PREs of the NMRSS estimators over estimators under NRSS

$n = 12$ Estimators	$\rho = 0.9$ PRE	$\rho = 0.8$ PRE	$\rho = 0.7$ PRE	$\rho = 0.6$ PRE
$\overline{y}_{[n]N}$	[109, 114]	[102, 113]	[107, 110]	[107, 109]
$\overline{y}_{rN}$	[107, 108]	[102, 104]	[108, 110]	[103, 107]
$\overline{y}_{pN}$	[101, 123]	[101, 119]	[101, 120]	[101, 114]
$\overline{y}_{\text{reg}N}$	[106, 107]	[103, 105]	[106, 108]	[101, 106]
$\overline{y}_{\text{expr}N}$	[102, 107]	[104, 106]	[106, 109]	[101, 104]
$\overline{y}_{\text{exppN}}$	[102, 120]	[101, 117]	[109, 115]	[1011, 111]
t_{p1}	[106, 107]	[103, 105]	[106, 108]	[101, 106]
t_{p2}	[102, 121]	[109, 119]	[102, 128]	[103, 122]
t_{p3}	[108, 109]	[102, 106]	[102, 107]	[103, 110]
$n = 16$ Estimators	$\rho = 0.9$ PRE	$\rho = 0.8$ PRE	$\rho = 0.7$ PRE	$\rho = 0.6$ PRE
$\overline{y}_{[n]N}$	[109, 113]	[101, 112]	[101, 105]	[107, 109]
$\overline{y}_{rN}$	[106, 109]	[102, 104]	[101, 108]	[103, 107]
$\overline{y}_{pN}$	[101, 123]	[101, 119]	[101, 119]	[101, 114]
$\overline{y}_{\text{reg}N}$	[105, 108]	[103, 104]	[101, 108]	[101, 106]
$\overline{y}_{\text{expr}N}$	[101, 107]	[103, 106]	[101, 108]	[101, 105]
$\overline{y}_{\text{exppN}}$	[109, 120]	[102, 117]	[101, 113]	[101, 111]
t_{p1}	[105, 108]	[103, 104]	[101, 1.08]	[101, 106]
t_{p2}	[102, 122]	[110, 119]	[102, 129]	[102, 122]
t_{p3}	[109, 111]	[102, 105]	[102, 107]	[103, 110]

The comparison between classical MRSS and NMRSS using MSEs and PREs is provided in Tables 8 and 9. Tables 8 and 9 demonstrate that the MSEs of the suggested estimators obtained through classical MRSS are between the lower and higher value of MSE obtained using NMRSS, and the same goes for the PREs, indicating that the latter method is more effective than the former.

The study highlights that classical ranked set sampling is ill-suited for dealing with vague or indeterminate data. NMRSS proves superior for estimating uncertain or interval data. The tables present dependable results for neutrosophic data compared to classical results.

9 Conclusion

This study aims to tackle the challenges posed by ambiguous or indeterminate data within the realm of classical statistics. Through the utilization of MRSS, an enhancement over SRS and RSS, the study addresses the complexities associated with expensive and asymmetric data. Given the absence of prior research addressing this specific issue, our work represents a novel and significant advancement in survey sampling methodologies, particularly in dealing with imprecise data featuring asymmetric distributions.

In this research paper, we have proposed a new modification of RSS: the very first neutrosophic median ranked set method in sampling theory. We've put forth some enhanced NMRSS estimators designed for estimating population means while making use of auxiliary information. To assess their accuracy, we calculated both bias and MSE for these suggested estimators, focusing on first-order approximations. We compared our recommended estimators against existing ones using a natural population data on natural growth rates and a simulated population.

Through a combination of numerical illustrations and simulation studies, we've found compelling evidence that our proposed estimators outperform existing ones within the framework of NMRSS. Among these estimators, t_{p2} emerged as the top performer. It's important to note that the sensitivity analysis of our recommended estimators under NMRSS mirrors that of classical MRSS. The MSE and PRE of the recommended estimator decrease and increase, respectively, with the increase in values of sample sizes and correlation coefficients.

Moreover, a comparison between the recommended estimators under NMRSS and the estimators under NSRS and NRSS revealed that NMRSS is a more effective alternative to NSRS and NRSS, much like classical MRSS to classical SRS and classical RSS. The MSEs of the proposed estimators derived from classical MRSS fall within the range between the lower and higher MSE values obtained through NMRSS. Similarly, the PREs exhibit a similar pattern. These findings

Table 8 MSEs and PREs of the estimators (neutrosophic vs classical) for odd case

$\rho = 0.9$	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\overline{y}_{[n]N}$	[25.8558, 66.8489]	30.3481	[100, 100]	100
$\overline{y}_{rN}$	[9.4335, 13.6782]	10.6122	[274, 489]	286
$\overline{y}_{pN}$	[82.3835, 248.6207]	97.9875	[27, 31]	31
$\overline{y}_{\text{reg}N}$	[8.3641, 11.7963]	9.2976	[309, 567]	326
$\overline{y}_{\text{expr}N}$	[12.6315, 24.1884]	14.4922	[205, 276]	209
$\overline{y}_{\text{exp}pN}$	[49.1065, 141.6597]	58.1799	[47, 53]	52
t_{p1}	[8.3489, 11.7476]	9.2816	[310, 569]	327
t_{p2}	[3.2519, 7.7395]	3.8547	[795, 864]	787
t_{p3}	[2.9685, 4.3086]	3.3317	[871, 1552]	911
$\rho = 0.8$	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\overline{y}_{[n]N}$	[29.1643, 63.5223]	35.6477	[100, 100]	100
$\overline{y}_{rN}$	[17.2001, 26.2012]	20.5729	[170, 242]	173
$\overline{y}_{pN}$	[83.7130, 234.4552]	98.9972	[27, 35]	36
$\overline{y}_{\text{reg}N}$	[14.4049, 20.4974]	17.6304	[202, 310]	202
$\overline{y}_{\text{expr}N}$	[17.8592, 28.1602]	22.0759	[163, 226]	161
$\overline{y}_{\text{exp}pN}$	[51.1156, 132.2873]	61.2881	[48, 57]	58
t_{p1}	[14.3792, 20.4354]	17.6007	[203, 311]	203
t_{p2}	[4.2796, 12.1591]	4.8312	[522, 681]	738
t_{p3}	[5.4381, 8.2076]	6.4180	[536, 774]	555
$\rho = 0.7$	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\overline{y}_{[n]N}$	[33.8702, 68.0986]	40.3408	[100, 100]	100
$\overline{y}_{rN}$	[26.6266, 41.3841]	29.8109	[127, 165]	135
$\overline{y}_{pN}$	[81.3494, 223.5612]	99.3064	[30, 42]	41
$\overline{y}_{\text{reg}N}$	[22.0309, 32.2804]	24.9118	[154, 211]	162
$\overline{y}_{\text{expr}N}$	[25.2190, 38.6479]	29.0213	[134, 176]	139
$\overline{y}_{\text{exp}pN}$	[52.5804, 129.7364]	63.7691	[52, 64]	63
t_{p1}	[21.9881, 32.1936]	24.8667	[154, 212]	162
t_{p2}	[4.3620, 13.3893]	5.2121	[509, 776]	774
t_{p3}	[8.2888, 13.2007]	9.2138	[409, 516]	438
$\rho = 0.6$	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\overline{y}_{[n]N}$	[35.8566, 67.8329]	44.1928	[100, 100]	100
$\overline{y}_{rN}$	[31.8830, 52.6924]	38.5595	[112, 129]	115
$\overline{y}_{pN}$	[82.8448, 211.6999]	98.3161	[32, 43]	45
$\overline{y}_{\text{reg}N}$	[25.1446, 38.9294]	31.2729	[143, 174]	141
$\overline{y}_{\text{expr}N}$	[28.4930, 44.1718]	35.3149	[126, 154]	125
$\overline{y}_{\text{exp}pN}$	[53.9739, 123.6756]	65.1932	[55, 66]	68
t_{p1}	[25.0939, 38.8304]	31.2119	[143, 175]	142
t_{p2}	[4.7355, 14.1430]	5.3729	[480, 757]	823
t_{p3}	[9.9709, 16.7169]	11.8420	[360, 406]	373

Table 9 MSEs and PREs of the estimators (neutrosophic vs classical) for even case

$\rho = 0.9$	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\overline{y}_{[n]N}$	[16.68563, 49.02427]	19.51716	[100, 100]	100
$\overline{y}_{rN}$	[6.85639, 9.91454]	7.66664	[243, 494]	255
$\overline{y}_{pN}$	[50.60409, 182.193]	59.9208	[27, 33]	33
$\overline{y}_{\text{reg}N}$	[6.26758, 8.80943]	6.96706	[266, 556]	280
$\overline{y}_{\text{expr}N}$	[8.75986, 17.71203]	10.02276	[190, 277]	195
$\overline{y}_{\text{exp}pN}$	[30.63371, 103.8513]	36.14984	[47, 54]	54
t_{p1}	[6.26095, 8.78384]	6.96016	[267, 558]	280
t_{p2}	[2.12389, 5.80765]	2.48653	[786, 844]	785
t_{p3}	[2.20148, 3.23414]	2.45599	[758, 1516]	795
ρ =0.8	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\overline{y}_{[n]N}$	[19.60108, 46.49099]	23.88907	[100, 100]	100
$\overline{y}_{rN}$	[12.42685, 19.12443]	14.83691	[158, 243]	161
$\overline{y}_{pN}$	[52.20264, 171.3741]	61.72381	[27, 38]	39
$\overline{y}_{\text{reg}N}$	[10.87602, 15.41875]	13.21414	[180, 302]	181
$\overline{y}_{\text{expr}N}$	[12.83555, 20.61814]	15.76516	[153, 225]	152
t_{p1}	[10.86386, 15.38587]	13.20018	[180, 302]	181
t_{p2}	[2.68475, 9.13607]	3.02011	[509, 730]	791
t_{p3}	[3.99515, 6.19232]	4.65817	[491, 751]	513
$\rho = 0.7$	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\overline{y}_{[n]N}$	[23.50022, 49.92603]	27.59529	[100, 100]	100
$\overline{y}_{rN}$	[18.98922, 30.17608]	21.4977	[124, 165]	128
$\overline{y}_{pN}$	[52.16444, 163.5474]	62.57538	[31, 45]	44
$\overline{y}_{\text{reg}N}$	[16.4401, 24.25845]	18.72936	[143, 206]	147
$\overline{y}_{\text{expr}N}$	[18.22557, 28.31713]	20.93618	[129, 176]	132
$\overline{y}_{\text{exp}pN}$	[34.81318, 95.00277]	41.47502	[53, 68]	67
t_{p1}	[16.41913, 24.21205]	18.70726	[143, 206]	148
t_{p2}	[2.70979, 10.04169]	3.23187	[497, 867]	854
t_{p3}	[5.89925, 9.93174]	6.65863	[398, 503]	414
$\rho = 0.6$	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\overline{y}_{[n]N}$	[25.08118, 49.50014]	30.90622	[100, 100]	100
$\overline{y}_{rN}$	[22.62167, 38.51675]	27.72227	[111, 129]	111
$\overline{y}_{pN}$	[53.17782, 154.5742]	63.02822	[32, 47]	49
$\overline{y}_{\text{reg}N}$	[18.8473, 29.17245]	23.55219	[133, 170]	131
$\overline{y}_{\text{expr}N}$	[20.64679, 32.24711]	25.69699	[121, 154]	120
$\overline{y}_{\text{exp}pN}$	[35.92486, 90.27583]	43.34997	[55, 70]	71
t_{p1}	[18.82196, 29.1197]	23.52152	[133, 170]	131
t_{p2}	[2.9217, 10.62878]	3.31264	[466, 858]	933
t_{p3}	[7.09854, 12.59245]	8.48045	[353, 393]	364

suggest that the suggested method surpasses the effectiveness of classical approaches. Our study underscores the efficiency and reliability of NMRSS for handling neutrosophic data, with our proposed NMRSS delivering superior mean estimations compared to existing methods.

Based on the numerical illustrations and simulation studies we've conducted, it's reasonable to recommend the use of our proposed estimators over the alternatives presented in this paper in various real-world scenarios, spanning fields like agriculture, mathematics, biology, poultry farming, economics, commerce, and the social sciences.s

Furthermore, given the limited availability of neutrosophic MRSS estimators, there's ample room for further exploration. Building upon this study, we can consider defining variations of NRSS, such as unbalanced NRSS, quantile NRSS, extreme NRSS, double NRSS, and percentile NRSS, akin to what exists in classical RSS. Additionally, we can explore the replacement of our proposed estimators with alternative methods or estimators. Expanding beyond sampling theory, further research avenues in statistics, encompassing fields like control charts, inference, reliability analysis, non-parametric estimation, hypothesis testing, and some other fields of science, present promising opportunities for exploration.

References

1. Al-Omari, A.I.: Ratio estimation of the population mean using auxiliary information in simple random sampling and median ranked set sampling. Stat. Probab. Lett. **82**, 1883–1890 (2012)
2. Al-Omari, A.I., Bouza, C.N.: Ratio estimators of the population mean with missing values using ranked set sampling. Environmetrics **26**(2), 67–76 (2015)
3. Bahl, S., Tuteja, R.K.: Ratio and product type exponential estimators. J. Inf. Optim. Sci. **12**(1), 159–164 (1991)
4. Bhushan, S., Kumar, A., Lone, S.A.: On some novel classes of estimators using RSS. Alex. Eng. J. **61**, 5465–5474 (2022)
5. Bouza, C.N.: Ranked set sub-sampling of the non-response strata for estimating the difference of means. Biom. J. J. Math. Methods Biosci. **44**(7), 903–915 (2002)
6. Bouza, C.N.: Ranked set sampling for the product estimator. Rev. Investig. Oper. **29**(3), 201–206 (2008)
7. Dell, T., Clutter, J.: Ranked set sampling theory with order statistics background. Biometrics **28**, 545–555 (1972)
8. Ganesh, S., Ganeslingam, S.: Ranked set sampling vs simple random sampling in the estimation of the mean and the ratio. J. Stat. Manag. Syst. **9**(2), 459–472 (2006)
9. Kadilar, C., Cingi, H.: Ratio estimators in simple random sampling. Appl. Math. Comput. **151**, 893–902 (2004)
10. Kadilar, C., Unyazici, Y., Cingi, H.: Ratio estimator for the population mean using ranked set sampling. Stat. Pap. **50**, 301–309 (2009)
11. Koyuncu, N.: New difference-cum-ratio and exponential type estimators in median ranked set sampling. Hacet. J. Math. Stat. **45**(1), 207–225 (2016)
12. Mandowara, V.L., Mehta, N.: Efficient generalized ratio-product type estimators for finite population mean with ranked set sampling. Aust. J. Stat. **42**(3), 137–148 (2013)
13. McIntyre, G.A.: A method for unbiased selective sampling using ranked sets. Crop. Pasture Sci. **3**, 385–390 (1952)
14. Mishra, P., Adichwal, N.K., Singh, R.: A new log-product-type estimator using auxiliary information. J. Sci. Res. **61**, 179–183 (2017)
15. Muttlak, H.A.: Median ranked set sampling. J. Appl. Stat. Sci. **6**, 245–255 (1997)
16. Pandey, B.N., Dubey, V.: Modified product estimator using coefficient of variation of the auxiliary variate. Assam Stat. Rev. **2**(2), 64–66 (1988)
17. Samawi, H.M., Muttlak, H.A.: Estimation of ratio using rank set sampling. Biom. J. **38**(6), 753–764 (1996)
18. Samawi, H.M., Muttlak, H.A.: On ratio estimation using median ranked set sampling. J. Appl. Stat. Sci. **10**(2), 89–98 (2002)
19. Singh, H.P., Tailor, R., Tailor, R., Kakran, M.S.: An improved estimator of population mean using power transformation. J. Indian Soc. Agric. Stat. **58**(2), 223–230 (2004)
20. Singh, H.P., Tailor, R., Singh, S.: General procedure for estimating the population means using ranked set sampling. J. Stat. Comput. Simul. **84**(5), 931–945 (2014)
21. Singh, P., Bouza, C., Singh, R.: Generalized exponential estimator for estimating the population mean using an auxiliary variable. J. Sci. Res. **63**, 273–280 (2019)
22. Singh, R., Mishra, R.: Neutrosophic transformed ratio estimators for estimating finite population mean in sample surveys. Adv. Sampl. Theory Appl. **1**, 39–47 (2021)
23. Singh, R., Kumari, A.: Improved estimators of population mean using auxiliary variables in ranked set sampling. Rev. Investig. Oper. **44**(2), 271–280 (2023)
24. Singh, R., Smarandache, F., Mishra, R.: Generalized robust-type neutrosophic ratio estimators of pharmaceutical daily stock prices.

Cognitive intelligence with neutrosophic statistics in bioinformatics 417–429 (2023)
25. Sisodia, B., Dwivedi, V.K.: A modified ratio estimator using coefficient of variation of auxiliary variable. J. Indian Soc. Agric. Stat. **33**(1), 13–18 (1981)
26. Smarandache, F.: Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. ProQuest Inf. Learn. **105**, 118–123 (1998)
27. Smarandache, F.: A unifying field in logics: neutrosophic logic. Philosophy. American Research Press, 1–141 (1999)
28. Smarandache, F.: A unifying field in logics: neutrosophic logic, neutrosophic set, neutrosophic probability, and statistics (2001) arXiv:math/0101228
29. Smarandache, F.: Neutrosophic set a generalization of the intuitionistic fuzzy set. Int. J. Pure Appl. Math. **24**(3), 287 (2005)
30. Smarandache, F.: Neutrosophic logic—a generalization of the intuitionistic fuzzy logic. Multispace Multi Struct. Neutrosophic Transdiscipl. **4**, 396 (2010)
31. Smarandache, F.: Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability. In: Infinite study (2013)
32. Smarandache, F.: Introduction to neutrosophic statistics. In: Infinite study (2014)
33. Stokes, L.: Ranked set sampling with concomitant variables. Commun. Stat. Theory Methods **6**, 1207–1211 (1977)
34. Tahir, Z., Khan, H., Aslam, M., Shabbir, J., Mahmood, Y., Smarandache, F.: Neutrosophic ratio-type estimators for estimating the population mean. Complex Intell. Syst. **7**, 2991–3001 (2021)
35. Takahasi, K., Wakimoto, K.: On unbiased estimates of the population mean based on the sample stratified by means of ordering. Ann. Inst. Stat. Math. **20**, 1–31 (1968)
36. Vishwakarma, G., Singh, A.: Computing the effect of measurement errors on ranked set sampling estimators of the population mean. Concurr. Comput. Pract. Exp. **34**(27), e7333 (2022)
37. Vishwakarma, G.K., Singh, A.: Generalized estimator for computation of population mean under neutrosophic ranked set technique: an application to solar energy data. Comput. Appl. Math. **41**(4), 144 (2022)

Neutrosophic Rhotrices for Improved Diagnostic Accuracy through Score Rhotrix Computation

B. Ananda Priya,[1] **P. Gnanachandra,**[2] **Saeid Jafari,**[3] **Florentin Smarandache**[4]

[1] Research Scholar, Ayya Nadar Janaki Ammal College (Affliated to Madurai Kamraj University); priyaraghul331@gmail.com

[2]Ayya Nadar Janaki Ammal College (Affliated to Madurai Kamraj University); pgchandra07@gmail.com

[3]Mathematical and Physical Science Foundation, 4200 Slagelse, Denmark; jafaripersia@gmail.com

[4]Department of Mathematics, University of New Mexico, Gallup, NM, USA; smarand@unm.edu

*Correspondence: pgchandra07@gmail.com

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Abstract. We introduce neutrosophic rhotrices, which serve as a novel extension of neutrosophic matrices. The primary objective is to establish the foundational structure for neutrosophic rhotrices and to define crucial operations that can be performed on them. We explore the concept of neutrosophic rhotrices in depth, outlining the fundamental operations necessary for their effective manipulation. Furthermore, we investigate the essential properties of neutrosophic rhotrices utilizing these newly established operations. In addition, we provide an algorithm designed to enhance decision-making processes in medical diagnostics, supported by an illustrative example to clarify its application.

Keywords: Neutrosophic Rhotrix, Rhotrix, Heart based Multiplication, Trace

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In 2003, Ajibade [1] introduced a method for representing arrays of numbers in a rhomboidal shape, which he named rhotrix and developed the structure of an n-dimensional rhotrix to fall between the $(n-1)\times(n-1)$ - dimensional matrix and the $n\times n$ - dimensional matrix. Ajibade [10] also proposed the first multiplication method for rhotrices, called heart-based multiplication. In 2004, B. Sani [2] introduced another multiplication method for rhotrices, known as row-column multiplication, which is similar to traditional matrix multiplication. These two methods are the primary techniques for rhotrix multiplication, though various other multiplications can be defined, none of which generate algebraic structures.

Sani [3] also introduced a technique for transforming a rhotrix into a matrix form, referred to as a coupled matrix. In 2008, A. O. Isere [4] expanded on Ajibade's work by introducing

the concept of even-dimensional rhotrix, since Ajibade's rhotrix is always of odd dimension. Additionally, numerous authors have contributed to the development of rhotrix theory, exploring concepts such as rhotrix groups [7, 11], rhotrix rings [5], rhotrix vector spaces [6], and the application of rhotrices in various fields, including cryptography [12–14].

The concept of neutrosophy was introduced by Florentin Smarandache [8, 16] in the 1990s. Neutrosophic logic extends classical and fuzzy logic by considering three components: truth (T), indeterminacy (I), and falsity (F). This triad is utilized to handle real-world problems where information is incomplete, inconsistent, or uncertain.

Kandasamy and Smarandache [17–19] expanded on this concept by introducing neutrosophic algebraic structures, including neutrosophic fields, vector spaces, groups, and rings. In linear algebra, matrices are essential for understanding vector spaces and linear transformations, prompting the creation of neutrosophic matrices. A neutrosophic matrix is an extension of the classical matrix concept, incorporating neutrosophic logic, which deals with indeterminacy. Recently, Mohammad Abobala and et al. [15] investigated the algebraic properties of these matrices, such as diagonalization, invertibility, determinants, and their algebraic representations through linear transformations. Neutrosophic matrices have found applications in various fields, particularly in dealing with uncertain, inconsistent, and incomplete information. Mamoni Dhar [9], introduced neutrosophic soft matrices and also a score matrix addressing patient-symptoms and symptoms-disease neutrosophic soft matrices is also proposed to aid in decision-making. Recently, the authors [20] introduced fuzzy rhotrices and its application in decision making of medical diagnostics has been studied.

We seek to introduce the concept of neutrosophic rhotrices and establish the fundamental operations required for their manipulation. We will define the basic operations and furthermore, conduct an in-depth examination of the fundamental properties of neutrosophic rhotrices, utilizing the newly defined operations to explore their characteristics. Additionally, an algorithm is proposed for medical diagnosis using neutrosophic rhotrices accompanied with an illustrative example.

Let n be positive odd integer. A rhotrix of size n is represented as

$$\left\langle \begin{array}{ccccccccc} & & & & a_1 & & & & \\ & & & a_2 & a_3 & a_4 & & & \\ & & .. & .. & .. & .. & .. & & \\ & .. & .. & .. & .. & .. & .. & .. & \\ a_{\{\frac{t+1}{2}\}-\frac{n}{2}} & .. & .. & .. & a_{\frac{t+1}{2}} & .. & .. & .. & a_{\{\frac{t+1}{2}\}+\frac{n}{2}} \\ & .. & .. & .. & .. & .. & .. & .. & \\ & & .. & .. & .. & .. & .. & & \\ & & & a_{t-3} & a_{t-2} & a_{t-1} & & & \\ & & & & a_t & & & & \end{array} \right\rangle,$$

where $t = \frac{n^2+1}{2}$. In th above representation, the entries in red represents the first row, entries in blue represents first column, and entries in purple represents the diagonal. Here the major horizontal axis is $a_{\{\frac{t+1}{2}\}-\frac{n}{2}} \cdots a_{\frac{t+1}{2}} \cdots a_{\{\frac{t+1}{2}\}+\frac{n}{2}}$ and the major vertical axis is $a_1\ a_3 \cdots a_{\frac{t+1}{2}} \cdots a_{t-2}\ a_t$. Heart of rhotrix is the entry intersecting both major horizontal axis and major vertical axis of the rhotrix and that is $a_{\frac{t+1}{2}}$.

Heart-based Multiplication, as introduced by A. O. Ajibade, involves multiplying each entry of the first rhotrix by the heart of the second rhotrix, and then adding the product of the corresponding entry of the second rhotrix with the heart of the first rhotrix. In addition, the process of adding two rhotrices is done by summing each corresponding entry. To transpose a rhotrix, one flips its entries over the major vertical axis, keeping the entries on this axis unchanged.

1. Neutrosophic Rhotrix

Within this section, we lay the groundwork by presenting neutrosophic rhotrices and elucidating fundamental operations, namely neutrosophic rhotrix addition and heart-based multiplication. Subsequently, we analyze the algebraic attributes of neutrosophic rhotrices under these operations, with a specific emphasis on exploring their trace properties.

Definition 1.1. A ***neutrosophic rhotrix*** is defined as a rhotrix with each of the entries are from a neutrosophic set $R = [(r_i^T, r_i^I, r_i^F)]$, where r_i^T, r_i^I, r_i^F characterized by a membership function $\mu_R : F \to [0, 1]$, indeterminate function $\mathcal{I}_R : F \to [0, 1]$, falsity function $\mathcal{F}_R : F \to [0, 1]$ respectively and F is any field.

Suppose $R_1 = \left\langle \begin{array}{ccc} & r_1 & \\ r_2 & r_3 & r_4 \\ & r_5 & \end{array} \right\rangle$ is a rhotrix in $R_n(F)$, then the neutrosophic rhotrix is represented

$$\text{as } (R_1)_N = \left\langle \begin{matrix} & (\mu_R(r_1), \mathcal{I}_R(r_1), \mathcal{F}_R(r_1)) & \\ (\mu_R(r_2), \mathcal{I}_R(r_1), \mathcal{F}_R(r_1)) & (\mu_R(r_3), \mathcal{I}_R(r_1), \mathcal{F}_R(r_1)) & (\mu_R(r_4), \mathcal{I}_R(r_1), \mathcal{F}_R(r_1)) \\ & (\mu_R(r_5), \mathcal{I}_R(r_1), \mathcal{F}_R(r_1)) & \end{matrix} \right\rangle$$

In simpler terms, it can be written as,

$$(R_1)_N = \left\langle \begin{matrix} & (r_1^T, r_1^I, r_1^F) & \\ (r_2^T, r_2^I, r_2^F) & (r_3^T, r_3^I, r_3^F) & (r_4^T, r_4^I, r_4^F) \\ & (r_5^T, r_5^I, r_5^F) & \end{matrix} \right\rangle$$

Definition 1.2. We define the **operations on neutrosophic rhotrices** as follows:

(i) The addition of two neutrosophic rhotrices is possible only if they are of the same size. Addition of two neutrosophic rhotrices $'A'_N$ and $'B'_N$ is defined as:

$$\text{The } i^{th} \text{ entry of } A_N + B_N = (max\{a_i^T, b_i^T\}, max\{a_i^I, b_i^I\}, min\{a_i^F, b_i^F\}), \text{ for all } i.$$

That is, for rhotrices $A_N = \left\langle \begin{matrix} & (a_1^T, a_1^I, a_1^F) & \\ (a_2^T, a_2^I, a_2^F) & (a_3^T, a_3^I, a_3^F) & (a_4^T, a_4^I, a_4^F) \\ & (a_5^T, a_5^I, a_5^F) & \end{matrix} \right\rangle$ of size 3, and $B_N = \left\langle \begin{matrix} & (b_1^T, b_1^I, b_1^F) & \\ (b_2^T, b_2^I, b_2^F) & (b_3^T, b_3^I, b_3^F) & (b_4^T, b_4^I, b_4^F) \\ & (b_5^T, b_5^I, b_5^F) & \end{matrix} \right\rangle$ of size 3,

$$A_N + B_N = \left\langle \begin{matrix} & (max\{a_1^T, b_1^T\}, max\{a_1^I, b_1^I\}, & \\ & min\{a_1^F, b_1^F\}) & \\ (max\{a_2^T, b_2^T\}, max\{a_2^I, b_2^I\} & (max\{a_3^T, b_3^T\}, max\{a_3^I, b_3^I\} & (max\{a_4^T, b_4^T\}, max\{a_4^I, b_4^I\} \\ , min\{a_2^F, b_2^F\}) & , min\{a_3^F, b_3^F\}) & , min\{a_4^F, b_4^F\}) \\ & (max\{a_5^T, b_5^T\}, max\{a_5^I, b_5^I\}, & \\ & min\{a_5^F, b_5^F\}) & \end{matrix} \right\rangle$$

(ii) The heart based neutrosophic multiplication of two neutrosophic rhotrices is possible only if they are of the same size. Multiplication of two neutrosophic rhotrices 'A' and 'B' is defined as:

$$\text{The } i^{th} \text{ entry of } A_N \circ B_N = (max\{min(a_i^T, e_b^T), min(b_i^T, e_a^T)\}, max\{min(a_i^I, e_b^I), min(b_i^I, e_a^I)\},$$
$$min\{max(a_i^F, e_b^F), max(b_i^F, e_a^F)\}), \text{ for all } i, \text{ except the } heart$$
$$Heart \text{ of } A_N \circ B_N = (min\{e_a, e_b\}, min\{e_a, e_b\}, max\{e_a, e_b\})$$

Definition 1.3 (Trace of Neutrosophic Rhotrix)**.** The trace of a neutrosophic rhotrix of size 'n' is adding the entries in major vertical axis and it is denoted as $Tr(.)$.

That is, for a rhotrix $A_N = \left\langle \begin{matrix} & (a_1^T, a_1^I, a_1^F) & \\ (a_2^T, a_2^I, a_2^F) & (a_3^T, a_3^I, a_3^F) & (a_4^T, a_4^I, a_4^F) \\ & (a_5^T, a_5^I, a_5^F) & \end{matrix} \right\rangle$ of size 3,

$$Tr(A_N) = (max\{a_1^T, a_3^T, a_5^T\}, max\{a_1^I, a_3^I, a_5^I\}, min\{a_1^F, a_3^F, a_5^F\})$$

Theorem 1.4. *For any neutrosophic rhotrices with the neutrosophic addition operation, the following axioms holds:*

- $(A_N + B_N) + C_N = A_N + (B_N + C_N)$ *(Associativity),*
- $A_N + O = A_N$, *where* $O = \left\langle (0,0,1) \begin{matrix} (0,0,1) \\ (0,0,1) \\ (0,0,1) \end{matrix} (0,0,1) \right\rangle$ *(Additive Identity),*
- $A_N + B_N = B_N + A_N$ *(Commutativity), where* $A_N, B_N,$ *and* C_N *are any neutrosophic rhotrices*

Proof. Let $A_N, B_N,$ and C_N of size 3 be any three neutrosophic rhotrices . Then,
$A_N + (B_N + C_N) = (A_N + B_N) + C_N =$

$$\left\langle (max\{a_2^T, b_2^T, c_2^T\}, max\{a_2^I, b_2^I, c_2^I\}, min\{a_2^F, b_2^F, c_2^F\}) \quad \begin{matrix} (max\{a_1^T, b_1^T, c_1^T\}, max\{a_1^I, b_1^I, c_1^I\}, \\ min\{a_1^F, b_1^F, c_1^F\}) \\ (max\{a_3^T, b_3^T, c_3^T\}, max\{a_3^I, b_3^I, c_3^I\}, \\ min\{a_3^F, b_3^F, c_3^F\}) \\ (max\{a_5^T, b_5^T, c_5^T\}, max\{a_5^I, b_5^I, c_5^I\}, \\ min\{a_5^F, b_5^F, c_5^F\}) \end{matrix} \quad (max\{a_4^T, b_4^T, c_4^T\}, max\{a_4^I, b_4^I, c_4^I\}, min\{a_4^F, b_4^F, c_4^F\}) \right\rangle$$

Suppose $O = \left\langle (x_2^T, x_2^I, x_2^F) \begin{matrix} (x_1^T, x_1^I, x_1^F) \\ (x_3^T, x_3^I, x_3^F) \\ (x_5^T, x_5^I, x_5^F) \end{matrix} (x_4^T, x_4^I, x_4^F) \right\rangle$. For $A_N + O = A_N$,

$$\left\langle (max\{a_2^T, x_2^T\}, max\{a_2^I, x_2^I\}, min\{a_2^F, x_2^F\}) \quad \begin{matrix} (max\{a_1^T, x_1^T\}, max\{a_1^I, x_1^I\}, \\ min\{a_1^F, x_1^F\}) \\ (max\{a_3^T, x_3^T\}, max\{a_3^I, x_3^I\}, \\ min\{a_3^F, x_3^F\}) \\ (max\{a_5^T, x_5^T\}, max\{a_5^I, x_5^I\}, \\ min\{a_5^F, x_5^F\}) \end{matrix} \quad (max\{a_4^T, x_4^T\}, max\{a_4^I, x_4^I\}, min\{a_4^F, x_4^F\}) \right\rangle$$

$$= \left\langle (a_2^T, a_2^I, a_2^F) \begin{matrix} (a_1^T, a_1^I, a_1^F) \\ (a_3^T, a_3^I, a_3^F) \\ (a_5^T, a_5^I, a_5^F) \end{matrix} (a_4^T, a_4^I, a_4^F) \right\rangle$$

Then, $max\{a_i^T, x_i^T\} = a_i^T, max\{a_i^I, x_i^I\} = a_i^I and min\{a_i^F, x_i^F\} = a_i^F,$ for all i, which implies $x_i^T = x_i^I = 0, x_i^F = 1$ for all i. Therefore, the additive identity for rhotrix size 3 is $\left\langle (0,0,1) \begin{matrix} (0,0,1) \\ (0,0,1) \\ (0,0,1) \end{matrix} (0,0,1) \right\rangle$. In general, the additive identity for rhotrix size 'n' is a rhotrix with all its entries as $(0,0,1)$.

Since each entry of $A_N + B_N = (max\{a_i^T, b_i^T\}, max\{a_i^I, b_i^I\}, min\{a_i^F, b_i^F\})$ and $max\{a_i, b_i\} =$

$max\{b_i, a_i\}$, and $min\{a_i, b_i\} = min\{b_i, a_i\}$ it is obvious that the commutative axiom holds for neutrosophic addition. □

Theorem 1.5. *The heart based multiplication operation among any neutrosophic rhotrices holds the following axioms:*

- $(A_N \circ B_N) \circ C_N = A_N \circ (B_N \circ C_N)$ *(Associativity),*
- $A_N \circ I = A_N$, *where* $I = \left\langle \begin{matrix} & (0,0,1) & \\ (0,0,1) & (1,1,0) & (0,0,1) \\ & (0,0,1) & \end{matrix} \right\rangle$ *(Multiplicative Identity),*
- $A_N \circ B_N = B_N \circ A_N$ *(Commutativity), where* $A_N, B_N,$ *and* C_N *are any neutrosophic rhotrices*

Proof. Let us consider any three neutrosophic rhotrices $A_N, B_N,$ and C_N of size 3, and let $e_a, e_b,$ and e_c be the entries of heart of the rhotrices $A, B,$ and C, respectively. Then, the i^{th} entry of $(A_N \circ B_N) \circ C_N$

$$= \begin{cases} \Big(max\{min(a_i^T, e_b^T, e_c^T), min(b_i^T, e_b^T, e_c^T), min(c_i^T, e_b^T, e_c^T)\}, \\ max\{min(a_i^I, e_b^I, e_c^I), min(b_i^I, e_b^I, e_c^I), min(c_i^I, e_b^I, e_c^I)\}, \\ min\{max(a_i^F, e_b^F, e_c^F), max(b_i^F, e_b^F, e_c^F), max(c_i^F, e_b^F, e_c^F)\}\Big) & , \forall i, \text{ except } heart \\ \Big(min\{e_a^T, e_b^T, e_c^T\}, min\{e_a^I, e_b^I, e_c^I\}, max\{e_a^F, e_b^F, e_c^F\}\Big) & , \textit{ for heart} \end{cases}$$

$= i^{th}$ entry of $A_N \circ (B_N \circ C_N)$

Suppose $I_N = \left\langle \begin{matrix} & (x_1^T, x_1^I, x_1^F) & \\ (x_2^T, x_2^I, x_2^F) & (x_3^T, x_3^I, x_3^F) & (x_4^T, x_4^I, x_4^F) \\ & (x_5^T, x_5^I, x_5^F) & \end{matrix} \right\rangle$

For $A_N \circ I_N = A_N$,

The i^{th} entry of $A_N \circ I_N = \begin{cases} \Big(max\{min(a_i^T, x_3^T), min(x_i^T, e_a^T)\}, \\ max\{min(a_i^I, x_3^I), min(x_i^I, e_a^I)\}, \\ min\{max(a_i^F, x_3^F), max(x_i^F, e_a^F)\}\Big) & , \forall i, \text{ except the } heart \\ (min\{e_a^T, x_3^T\}, min\{e_a^I, x_3^I\}, max\{e_a^F, x_3^F\}) & , \textit{ for heart} \end{cases}$

Then, $min\{e_a^T, x_3^T\} = e_a^T$, implies $x_3^T = 1$ and similarly $x_3^I = 1$. Also, $max\{e_a^F, x_3^F\} = e_a^F$, implies $x_3^F = 0$. Since $x_3^T = 1$ and $max\{min(a_i^T, x_3^T), min(x_i^T, e_a^T)\} = a_i^T$, $x_i^T = 0$. Similarly, $x_i^I = 0$ and $x_i^F = 1$

Therefore, the additive identity for rhotrix size 3 is $I = \left\langle \begin{matrix} & (0,0,1) & \\ (0,0,1) & (1,1,0) & (0,0,1) \\ & (0,0,1) & \end{matrix} \right\rangle$. In general, the multiplicative identity for rhotrix size 'n' is a rhotrix of size 'n'with all entries (0,0,1) and heart of the rhotrix as (1,1,0).

Since the heart based rhotrix multiplication is commutative, it is obvious that the commutative axiom holds for heart based neutrosophic multiplication. □

Theorem 1.6. *Heart based multiplication of neutrosophic rhotrices is distributive with respect to addition of neutrosophic rhotrices. That is,* $A_N \circ (B_N + C_N) = A_N \circ B_N + A_N \circ C_N$*, where* A_N, B_N, C_N *are neutrosophic rhotrices.*

Proof. Let $A_N = \left\langle \begin{matrix} & (a_1^T, a_1^I, a_1^F) & \\ (a_2^T, a_2^I, a_2^F) & (e_a^T, e_a^I, e_a^F) & (a_4^T, a_4^I, a_4^F) \\ & (a_5^T, a_5^I, a_5^F) & \end{matrix} \right\rangle$,

$B_N = \left\langle \begin{matrix} & (b_1^T, b_1^I, b_1^F) & \\ (b_2^T, b_2^I, b_2^F) & (e_b^T, e_b^I, e_b^F) & (b_4^T, b_4^I, b_4^F) \\ & (b_5^T, b_5^I, b_5^F) & \end{matrix} \right\rangle$ be neutrosophic rhotrices.

Then, the i^{th} entry of

$$\begin{aligned} A_N \circ (B_N + C_N) =& \Big(max\Big\{ min(a_i^T, max\{e_b^T, e_c^T\}), min(e_a^T, max\{b_i^T, c_i^T\}) \Big\}, max\Big\{ min(a_i^I, max\{e_b^I, e_c^I\}), \\ & min(e_a^I, max\{b_i^I, c_i^I\}) \Big\}, min\Big\{ max(a_i^F, min\{e_b^F, e_c^F\}), max(a_i^F, max\{e_b^F, e_c^F\}) \Big\} \Big) \\ =& \Big(max\{min(a_i^T, e_b^T, e_c^T), min(b_i^T, e_a^T, e_c^T), min(c_i^T, e_a^T, e_b^T)\}, max\{min(a_i^I, e_b^I, e_c^I), \\ & min(b_i^I, e_a^I, e_c^I), min(c_i^I, e_a^I, e_b^I)\}, min\{min(a_i^F, e_b^F, e_c^F), min(b_i^F, e_a^F, e_c^F), \\ & min(c_i^F, e_a^F, e_b^F)\} \Big) \\ =& i^{th} \text{ entry of } (A_N \circ B_N) + (A_N \circ C_N) \end{aligned}$$

Therefore, $A_N \circ (B_N + C_N) = (A_N \circ B_N) + (A_N \circ C_N)$. This means that, we have proved, for any rhotrices of size 3, distributive property holds. In similar manner, we also can prove, distributive axiom holds for any rhotrices of size 'n'. □

Theorem 1.7. *For two neutrosophic rhotrices* A_N *and* B_N *and any scalar* λ *such that* $0 \leq \lambda \leq 1$*, the following properties hold:*

(i) $Tr(A_N + B_N) = Tr(A_N) + Tr(B_N)$

(ii) $Tr(\lambda A_N) = \lambda Tr_f(A_N)$

(iii) $Tr(A_N) = Tr(A_N^T)$

Proof. Consider two neutrosophic rhotrices of size 3, $A_N = \left\langle \begin{matrix} & (a_1^T, a_1^I, a_1^F) & \\ (a_2^T, a_2^I, a_2^F) & (a_3^T, a_3^I, a_3^F) & (a_4^T, a_4^I, a_4^F) \\ & (a_5^T, a_5^I, a_5^F) & \end{matrix} \right\rangle$ and $B_N = \left\langle \begin{matrix} & (b_1^T, b_1^I, b_1^F) & \\ (b_2^T, b_2^I, b_2^F) & (b_3^T, b_3^I, e_b^F) & (b_4^T, b_4^I, b_4^F) \\ & (b_5^T, b_5^I, b_5^F) & \end{matrix} \right\rangle$.

(i) Since the maximum operation is associative, we have the following:

$$\begin{aligned} Tr(A_N + B_N) &= (max\{max(a_1^T, b_1^T), max(a_3^T, b_3^T), max(a_5^T, b_5^T)\}, max\{max(a_1^I, b_1^I), max(a_3^I, b_3^I), \\ &\quad max(a_5^I, b_5^I)\}, min\{min(a_1^F, b_1^F), min(a_3^F, b_3^F), min(a_5^F, b_5^F)\}) \\ &= (max\{max(a_1^T, a_3^T, a_5^T), max(b_1^T, b_3^T, b_5^T)\}, max\{max(a_1^I, a_3^I, a_5^I), \\ &\quad max(b_1^I, b_3^I, b_5^I)\}, min\{min(a_1^F, a_3^F, a_5^F), min(b_1^F, b_3^F, b_5^F)\}) \\ &= Tr_f(A_N) + Tr_f(B_N) \end{aligned}$$

Therefore, $Tr(A_N + B_N) = Tr(A_N) + Tr(B_N)$, for any neutrosophic rhotrices of size 'n'.

(ii) Now, $Tr(\lambda A_N) = (max\{min(\lambda, a_1^T), min(\lambda, a_3^T), min(\lambda, a_5^T)\}, max\{min(\lambda, a_1^I), min(\lambda, a_3^I), min(\lambda, a_5^I)\}, min\{max(\lambda, a_1^F), max(\lambda, a_3^F), max(\lambda, a_5^F)\})$=$(min\{\lambda, max(a_1^T, a_3^T, a_5^T)\}), min\{\lambda, max(a_1^I, a_3^I, a_5^I)\}, max\{\lambda, min(a_1^F, a_3^F, a_5^F)\}$ ($\because$ (A$\wedge$B)$\vee$(A$\wedge$C)$\vee$(A$\wedge$D)=A$\wedge$(B$\vee$C$\vee$D), where $\wedge$ and $\vee$ represents min and max operation resp.) and $\lambda Tr_f(A_f) = (min\{\lambda, max(a_1^T, a_3^T, a_5^T)\}), min\{\lambda, max(a_1^I, a_3^I, a_5^I)\}, max\{\lambda, min(a_1^F, a_3^F, a_5^F)\}$. Therefore, $Tr(\lambda A_N) = \lambda Tr_f(A_N)$.

(iii) Since there is no changes in the major horizontal axis in A_N^T and A_N, $Tr(A_N) = Tr(A_N^T)$.

□

2. **Elevating Diagnostic Precision with Neutrosophic Rhotrices and Fuzzy rhotrices**

In the context of decision-making, when using matrices for medical diagnosis, if an individual does not exhibit a particular symptom, the value corresponding to that symptom is set to zero, and computations proceed accordingly, accounting for the absence of that symptom. However, when applying rhotrices, individuals can be categorized based on the number of symptoms they exhibit. For example, patients with three symptoms can be grouped together, as can those with two symptoms, which simplifies the overall diagnostic process by creating more manageable groups.

Method 1 (Using Neutrosophic Rhotrices)

In this method, we incorporate neutrosophic rhotrices into medical diagnosis, this framework advances healthcare by effectively managing uncertainty, indeterminacy, and imprecision.

Let S represent the set of symptoms associated with certain diseases, D denote the set of

diseases, and P refer to the set of patients.

Step 1: Construct a symptom-disease neutrosophic rhotrix $A = [a_ij]$ of size n (=number of symptoms and diseases) in the following manner.
Here, $s_i d_j^T$ represents the membership value indicating how much symptom s_i contributes to the occurrence of disease d_j, $s_i d_j^I$ represents the membership value indicating how much symptom s_i contributes to the indeterminacy of disease d_j, and $s_i d_j^F$ represents the membership value indicating how much symptom s_i contributes to the falsity of disease d_j.Thus, the symptom-disease neutrosophic rhotrix A can be formed as

$$\left\langle \begin{array}{ccccc} & & (s_1d_1^T, s_1d_1^I, s_1d_1^F) & & \\ & (s_3d_1^T, s_3d_1^I, s_3d_1^F) & (s_2d_2^T, s_2d_2^I, s_2d_2^F) & (s_1d_3^T, s_1d_3^I, s_1d_3^F) & \\ (s_5d_1^T, s_5d_1^I, s_5d_1^F) & (s_4d_2^T, s_4d_2^I, s_4d_2^F) & (s_3d_3^T, s_3d_3^I, s_3d_3^F) & (s_2d_4^T, s_2d_4^I, s_2d_4^F) & (s_1d_5^T, s_1d_5^I, s_1d_5^F) \\ & (s_5d_3^T, s_5d_3^I, s_5d_3^F) & (s_4d_4^T, s_4d_4^I, s_4d_4^F) & (s_3d_5^T, s_3d_5^I, s_3d_5^F) & \\ & & (s_5d_5^T, s_5d_5^I, s_5d_5^F) & & \end{array} \right\rangle$$

Step 2: Construct a patient-symptom neutrosophic matrix $B = [b_ij]$ of size n (=number of symptoms and patients) in the following manner.

Here, $p_i s_j^T$ represents the membership value indicating how much the patient p_i suffers from the symptom s_j, $p_i s_j^I$ represents the membership value indicating how much the patient p_i contributes to the indeterminacy of symptom s_j, and $p_i s_j^F$ represents the membership value indicating how much the patient p_i contributes to the falsity of the symptom s_j.Thus, the symptom-disease neutrosophic rhotrix B can be formed as

$$\left\langle \begin{array}{ccccc} & & (p_1s_1^T, p_1s_1^T, p_1s_1^T) & & \\ & (p_3s_1^T, p_3s_1^T, p_3s_1^T) & (p_2s_2^T, p_2s_2^I, p_2s_2^F) & (p_1s_3^T, p_1s_3^I, p_1s_3^F) & \\ (p_5s_1^T, p_5s_1^I, p_5s_1^F) & (p_4s_2^T, p_4s_2^I, p_4s_2^F) & (p_3s_3^T, p_3s_3^I, p_3s_3^F) & (p_2s_4^T, p_2s_4^I, p_2s_4^F) & (p_1s_5^T, p_1s_5^I, p_1s_5^F) \\ & (p_5s_3^T, p_5s_3^I, p_5s_3^F) & (p_4s_4^T, p_4s_4^I, p_4s_4^F) & (p_3s_5^T, p_3s_5^I, p_3s_5^F) & \\ & & (p_5s_5^T, p_5s_5^I, p_5s_5^F) & & \end{array} \right\rangle$$

Step 3: Compute the rhotrix C by performing heart-based neutrosophic multiplication between rhotrices A and B, where the operation $\circ$ denotes the heart based neutrosophic multiplication.

Step 4: Construct the complement rhotrices A^c and B^c for rhotrices A and B respectively, both having the same dimensions. Afterward, calculate the composition rhotrix D by applying the heart-based neutrosophic multiplication $\circ$ to the complement matrices A^c and B^c.

Step 5: Transform the rhotrix A and B to A_S and B_S by adding the truth and indeterminacy values, then subtracting the falsity value for each neutrosophic number in the rhotrix A and B.

Step 6: Derive the score rhotrix M by applying the minimum operator, denoted as $(-)$, to the rhotrices A_S and B_S. This results in $M = A_S(-)B_S$.
This will be used to analyze and interpret the relationships or differences between the elements under study.

Illustrative Analysis: Assume there are five symptoms and five diseases. Out of these, three symptoms (s_1, s_3, s_5) are associated with three specific diseases (d_1, d_3, d_5), while the remaining two symptoms (s_2, s_4) correspond to two other diseases (d_2, d_4).

Let us, for illustration, take symptom disease neutrosophic rhotrix as

$$A = \left\langle \begin{array}{ccccc} & & (0.2,0.4,0.6) & & \\ & (0.5,0.1,0.3) & (0.2,0.5,0.2) & (0.4,0.4,0.1) & \\ (0.9,0.6,0.3) & (0.2,0.9,0.1) & (0.1,0.7,0.4) & (0.2,0.6,0.3) & (0.1,0.5,0.4) \\ & (0.3,0.4,0.8) & (0.7,0.9,0.1) & (0.5,0.5,0.5) & \\ & & (0.1,0.4,0.1) & & \end{array} \right\rangle$$

Consider a scenario with five symptoms and five diseases. Three patients (p_1, p_3, p_5) experience three symptoms (s_1, s_3, s_5), while the other two patients (p_2, p_4) exhibit the remaining two symptoms (s_2, s_4).

Let us, for illustration, take the patient symptom neutrosophic rhotrix as

$$B = \left\langle \begin{array}{ccccc} & & (0.3,0.4,0.1) & & \\ & (0.2,0.1,0.2) & (0.1,0.9,0.4) & (0.7,0.3,0.1) & \\ (0.4,0.3,0.3) & (0.7,0.1,0.3) & (0.4,0.1,0.3) & (0.7,0.7,0.7) & (0.4,0.2,0.7) \\ & (0.4,0.4,0.1) & (0.3,0.4,0.7) & (0.9,0.9,0.9) & \\ & & (0.4,0.4,0.2) & & \end{array} \right\rangle$$

For the above illustration, we heart based multiply the two rhotrices A and B and get

$$C = \left\langle \begin{array}{ccccc} & & (0.2,0.4,0.4) & & \\ & (0.4,0.1,0.3) & (0.2,0.7,0.3) & (0.4,0.3,0.3) & \\ (0.4,0.3,0.3) & (0.2,0.1,0.3) & (0.1,0.1,0.3) & (0.2,0.7,0.3) & (0.1,0.2,0.4) \\ & (0.3,0.4,0.4) & (0.4,0.4,0.3) & (0.4,0.7,0.5) & \\ & & (0.1,0.4,0.3) & & \end{array} \right\rangle$$

By using,

$$A^c = \left\langle (0.1,0.4,0.7) \begin{matrix} (0.5,0.9,0.7) \\ (0.8,0.1,0.9) \\ (0.7,0.6,0.2) \end{matrix} \begin{matrix} (0.8,0.6,0.4) \\ (0.8,0.5,0.8) \\ (0.9,0.3,0.6) \\ (0.3,0.1,0.9) \\ (0.9,0.6,0.9) \\ (0.7,0.6,0.9) \end{matrix} \begin{matrix} (0.6,0.6,0.9) \\ (0.8,0.4,0.7) \\ (0.5,0.5,0.5) \end{matrix} (0.9,0.5,0.6) \right\rangle \text{ and}$$

$$B^c = \left\langle (0.6,0.7,0.7) \begin{matrix} (0.8,0.9,0.8) \\ (0.3,0.9,0.7) \\ (0.6,0.6,0.9) \end{matrix} \begin{matrix} (0.9,0.1,0.6) \\ (0.6,0.9,0.7) \\ (0.7,0.6,0.3) \\ (0.6,0.6,0.8) \\ (0.7,0.6,0.7) \end{matrix} \begin{matrix} (0.3,0.7,0.9) \\ (0.3,0.3,0.3) \\ (0.1,0.1,0.1) \end{matrix} (0.6,0.8,0.3) \right\rangle, \text{ we get } D =$$

$$\left\langle (0.6,0.4,0.7) \begin{matrix} (0.8,0.9,0.7) \\ (0.6,0.3,0.7) \\ (0.6,0.6,0.7) \end{matrix} \begin{matrix} (0.9,0.5,0.6) \\ (0.6,0.3,0.7) \\ (0.7,0.3,0.6) \\ (0.6,0.6,0.8) \end{matrix} \begin{matrix} (0.6,0.6,0.9) \\ (0.6,0.4,0.6) \\ (0.5,0.5,0.6) \end{matrix} (0.6,0.5,0.6) \right\rangle$$

Then, by the aforementioned algorithm, we transform A to

$$A_S = \left\langle 0.4 \begin{matrix} 0.2 \\ 0 \\ 0.3 \end{matrix} \begin{matrix} 0.2 \\ 0.6 \\ -0.1 \\ 0.5 \\ 0.2 \end{matrix} \begin{matrix} 0.4 \\ 0.6 \\ 0.6 \end{matrix} -0.1 \right\rangle$$

and B to $B_S = \left\langle 0.3 \begin{matrix} 1 \\ 0.2 \\ 0.5 \end{matrix} \begin{matrix} 0.6 \\ 0.8 \\ 0.2 \\ 0.4 \end{matrix} \begin{matrix} 0.3 \\ 0.4 \\ 0.4 \end{matrix} 0.5 \right\rangle$.

Then, the score rhotrix thus obtained is, $\left\langle \boxed{0.1} \begin{matrix} -0.4 \\ -0.8 \\ -0.2 \\ -0.8 \\ -0.2 \end{matrix} \begin{matrix} -0.2 \\ -0.3 \\ \boxed{0.1} \end{matrix} \begin{matrix} \boxed{0.1} \\ \boxed{0.2} \\ \boxed{0.2} \end{matrix} -0.6 \right\rangle$

We can observe that, p_1 is affected by d_3, p_2 is affected by d_4, p_3 is affected by d_5, p_4 is affected by d_4 and p_5 is affected by d_1, by finding maximum of each row.

Method 2 (Using Fuzzy Rhotrices)

In optimizing diagnostics with fuzzy rhotrices, let S represent the set of symptoms associated with certain diseases, D denote the set of diseases, and P refer to the set of patients.

Step 1: Construct a symptom-disease fuzzy rhotrix $A = [a_ij]$ of size n (=number of symptoms and diseases) in the following manner, based on the following example. Consider there are 5 symptoms and 5 diseases, in which 3 symptoms(s_1, s_3, s_5) can indicate 3 diseases (d_1, d_3, d_5) and another 2 symptoms (s_2, s_4) indicate another 2 diseases (d_2, d_4).Here, s_id_j represents the membership value indicating how much symptom s_i contributes to the occurrence of disease d_j.

Then the symptom-disease fuzzy rhotrix A can be formed as $A = \left\langle \begin{matrix} & & s_1d_1 & & \\ & s_3d_1 & s_2d_2 & s_1d_3 & \\ s_5d_1 & s_4d_2 & s_3d_3 & s_2d_4 & s_1d_5 \\ & s_5d_3 & s_4d_4 & s_3d_5 & \\ & & s_5d_5 & & \end{matrix} \right\rangle$

Suppose, for illustration, $A = \left\langle \begin{matrix} & & 0.2 & & \\ & 0.5 & 0.7 & 0.4 & \\ 0.6 & 0.1 & 0.9 & 0.6 & 0.4 \\ & 0.2 & 0.1 & 0.4 & \\ & & 0.5 & & \end{matrix} \right\rangle$

Step 2: Construct a patient-symptom fuzzy matrix $B = [b_ij]$ of size n (=number of symptoms and patients) in the following manner, based on the following example. Consider there are 5 symptoms and 5 diseases, in which 3 patients (p_1, p_3, p_5) has 3 symptoms(s_1, s_3, s_5) and another 2 patients (p_2, p_4) indicate another 2 symptoms (s_2, s_4).

Here, p_is_j represents the membership value indicating how much a patient p_i have affected by the symptom s_j.

Then the patient-symptom fuzzy rhotrix can be formed as $B = \left\langle \begin{matrix} & & p_1s_1 & & \\ & p_3s_1 & p_2s_2 & p_1s_3 & \\ p_5s_1 & p_4s_2 & p_3s_3 & p_2s_4 & p_1s_5 \\ & p_5s_3 & p_4s_4 & p_3s_5 & \\ & & p_5s_5 & & \end{matrix} \right\rangle$

Suppose, for illustration, $B = \left\langle \begin{matrix} & & 0.5 & & \\ & 0.7 & 0.9 & 0.8 & \\ 0.2 & 0.5 & 0.9 & 0.2 & 0.3 \\ & 0.4 & 0.5 & 0.1 & \\ & & 0.2 & & \end{matrix} \right\rangle$

Step 3: Evaluate $C = A \circ B$, where $\circ$ is heart based fuzzy multiplication

For the above illustration, $C = \left\langle \begin{matrix} & 0.5 & \\ 0.7 & 0.9 & 0.8 \\ 0.6 \quad 0.5 & 0.9 & 0.6 \quad 0.4 \\ 0.4 & 0.5 & 0.4 \\ & 0.5 & \end{matrix} \right\rangle$

Step 4: Build the complement matrix A^c and B^c of A and B of size n and form the composition matrix D by computing $A^c \circ B^c$.

Then, $D = \left\langle \begin{matrix} & 0.1 & \\ 0.1 & 0.1 & 0.1 \\ 0.1 \quad 0.1 & 0.1 & 0.1 \quad 0.1 \\ 0.1 & 0.1 & 0.1 \\ & 0.1 & \end{matrix} \right\rangle$

Step 5: Compute $M = C(-)D$, where (-) denotes min operator Then, M for the above illustration is $\left\langle \begin{matrix} & 0.4 & \\ 0.6 & 0.8 & 0.7 \\ 0.5 \quad 0.4 & 0.8 & 0.5 \quad 0.3 \\ 0.3 & 0.4 & 0.3 \\ & 0.4 & \end{matrix} \right\rangle$

Step 6: Calculating the relativity values and form the comparison matrix

Then, the comparison matrix for the above illustration is $\left\langle \begin{matrix} & 0 & \\ -0.143 & 0 & 0.143 \\ 0.4 \quad -0.2 & 0 & 0.2 \quad -0.4 \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle$,

We can observe that, p_1 is affected by d_3, p_2 is affected by d_4, p_3 is affected by d_3 and d_5, p_4 is affected by d_4 and p_5 is affected by d_1, by finding maximum of each row.

Comparative Analysis: The method applied for medical diagnostics using fuzzy rhotrices, observe that it only deals with truthfulness and do not explicitly address indeterminacy, which limits their ability to manage situations where medical data is incomplete or contradictory. But, our current algorithm uses neutrosophic rhotrices which are designed to handle three types of uncertainties: truth, indeterminacy, and falsity. In medical diagnosis, this means that a symptom can be partially present (truth), uncertain (indeterminacy), or absent (falsity), offering a richer framework for capturing ambiguity in symptoms.

Neutrosophic rhotrices, due to their sophisticated structure, provide significant advantages in decision-making processes, especially in situations where data can be organized into distinct categories or groups. For instance, in the field of medical diagnostics, neutrosophic rhotrices can facilitate the classification of patients based on their symptoms. This grouping allows healthcare professionals to make more precise and personalized decisions regarding diagnosis and treatment.

Compared to neutrosophic matrices, computations in neutrosophic rhotrices are much simpler. Rhotrices reduce computational complexity by handling multi-dimensional data in a more streamlined manner, enabling faster execution of operations like matrix multiplication and inversion. This efficiency is particularly valuable when dealing with large datasets in medical diagnosis.

In summary, the use of neutrosophic rhotrices in medical diagnostics not only streamlines the decision-making process but also promotes a more informed and personalized approach to patient care.

Conclusion

We introduced neutrosophic rhotrices, a novel extension of neutrosophic matrices, designed to effectively manage the complexities of uncertainty, indeterminacy, and falsity in decision-making processes, particularly in the context of medical diagnostics. By extending the traditional matrix structure with an additional dimension, neutrosophic rhotrices provide a more flexible and comprehensive framework for representing and processing medical data where ambiguity is inherent. Further, we explored the basic properties of neutrosophic rhotrices, such as their algebraic properties and their computational simplicity compared to neutrosophic matrices. Additionally, we compared the medical diagnostic method using neutrosophic rhotrices with that of fuzzy rhotrices and found that neutrosophic rhotrices provide significantly better results in the decision-making process.

References

1. AO Ajibade, The concept of rhotrix in mathematical enrichment, *International Journal of Mathematical Education in Science and Technology*, 34(2):175-179, 2003
2. B Sani, An alternative method for multiplication of rhotrices, *International Journal of Mathematical Education in Science and Technology*, 35(5):777-781, 2004
3. B Sani, Conversion of a rhotrix to a 'coupled matrix', *International Journal of Mathematical Education in Science and Technology*, 39(2):244-249, 2008
4. A O Isere, Even dimensional rhotrix, *Notes on Number Theory and Discrete Mathematics*, 24(2):125-133, 20

5. A Mohammed. The non-commutative full rhotrix ring and its subrings, *Science World Journal*, 13(2):24–36, 2018
6. Abdulhadi Aminu, Rhotrix vector spaces, *International Journal of Mathematical Education in Science and Technology*, 41(4):531–538, 2010
7. A Mohammed and UE Okon, On subgroups of non-commutative general rhotrix group, *Notes on Number Theory and Discrete Mathematics*, 22(2):72–90, 2016
8. F. Smarandache , Neutrosophic set - a generalization of the intuitionistic fuzzy set, Granular Computing, 2006 IEEE International Conference, 38 – 42,2006,DOI:10.1109/GRC.2006.1635754.
9. Dhar, M., Neutrosophic soft matrices and its application in medical diagnosis. Journal of fuzzy extension and application, 2 (1), 23-32, 2021.
10. Absalom, Ezugwu E and Ajibade, Abiodun O and Mohammed, A, Generalization of heart-oriented rhotrix multiplication and its algorithm implementation, *International Journal of Computer Applications*, Vol. 975,pg. no.8887, 2011
11. Tudunkaya SM, An Extension of Certain Construction of Finite Groups, *Nigerian Journal of Scientific Research*,vol. 17(3), 367-371, 2018
12. Bhuvaneshwari R and Kaladevi V, Energy Of Chemical Graphs With Adjacency Rhotrix, *European Journal of Molecular & Clinical Medicine*, vol. 7(08),2020
13. P Gnanachandra, B Ananda Priya and Seenivasan M(2023), AI-Driven Innovations in Cryptography: Enhancing Key Generation and Security, *E3S Web Conferences*, 399
14. Ananda Priya B, Gnanachandra P, Seenivasan M, Classification of the Commutative General Rhotrix Group: An Algebraic Approach, 6th International Conference on Mathematical Modelling, Applied Analysis and Computation-2023.
15. Mohammad Abobala, On the Representation of Neutrosophic Matrices by Neutrosophic Linear Transformations, Journal of Mathematics, 2021, DOI: 10.1155/2021/5591576.
16. F. Smarandache, A geometric interpretation of the neutrosophic set — A generalization of the intuitionistic fuzzy set Granular Computing (GrC), 2011 IEEE International Conference , 602 – 606, 2011, DOI: 10.1109/GRC.2011.6122665.
17. W.B. Vasantha Kandasamy & F. Smarandache. Neutrosophic Super Matrices and Quasi Super Matrices. *Ohio: Zip Publishing*, 2012.
18. WB, Vasantha Kandasamy, Ilanthenral Kandasamy, and Florentin Smarandache, Neutrosophic quadruple vector spaces and their properties, *Mathematics*, 7(8), 2019, 758.
19. Vasantha Kandasamy W.B,Ilanthenral Kandasamy and Florentin Smarandache, Semi-Idempotents in Neutrosophic Rings, *Mathematics*, 2019, 7(6), 507, https://doi.org/10.3390/math7060507
20. Ananda Priya, B. Gnanachandra P., Harnessing Fuzzy Rhotrices for Accurate Medical Diagnosis through Symptom – Disease Rhotrices, Preprint

Construction Of Almost Unbiased Estimator For Population Mean Using Neutrosophic Information

Rajesh Singh[1], Anamika Kumari[2], Florentin Smarandache[3] and Shobh Nath Tiwari[1*]

[1]Department of Statistics, Institute of Science, Banaras Hindu University, Varanasi-221005, Uttar Pradesh, India

[2]Department of Data Science and Computing, Manipal Academy of Higher Education

[3]Department of Mathematics, University of New Mexico, 705 Gurley Ave, Gallup, USA

* Corresponding: shobh7285@gmail.com

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Abstract: In classical statistics, the population mean is estimated using determinate, crisp data value when auxiliary information is known. These estimates can often be biased. The main objective of this study is to introduce the neutrosophic estimator with the minimum mean squared error (MSE) for the unknown value of the population mean as well as overcome the limitations of classical statistics when dealing with ambiguous or indeterminate data. Neutrosophic statistics was introduced by Florentin Smarandache. It is a generalisation of classical statistics that addresses ambiguous, unclear, vague, and indeterminate data. In this study, we have proposed neutrosophic almost unbiased estimator that use known neutrosophic auxiliary parameters to estimate the neutrosophic population mean of the primary variable. Equations for bias and mean squared error are calculated for the suggested estimators up to the first order of approximation. The proposed estimator performs better than the other existing estimators with respect to the MSE and percent relative efficiency (PRE) criteria. The estimator with the highest PRE or lowest MSE is advised for practical utility in various kinds of application areas. The theoretical conclusions are validated by the empirical analysis, which made use of the real data sets.

Keywords: Classical statistics, neutrosophic statistics, auxiliary information, bias, mean squared error, percent relative efficiency.

1. Introduction

One of the purpose of sampling theory is to estimate study variable on the basis of auxiliary information with more precision. In sampling theory, we estimate population parameter on the basis of auxiliary variable that has a correlation with study variable. Some of the popular methods used to obtain these objectives to estimate population parameter using ratio, product, regression method of estimation.

Ratio and product estimators are frequently used in survey sampling for getting an improved estimator of population mean given that the auxiliary information is known. The concept of ratio estimator was introduced by [1] as a method for utilising auxiliary information. Ratio estimator compute ratios between the sample means of the study variable and the auxiliary variable and using

known auxiliary data on a variable that is linearly related to the variable under study, is used to estimate the population mean.

Whereas, the product estimator, an additional technique for adding auxiliary information into estimation, was put up by [2]. In the product method of estimation, the sample means of auxiliary variable and study variable are multiplied, and the result is used as the foundation for estimation. Over time, several improvements have been made to the classical ratio and product estimators.

One improvement was proposed by [3], they proposed exponential ratio and product type estimators for estimating unknown population mean of the study variable using auxiliary information.

Several researchers such as [1,5,6,7] accomplished some outstanding work in this area. However, our primary focus here is on proposing almost unbiased estimator using neutrosophic information.

Classical statistics deals with data form by crisp numbers and there is no uncertainty in measurement of the observation. But there are so many cases when indeterminacy, vagueness, ambiguousness exists in the data. In day-to-day life where we have indeterminate data, for illustration, the measurement of temperature anywhere, decision support systems, information systems, the weather forecast, economy growth and more. In such cases we need to introduce new techniques to handle with such type of indeterminate data. Fuzzy logic is one method for resolving these kinds of issues regarding ambiguous, imprecise or uncertain observation but still, it ignores indeterminacy. In this situation, neutrosophic techniques are more suitable which address both randomness and indeterminacy.

While classical statistics deals with determinate and crisp values, neutrosophic statistics deals with sets values. As an extension of classical statistics, neutrosophic statistics is identical as classical statistics in the absence of indeterminacy, [9]. The evaluation of both determinate and indeterminate components of the observations is possible with neutrosophic logic, which is an extension of fuzzy logic. It is particularly useful for analysing data that is imprecise or unclear, as noted by [9].

Neutrosophic statistics can be used to solve a number of real-world issues as neutrosophic statistics to the analysis of road traffic accidents, [10]. [11] introduced a goodness-of-fit test within the framework of neutrosophic statistics and [12] concentrated on using neutrosophic analysis of variance on data from university students. In neutrosophic statistics, the sample size may not be precisely determined and not known as the exact number, [13]. Methods based on fuzzy logic are rapidly evolving and frequently used in decision-making settings, [14]. Further advancement of fuzzy sets is complex fuzzy sets, and a complex neutrosophic set is their generalized form. Estimation using neutrosophic techniques are a relatively new and mostly unexplored area compared to estimation challenges in classical probability sampling designs with definite and well-defined data, [15].

Classical statistics struggled to handle indeterminacies that arise from ambiguous, imprecise or incomplete frequency-related data, because it involved values that were uncertain and could include a range of possible measurements. Therefore, neutrosophic statistics is used as a more adaptable and extension of classical statistics in uncertain or indeterminacy situations. However, there has been limited attention given to neutrosophic sampling as compare to classical statistics for estimating the population parameters.

Frame of our paper is constructed in the following manner: Section 1 mentioned Introduction. Section 2 discussed about observation and terminology of neutrosophic statistics. Section 3 outlines some existing neutrosophic estimators. Section 4 presents the proposed neutrosophic almost

unbiased estimator. Section 5 mentioned some member of the proposed estimator. Section 6 presents empirical study. Section 7 is devoted to discussion whereas section 8 discusses the conclusion.

2. Neutrosophic observation and terminology

Numerous kinds of neutrosophic observations were provided, such as quantitative neutrosophic data that suggested a number might fall within an unknown interval (a,b), [9]. There are several ways to represent the interval value of a neutrosphic number. [16] and [17] have used neutrosophic interval values as $Z_N = Z_L + Z_U I_N$, where, $I_N \in [I_L, I_U]$. In addition, we apply the same notations from [17] to the considered neutrosophic data, which are represented in the intervals form as $Z_N \in [Z_L, Z_U]$, where, 'Z_L'is the lower value and ‘ Z_U ’ is the upper value of the neutrosophic variable Z_N. We choose a neutrosophic sample of size $n_N \in [n_L, n_U]$ using the simple random sampling without replacement (SRSWOR) technique from a finite population of 'N' units $(P_1, P_2, ..., P_N)$. Let $y_N(i) \in [y_L, y_U]$ be the ith sample observation of our neutrosophic data. The auxiliary variable $x_N(i)$ is similarly represented as $x_N(i) \in [x_L, x_U]$.

Let $\bar{y}_N(i) \in [\bar{y}_L, \bar{y}_U]$ and $\bar{x}_N(i) \in [\bar{x}_L, \bar{x}_U]$ be sample mean for the neutrosophic y_N and x_N respectively and neutrosophic auxiliary variable x_N is correlated to our neutrosophic study variable y_N. The population means for the neutrosophic variables y_N and x_N, which represent the overall averages of the neutrosophic data set, are represented by $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$ and $\bar{X}_N \in [\bar{X}_L, \bar{X}_U]$ respectively. For y_N and x_N, the neutrosophic coefficients of variation are $C_{yN} \in [C_{yNL}, C_{yNU}]$ and $C_{xN} \in [C_{xNL}, C_{xNU}]$ respectively. Furthermore, the neutrosophic correlation coefficient between neutrosophic variables x_N and y_N is denoted as $\rho_{yxN} \in \left[\rho_{yxL}, \rho_{yxU}\right]$. The neutrosophic coefficient of kurtosis for auxiliary variable x_N is denoted by $\beta_{2(x)N} \in [\beta_{2(x)\mathrm{L}}, \beta_{2(x)\mathrm{U}}]$. The neutrosophic mean errors are $\bar{\mathrm{e}}_{yN} \in [\bar{\mathrm{e}}_{yL}, \bar{\mathrm{e}}_{yU}]$ and $\bar{\mathrm{e}}_{xN} \in [\bar{\mathrm{e}}_{xL}, \bar{\mathrm{e}}_{xU}]$. In order to find the estimator's bias and MSE, we write

$\bar{\mathrm{y}}_N(i) = \bar{Y}_N\left(1 + \bar{e}_{yN}(i)\right)$, $\bar{x}_N(i) = \bar{X}_N\left(1 + \bar{e}_{xN}(i)\right)$,

$E(\bar{e}_{xN}) = E(\bar{e}_{yN}) = 0$ and

$E(\bar{e}_{yN}^2) = \theta_N C_{yN}^2$, $E(\bar{e}_{xN}^2) = \theta_N C_{xN}^2$, $E(\bar{e}_{xN}\bar{e}_{yN}) = \theta_N \rho_{yxN} C_{yN} C_{xN}$.

Where, $\bar{e}_{yN} \in [\bar{e}_{yL}, \bar{e}_{yU}]$, $\bar{e}_{xN} \in [\bar{e}_{xL}, \bar{e}_{xU}]$, $\bar{e}_{yN}\bar{e}_{xN} \in [\bar{e}_{yL}\bar{e}_{xL}, \bar{e}_{yU}\bar{e}_{xU}]$, $\bar{e}_{yN}^2 \in [\bar{e}_{yL}^2, \bar{e}_{yU}^2]$, $\bar{e}_{xN}^2 \in [\bar{e}_{xL}^2, \bar{e}_{xU}^2]$.

$C_{xN}^2 = \dfrac{\sigma_{xN}^2}{\bar{X}_N^2}$, $C_{yN}^2 = \dfrac{\sigma_{yN}^2}{\bar{Y}_N^2}$, $C_{yxN} = \rho_{yxN} C_{yN} C_{xN}$,

where, $C_{xN}^2 \in [C_{xL}^2, C_{xU}^2]$, $C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]$.

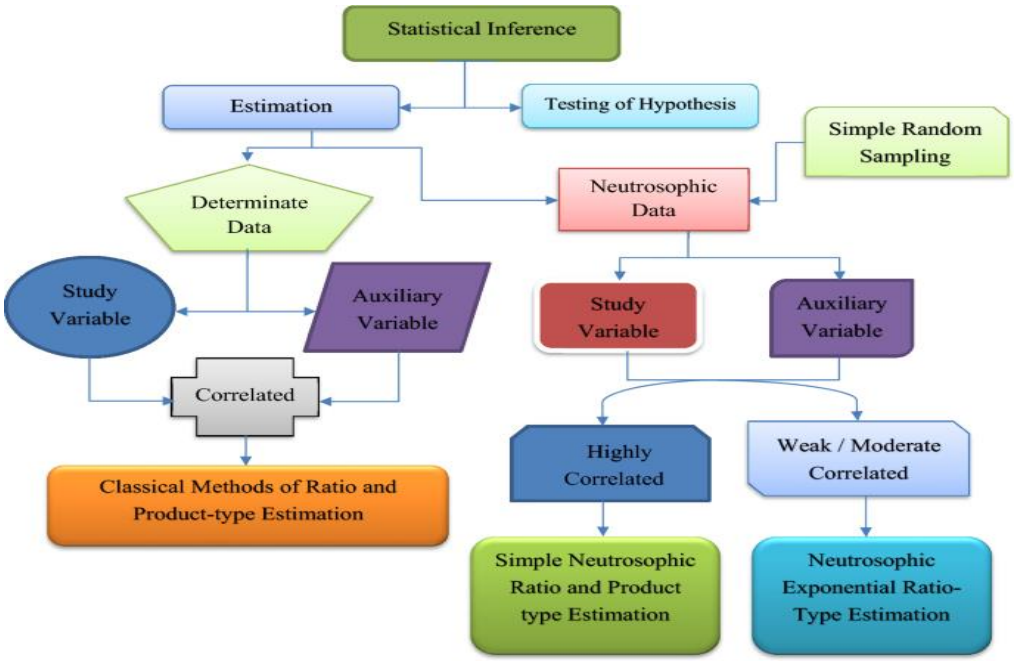
$$\rho_{yxN} = \frac{\sigma_{yxN}}{\sigma_{yN}\sigma_{xN}} \ , \ \rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}],$$

$$\theta_N = \frac{(1-f_N)}{n_N}, \text{ where, } \theta_N \in [\theta_L, \theta_U], \ n_N \in [n_L, n_U] \ , \ \sigma_{xN}^2 \in [\sigma_{xL}^2, \sigma_{xU}^2], \ \sigma_{yN}^2 \in [\sigma_{yL}^2, \sigma_{yU}^2],$$

$$\sigma_{yxN} \in [\sigma_{yxL}, \sigma_{yxU}].$$

Likewise, in a neutrosophic environment, bias and MSE can be defined as $Bias(\bar{y}_N) \in [Bias_L, Bias_U]$, $MSE(\bar{y}_N) \in [MSE_L, MSE_U]$ respectively.

The graph below illustrates the flowchart of the proposed study employing neutrosophic numbers. The flowchart below has been recommended by [16].



3. Some existing neutrosophic estimators

The estimation of neutrosophic population mean $\bar{Y}_N$ using neutrosophic sample mean which is provided by,

$$t_{1N} = \bar{y}_N \tag{3.1}$$

The variance of the estimator t_{1N} to the first order of approximation is provided by,

$$V(t_{1N}) = \theta_N \bar{Y}_N^2 C_{yN}^2 \ , \text{ where, } t_{1N} \in [t_{1L}, t_{1U}]. \tag{3.2}$$

[16] recommended the usual neutrosophic ratio estimator of neutrosophic population mean $\bar{Y}_N$ utilising the known neutrosophic population mean of X as,

$$t_{2N} = \bar{y}_N \left(\frac{\bar{X}_N}{\bar{x}_N} \right) \tag{3.3}$$

The bias term and MSE of the estimator t_{2N} are given by, respectively

$$Bias(t_{2N}) = \theta_N \bar{Y}_N (C_{xN}^2 - C_{yxN}) \tag{3.4}$$

$$MSE(t_{2N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + C_{xN}^2 - 2C_{yxN} \right) \tag{3.5}$$

Where, $C_{yxN} = \rho_{yxN} C_{xN} C_{yN}$ and $t_{2N} \in [t_{2L}, t_{2U}]$.

Motivated by [2], the neutrosophic product estimators is given by,

$$t_{3N} = \bar{y}_N \left(\frac{\bar{x}_N}{\bar{X}_N} \right) \tag{3.6}$$

The bias and MSE of the estimator's (t_{3N}) are provided by,

$$Bias(t_{3N}) = \theta_N \bar{Y}_N C_{yxN} \tag{3.7}$$

$$MSE(t_{3N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + C_{xN}^2 + 2C_{yxN} \right) \text{ , where, } t_{3N} \in [t_{3L}, t_{3U}]. \tag{3.8}$$

Inspired by [4], [16] proposed the following neutrosophic ratio estimator that utilises the coefficient of variation (CV) of the neutrosophic variable X as,

$$t_{4N} = \bar{y}_N \left(\frac{\bar{X}_N + C_{xN}}{\bar{x}_N + C_{xN}} \right) \tag{3.9}$$

The bias and MSE of the estimator t_{4N}, up to the first order of approximation are respectively,

$$Bias(t_{4N}) = \theta_N \bar{Y}_N (\lambda_{1N}^2 C_{xN}^2 - \lambda_{1N} C_{yxN}) \tag{3.10}$$

$$MSE(t_{4N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \lambda_{1N}^2 C_{xN}^2 - 2\lambda_{1N} C_{yxN} \right) \tag{3.11}$$

where, $\lambda_{1N} = \dfrac{\bar{X}_N}{\bar{X}_N + C_{xN}}$ and $t_{4N} \in [t_{4L}, t_{4U}]$.

Motivated by [3], [16] suggested the following neutrosophic exponential ratio-type estimator as follows:

$$t_{5N} = \bar{y}_N \exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{x}_N + \bar{X}_N} \right) \tag{3.12}$$

The estimator's (t_{5N}) bias and MSE are provided by,

$$Bias(t_{5N}) = \theta_N \bar{Y}_N \left(\frac{3}{8} C_{xN}^2 - \frac{1}{2} C_{yxN} \right) \tag{3.13}$$

$$MSE(t_{5N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{C_{xN}^2}{4} - C_{yxN} \right) \text{ , where, } t_{5N} \in [t_{5L}, t_{5U}]. \tag{3.14}$$

Motivated by [3], [18] suggested the neutrosophic product-type exponential estimator as,

$$t_{6N} = \bar{y}_N \exp\left(\frac{\bar{x}_N - \bar{X}_N}{\bar{x}_N + \bar{X}_N} \right) \tag{3.15}$$

The bias and MSE of the estimator t_{6N} are given by,

$$Bias(t_{6N}) = \theta_N \bar{Y}_N \left(\frac{3}{8} C_{xN}^2 + \frac{1}{2} C_{yxN} \right) \tag{3.16}$$

$$MSE(t_{6N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{C_{xN}^2}{4} + C_{yxN} \right) \text{ , where, } t_{6N} \in [t_{6L}, t_{6U}]. \tag{3.17}$$

Motivated by [19], [17] suggested the neutrosophic ratio estimator t_{7N}, utilising the known population coefficient of correlation ρ_{yxN} is provided as,

$$t_{7N} = \bar{y}_N \left(\frac{\bar{X}_N + \rho_{yxN}}{\bar{x}_N + \rho_{yxN}} \right) \tag{3.18}$$

The bias and MSE of the estimator t_{7N} are respectively given as,

$$Bias(t_{7N}) = \theta_N \bar{Y}_N (\lambda_{2N}^2 C_{xN}^2 - \lambda_{2N} C_{yxN}) \tag{3.19}$$

$$MSE(t_{7N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \lambda_{2N}^2 C_{xN}^2 - 2\lambda_{2N} C_{yxN} \right) \tag{3.20}$$

where, $\lambda_{2N} = \dfrac{\bar{X}_N}{\bar{X}_N + \rho_{yxN}}$ and $t_{7N} \in [t_{7L}, t_{7U}]$.

Inspired by [20], [17] proposed the neutrosophic ratio estimators utilising CV and the coefficient of kurtosis of auxiliary variable X, expressed as follows:

$$t_{8N} = \bar{y}_N \left(\frac{\beta_{2(x)N} \bar{X}_N + C_{xN}}{\beta_{2(x)N} \bar{x}_N + C_{xN}} \right) \tag{3.21}$$

The estimator's (t_{8N}) bias and MSE are defined as,

$$Bias(t_{8N}) = \theta_N \bar{Y}_N (\lambda_{3N}^2 C_{xN}^2 - \lambda_{3N} C_{yxN}) \tag{3.22}$$

$$MSE(t_{8N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \lambda_{3N}^2 C_{xN}^2 - 2\lambda_{3N} C_{yxN} \right) \tag{3.23}$$

where, $\lambda_{3N} = \dfrac{\beta_{2(x)N}\bar{X}_N}{\beta_{2(x)N}\bar{X}_N + C_{xN}}$ and $t_{8N} \in [t_{8L}, t_{8U}]$.

Motivated by [8], [16] recommended the neutrosophic modified exponential ratio estimator defined as,

$$t_{9N} = \bar{y}_N \exp\left(\frac{(a\bar{X}_N + b) - (a\bar{x}_N + b)}{(a\bar{X}_N + b) + (a\bar{x}_N + b)}\right) \tag{3.24}$$

where a and b denotes the auxiliary neutrosophic parameters.

The bias and MSE of the neutrosophic exponential ratio estimator t_{9N} are respectively given as,

$$Bias(t_{9N}) = \theta_N \bar{Y}_N \left(\frac{3}{8}\lambda_{4N}^2 C_{xN}^2 - \frac{1}{2}\lambda_{4N} C_{yxN}\right) \tag{3.25}$$

$$MSE(t_{9N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \lambda_{4N}^2 C_{xN}^2 - 2\lambda_{4N} C_{yxN}\right) \tag{3.26}$$

where, $\lambda_{4N} = \dfrac{a\bar{X}_N}{2(a\bar{X}_N + b)}$ and $t_{9N} \in [t_{9L}, t_{9U}]$.

From (3.22) and (3.25), we see that the estimators t_{8N} and t_{9N} recommended by [17], [16] respectively, are biased estimator. Bias is disadvantageous in certain applications.

Motivated by [8] and [17], we have suggested an almost unbiased estimator for the population mean $\bar{Y}_N$ using neutrosophic auxiliary information in SRSWOR.

4. The proposed neutrosophic almost unbiased estimator

Suppose $t_0 = \bar{y}_N$, $t_1 = \left[\bar{y}_N \left(\dfrac{a\bar{X}_N + b}{a\bar{x}_N + b}\right)^K\right]$, $t_2 = \left\lfloor \bar{y}_N \exp\left(K\left(\dfrac{(a\bar{X}_N + b) - (a\bar{x}_N + b)}{(a\bar{X}_N + b) + (a\bar{x}_N + b)}\right)\right)\right\rfloor$

such that $t_0, t_1, t_2 \in A$, where A is the collection of all possible estimators for estimating the neutrosophic population mean $\bar{Y}_N$. Justification of using a and b in proposing estimators is given by [22]. By definition, the set A is a linear variety if

$$t^* = \sum_{i=0}^{2} \alpha_i t_i \in A. \tag{4.1}$$

$$\text{for } \sum_{i=0}^{2} \alpha_i = 1 \text{ and } \alpha_i \in R. \tag{4.2}$$

where $\alpha_i (i = 0,1,2)$ refers to the statistical constants, R stands for the set of real numbers and K takes distinct values, that is, $K = 1$ for the ratio estimators and $K = -1$ for the product estimators.

The bias and MSE of the estimators t_1 and t_2 are, respectively, given by (up to the first order of approximation)

$$B(t_1) = \theta_N \bar{Y}_N \left[\lambda^2 C_{xN}^2 \left(\frac{K(K+1)}{2} \right) - \lambda K \rho_{yxN} C_{yN} C_{xN} \right] \tag{4.3}$$

$$MSE(t_1) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \lambda^2 K^2 C_{xN}^2 - 2\lambda K \rho_{yxN} C_{yN} C_{xN} \right] \tag{4.4}$$

$$B(t_2) = \theta_N \bar{Y}_N \left[\lambda^2 C_{xN}^2 \left(\frac{2K + K^2}{8} \right) - \frac{\lambda K}{2} \rho_{yxN} C_{yN} C_{xN} \right] \tag{4.5}$$

$$MSE(t_2) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{\lambda^2 K^2}{4} C_{xN}^2 - \lambda K \rho_{yxN} C_{yN} C_{xN} \right] \tag{4.6}$$

where, $\lambda = \dfrac{a\bar{X}_N}{a\bar{X}_N + b}$; a and b denotes the auxiliary neutrosophic parameters.

The bias and MSE of the estimator t^* are determined using,

$\bar{y}_N(i) = \bar{Y}_N \left(1 + \bar{e}_{yN}(i)\right)$, $\bar{x}_N(i) = \bar{X}_N \left(1 + \bar{e}_{xN}(i)\right)$.

Such that, $E(\bar{e}_{xN}) = E(\bar{e}_{yN}) = 0$.

Defining the estimator t^* in terms of $\bar{e}_{xN}$ and $\bar{e}_{yN}$, we obtain,

$$\begin{aligned} t^* &= \alpha_0 \bar{Y}_N \left(1 + \bar{e}_{yN}\right) + \alpha_1 \left[\bar{Y}_N \left(1 + \bar{e}_{yN}\right) \left(\frac{a\bar{X}_N + b}{a\bar{X}_N \left(1 + \bar{e}_{xN}\right) + b} \right)^K \right] \\ &+ \alpha_2 \left[\bar{Y}_N \left(1 + \bar{e}_{yN}\right) \exp\left(K \left(\frac{(a\bar{X}_N + b) - \left(a\bar{X}_N \left(1 + \bar{e}_{xN}\right) + b\right)}{(a\bar{X}_N + b) + \left(a\bar{X}_N \left(1 + \bar{e}_{xN}\right) + b\right)} \right) \right) \right] \\ &= \alpha_0 \bar{Y}_N \left(1 + \bar{e}_{yN}\right) + \alpha_1 \left[\bar{Y}_N \left(1 + \bar{e}_{yN}\right) \left(1 + \lambda \bar{e}_{xN}\right)^{-K} \right] \\ &+ \alpha_2 \bar{Y}_N \left(1 + \bar{e}_{yN}\right) \exp\left[K \left(-\frac{\lambda}{2} \bar{e}_{xN} \left(1 + \frac{\lambda}{2} \bar{e}_{xN} \right)^{-1} \right) \right] \end{aligned} \tag{4.7}$$

where, $\lambda = \dfrac{a\bar{X}_N}{a\bar{X}_N + b}$.

Expanding the terms on the right side of the equation (4.7) and having the terms up to the first order of approximation.

$$t^* = \overline{Y}_N \left[\begin{array}{l} (\alpha_0 + \alpha_1 + \alpha_2) + (\alpha_0 + \alpha_1 + \alpha_2)\overline{e}_{yN} - \lambda K \overline{e}_{xN} \left(\alpha_1 + \frac{\alpha_2}{2} \right) + \alpha_1 \lambda^2 \overline{e}_{xN}{}^2 \left(\frac{K(K+1)}{2} \right) \\ + \alpha_2 \lambda^2 \overline{e}_{xN}{}^2 \left(\frac{2K + K^2}{8} \right) - K\alpha_1 \lambda \overline{e}_{xN} \overline{e}_{yN} - \frac{K}{2} \alpha_2 \lambda \overline{e}_{xN} \overline{e}_{yN} \end{array} \right] \tag{4.8}$$

The bias of the estimator t^*, up to the second power of $e's$, can be computed by subtracting $\overline{Y}_N$ from both sides of equation (4.8) after taking the expectation of each side.

$$B(t^*) = \theta_N \overline{Y}_N C_{xN}^2 \left[\alpha_1 \lambda^2 \frac{K(K+1)}{2} + \alpha_2 \lambda^2 \left(\frac{2K + K^2}{8} \right) - \left(K\alpha_1 \lambda + \frac{K}{2} \alpha_2 \lambda \right) \rho_{yxN} \frac{C_{yN}}{C_{xN}} \right] \tag{4.9}$$

From equation (4.8), we have

$$t^* - \overline{Y}_N \cong \overline{Y}_N \left[\overline{e}_{yN} - \lambda \alpha K \overline{e}_{xN} \right] \tag{4.10}$$

where, $\alpha = \left(\alpha_1 + \frac{\alpha_2}{2} \right)$. (4.11)

By taking square of each side of (4.10) and then taking expectations, we get MSE of the estimator t^*, to the first order of approximation as,

$$MSE(t^*) = \theta_N \overline{Y}_N^2 \left[C_{yN}^2 + C_{xN}^2 \lambda K \alpha \left(\lambda K \alpha - 2\delta_N \right) \right] \tag{4.12}$$

where, $\delta_N = \rho_{yxN} \frac{C_{yN}}{C_{xN}}$.

Minimum MSE of the estimator t^* would occur when

$$\alpha = \frac{\delta_N}{\lambda K}. \tag{4.13}$$

Applying this value of $\alpha = \frac{\delta_N}{\lambda K}$ in equation (4.1), we get the optimum value of estimator t^* (optimum).

Hence, the minimum MSE of the estimator t^* is provided by

$$\min . MSE\left(t^*\right) = \theta_N \overline{Y}_N^2 C_{yN}^2 \left(1 - \rho_{yxN}^2\right) \tag{4.14}$$

This MSE is equivalent to the classical linear regression estimator.

By using equation (4.11) and (4.13) we have

$$\alpha_1 + \frac{\alpha_2}{2} = \alpha = \frac{\delta_N}{\lambda K}. \tag{4.15}$$

There are only two equations with three unknowns in equations (4.2) and (4.15). The unique values for $\alpha_i's$, $(i=0,1,2)$ can't be found. In order to get unique values for $\alpha_i's$, we will apply the linear constraint

$$\sum_{i=0}^{2}\alpha_i B(t_i)=0. \tag{4.16}$$

where $B(t_i)$ represents the bias of i^{th} estimator.

The matrix form of equations (4.2), (4.15), and (4.16) is as follows:

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & \frac{1}{2}\\ 0 & B(t_1) & B(t_2)\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix}=\begin{pmatrix}1\\ \alpha\\ 0\end{pmatrix} \tag{4.17}$$

where, $\alpha=\dfrac{\delta_N}{\lambda K}$.

By using equation (4.17), The unique values of $\alpha_i's$ $(i=0,1,2)$ are obtained as,

$$\left.\begin{aligned}\alpha_0&=\frac{B(t_2)-\frac{B(t_1)}{2}-\alpha B(t_2)+\alpha B(t_1)}{B(t_2)-\frac{B(t_1)}{2}}\\ \alpha_1&=\frac{\alpha B(t_2)}{B(t_2)-\frac{B(t_1)}{2}}\\ \alpha_2&=\frac{-\alpha B(t_1)}{B(t_2)-\frac{B(t_1)}{2}}\end{aligned}\right] \tag{4.18}$$

such that,

$\sum_{i=0}^{2}\alpha_i=1$ and $\alpha_i\in R$.

By using these $\alpha_i's$, $\left(i=0,1,2\right)$ eliminate the bias up to terms of order $o(n^{-1})$ at (4.1).

5. Some member of the suggested estimator 't^*'

Several estimators of the population mean are provided by choosing the appropriate combination of constants a, b and $\alpha_i's$,$\left(i=0,1,2\right)$, as shown in the following scheme.

Estimator	Values of

	α_0	α_1	α_2	a	b	K
$t_{1N}=\bar{y}_N$ The mean per unit estimator	1	0	0	0	0	1
$t_{2N}=\bar{y}_N\left(\frac{\bar{X}_N}{\bar{x}_N}\right)$ [16] estimator	0	1	0	1	0	1
$t_{3N}=\bar{y}_N\left(\frac{\bar{x}_N}{\bar{X}_N}\right)$ Neutrosophic product estimator	0	1	0	1	0	-1
$t_{4N}=\bar{y}_N\left(\frac{\bar{X}_N+C_{xN}}{\bar{x}_N+C_{xN}}\right)$ [16] estimator	0	1	0	1	C_{xN}	1
$t_{5N}=\bar{y}_N\exp\left(\frac{\bar{X}_N-\bar{x}_N}{\bar{x}_N+\bar{X}_N}\right)$	0	0	1	1	0	1
$t_{6N}=\bar{y}_N\exp\left(\frac{\bar{x}_N-\bar{X}_N}{\bar{x}_N+\bar{X}_N}\right)$ [18] estimator	0	0	1	1	0	-1
$t_{7N}=\bar{y}_N\left(\frac{\bar{X}_N+\rho_{yxN}}{\bar{x}_N+\rho_{yxN}}\right)$	0	1	0	1	ρ_{yxN}	1
$t_{8N}=\bar{y}_N\left(\frac{\beta_{2(x)N}\bar{X}_N+C_{xN}}{\beta_{2(x)N}\bar{x}_N+C_{xN}}\right)$ [17] estimator	0	1	0	$\beta_{2(x)N}$	C_{xN}	1
$t_{9N}=\bar{y}_N\exp\left(\frac{(a\bar{X}_N+b)-(a\bar{x}_N+b)}{(a\bar{X}_N+b)+(a\bar{x}_N+b)}\right)$	0	0	1	a	b	1

In addition, the estimators mentioned above, by simply changing the values of a, b and $\alpha_i's$ ($i=0,1,2$). From the proposed estimator t^* at (4.1), several estimators can be created.

6. Empirical Study

Population 1 : Here, we use the data provided by [21], which is birth rate and data on the natural growth rate obtained from the sample registration system (SRS)(2020) is used to demonstrate the performance of different estimators of neutrosophic study variable $Y_N \in [Y_L, Y_U]$. The neutrosophic auxiliary variable and neutrosophic study variable are given by,

X_N : Birth rate Y_N : Natural growth rate

The PRE formula is given by

$$PRE = \frac{MSE(\bar{y}_N)}{MSE(\text{estimator})} \times 100 \tag{6.1}$$

Table 1. Descriptive statistics of birth rate and natural growth rate data as per SRS 2020.

Parameter	Neutrosophic Value	Parameter	Neutrosophic Value
N_N	[36, 36]	n_N	[10, 10]
μ_{xN}	[14.472, 18.053]	C_{yN}	[0.333, 0.411]
μ_{yN}	[10.894, 10.631]	ρ_{yxN}	[0.620, 0.849]
C_{xN}	[0.213, 0.236]	$\beta_{2(x)N}$	[3.034, 2.173]

Table 2. Neutrosophic MSEs and PRE of the different estimators.

Estimator	MSE	I_N	PRE
t_{1N}	[0.951886, 1.378714]	[0, 0.31]	[100, 100]
t_{2N}	[0.490003, 0.586746]	[0, 0.16]	[194.26, 234.98]
t_{3N}	[2.092575, 3.174567]	[0, 0.34]	[43.43, 45.49]
t_{4N}	[2.070519, 3.145647]	[0, 0.34]	[43.83, 45.97]
t_{5N}	[0.672372, 0.820965]	[0, 0.18]	[141.57, 167.94]
t_{6N}	[1.425287, 2.163247]	[0, 0.34]	[63.73, 66.79]
t_{7N}	[0.510456, 0.586471]	[0, 0.13]	[186.48, 235.09]
t_{8N}	[0.492618, 0.586646]	[0, 0.16]	[193.23, 235.02]
t_{9N} $(a=1, b=1)$	[0.684577, 0.844600]	[0, 0.19]	[139.05, 163.24]
t^*	**[0.385639, 0.586416]**	[0, 0.34]	**[235.11, 246.83]**

Population 2 : Here, we use the data sets considered by [18]. Numerous data points, such as the minimum, maximum and mean temperatures over a wide time duration, are included in this dataset regarding the All India Seasonal and Annual Temperature Series. The neutrosophic auxiliary variable and neutrosophic study variable are given by,

X_N : Minimum and maximum temperatures in January and February.

Y_N : Temperatures from March through May.

Table 3. Descriptive statistics of the population 2.

Parameter	Neutrosophic Value	Parameter	Neutrosophic Value
N_N	[119, 119]	n_N	[20, 20]
μ_{xN}	[13.908,24.674]	C_{yN}	[0.0007, 00.0006]
μ_{yN}	[20.694, 31.560]	ρ_{yxN}	[0.6274, 0.7202]

C_{xN}	[0.0017, 0.0015]	$\beta_{2(x)N}$	[4.973, 5.583]

Table 4. Neutrosophic MSEs and PRE of the different estimators.

Estimator	MSE	I_N	PRE
t_{1N}	[8.87822E-06, 1.72193E-05]	[0, 0.48]	[100.00, 100.00]
t_{2N}	[3.39108E-05, 5.58304E-05]	[0, 0.39]	[26.18, 30.84]
t_{3N}	[8.78263E-05, 0.000173977]	[0, 0.50]	[10.11, 9.90]
t_{4N}	[3.39013E-05, 5.58219E-05]	[0, 0.39]	[26.19, 30.85]
t_{5N}	[8.39692E-06, 1.21037E-05]	[0, 0.31]	[105.73, 142.26]
t_{6N}	[3.53547E-05, 7.1177E-05]	[0, 0.50]	[25.11, 24.19]
t_{7N}	[3.06832E-05, 5.20435E-05]	[0, 0.41]	[28.94, 33.09]
t_{8N}	[3.39089E-05, 5.58288E-05]	[0, 0.39]	[26.18, 30.84]
t_{9N} $(a=1, b=1)$	[7.61584E-06, 1.13888E-05]	[0, 0.33]	[116.58, 151.19]
t^*	**[5.38372E-06, 8.28831E-06]**	[0, 0.35]	**[164.91, 207.75]**

7. Discussion

In Table 2, as can be shown that the sampling variance of the neutrosophic estimator t_{1N} is [0.951886, 1.378714] and the neutrosophic exponential ratio estimator's (t_{5N}) MSE is $[0.672372, 0.820965]$ whereas the range of the neutrosophic MSEs (t_{4N}) is [2.070519, 3.145647]. The reason behind the higher MSEs of the neutrosophic estimators t_{4N} over the neutrosophic estimator t_{1N} is that the low value of the neutrosophic CV of the neutrosophic auxiliary variable x. In comparison to the other population mean $(\bar{Y}_N)$ estimators in competition, the MSE of the proposed class of neutrosophic estimators (t^*) is the lowest at [0.385639, 0.586416] with higher PRE.

Similarly, the proposed and existing estimators' MSEs and PRE are given by Table 4. Additionally, it has shown that the suggested estimators' MSEs and PREs are lower and higher respectively, in comparison to those of the other estimators already in use. Its value is highlighted by the bolded text. Hence, Tables 2 and 4 confirmed these results, showing that the suggested estimators are preferable to the existing ones.

8. Conclusion

In this paper, we have recommended an almost unbiased neutrosophic estimator to estimate the neutrosophic population mean $\bar{Y}_N$ using some known value of auxiliary information. In order to evaluate their precision, we computed bias and MSE for the suggested estimators, concentrating on first-order approximations. We evaluated our suggested estimator with existing estimators using data from a natural environment. Via conducting empirical study, we have identified strong evidence that our suggested estimator performs better than the existing ones within the context of neutrosophic sampling. It should be emphasised that in cases where the study variable's observations are nondeterministic or uncertain, the neutrosophic estimators are best suited for enhanced population mean estimate. Therefore, it might be advised to use the proposed class of estimators to

estimate the neutrosophic population mean $\overline{Y}_N$ in a variety of real-world settings, including those in the social sciences, economics, agriculture, biology and mathematics.

References

[1] W. G. Cochran, "The estimation of the yields of cereal experiments by sampling for the ratio of grain to total produce," J. Agric. Sci., vol. 30, no. 2, pp. 262–275, Apr. 1940, doi: 10.1017/S0021859600048012.

[2] M. N. Murthy, "Product method of estimation," Sankhyā Indian J. Stat. Ser. A. (1961-2002), Vol. 26, No. 1, pp. 69-74, Jul.1964, https://www.jstor.org/stable/25049308.

[3] S. Bahl and R. K. Tuteja, "Ratio and product type exponential estimators," J. Inf. Optim. Sci., vol. 12, no. 1, pp. 159–164, Jan. 1991, doi:10.1080/02522667.1991.10699058.

[4] B. V. S. Sisodia and V. K. Dwivedi, "A modified ratio estimator using coefficient of variation of auxiliary variable," J Indian Soc Agric Stat, 33(1), 13–18, 1981.

[5] C. Kadilar and H. Cingi, "Ratio estimators in simple random sampling," Appl. Math. Comput., vol.151, no. 3, pp. 893–902, Apr. 2004, doi:10.1016/S0096-3003(03)00803-8.

[6] R. Singh, R. Mishra, A. Kumari, and S. K. Yadav, "Almost unbiased estimators for population coefficient of variation using auxiliary information," Asian J. Probab. Stat., vol. 26, no. 5, pp. 1–18, May 2024, doi:10.9734/ajpas/2024/v26i5614.

[7] H. P. Singh and R. S. Solanki, "A new procedure for variance estimation in simple random sampling using auxiliary information," Stat. Pap., vol. 54, no. 2, pp. 479–497, May 2013, doi:10.1007/s00362-012-0445-2.

[8] R. Singh, P. Chauhan, F. Smarandache, and N. Sawan, "Auxiliary information and a priori values in construction of improved estimators," Infinite Study, 2007.

[9] F. Smarandache, "Introduction to neutrosophic statistics," Infinite Study, 2014.

[10] M. Aslam, "Monitoring the road traffic crashes using NEWMA chart and repetitive sampling," International Journal of Injury Control and Safety Promotion, 28(1), pp. 39–45. 2020, doi:10.1080/17457300.2020.1835990.

[11] M. Aslam, "A new goodness of fit test in the presence of uncertain parameters," Complex Intell. Syst., Oct. 2020, doi:10.1007/s40747-020-00214-8.

[12] M. Aslam, "Neutrosophic analysis of variance: application to university students," Complex Intell. Syst., vol. 5, no. 4, pp. 403–407, Dec. 2019, doi:10.1007/s40747-019-0107-2.

[13] F. Smarandache, "A unifying field in logics: Neutrosophic logic, neutrosophic set, neutrosophic probability, and statistics. arXiv preprint math/0101228 - 6th ed.". 2001.

[14] N. Jan, L. Zedam, T. Mahmood, K. Ullah, and Z. Ali, "Multiple attribute decision making method under linguistic cubic information," J. Intell. Fuzzy Syst., vol. 36, no. 1, pp. 253–269, Jan. 2019, doi:10.3233/JIFS-181253.

[15] R. Varshney, A. Pal, Mradula, and I. Ali, "Optimum allocation in the multivariate cluster sampling design under gamma cost function," J. Stat. Comput. Simul., vol. 93, no. 2, pp. 312–323, Jan. 2023, doi:10.1080/00949655.2022.2104845.

[16] Z. Tahir, H. Khan, M. Aslam, J. Shabbir, Y. Mahmood, and F. Smarandache, "Neutrosophic ratio-type estimators for estimating the population mean," Complex Intell. Syst., vol. 7, no. 6, pp. 2991–3001, Dec. 2021, doi:10.1007/s40747-021-00439-1.

[17] S. K. Yadav and F. Smarandache, "Generalized neutrosophic sampling strategy for elevated estimation of population mean," Neutrosophic Sets and Systems, Vol. 53, 2023 SSRN doi:10.2139/ssrn.4652035.

[18] V. K. Yadav and S. Prasad, "Neutrosophic estimators for estimating the population mean in survey sampling," Meas. Interdiscip. Res. Perspect., pp. 1–25, Mar. 2024, doi: 10.1080/15366367.2023.2267835.

[19] H. P. Singh and R. Tailor, "Estimation of finite population mean using known correlation coefficient between auxiliary characters," Statistica, vol. 65, no. 4, Art. no. 4, 2005, doi: 10.6092/issn.1973-2201/1300.

[20] L. N. Upadhyaya and H. P. Singh, "Use of transformed auxiliary variable in estimating the finite population mean," Biom. J., vol. 41, no. 5, pp. 627–636, 1999, doi: 10.1002/(SICI)1521-4036(199909)41:5<627::AID-BIMJ627>3.0.CO;2-W.

[21] R. Singh and A. Kumari, "Neutrosophic ranked set sampling scheme for estimating population mean: An application to demographic data," Neutrosophic Sets and Systems, Vol. 68, 2024, doi:10.5281/zenodo.11479519.

[22] R. Singh and M. Kumar, "A note on transformations on auxiliary variable in survey sampling," Model Assist. Stat. Appl., vol. 6, no. 1, pp. 17–19, Mar. 2011, doi: 10.3233/MAS-2011-0154.

Neutrosophic Geometric Distribution: A Novel Approach for Modeling Indeterminate Discrete Outcomes

Rehan Ahmad Khan Sherwani[1], Sadia Iqbal[1], Shumaila Abbas[1], Muhammad Saleem[2], Muhammad Aslam[3,*] and Florentin Smarandache[4]

[1]College of Statistical Sciences
University of the Punjab Lahore
Pakistan
e-mail: rehan.stat@pu.edu.pk
sadiaiqbal905@gmail.com
shumaila.stat@pu.edu.pk

[2]Department of Industrial Engineering
Faculty of Engineering-Rabigh
King Abdulaziz University
Jeddah 21589, Saudi Arabia
e-mail: msaleim1@kau.edu.sa

[3]Department of Statistics
Faculty of Science
King Abdulaziz University
Jeddah 21551, Saudi Arabia
e-mail: aslam_ravian@hotmail.com

[4]Mathematics, Physics, and Natural Science Division
University of New Mexico Gallup
NM, U. S. A.
e-mail: fsmarandache@gmail.com

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Abstract

Uncertainty, vagueness, and ambiguity surround us in many real-life problems and, therefore, always remain under consideration for researchers to quantify them. This study proposed neutrosophic discrete probability distribution as a generalization of classical or existing probability distributions, named neutrosophic geometric distribution. Case studies presented in the paper will help understand the concept and application of the proposed distribution. Several properties are derived, like the proposed distribution's moment, characteristic, and probability-generating functions. Furthermore, the newly proposed distribution derives properties from the reliability analysis, such as survival function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, mills ratio, and odds ratio. In addition, order statistics for NGD, including *w*th, the largest, and the smallest order statistics, are also derived from joint, median, minimum, and maximum order statistics. This examination opens the path for managing issues that follow traditional conveyances and simultaneously contain information that is not determined precisely.

1. Introduction

Smarandache first initiated neutrosophy in [1]. It is a new branch of the philosophy presented as a generalization of fuzzy logic and a generalization of intuitionistic fuzzy logic [2, 3]. The modern world is overfilled with

uncertainty, ambiguity, fuzzy (problems, circumstances, and ideas) [4]. The classical probability ignores extreme, aberrant, unclear values, so a new suitable instrument had to be used [5]. The basic concept of neutrosophic sets was introduced by [6] in a well-defined book in 2014 purely in a statistics scenario, which presents a new base for dealing with several issues containing indeterminate data. The primary objective of neutrosophic logic is to characterize a logical statement in a 3-D neutrosophic space. Each dimension of the nutrosophic space represents, respectively, Truth (T), False (U), and Indeterminacy (I) of the statements under consideration. The terms T, I, and U are the standard, non-standard real subsets $(-0, 1+)$ without any specific connection. Many researchers extended the classical distributions neutrosophically, including neutrosophic binomial distribution and neutrosophic normal distribution, neutrosophic multinomial distribution, neutrosophic Poisson distribution, neutrosophic exponential distribution and neutrosophic uniform distribution, neutrosophic gamma distribution, neutrosophic Weibull distribution and its several families, etc [7-10]. The readers can see further information on neutrosophic statistics vs classical statistics in [5, 8, 11-15]. This study extended classical geometric distribution, which is neutrosophically named neutrosophic geometric distribution, using neutrosophic logic.

2. Neutrosophic Geometric Distribution (NGD)

Geometric distribution from the distribution theory is a classical discrete probability distribution [16]. The distribution is related to the binomial distribution in terms of the nature of the experimental trial, i.e., the trials are independent and have two possible outcomes (success or failure). The random variable used in the geometric distribution is the number of trials needed to attain a first success. The geometric distribution requires the exact number of failure attempts before obtaining a first success, but the precise number of failures is hard to find in some situations. In such situations, the experimenter is not sure about the classification of the outcome, i.e., either classify the outcome as a success or failure. This indecisive situation is

called the *indeterminate state* of the experiment. It requires some specialized form of geometric distribution because the methodology of the classical geometric distribution does not support handling such situations. This section proposed a modified form of the classical geometric distribution called *neutrosophic geometric distribution*.

The neutrosophic geometric distribution of discrete r.v. X is well-defined as the extension of the classical geometric distribution of X with some indeterminacy or vagueness in the experiment. A neutrosophic geometric experiment may result in some failures as well as indeterminate outcomes till the occurrence of the first success; for example, tossing a coin on an unstable surface that may have cracks, a coin may fall on its edge inside the crack, and one may get neither head nor tale but some indeterminacy. The probability of success is labeled as $p_r(S)$, a failure $p_r(U)$ and indeterminate outcome as $p_r(I)$.

Neutrosophic geometric r.v. X represents the number of trials needed for first success when experimenting with a variable number of times - the neutrosophic probability distribution of r.v. X is called a *neutrosophic geometric probability distribution*.

For $x \in \{1, 2, 3, ..., \infty\}$, Np_r (occurrences of first success after x trials) $= (T_x, U_x, I_x)$.

For cases where *threshold* $> th^*$ will belong to indeterminate part and when *threshold* $< th^*$, the cases will belong to a determinate part. The probability mass function (T_x) and the cumulative distribution function $(F(x))$ of neutrosophic geometric distribution are given in equations (2.1) and (2.4):

$$T_x = p_r(S) \sum_{h=0}^{th^*} \binom{x-1}{h} (p_r(I))^h (p_r(U))^{x-h-1}. \quad (2.1)$$

The same will be true when trials result in failures as there is only one success, not a fixed number of successes, like in binomial distribution:

$$U_x = p_r(S)\sum_{h=0}^{th^*}\binom{x-1}{h}(p_r(I))^h(p_r(U))^{x-h-1} \tag{2.2}$$

and

$$I_x = p_r(S)\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*}$$

$$\times\sum_{x=1}^{\infty}\sum_{h=0}^{x-z^*-1}\binom{x-z^*-1}{h}(p_r(U))^{x-z^*-h-1}. \tag{2.3}$$

The CDF corresponding to equation (2.1) is given by

$$F(x) = 1 - p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{m'=0}^{\infty}\binom{x+m'}{m'}\left(\frac{p_r(I)}{p_r(U)}\right)^{m'}. \tag{2.4}$$

The CDF corresponding to equation (2.3) is given by

$$F(x) = 1 - p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\cdot\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}. \tag{2.5}$$

2.1. Physical conditions

(i) Each trial results in three mutually exclusive and exhaustive outcomes: success, failure, and indeterminacy.

(ii) All the trials must be independent.

(iii) The probability of success remains fixed or constant for every trial.

(iv) The experiment performs a variable number of times until the first success occurs.

2.2. Case studies

Case study 1. A physician is seeking an anti-depressant for five patients newly diagnosed. Assume that, of all available anti-depressant drugs, 60% were found effective, 20% were found ineffective, and 30% have no

evidence about their effectiveness. What is the probability that the first drug is effective for the fifth patient?

Using the information in the problem mentioned above, we compute the probability for all three parts of PMF.

$X \rightarrow$ No of patients until we get the one for which ant-depressant drug found effective.

So $x = 1, 2, 3, 4, 5$ and we have to find $NGp_r\ (x = 5)$.

p_r (effective drug) $= p_r(S) = 0.6,$

p_r (ineffective drug) $= p_r(U) = 0.2,$

p_r (neither effective nor ineffective) $= p_r(I) = 0.3.$

Let indeterminacy threshold be 2. Then

$$T_5 = (0.6)\sum_{h=0}^{2}\binom{5-1}{h}(0.3)^h(0.2)^{5-h-1}$$

$$= (0.6)\left[\binom{5-1}{0}(0.3)^0(0.2)^{5-0-1} + \binom{5-1}{1}(0.3)^1(0.2)^{5-1-1} + \binom{5-1}{2}(0.3)^2(0.2)^{5-2-1}\right]$$

$$= (0.6)\left(\frac{1}{625} + \frac{6}{625} + \frac{27}{1250}\right)$$

$$= 0.01968,$$

$$I_5 = (0.6)\sum_{z^*=3}^{5-1=4}\frac{(5-1)!}{z^*!}(0.3)^{z^*}\sum_{h=0}^{5-z^*-1}\frac{(0.2)^{5-z^*-h-1}}{h!(5-z^*-h-1)!}$$

$$= (0.6)\left[\left\{\frac{(5-1)!}{3!}(0.3)^3\sum_{h=0}^{5-3-1=1}\frac{(0.2)^{5-3-h-1}}{h!(5-3-h-1)!}\right\}\right.$$

$$+\left\{\frac{(5-1)!}{4!}(0.3)^4\sum_{h=0}^{5-4-1=0}\frac{(0.2)^{5-4-h-1}}{h!(5-4-h-1)!}\right\}\Bigg]$$

$$=(0.6)\Bigg[4(0.3)^3\left\{\frac{(0.2)^{2-0-1}}{0!(2-0-1)!}+\frac{(0.2)^{2-1-1}}{1!(2-1-1)!}\right\}$$

$$+(0.3)^4\left\{\frac{(0.2)^{1-0-1}}{0!(1-0-1)!}\right\}\Bigg]$$

$$=(0.6)(0.1296+8.1\times10^{-3})$$

$$=0.08262,$$

$$U_5=(p_r(S)+p_r(I)+p_r(U))^x-T_5-I_5$$

$$=(0.6+0.3+0.2)^5-0.01968-0.08262$$

$$=1.50821.$$

If normalized, the computed vector becomes

$$(T_5,\ I_5,\ U_5)=(0.01968,\ 0.08262,\ 1.50821).$$

By dividing each component of the vector by its total sum, we have

$$0.01968+0.08262+1.50821=1.61051.$$

Hence we get

$$(T_5,\ I_5,\ U_5)=(0.012219,\ 0.051301,\ 0.93648).$$

Case study 2. Suppose an investigator is searching for a student who lives five miles away from the investigator. The investigator knows that 52% of the 25000 live within five miles of him, 33% do not, and there is no information about 10% of students where they live. The investigator randomly calls college students until one says the student lives within 5 miles of him. What is the probability that six people need a contact?

Using the information in the abovementioned problem, we compute the probability for all three parts of PMF.

$X \rightarrow$ Number of students the investigator must contact till one says yes.

So, $x = 1, 2, 3, \ldots$ (total number of students), we need to find the $NGp_r(x = 6)$.

p_r (do live within five miles) $= p_r(S) = 0.52,$

p_r (do not live within five miles) $= p_r(U) = 0.33,$

p_r (no information about where they live) $= p_r(I) = 0.1.$

Let indeterminacy threshold be 2. Then

$$T_6 = (0.52)\sum_{h=0}^{2}\binom{6-1}{h}(P(I))^h(P(U))^{6-h-1}$$

$$= (0.52)\left\{\binom{6-1}{0}(0.1)^0(0.33)^{6-0-1} + \binom{6-1}{1}(0.1)^1(0.33)^{6-1-1} + \binom{6-1}{2}(0.1)^2(0.33)^{6-2-1}\right\}$$

$$= (0.52)\{(3.9135\times10^{-3}) + (5.9296\times10^{-3}) + (3.5937\times10^{-3})\}$$

$$= 0.006987,$$

$$I_6 = (0.52)\sum_{z^*=3}^{6-1=5}\frac{(6-1)!}{z^*!}(0.1)^{z^*}\sum_{h=0}^{6-z^*-1}\frac{(0.33)^{6-z^*-h-1}}{h!(6-z^*-h-1)!}$$

$$= (0.52)\left[\left\{\frac{(6-1)!}{3!}(0.1)^3\sum_{k=0}^{6-3-1=2}\frac{(0.33)^{6-3-h-1}}{h!(6-3-h-1)!}\right\} + \left\{\frac{(6-1)!}{4!}(0.1)^4\sum_{h=0}^{6-4-1=1}\frac{(0.33)^{6-4-h-1}}{h!(6-4-h-1)!}\right\}\right.$$

$$+\left\{\frac{(6-1)!}{5!}(0.1)^5\sum_{h=0}^{6-5-1=0}\frac{(0.33)^{6-5-h-1}}{h!(6-5-h-1)!}\right\}\Bigg]$$

$$= (0.52)\Bigg[(0.02)\left\{\frac{(0.33)^{3-0-1}}{0!(3-0-1)!}+\frac{(0.33)^{3-1-1}}{1!(3-1-1)!}+\frac{(0.33)^{3-2-1}}{2!(3-2-1)!}\right\}$$

$$+(5\times10^{-4})\cdot\left\{\frac{(0.33)^{2-0-1}}{0!(2-0-1)!}+\frac{(0.33)^{2-1-1}}{1!(2-1-1)!}\right\}$$

$$+(1\times10^{-5})\left\{\frac{(0.33)^{1-0-1}}{1!(1-0-1)!}\right\}\Bigg]$$

$$= (0.52)(0.01769+6.65\times10^{-4}+0.1^5)$$

$$= 9.5498\times10^{-3},$$

$$U_6 = (p_r(S)+p_r(I)+p_r(U))^x - T_6 - I_6$$

$$= (0.52+0.1+0.33)^6 - 0.006987 - 9.5498\times10^{-3}$$

$$= 0.71856.$$

If normalized, the computed vector becomes

$$(T_6,\ I_6,\ U_6) = (0.006987,\ 0.71856,\ 9.5498\times10^{-3}).$$

By dividing each component of the vector by its total sum, we have

$$0.006987 + 9.5498\times10^{-3} + 0.71856 = 0.735097.$$

Hence we get

$$(T_6,\ I_6,\ U_6) = (9.50487\times10^{-3},\ 0.012991,\ 0.97750).$$

Case study 3. Suppose that different computer components have a random selection, from which 30% were defective, 60% were non-defective, and 10% were unclear whether defective or not. How likely is the seventh component tested to cause the first defect?

Using the information in the abovementioned problem, we compute the probability for all three parts of PMF.

X takes on values 1, 2, 3, ..., and we have to find $NGp_r\ (x = 7)$.

p_r (defective components) $= p_r(S) = 0.3$,

p_r (non-defective components) $= p_r(U) = 0.6$,

p_r (neither defective nor non-defective) $= p_r(I) = 0.1$.

Let indeterminacy threshold be 3. Then

$$T_7 = (0.3)\sum_{h=0}^{3}\binom{7-1}{h}(0.1)^h(0.6)^{7-h-1}$$

$$= (0.3)\left[\binom{7-1}{0}(0.1)^0(0.6)^{7-0-1} + \binom{7-1}{1}(0.1)^1(0.6)^{7-1-1} + \binom{7-1}{2}(0.1)^2(0.6)^{7-2-1} + \binom{7-1}{3}(0.1)^3(0.6)^{7-3-1}\right]$$

$$= (0.3)\{0.046656 + 0.046656 + 0.01944 + (4.32\times10^{-3})\}$$

$$= 0.0351216,$$

$$I_7 = (0.3)\sum_{z^*=4}^{7-1=5}\frac{(7-1)!}{z^*!}(0.1)^{z^*}\sum_{h=0}^{7-z^*-1}\frac{(0.6)^{7-z^*-h-1}}{h!(7-z^*-h-1)!}$$

$$= (0.3)\left[\left\{\frac{(7-1)!}{4!}(0.1)^4\sum_{h=0}^{7-4-1=2}\frac{(0.6)^{7-4-h-1}}{h!(7-4-h-1)!}\right\} + \left\{\frac{(7-1)!}{5!}(0.1)^5\sum_{h=0}^{7-5-1=1}\frac{(0.6)^{7-5-h-1}}{h!(7-5-h-1)!}\right\} + \left\{\frac{(7-1)!}{6!}(0.1)^6\sum_{h=0}^{7-6-1=0}\frac{(0.6)^{7-6-h-1}}{h!(7-6-h-1)!}\right\}\right]$$

$$= (0.3)\left[(30\times 0.1^4)\left\{\frac{(0.6)^{3-0-1}}{0!(3-0-1)!}+\frac{(0.6)^{3-1-1}}{1!(3-1-1)!}+\frac{(0.6)^{3-2-1}}{2!(3-2-1)!}\right\}\right.$$

$$+ (6\times 0.1^5)\left\{\frac{(0.6)^{2-0-1}}{0!(2-0-1)!}+\frac{(0.6)^{2-1-1}}{1!(2-1-1)!}\right\}$$

$$\left.+ (0.1^6)\left\{\frac{(0.6)^{1-0-1}}{1!(1-0-1)!}\right\}\right]$$

$$= (0.3)(3.84\times 10^{-3} + 9.6\times 10^{-5} + 0.1^6)$$

$$= 1.1811\times 10^{-3},$$

$$U_7 = (p_r(S) + p_r(I) + p_r(U))^x - T_7 - I_7$$

$$= (0.3 + 0.1 + 0.6)^7 - 0.0351216 - 1.1811\times 10^{-3}$$

$$= 0.96530.$$

Hence we get

$$(T_7,\ I_7,\ U_7) = (0.0351216,\ 0.96530,\ 1.1811\times 10^{-3}).$$

3. Main Properties of NGD

3.1. Moment generating function

Moment generating function (m.g.f) for the true part of *PMF* $X \sim NGD(x;\ p_r(S))$ is given by

$$M_0(t) = \left(1 - \frac{p_r(I)}{p_r(U)}\right)^{-1} \cdot \frac{p_r(S)}{(e^{-t} - p_r(U))}. \tag{3.1}$$

The indeterminate part of PMF is given as

$$M_0(t) = e^t \cdot p_r(S) \cdot B^* \cdot \left(\sum_{n=1}^{\infty} -\frac{(p_r(I))^n}{n}\right) \cdot \{1 - p_r(U) \cdot e^t\}^{-1}, \tag{3.2}$$

where

$$B^* = -\frac{1}{p_r(U)} + \frac{1}{2!(p_r(U))^3} - \frac{1}{2!3!(p_r(U))^5} + \cdots.$$

3.2. Characteristic function

The characteristic function of $X \sim NGD(x;\ p_r(S))$ the true part of PMF is given by

$$\phi_i(t) = \frac{p_r(S)}{p_r(U)} \cdot \left(1 - \frac{p_r(I)}{p_r(U)}\right)^{-1} \cdot (1 - p_r(U) \cdot e^{it})^{-1}. \tag{3.3}$$

The indeterminate part of PMF is given as

$$\phi_i(t) = p_r(S) \cdot B^* \cdot \left\{\sum_{n=1}^{\infty} -\frac{(p_r(I))^n}{n}\right\} \cdot \{1 - p_r(U) \cdot e^{it}\}^{-1}. \tag{3.4}$$

3.3. Probability generating function

The probability generating function for the true part of PMF $X \sim NGD(x;\ p_r(S))$ is given by

$$G(\theta) = \theta p_r(S)\left(1 - \frac{P(I)}{p_r(U)}\right)^{-1} \cdot (1 - \theta p_r(U))^{-1}. \tag{3.5}$$

The indeterminate part of PMF is given as

$$G(\theta) = \theta p_r(S)\left\{\sum_{n=1}^{\infty} -\frac{(p_r(I))^n}{n}\right\}\{1 - (p_r(U)\theta)\}^{-1}. \tag{3.6}$$

4. Reliability Analysis

This section finds various reliability properties like survival function, hazard rate function, reversed hazard rate function, and cumulative hazard rate function. In addition, the Mills ratio and the odd ratio for the new proposed distribution are derived.

4.1. Survival function

The survival function of r.v. $X \sim NGD(x;\ p_r(S))$ for the true part of PMF is given as:

$$S(x) = p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i. \tag{4.1}$$

The indeterminate part of PMF is given as:

$$S(x) = p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}. \tag{4.2}$$

4.2. Hazard rate or failure rate function

The hazard rate function of r.v. $X \sim NGD(x;\ p_r(S))$ for the true part of PMF is given as:

$$h(x) = \frac{\sum_{h=0}^{th^*}\binom{x-1}{h}p_r(I)^h\,p_r(U)^{x-h-1}}{(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i}. \tag{4.3}$$

The indeterminate part of PMF is given as:

$$h(x) = \frac{\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*} \times \sum_{x=1}^{\infty}\sum_{h=0}^{x-z^*-1}\binom{x-z^*-1}{h}(p_r(U))^{x-z^*-h-1}}{\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}}. \tag{4.4}$$

4.3. Reversed hazard rate function

The reversed hazard rate function (RHRF) of r.v. $X \sim NGD(x;\ p_r(S))$ for the true part of PMF is given as:

$$\partial(x) = \frac{p_r(S)\sum_{h=0}^{th^*}\binom{x-1}{h} p_r(I)^h\, p_r(U)^{x-h-1}}{1-\, p_r(S)(p_r(U))^x(1-\, p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i}. \quad (4.5)$$

For the indeterminate part of PMF, RHRF is given as:

$$\partial(x) = \frac{\begin{matrix} p_r(S)\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*} \\ \times\sum_{x=1}^{\infty}\sum_{h=0}^{x-z^*-1}\binom{x-z^*-1}{h}(p_r(U))^{x-z^*-h-1} \end{matrix}}{1-\, p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}}.$$

(4.6)

4.4. Cumulative hazard rate function

The cumulative hazard rate function (CHRF) of r.v. $X \sim NGD(x;\, p_r(S))$ for true part of PMF is given as:

$$H(x) = -\ln\left\{p_r(S)(p_r(U))^x(1-\, p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(F)}\right)^i\right\}. \quad (4.7)$$

For the indeterminate part of PMF, CHRF is given as:

$$H(x) = -\ln\left\{p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}\right\}.$$

(4.8)

4.5. Mills ratio

The Mills ratio of r.v. $X \sim NGD(x;\, p_r(S))$ for the true part of PMF is given as:

$$m(x)=\frac{\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}}{\sum_{h=0}^{th^*}\binom{x-1}{h}p_r(I)^h\,p_r(U)^{x-h-1}}. \quad (4.9)$$

For the indeterminate part of PMF, the Mills ratio is given as:

$$m(x)=\frac{\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}}{\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*}\times\sum_{x=1}^{\infty}\sum_{h=0}^{x-z^*-1}\binom{x-z^*-1}{h}(p_r(U))^{x-z^*-h-1}}. \quad (4.10)$$

4.6. Odds ratio

An odds ratio of r.v. $X \sim NGD(x;\ p_r(S))$ for the true part of PMF is given as:

$$\varphi(x)=\frac{1-p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{P(F)}\right)^i}{p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i}. \quad (4.11)$$

For an indeterminate part of PMF, the odds ratio is given as:

$$\varphi(x)=\frac{1-p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}}{p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}}. \quad (4.12)$$

5. Order Statistics

In this section, we derived the order statistics for the new proposed distribution NGD, such as *w*th order statistics, joint, largest, and smallest

order statistics, maximum and minimum, median order statistics, and smallest and largest order statistics.

5.1. *w*th order statistics

Let $X_1, X_2, ..., X_w$ be the random sample from NGD, and let $X_{(1)}, X_{(2)}, ..., X_{(w)}$ be the corresponding order statistics. The proposed *w*th order statistic for the true part of the NGD is:

$$f_{w:n}(x) = \frac{n!}{(w-1)!(n-w)!}(p_r(S))^{n-w+1}\sum_{h=0}^{th^*}\binom{x-1}{h}(p_r(I))^h(p_r(U))^{x-h-1}$$

$$\times\left[1-p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i\right]^{w-1}$$

$$\times\left[(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i\right]^{n-w}. \qquad (5.1)$$

*w*th order statistics for the indeterminate part of NGD can be given as:

$$f_{w:n}(x) = \frac{n!}{(w-1)!(n-w)!}(p_r(S))^{n-w+1}$$

$$\times\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*}\sum_{x=1}^{\infty}\sum_{h=0}^{x-z^*-1}\binom{x-z^*-1}{h}(p_r(U))^{x-z^*-h-1}$$

$$\times\left[-p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}\right]^{w-1}$$

$$\times\left[\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}\right]^{n-w}. \qquad (5.2)$$

5.2. Joint order statistics

Joint order statistics of $y_{1:b}$ and $y_{u:b}$ for true part of NGD is derived as follows:

$$
\begin{aligned}
& f_{a:u:b}(y) \\
&= \frac{b!}{(a-1)!(u-a-1)!(b-u)!}(p_r(S))^{n-a+1}(1-p_r(U))^{m-n+1} \\
&\times \left\{ \sum_{h=0}^{th^*} \binom{y-1}{h} (p_r(I))^h (p_r(U))^{y-h-1} \right\} \\
&\times \left\{ \sum_{h=0}^{th^*} \binom{z-1}{h} (p_r(I))^h (p_r(U))^{z-h-1} \right\} \\
&\times \left[1-p_r(S)(p_r(U))^x (1-p_r(U))^{-1} \sum_{i=0}^{\infty} \binom{x+i}{i} \left(\frac{p_r(I)}{p_r(U)} \right)^i \right]^{m-1} \\
&\times \left[(p_r(U))^y \sum_{i=0}^{\infty} \binom{y+i}{i} \left(\frac{p_r(I)}{p_r(U)} \right)^i - (p_r(U))^z \sum_{i=0}^{\infty} \binom{z+i}{i} \left(\frac{p_r(I)}{p_r(U)} \right)^i \right]^{u-a-1} \\
&\times \left[(p_r(U))^y \sum_{i=0}^{\infty} \binom{y+i}{i} \left(\frac{p_r(I)}{p_r(U)} \right)^i \right]^{b-u} . \qquad (5.3)
\end{aligned}
$$

Joint order statistic for the indeterminate part of NGD is given as follows:

$$
\begin{aligned}
& f_{a:u:b}(x) \\
&= \frac{b!}{(a-1)!(u-a-1)!(b-u)!}(p_r(S))^{s'(b-a+1)}
\end{aligned}
$$

$$\times\left[\sum_{z^*=th+1}^{y-s'}\binom{y-s'}{z^*}(p_r(I))^{z^*}\sum_{x=s'}^{\infty}\sum_{t=0}^{y-s'-z^*}\binom{y-s'-z^*}{t}(p_r(U))^{y-s'-z^*-t}\right]$$

$$\times\left[\sum_{z^*=th^*+1}^{z-s'}\binom{z-s'}{z^*}(p_r(I))^{z^*}\sum_{z=s'}^{\infty}\sum_{t=0}^{z-s'-z^*}\binom{z-s'-z^*}{t}(p_r(U))^{z-s'-z^*-t}\right]$$

$$\times\left[\sum_{m'=1}^{\infty}\binom{y-s'+m'}{m'}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\binom{y-s'}{m'}(p_r(U))^{y-s'-m'}\right.$$

$$\left.-\sum_{m'=1}^{\infty}\binom{z-s'+m'}{m'}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\binom{z-s'}{m'}(p_r(U))^{x-s'-m'}\right]^{u-a-1}$$

$$\times\left[\sum_{m'=1}^{\infty}\binom{y-s'+m'}{m'}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\binom{y-s'}{m'}(p_r(U))^{y-s'-m'}\right]^{b-u}. \quad (5.4)$$

5.3. Largest order statistics

For $u=\eta'$, the largest order statistic for the true part of NGD is given:

$$f_{\eta':\eta'}(x)=\eta' p_r(S)\sum_{h=0}^{th^*}\binom{x-1}{h}(p_r(I))^h(p_r(U))^{x-h-1}$$

$$\times\left[1-p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i\right]^{\eta'-1}. \quad (5.5)$$

Largest order statistic for an indeterminate part of NGD is given as:

$$f_{\eta':\eta'}(x)$$

$$=\eta' p_r(S)\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*}$$

$$\times \sum_{x=1}^{\infty} \sum_{h=0}^{x-z^*-1} \binom{x-z^*-1}{h} (p_r(U))^{x-z^*-h-1}$$

$$\times \left[1 - p_r(S) \sum_{m'=0}^{\infty} \binom{x+m'}{m'+1} (p_r(I))^{m'+1} \sum_{m'=0}^{\infty} \binom{x-1}{m'} (p_r(U))^{x-m'-1} \right]^{\eta'-1}.$$

(5.6)

5.4. Smallest order statistics

For $u = 1$, smallest order statistics for the true part of NGD is given as:

$$f_{1:n}(x) = n(p_r(S))^n \sum_{h=0}^{th^*} \binom{x-1}{h} (p_r(I))^h (p_r(U))^{x-h-1}$$

$$\times \left[(p_r(U))^x (1 - p_r(U))^{-1} \sum_{i=0}^{\infty} \binom{x+i}{i} \left(\frac{p_r(I)}{p_r(U)} \right)^i \right]^{n-1}. \quad (5.7)$$

Smallest order statistics for the indeterminate part of NGD is given as:

$$f_{1:n}(x) = n(p_r(S))^n \sum_{z^*=th^*+1}^{x-1} \binom{x-1}{z^*} (p_r(I))^{z^*}$$

$$\times \sum_{x=1}^{\infty} \sum_{h=0}^{x-z^*-1} \binom{x-z^*-1}{h} (p_r(U))^{x-z^*-h-1}$$

$$\times \left[\sum_{m'=0}^{\infty} \binom{x+m'}{m'+1} (p_r(I))^{m'+1} \sum_{m'=0}^{\infty} \binom{x-1}{m'} (p_r(U))^{x-m'-1} \right]^{n-1}. \quad (5.8)$$

5.5. Median order statistics

For $u = m + 1$, median order statistics for the true part of NGD is given as:

$$f_{m+1,n}(x) = \frac{(2m+1)!}{m!n!}(p_r(S))^{m+1}$$

$$\times\left[1 - p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{m'=0}^{\infty}\binom{x+m'}{m'}\left(\frac{p_r(I)}{p_r(U)}\right)^{m'}\right]^m$$

$$\times\left[(p_r(U))^x(1-p_r(U))^{-1}\sum_{m'=0}^{\infty}\binom{x+m'}{m'}\left(\frac{p_r(I)}{p_r(U)}\right)^{m'}\right]^m$$

$$\times\sum_{h=0}^{th^*}\binom{x-1}{h}(p_r(I))^h(p_r(U))^{x-h-1}. \tag{5.9}$$

The median order statistic for the indeterminate part of NGD is given as:

$$f_{m+1,n}(x)$$

$$= \frac{(2m+1)!}{m!n!}(p_r(S))^{m+1}$$

$$\times\left[1 - p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(P(I))^{j+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}\right]^m$$

$$\times\left[\sum_{j=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}\right]^m$$

$$\times\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*}\sum_{x=1}^{\infty}\sum_{h=0}^{x-z^*-1}\binom{x-z^*-1}{h}(p_r(U))^{x-z^*-h-1}.$$

(5.10)

5.6. Minimum and maximum joint order statistics

Minimum and maximum joint order statistics for the true part of NGD are given as follows:

$$
\begin{aligned}
& f_{1:a:a}(y) \\
&= a(a-1)(p_r(S))^a (1-p_r(U))^{-1} \\
&\times \left\{ \sum_{h=0}^{th^*} \binom{y-1}{h} (p_r(I))^h (p_r(U))^{y-h-1} \right\} \\
&\times \left\{ \sum_{h=0}^{th^*} \binom{z-1}{h} (p_r(I))^h (p_r(U))^{z-h-1} \right\} \\
&\times \left[(p_r(U))^y \sum_{i=0}^{\infty} \binom{y+i}{i} \left(\frac{p_r(I)}{p_r(U)} \right)^i - (p_r(U))^z \sum_{i=0}^{\infty} \binom{z+i}{i} \left(\frac{p_r(I)}{p_r(U)} \right)^i \right]^{a-2}.
\end{aligned}
\tag{5.11}
$$

Minimum and maximum joint order statistics for the indeterminate part of NGD are given as:

$$
\begin{aligned}
& f_{1:a:a}(y) \\
&= a(a-1)(p_r(S))^n \\
&\times \left\{ \sum_{z^*=th^*+1}^{y-1} \binom{y-1}{z^*} (p_r(I))^{z^*} \sum_{x=1}^{\infty} \sum_{h=0}^{y-z^*-1} \binom{y-z^*-1}{h} (p_r(U))^{y-z^*-h-1} \right\} \\
&\times \left\{ \sum_{z^*=th^*+1}^{z-1} \binom{z-1}{z^*} (p_r(I))^{z^*} \sum_{x=1}^{\infty} \sum_{h=0}^{z-z^*-1} \binom{z-z^*-1}{h} (p_r(U))^{z-z^*-h-1} \right\} \\
&\times \left[\sum_{m'=0}^{\infty} \binom{y+m'}{m'+1} (p_r(I))^{m'+1} \sum_{m'=0}^{\infty} \binom{y-1}{m'} (p_r(U))^{y-m'-1} \right. \\
&\left. - \sum_{m'=0}^{\infty} \binom{z+m'}{m'+1} (p_r(I))^{m'+1} \sum_{m'=0}^{\infty} \binom{z-1}{m'} (p_r(U))^{z-m'-1} \right]^{a-2}.
\end{aligned}
\tag{5.12}
$$

6. Conclusion

This paper proposed one parameter discrete probability distribution named neutrosophic geometric distribution (NGD). One of the reasons for generalizing a classical geometric distribution to a neutrosophic geometric distribution is the property of dealing with the uncertain situation that classical geometric distribution fails to deal with, e.g., consider a coin toss problem where tossing a coin on an irregular surface in which each trial may result in success or failure. There may be an outcome we cannot examine whether its success or failure (almost pointing towards both); such outcome will be considered indeterminacy, which classical statistics exclude\ignored during the experiment. The proposed modified form of the geometric distribution is explained with the help of some case studies. Moreover, several distributional properties and characteristics were explored for the newly developed neutrosophic geometric distribution.

References

[1] F. Smarandache et al., New challenges in neutrosophic theory and applications, Mathematics (2020), 1-350. https://digitalrepository.unm.edu/math_fsp/453.

[2] L. A. Zadeh, Fuzzy logic - a personal perspective, Fuzzy Sets and Systems 281 (2015), 4-20.

[3] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1) (1986), 87-96.

[4] M. Aslam et al., Monitoring mortality caused by Covid-19 using gamma-distributed variables based on generalized multiple dependent state sampling, Computational and Mathematical Methods in Medicine 2021 (2021), Article ID 6634887. https://doi.org/10.1155/2021/6634887.

[5] R. A. K. Sherwani et al., Neutrosophic beta distribution with properties and applications, Neutrosophic Sets and Systems 41 (2021), 209-214.

[6] F. Smarandache, Introduction to Neutrosophic Statistics, Sytech Publishing House, Craiava, 2014.

[7] R. Alhabib et al., Some neutrosophic probability distributions, Neutrosophic Sets and Systems 22 (2018), 30-38.

[8] K. F. H. Alhasan and F. Smarandache, Neutrosophic Weibull distribution and neutrosophic family Weibull distribution. Neutrosophic Sets and Systems 28(1) (2019), 191-199.

[9] M. A. Aslam, Neutrosophic Rayleigh distribution with some basic properties and application, Neutrosophic Sets in Decision Analysis and Operations Research, IGI Global, 2020, pp. 119-128.

[10] S. Patro and F. Smarandache, The neutrosophic statistical distribution, more problems, Neutrosophic Sets and Systems 12 (2016), 73-79.

[11] M. Albassam, N. Khan and M. Aslam, Neutrosophic D'Agostino test of normality: an application to water data, Journal of Mathematics 2021 (2021), 1-15.

[12] J. C. S. Morán, J. F. E. Chuga and W. M. Arias, Neutrosophic statistics applied to the analysis of socially responsible participation in the community, Neutrosophic Sets and Systems 26 (2019), 18-28.

[13] R. A. K. Sherwani et al., Neutrosophic entropy measures for the normal distribution: theory and applications, Pakistan J. Statist. 37(3) (2021), 303-313.

[14] F. Smarandache et al., Introduction to neutrosophy and neutrosophic environment, Neutrosophic Set in Medical Image Analysis, Academic Press, 2019, pp. 3-29.

[15] M. B. Zeina and A. Hatip, Neutrosophic random variables, Neutrosophic Sets and Systems 39(1) (2021), 43-52.

[16] B. Milošević, M. D. Jiménez-Gamero and M. V. Alba-Fernández, Quantifying the ratio-plot for the geometric distribution, J. Stat. Comput. Simul. 91(11) (2021), 2153-2173.

Single-Valued Neutrosophic Ideal Approximation Spaces

Yaser Saber[1,2,*], Mohamed Abusalih[1], Esam Bader[1], Tawfik Elmasry[1,3], Abdelaziz Babiker[1], Florentin Smarandache[4]

[1]*Department of Business Administration, College of Science and Human Studies, Hotat Sudair, Majmaah University, Majmaah, 11952, Saudi Arabia*

[2]*Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut, 71524, Egypt*

[3]*Department of Operations Research, Faculty of Graduate Studies Statistical Research, Cairo University, Egypt*

[4]*Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA*

Corresponding author: y.sber@mu.edu.sa and m.abusalih@mu.edu.sa

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Abstract. In this paper, we defined the basic idea of the single-valued neutrosophic upper $(\alpha_n)^{\delta}$, single-valued neutrosophic lower $(\alpha_n)_{\delta}$ and single-valued neutrosophic boundary sets $(\alpha_n)^{B}$ of a rough single-valued neutrosophic set α_n in a single-valued neutrosophic approximation space $(\tilde{\mathcal{F}}, \delta)$. Based on α_n and δ, we introduced the single-valued neutrosophic ideal approximation interior operator $\text{int}^{\delta}_{\alpha_n}$ and the single-valued neutrosophic ideal approximation closure operator $\text{Cl}^{\delta}_{\alpha_n}$. We joined the single-valued neutrosophic ideal notion with the single-valued neutrosophic approximation spaces and then introduced the single-valued neutrosophic ideal approximation closure and interior operators associated with a rough single-valued neutrosophic set α_n. single-valued neutrosophic ideal approximation connectedness and the single-valued neutrosophic ideal approximation continuity between single-valued neutrosophic ideal approximation spaces are introduced. The concepts of single-valued neutrosophic groups and their approximations have also been applied in the development of fuzzy systems, enhancing their ability to model and reason using uncertain and imprecise information.

1. Introduction

Sometimes, it is not convenient to apply practical problems to real-life applications. Data in medical sciences, economics, weather, climate changes, etc always involve various types of uncertainties. Moreover, fuzzy systems have been extensively studied and applied in various

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domains due to their ability to handle uncertainty and imprecision. Fuzzy sets and fuzzy logic, provide a flexible framework for modeling and reasoning with vague and uncertain information. Fuzzy systems have found applications in areas such as control systems, decision support systems, and pattern recognition. The integration of neutrosophic set theory with fuzzy systems offers an intriguing avenue for further exploration. By incorporating the three membership functions of truth, indeterminacy, and falsity, neutrosophic sets can provide an enhanced representation of uncertainty. The combination of neutrosophic sets and fuzzy systems has the potential to improve the modeling and decision-making capabilities by capturing and managing a wider range of uncertainties that arise in real-world applications.

The notion of rough sets was given by Pawlak [1] referring to the uncertainty of intelligent systems characterized by insufficient and incomplete information. Rough sets are defined depending on some equivalence relation δ on a universal finite set $\tilde{\mathcal{F}}$. The pair $(\tilde{\mathcal{F}}, \delta)$ was called an approximation space based on an equivalence relation imposed on $\tilde{\mathcal{F}}$. In any approximation space, the notions of lower approximation, upper approximation, and boundary region operators of some subset could be induced. Many types of generalizations of Pawlak's rough set have been obtained by replacing equivalence relation with an arbitrary binary relation. On the other hand, the relationships between rough sets and topological spaces were studied in [2]. A lot of fuzzy generalizations of rough approximation have been proposed in the literature [3–7]. Irfan in [8] studied the connections between fuzzy set, rough set, and soft set ([9]) notions. Many papers studied the relationship between fuzzy rough set notions and fuzzy topologies [10, 11]. Recently, many researchers have used topological approaches in the study of rough sets and their applications. In [12], it was used the notion of ideal in soft rough ordinary topological space, and in [13], the authors introduced fuzzy soft connectedness in the sense of Chang [14].

To exceed the difficulties in using the traditional classical methods the word neutrosophy is understood to be a tool for handling problems involving incomplete, indeterminate and inconsistent information and the theory was initiated by Smarandache [15] as a generalization of fuzzy sets and intuitionistic fuzzy sets. He defined the neutrosophic sets to be characterized by three membership functions independently: truth, indeterminacy and falsity. For this reason, the neutrosophic set theory becomes an attractive field for scientists and researchers who like to develop their concerns and match up their works in this scope, such as Wang et al [16] who formed the single-valued neutrosophic sets, Yang et al., and Qiu at al., [17, 18] who proposed the single-valued neutrosophic relations and single-Valued neutrosophic rough sets, and also Saber et al. [19–30] who familiarized the concepts of single-valued neutrosophic ideal open local function and single-valued neutrosophic topological space.

The incentive of this paper is to present a new better single-valued neutrosophic lower and single-valued neutrosophic upper sets through which we get a more consistent single-valued neutrosophic boundary region set off a single-valued neutrosophic set α_n. From these single-valued neutrosophic lower and single-valued neutrosophic upper sets, we introduced concepts of new

single-valued neutrosophic interior and single-valued neutrosophic closure operators related to a specific single-valued neutrosophic set $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$. and that single-valued neutrosophic relation δ on $\tilde{\mathcal{F}}$. In the single-valued neutrosophic approximation space $(\tilde{\mathcal{F}}, \delta)$, based on this single-valued neutrosophic interior and single-valued neutrosophic closure operators, we defined fuzzy approximation connectedness. Defining a single-valued neutrosophic ideal on $\tilde{\mathcal{F}}$ generates a single-valued neutrosophic ideal approximation space in which a single-valued neutrosophic local function was defined and many results are proved. Connectedness in single-valued neutrosophic ideal approximation spaces $(\tilde{\mathcal{F}}, \delta, \hbar)$ is defined and related with connectedness in single-valued neutrosophic approximation spaces. Also, single-valued neutrosophic ideal approximation continuity among two single-valued neutrosophic ideal approximation spaces were discussed.

In this study, $\tilde{\mathcal{F}}$ denotes to an initial universe, $\xi^{\tilde{\mathcal{F}}}$ is the collection of all single valued neutrosophic sets (simply, $\mathcal{SVNS}$) on $\tilde{\mathcal{F}}$ (where, $\xi = [0,1], \xi_0 = (0,1]$ and $\xi_1 = [0,1)$)

2. Preliminaries

Definition 2.1. *Let $\tilde{\mathcal{F}}$ be a finite set, with a generic element in $\tilde{\mathcal{F}}$ indicated by v. A $\mathcal{SVNS}$ [4] is defined as:*

$$\alpha_n = \{\langle v, \tilde{\varrho}_{\alpha_n}(v), \tilde{\sigma}_{\sigma_n}(v), \tilde{\varsigma}_{\alpha_n}(v)\rangle : v \in \tilde{\mathcal{F}}\}$$

where $\tilde{\varrho}_{\alpha_n} : \tilde{\mathcal{F}} \to \xi$ ($\tilde{\varrho}_{\alpha_n}$ called membership function), $\tilde{\sigma}_{\alpha_n} : \tilde{\mathcal{F}} \to \xi$ ($\tilde{\sigma}_{\alpha_n}$ called nonmembership function) and $\tilde{\varsigma}_{\alpha_n} : \tilde{\mathcal{F}} \to \xi$ ($\tilde{\varsigma}_{\alpha_n}$ called nonmembership function) of v to α_n with

$$0 \leq \tilde{\varrho}_{\alpha_n}(v) + \tilde{\sigma}_{\alpha_n}(v) + \tilde{\varsigma}_{\alpha_n}(v) \leq 3$$

All characterizations and concepts of $\mathcal{SVNS}$ originate in found [19] and [17].

Definition 2.2. *For any $\omega \in \tilde{\mathcal{F}}$, define a single valued neutrosophic coset (sample, $\mathcal{SVN}-$coset) $[\omega] : \tilde{\mathcal{F}} \to [0,1]$ as:*

$$\tilde{\sigma}_{[\omega]}(v) = \tilde{\sigma}_\delta(\omega, v), \quad \tilde{\varsigma}_{[\omega]}(v) = \tilde{\zeta}_\delta(\omega, v) \quad \tilde{\varrho}_{[\omega]}(v) = \tilde{\sigma}_\delta(\omega, v), \quad \forall v \in \tilde{\mathcal{F}}, \tag{2.1}$$

All elements $v \in \tilde{\mathcal{F}}$ with $\mathcal{SVNS}$ value $\tilde{\sigma}_\delta(\omega, v) \leq 1, \tilde{s}_\delta(\omega, v) \leq 1, \tilde{\varrho}_\delta(\omega, v) > 0$ are elements having a membership value in the $\mathcal{SVN}-$coset $[\omega]$, and any element $v \in \tilde{\mathcal{F}}$ with $\tilde{\sigma}_\delta(\omega, \omega) = 0, \tilde{\varsigma}_\delta(\omega, \omega) = 0, \tilde{\varrho}_\delta(\omega, \omega) = 1$ is not included in the $\mathcal{SVN}-$coset $[\omega]$. Any $\mathcal{SVN}-$coset $[\omega]$ surely include the element $\omega \in \tilde{\mathcal{F}}$, and consequently

$$\tilde{\sigma}_{\wedge_{\mu\in\tilde{\mathcal{F}}}}([\omega](\mu)) = 0, \quad \tilde{\varsigma}_{\wedge_{\mu\in\tilde{\mathcal{F}}}}([\omega](\mu)) = 0, \quad \forall \omega \in \tilde{\mathcal{F}}, \quad \tilde{\varrho}_{\vee_{\mu\in\tilde{\mathcal{F}}}}([\omega](\mu)) = 1.$$

$$\tilde{\sigma}_{\wedge_{\mu\in\tilde{\mathcal{F}}}}([\omega](v)) = 0, \quad \tilde{\varsigma}_{\wedge_{\mu\in\tilde{\mathcal{F}}}}([\omega](v)) = 0, \quad \forall v \in \tilde{\mathcal{F}}, \quad \tilde{\varrho}_{\vee_{\mu\in\tilde{\mathcal{F}}}}([\omega](v)) = 1,$$

such that $\bigvee_{v\in\tilde{\mathcal{F}}}([v]) = \langle 0,1,1\rangle$.

Definition 2.3. *Let us define the single-valued neutrosophic difference between two `SVNSs` as next:*

$$\tilde{\varrho}_{\alpha_n} \overline{\wedge} \tilde{\varrho}_{\varepsilon_n}(\omega) = \begin{cases} 0, & \text{if} \quad \tilde{\varrho}_{\alpha_n}(\omega) \leq \tilde{\varrho}_{\varepsilon_n}(\omega), \\ \tilde{\varrho}_{\alpha_n} \wedge \tilde{\varrho}_{(\varepsilon_n)^c}(\omega), & \text{otherwise.} \end{cases}$$

$$\tilde{\sigma}_{\alpha_n}\overline{\vee}\tilde{\sigma}_{\varepsilon_n}(\omega)=\begin{cases}1, & \text{if} \quad \tilde{\sigma}_{\alpha_n}(\omega)\geq\tilde{\sigma}_{\varepsilon_n}(\omega),\\ \tilde{\sigma}_{\alpha_n}\vee\tilde{\sigma}_{(\varepsilon_n)^c}(\omega), & \text{otherwise.}\end{cases} \tag{2.2}$$

$$\tilde{\xi}_{\alpha_n}\overline{\vee}\tilde{\xi}_{\varepsilon_n}(\omega)=\begin{cases}1, & \text{if} \quad \tilde{\xi}_{\alpha_n}(\omega)\geq\tilde{\xi}_{\varepsilon_n}(\omega),\\ \tilde{\xi}_{\alpha_n}\vee\tilde{\xi}_{(\varepsilon_n)^c}(\omega), & \text{otherwise.}\end{cases}$$

3. Single-Valued Neutrosophic Ideal Approximation Spaces

Definition 3.1. *A nonempty collection of SVNSs $\hbar$ of a set $\tilde{\mathcal{F}}$ is called single valued neutrosophic ideal [8] (briefly, SVNI) on $\tilde{\mathcal{F}}$ if it satisfies the following:*

1. $\langle 0,1,1\rangle\in\hbar$,
2. *If $\tilde{\varrho}_{\alpha_n}(\omega)\leq\tilde{\varrho}_{\varepsilon_n}(v),\tilde{\sigma}_{\alpha_n}(\omega)\geq\tilde{\sigma}_{\varepsilon_n}(\omega),\tilde{\zeta}_{\alpha_n}(\omega)\geq\tilde{\xi}_{\varepsilon_n}(\omega)$ and $\alpha_n\in\hbar$, then, $\varepsilon_n\in\hbar,\forall\omega\in\tilde{\mathcal{F}}$ and $\alpha_n,\varepsilon_n\in\zeta^{\tilde{\mathcal{F}}}$*
3. *If $\alpha_n,\varepsilon_n\in\hbar$, then $\langle(\tilde{\varrho}_{\alpha_n}\vee\tilde{\varrho}_{\varepsilon_n})(\omega),(\tilde{\sigma}_{\alpha_n}\wedge\tilde{\sigma}_{\varepsilon_n})(\omega)(\tilde{\zeta}_{\alpha_n}\wedge\tilde{\xi}_{\varepsilon_n})(\omega)\rangle\in\hbar$, $\forall\ \omega\in\tilde{\mathcal{F}}$ and $\alpha_n,\varepsilon_n\in\zeta^{\tilde{\mathcal{F}}}$.*

If $\hbar_1$ and $\hbar_2$ are SVNIs on $\tilde{\mathcal{F}}$, we obtain $\hbar_1$ is finer than $\hbar_1$ [$\hbar_2$ is coarser than $\hbar_1$] if $\hbar_1\supseteq\hbar_2$. The triple $(\tilde{\mathcal{F}},\delta,\hbar)$ is said to be a single-valued neutrosophic ideal approximation space (briefly, SVNIAS). Denote the trivial SVNI $\hbar^{\circ}$ as a SVNI including only $\langle 0,1,1\rangle$.

Definition 3.2. *Let $(\tilde{\mathcal{F}},\delta,\hbar)$ be a SVNIAS and $\alpha_n\in\zeta^{\tilde{\mathcal{F}}}$. Then, the single valued neutrosophic local set $[\varepsilon_n]^{\star}_{\alpha_n}(\delta,\hbar)$ of a set $\varepsilon_n\in\zeta^{\tilde{\mathcal{F}}}$ is defined by:*

$$\begin{aligned}[\varepsilon_n]^{\star}_{\alpha_n}(\delta,\hbar)=\ &\bigwedge\{\pi_n\in\zeta^{\tilde{\mathcal{F}}}:\varepsilon_n\overline{\wedge}\pi_n=\langle\tilde{\varrho}_{\varepsilon_n}\overline{\wedge}\tilde{\varrho}_{\pi_n}(\omega),\tilde{\sigma}_{\sigma_n}\overline{\vee}\tilde{\sigma}_{\pi_n}(\omega),\\ &\tilde{\zeta}_{\varepsilon_n}\overline{\vee}\tilde{\zeta}_{\pi_n}(\omega)\rangle\in\hbar,\mathrm{Cl}^{\alpha_n}_{\delta}(\pi_n)=\pi_n\}\end{aligned}$$

Briefly, we will write $[\varepsilon_n]^{\star}_{\alpha_n}$ or $[\varepsilon_n]^{\star}_{\alpha_n}(\hbar)$ instead of $[\varepsilon_n]^{\star}_{\alpha_n}(\delta,\hbar)$.

Corollary 3.1. *Let $(\tilde{\mathcal{F}},\delta,\hbar)$ be a SVNIAS and $\alpha_n\in\zeta^{\tilde{\mathcal{F}}}$, where $\hbar^{\diamond}$ is the trivial single valued neutrosophic ideal on $\tilde{\mathcal{F}}$. Then, for each $\varepsilon_n\in\zeta^{\tilde{\mathcal{F}}}$, we have $[\varepsilon_n]^{\star}_{\alpha_n}=\mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n)$.*

Proof. Since, $\hbar^{\diamond}=\langle 0,1,1\rangle$, we obtain

$$[\varepsilon_n]^{\star}_{\alpha_n}(\delta,\hbar)=\bigwedge\{\pi_n\in\zeta^{\tilde{\mathcal{F}}}:\varepsilon_n\overline{\wedge}\pi_n=\langle\tilde{\varrho}_{\varepsilon_n}\overline{\wedge}\tilde{\varrho}_{\pi_n}(\omega),\tilde{\sigma}_{\sigma_n}\overline{\vee}\tilde{\sigma}_{\pi_n}(\omega),\tilde{\varsigma}_{\varepsilon_n}\overline{\vee}\tilde{\varsigma}_{\pi_n}(\omega)\rangle=\left\langle 0,1,1\right\rangle,\mathrm{Cl}^{\alpha_n}_{\delta}(\pi_n)=\pi_n\Big\}$$

this implies

$$[\varepsilon_n]^{\star}_{\alpha_n}(\delta,\hbar)=\bigwedge\{\pi_n\in\zeta^{\tilde{\mathcal{F}}}:\tilde{\varrho}_{\varepsilon_n}(\omega)\leq\tilde{\varrho}_{\pi_n}(\omega),\tilde{\sigma}_{\varepsilon_n}(\omega)\geq\tilde{\sigma}_{\pi_n}(\omega)\tilde{\zeta}_{\varepsilon_n}(\omega)\geq\tilde{\zeta}_{\pi_n}(\omega),\mathrm{Cl}^{\alpha_n}_{\delta}(\pi_n)=\pi_n\}.$$

Since $\varepsilon_n\leq\mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n),\mathrm{Cl}^{\alpha_n}_{\delta}\left(\mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n)\right)=\mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n)$, then $[\varepsilon_n]^{\star}_{\alpha_n}\leq\mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n)$. Let $\mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n)\not\leq[\varepsilon_n]^{\star}_{\alpha_n}$, then there exists $\pi_n\in\zeta^{\tilde{\mathcal{F}}},\tilde{\varrho}_{\varepsilon_n}(\omega)\leq\tilde{\varrho}_{\pi_n}(\omega),\tilde{\sigma}_{\varepsilon_n}(\omega)\geq\tilde{\sigma}_{\pi_n}(\omega),\tilde{\zeta}_{n_n}(\omega)\geq\tilde{\varsigma}_{\pi_n}(\omega),\mathrm{Cl}^{\alpha_n}_{\delta}(\pi_n)=\pi_n$, so that $\mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n)>\pi_n$ But $\tilde{\varrho}_{\varepsilon_n}(\omega)\leq\tilde{\varrho}_{\pi_n}(\omega),\tilde{\sigma}_{\varepsilon_n}(\omega)\geq\tilde{\sigma}_{\pi_n}(\omega),\tilde{\zeta}_{\varepsilon_n}(\omega)\geq\tilde{\zeta}_{\pi_n}(\omega)$ implies that $(\varepsilon_n)^{\delta}\leq(\pi_n)^{\delta}$ and then

$$\mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n)=[(\alpha_n)_{\delta}]^{c}\vee[\varepsilon_n]^{\delta}\leq[(\alpha_n)_{\delta}]^{c}\vee[\pi_n]^{\delta}=\mathrm{Cl}^{\alpha_n}_{\delta}(\pi_n)=\pi_n.$$

Contradiction, and then $[\varepsilon_n]^{\star}_{\alpha_n}=\mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n)$. □

δ	ω_1	ω_2	ω_3	ω_4	ω_5
ω_1	*(1, 0, 0)*	*(0, 1, 1)*	*(0, 1, 1)*	*(0, 1, 1)*	*(0, 1, 1)*
ω_2	*(0, 1, 1)*	*(1, 0, 0)*	*(0, 1, 1)*	*(0, 1, 1)*	*(0, 1, 1)*
ω_3	*(0, 1, 1)*	*(0, 1, 1)*	*(1, 0, 0)*	*(0, 1, 1)*	*(0, 1, 1)*
ω_4	*(0, 1, 1)*	*(0, 1, 1)*	*(0, 1, 1)*	*(1, 0, 0)*	*(0.2, 0.2, 0.2)*
ω_5	*(0, 1, 1)*	*(0, 1, 1)*	*(0, 1, 1)*	*(0.2, 0.2, 0.2)*	*(1, 0, 0)*

TABLE 1. SVNRof δ

Example 3.1. *This is an example that shows that* $[\varepsilon_n]^{\star}_{\alpha_n} \leq \mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n)$.

Let δ *be* SVNR *on a set* $\tilde{\mathcal{F}} = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ *as shown down.*

Assume that $\alpha_n = \langle (1,0,0), (1,0,0), (1,0,0), (0.1, 0.1, 0.1), (0.2, 0.2, 0.2) \rangle$. *Then,*

$$\tilde{\varrho}_{(\alpha_n)^{\delta}}(\omega_1) = \tilde{\varrho}_{\alpha_n}(\omega_1) \wedge \bigvee_{\alpha_n(\mu)>0, \mu \neq \omega_1} \tilde{\varrho}_{[\omega]}(\mu) = 0,$$

$$\tilde{\sigma}_{(\alpha_n)^{\delta}}(\omega_1) = \tilde{\sigma}_{\alpha_n}(\omega_1) \vee \bigwedge_{\alpha_n(\mu)>0, \mu \neq \omega_1} (\tilde{\sigma}_{[\omega]})(\mu) = 1,$$

$$\tilde{\xi}_{(\alpha_n)^{\delta}}(\omega_1) = \tilde{\varrho}_{\alpha_n}(\omega_1) \vee \bigwedge_{\alpha_n(\mu)>0, \mu \neq \omega_1} \tilde{\xi}_{[\omega]}(\mu) = 1,$$

Hence, $(\alpha_n)^{\delta}(\omega_1) = (0,1,1)$. *Similarly, we can obtain* $(\alpha_n)^{\delta}(\omega_2) = (0,1,1)$ *and* $(\alpha_n)^{\delta}(\omega_3) = (0,1,1)$. *Also,*

$$\tilde{\varrho}_{(\alpha_n)^{\delta}}(\omega_4) = \tilde{\varrho}_{\alpha_n}(\omega_4) \wedge \bigvee_{\alpha_n(\mu)>0, \mu \neq \omega_4} \tilde{\varrho}_{[\omega]}(\mu) = 0.1,$$

$$\tilde{\sigma}_{(\alpha_n)^{\delta}}(\omega_4) = \tilde{\sigma}_{\alpha_n}(\omega_4) \vee \bigwedge_{\alpha_n(\mu)>0, \mu \neq \omega_4} (\tilde{\sigma}_{[\omega]})(\mu) = 0.2,$$

$$\tilde{\xi}_{(\alpha_n)^{\delta}}(\omega_4) = \tilde{\varrho}_{\alpha_n}(\omega_4) \vee \bigwedge_{\alpha_n(\mu)>0, \mu \neq \omega_4} \tilde{\xi}_{[\omega]}(\mu) = 0.2,$$

Hence, $(\alpha_n)^{\delta}(\omega_4) = (0.1, 0.2, 0.2)$, *and*

$$\tilde{\varrho}_{(\alpha_n)^{\delta}}(\omega_5) = \tilde{\varrho}_{\alpha_n}(\omega_5) \wedge \bigvee_{\alpha_n(\mu)>0, \mu \neq \omega_4} \tilde{\varrho}_{[\omega]}(\mu) = 0.2,$$

$$\tilde{\sigma}_{(\alpha_n)^{\delta}}(\omega_5) = \tilde{\sigma}_{\alpha_n}(\omega_5) \vee \bigwedge_{\alpha_n(\mu)>0, \mu \neq \omega_4} (\tilde{\sigma}_{[\omega]})(\mu) = 0.2,$$

$$\tilde{\xi}_{(\alpha_n)^{\delta}}(\omega_5) = \tilde{\varrho}_{\alpha_n}(\omega_5) \vee \bigwedge_{\alpha_n(\mu)>0, \mu \neq \omega_4} \tilde{\xi}_{[\omega]}(\mu) = 0.2,$$

Thus $(\alpha_n)^{\delta}(\omega_5) = (0.2, 0.2, 0.2)$ *and than,*

$$(\alpha_n)^{\delta} = \langle (0,1,1), (0,1,1), (0,1,1), (0.1, 0.2, 0.2), (0.2, 0.2, 0.2) \rangle,$$

$$(\alpha_n)_\delta = \langle (1,0,0),(1,0,0),(1,0,0),(0.2,0.1,0.1),(0.2,0.2,0.2)\rangle,$$

$$[(\alpha_n)_\delta]^c = \langle (0,1,1),(0,1,1),(0,1,1),(0.1,0.9,0.2),(0.2,0.8,0.2)\rangle.$$

Let, $\varepsilon_n = \langle (0.3,0.3,0.3),(0,1,1),(0,1,1),(0,0.8,0.2),(0,0.8,0.2)\rangle$*, then* $(\varepsilon_n)^\delta = \langle (0,1,1),(0,1,1),(0,1,1),(0,0.8,0.2),(0,0.8,0.2)\rangle$ *and thus*

$$\mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n) = [(\alpha_n)_\delta]^c \vee (\varepsilon_n)^\delta = \langle (0,1,1),(0,1,1),(0,1,1),(0.1,0.8,0.2),(0.2,0.8,0.2)\rangle.$$

Assume that a `SVNI` *is defined on* $\tilde{\mathcal{F}}$ *as follows*

$$\hbar = \{\pi_n \in \zeta^{\tilde{\mathcal{F}}} : \pi_n \leq \langle (0.5,0.3,0.3),(0.5,0.5,0.5),(0.5,0.5,0.5),(0.5,0.5,0.5),(0.5,0.5,0.5)\rangle\}$$

Note that every $\pi_n \in \zeta^{\tilde{\mathcal{F}}}$*, we have* $\mathrm{Cl}_\delta^{\alpha_n}(\pi_n) \geq \langle (0,1,1),(0,1,1),(0,1,1),(0.1,0.9,0.2),(0.2,0.8,0.2)\rangle$*, and recall that if* $\varepsilon_n \in \hbar$*, then* $[\varepsilon_n]^\star_{\alpha_n} = \langle 0,1,1\rangle$ *[By equations 2.1.,* $\varepsilon_n \overline{\wedge} \langle 0,1,1\rangle = \varepsilon_n \in \hbar$*,* $\mathrm{Cl}_\delta^{\alpha_n}(\langle 0,1,1\rangle) = \langle 0,1,1\rangle$*] and if* $\mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n) = \varepsilon_n$ *and* $\varepsilon_n \notin \hbar$*, then* $[\varepsilon_n]^\star_{\alpha_n} = \varepsilon_n$*.*

Now, we get that $\forall\ \pi_n = \langle \rho,(0,1,1),(0,1,1),(0.1,0.9,0.2),(0.2,0.8,0.2)\rangle$ *for all* $\rho \in I$*, we get*

$$\mathrm{Cl}_\delta^{\alpha_n}(\pi_n) = \pi_n,$$

Form equations 2.1, we have $\varepsilon_n \overline{\wedge} \pi_n \in \hbar$*, and thus*

$$[\varepsilon_n]^\star_{\alpha_n} = \langle (0,1,1),(0,1,1),(0,1,1),(0.1,0.9,0.2),(0.2,0.8,0.2)\rangle \not\geq \mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n).$$

Hence, $[\varepsilon_n]^\star_{\alpha_n} \leq \mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n)$*.*

Proposition 3.1. *Let* $(\tilde{\mathcal{F}},\delta,\hbar)$ *be a* `SVNIAS` *and* $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$*. Then,*

1. $\varepsilon_n \leq \upsilon_n$ *implies* $[\varepsilon_n]^\star_{\alpha_n} \leq [\upsilon_n]^\star_{\alpha_n}$*.*
2. *If* $\hbar_1,\hbar_2$ *are* `SVNIs` *ideals on* $\tilde{\mathcal{F}}$ *and* $\hbar_1 \subseteq \hbar_2$*, then* $[\varepsilon_n]^\star_{\alpha_n}(\hbar_1) \geq [\varepsilon_n]^\star_{\alpha_n}(\hbar_2)$*.*
3. $\mathrm{int}_\delta^{\alpha_n}\left([\varepsilon_n]^\star_{\alpha_n}\right) \leq [\varepsilon_n]^\star_{\alpha_n} = \mathrm{Cl}_\delta^{\alpha_n}\left([\varepsilon_n]^\star_{\alpha_n}\right) \leq \mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n)$*.*
4. $([\varepsilon_n]^\star_{\alpha_n})^\star_{\alpha_n} \leq \mathrm{Cl}_\delta^{\alpha_n}\left([\varepsilon_n]^\star_{\alpha_n}\right)$*.*
5. $[\varepsilon_n]^\star_{\alpha_n} \vee [\upsilon_n]^\star_{\alpha_n} \leq [\varepsilon_n \vee \upsilon_n]^\star_{\alpha_n}$ *and* $[\varepsilon_n]^\star_{\alpha_n} \wedge [\upsilon_n]^\star_{\alpha_n} \geq [\varepsilon_n \wedge \upsilon_n]^\star_{\alpha_n}$*.*

Proof. Suppose that $[\varepsilon_n]^\star_{\alpha_n} \not\leq [\upsilon_n]^\star_{\alpha_n}$, then there exists $\eta_n \in \zeta^{\tilde{\mathcal{F}}}$ with $\mathrm{Cl}_\delta^{\alpha_n}(\eta_n) = \eta_n$ and $\upsilon_n \overline{\wedge} \eta_n = \left\langle \tilde{\varrho}_{\upsilon_n} \overline{\wedge} \tilde{\varrho}_{\eta_n}(\omega), \tilde{\sigma}_{\upsilon_n} \overline{\vee} \tilde{\sigma}_{\eta_n}(\omega), \tilde{\xi}_{\upsilon_n} \overline{\vee} \tilde{\xi}_{\eta_n}(\omega)\right\rangle \in \hbar$ such that

$$\tilde{\varrho}_{[\varepsilon_n]^\star_{\alpha_n}}(\omega) > \tilde{\varrho}_{\eta_n}(\omega) \geq \tilde{\varrho}_{[\upsilon_n]^\star_{\alpha_n}}(\omega),$$

$$\tilde{\sigma}_{[\varepsilon_n]^\star_{\alpha_n}}(\omega) < \tilde{\sigma}_{\eta_n}(\omega) \leq \tilde{\sigma}_{[\upsilon_n]^\star_{\alpha_n}}(\omega), \tag{3.1}$$

$$\tilde{\xi}_{[\varepsilon_n]^\star_{\alpha_n}}(\omega) < \tilde{\xi}_{\eta_n}(\omega) \leq \tilde{\xi}_{[\upsilon_n]^\star_{\alpha_n}}(\omega).$$

Since $\varepsilon_n \leq \upsilon_n$, then $\tilde{\varrho}_{\varepsilon_n} \overline{\wedge} \tilde{\varrho}_{\eta_n}(\omega) \leq \tilde{\varrho}_{\upsilon_n} \overline{\wedge} \tilde{\varrho}_{\eta_n}(\omega)$, $\tilde{\sigma}_{\varepsilon_n} \overline{\vee} \tilde{\sigma}_{\eta_n}(\omega) \geq \tilde{\sigma}_{\upsilon_n} \overline{\vee} \tilde{\sigma}_{\eta_n}(\omega)$, $\tilde{\xi}_{\varepsilon_n} \overline{\vee} \tilde{\xi}_{\eta_n}(\omega) \geq \tilde{\xi}_{\upsilon_n} \overline{\vee} \tilde{\xi}_{\eta_n}(\omega)$ and $\left\langle \tilde{\varrho}_{\varepsilon_n} \overline{\wedge} \tilde{\varrho}_{\eta_n}(\omega), \tilde{\sigma}_{\varepsilon_n} \overline{\vee} \tilde{\sigma}_{\eta_n}(\omega), \tilde{\xi}_{\varepsilon_n} \overline{\vee} \tilde{\xi}_{\eta_n}(\omega)\right\rangle \in \hbar$ with $\mathrm{Cl}_\delta^{\alpha_n}(\eta_n) = \eta_n$. Thus,

$$\tilde{\varrho}_{[\varepsilon_n]^\star_{\alpha_n}}(\omega) \leq \tilde{\varrho}_{[\eta_n]}(\omega), \quad \tilde{\sigma}_{[\varepsilon_n]^\star_{\alpha_n}}(\omega) \geq \tilde{\sigma}_{[\eta_n]}(\omega), \quad \tilde{\xi}_{[\varepsilon_n]^\star_{\alpha_n}}(\omega) \geq \tilde{\xi}_{[\eta_n]}(\omega),$$

which is a contradiction for equation (3) and hence $[\varepsilon_n]^{\star}_{\alpha_n} \leq [\upsilon_n]^{\star}_{\alpha_n}$.
(2): Suppose that $[\varepsilon_n]^{\star}_{\alpha_n}(\hbar_1) \not\geq [\varepsilon_n]^{\star}_{\alpha_n}(\hbar_2)$, then there exists $\eta_n \in \zeta^{\tilde{\mathcal{F}}}$ with $\mathrm{Cl}^{\alpha_n}_{\delta}(\eta_n) = \eta_n$ and $\upsilon_n \overline{\wedge} \eta_n = \left\langle \tilde{\varrho}_{\upsilon_n} \overline{\wedge} \tilde{\varrho}_{\eta_n}(\omega), \tilde{\sigma}_{\upsilon_n} \overline{\vee} \tilde{\sigma}_{\eta_n}(\omega), \tilde{\xi}_{\upsilon_n} \overline{\vee} \tilde{\xi}_{\eta_n}(\omega) \right\rangle \in \hbar_1$ such that

$$\begin{aligned} \tilde{\varrho}_{[\varepsilon_n]^{\star}_{\alpha_n}(\hbar_1)}(\omega) &\leq \tilde{\varrho}_{\eta_n}(\omega) < \tilde{\varrho}_{[\varepsilon_n]^{\star}_{\alpha_n}(\hbar_2)}(\omega), \\ \tilde{\sigma}_{[\varepsilon_n]^{\star}_{\alpha_n}(\hbar_1)}(\omega) &\geq \tilde{\sigma}_{\eta_n}(\omega) > \tilde{\sigma}_{[\varepsilon_n]^{\star}_{\alpha_n}(\hbar_2)}(\omega) \\ \tilde{\xi}_{[\varepsilon_n]^{\star}_{\alpha_n}(\hbar_1)}(\omega) &\geq \tilde{\xi}_{\eta_n}(\omega) > \tilde{\xi}_{[\varepsilon_n]^{\star}_{\alpha_n}(\hbar_2)}(\omega) \end{aligned} \tag{3.2}$$

Since $\hbar_1 \subseteq \hbar_2$, then $\left\langle \tilde{\varrho}_{\varepsilon_n} \overline{\wedge} \tilde{\varrho}_{\eta_n}(\omega), \tilde{\sigma}_{\varepsilon_n} \overline{\vee} \tilde{\sigma}_{\eta_n}(\omega), \tilde{\xi}_{\varepsilon_n} \overline{\vee} \tilde{\xi}_{\eta_n}(\omega) \right\rangle \in \hbar_2$ and $\mathrm{Cl}^{\alpha_n}_{\delta}(\eta_n) = \eta_n$. Thus,

$$\tilde{\varrho}_{[\varepsilon_n]^{\star}_{\alpha_n}(\hbar_2)}(\omega) \leq \tilde{\varrho}_{[\eta_n]}(\omega), \quad \tilde{\sigma}_{[\varepsilon_n]^{\star}_{\alpha_n}(\hbar_2)}(\omega) \geq \tilde{\sigma}_{[\eta_n]}(\omega), \quad \tilde{\xi}_{[\varepsilon_n]^{\star}_{\alpha_n}(\hbar_2)}(\omega) \geq \tilde{\xi}_{[\eta_n]}(\omega),$$

it is a contradiction for equation (4). Hence $[\varepsilon_n]^{\star}_{\alpha_n}(\hbar_1) \geq [\varepsilon_n]^{\star}_{\alpha_n}(\hbar_2)$.
(3): $\mathrm{int}^{\alpha_n}_{\delta}\left([\varepsilon_n]^{\star}_{\alpha_n}\right) \leq [\varepsilon_n]^{\star}_{\alpha_n} \leq \mathrm{Cl}^{\alpha_n}_{\delta}\left([\varepsilon_n]^{\star}_{\alpha_n}\right)$ direct. Since,

$$[\varepsilon_n]^{\star}_{\alpha_n} \leq \mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n),$$

then $\mathrm{Cl}^{\alpha_n}_{\delta}\left([\varepsilon_n]^{\star}_{\alpha_n}\right) \leq \mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n)$.
(4): Since $[\varepsilon_n]^{\star}_{\alpha_n} \leq \mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n)$, then $\left[[\varepsilon_n]^{\star}_{\alpha_n}\right]^{\star}_{\alpha_n} \leq \mathrm{Cl}^{\alpha_n}_{\delta}([\varepsilon_n]^{\star}_{\alpha_n})$.
(5): From (1), we have $\varepsilon_n \leq \upsilon_n \Longrightarrow [\varepsilon_n]^{\star}_{\alpha_n} \leq [\upsilon_n]^{\star}_{\alpha_n}$, and so (5) is satisfied. □

Definition 3.3. *Let $(\tilde{\mathcal{F}}, \delta, \hbar)$ be a* SVNIAS *and $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$. Then,*

$$(\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n) = \mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n) \vee ((\alpha_n)^{\delta})^{\star}_{\alpha_n} \quad \forall \varepsilon \in \zeta^{\tilde{\mathcal{F}}}. \tag{3.3}$$

$$(\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n) = \mathrm{int}^{\alpha_n}_{\delta}(\varepsilon_n) \wedge (((\alpha_n)^{\delta})^{\star}_{\alpha_n})^{c} \quad \forall \varepsilon \in \zeta^{\tilde{\mathcal{F}}}. \tag{3.4}$$

$(\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}$ and $(\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}$ are single valued neutrosophic operators from $\zeta^{\tilde{\mathcal{F}}}$ into $\zeta^{\tilde{\mathcal{F}}}$ based on a specific SVNS *α_n and* SVNI *$\hbar$ in the* SVNAS *$(\tilde{\mathcal{F}}, \delta)$.*

Now, if $\hbar = \hbar^{\diamond}$, then

$$(\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n) = \mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n \vee \alpha_n) \geq \mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n) = \mathrm{Cl}^{\alpha_n}_{\delta}((\varepsilon_n)^{\star}_{\alpha_n}) = (\varepsilon_n)^{\star}_{\alpha_n} \text{ and}$$

$$(\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n) = \mathrm{int}^{\alpha_n}_{\delta}(\varepsilon_n \vee (\alpha_n)^{c}) \geq \mathrm{int}^{\alpha_n}_{\delta}(\varepsilon_n) = \mathrm{int}^{\alpha_n}_{\delta}((((\varepsilon_n)^{c})^{\star}_{\alpha_n})^{c}) = (((\varepsilon_n)^{c})^{\star}_{\alpha_n})^{c}.$$

Proposition 3.2. *Let $(\tilde{\mathcal{F}}, \delta, \hbar)$ be a* SVNIAS *with $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ fixed. Then, for any $\varepsilon_n, \upsilon \in \zeta^{\tilde{\mathcal{F}}}$, we have:*

1. $(\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n) \leq \mathrm{int}^{\alpha_n}_{\delta}(\varepsilon_n) \leq \varepsilon_n \leq \mathrm{Cl}^{\alpha_n}_{\delta}(\varepsilon_n) \leq (\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n)$.
2. $(\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}\left((\varepsilon_n)^{c}\right) = \left((\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n)\right)^{c}$ *and* $(\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}\left((\varepsilon_n)^{c}\right) = \left((\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n)\right)^{c}$.
3. $(\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n \vee \pi_n) = (\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n) \vee (\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\pi_n)$ *and* $(\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n \wedge \pi_n) = (\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n) \wedge (\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\pi_n)$.
4. $(\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n \wedge \pi_n) = (\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n) \wedge (\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\pi_n)$ *and* $(\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n \vee \pi_n) = (\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n) \vee (\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\pi_n)$.
5. $(\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}((\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n)) \geq (\mathrm{Cl}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n)$ *and* $(\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}((\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n)) \leq (\mathrm{int}^{\alpha_n}_{\delta})^{\star}_{\alpha_n}(\varepsilon_n)$.

6. *If* $\varepsilon_n \leq \pi_n$, *then* $(\mathrm{Cl}_\delta^{\alpha_n})^\star_{\alpha_n}(\varepsilon_n) \leq (\mathrm{Cl}_\delta^{\alpha_n})^\star_{\alpha_n}(\pi_n)$ *and* $(\mathrm{int}_\delta^{\alpha_n})^\star_{\alpha_n}(\varepsilon_n) \leq (\mathrm{int}_\delta^{\alpha_n})^\star_{\alpha_n}(\pi_n)$.

Proof. (1): From equations 3.3 and 3.4, we get the proof.

(2): From the Definition 3.3, we get that

$$\begin{aligned}\left[\left(\mathrm{int}_\delta^{\alpha_n}\right)^\star_{\alpha_n}(\varepsilon_n)\right]^c &= \left[\left(\mathrm{int}_\delta^{\alpha_n}\right)(\varepsilon_n) \wedge \left(\left((\alpha_n)^\delta\right)^\star_{\alpha_n}\right)^c\right]^c \\ &= \left[\mathrm{int}_\delta^{\alpha_n}(\varepsilon_n)\right]^c \vee \left((\alpha_n)^\delta\right)^\star_{\alpha_n} \\ &= \mathrm{Cl}_\delta^{\alpha_n}\left((\varepsilon_n)^c\right) \vee \left((\alpha_n)^\delta\right)^\star_{\alpha_n} \\ &= \left(\mathrm{Cl}_\delta^{\alpha_n}\right)^\star_{\alpha_n}((\varepsilon_n)^c).\end{aligned}$$

By the same way, it can be shown that $(\mathrm{int}_\delta^{\alpha_n})^\star_{\alpha_n}\left((\varepsilon_n)^c\right) = \left((\mathrm{Cl}_\delta^{\alpha_n})^\star_{\alpha_n}(\varepsilon_n)\right)^c$.

(3)-(5): from the Definition $(\mathrm{int}_\delta^{\alpha_n})^\star_{\alpha_n}$ and $(\mathrm{Cl}_\delta^{\alpha_n})^\star_{\alpha_n}$, we get the proof.

(6): From the Definition $(\mathrm{int}_\delta^{\alpha_n})^\star_{\alpha_n}$ and $(\mathrm{Cl}_\delta^{\alpha_n})^\star_{\alpha_n}$, we get $\varepsilon_n \leq \pi_n \Rightarrow \mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n) \leq \mathrm{Cl}_\delta^{\alpha_n}(\pi_n), \forall \varepsilon_n, \pi_n \in \zeta^{\tilde{\mathcal{F}}}$, and then $\left(\mathrm{Cl}_\delta^{\alpha_n}\right)^\star_{\alpha_n}(\varepsilon_n) \leq \left(\mathrm{Cl}_\delta^{\alpha_n}\right)^\star_{\alpha_n}(\pi_n)$. Similarly, $\left(\mathrm{int}_\delta^{\alpha_n}\right)^\star_{\alpha_n}(\varepsilon_n) \leq \left(\mathrm{int}_\delta^{\alpha_n}\right)^\star_{\alpha_n}(\pi_n)$. □

4. Connectedness in Single Valued Neutrosophic Ideal Approximation Spaces

We begin this section by defining the notion of connectedness in single valued neutrosophic ideal approximation spaces. Some of its characteristic properties are considered.

Definition 4.1. *Let* $(\tilde{\mathcal{F}}, \delta)$ *be a* `SVNAS` *and* $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$. *Then,*

1. *The* `SVNSs` $\varepsilon_n, \upsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ *are called single valued neutrosophic approximation separated (briefly, SVNA-separated) if*

$$\mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n) \wedge \upsilon_n = \varepsilon_n \wedge \mathrm{Cl}_\delta^{\alpha_n}(\upsilon_n) = \langle 0, 1, 1\rangle.$$

2. *A* `SVNS` $\pi_n \in \zeta^{\tilde{\mathcal{F}}}$ *is called single valued neutrosophic approximation disconnected set (briefly, SVNA-disconnected) if there exist SVNA-separated sets* $\varepsilon_n, \upsilon_n \in \zeta^{\tilde{\mathcal{F}}}$, *such that* $\varepsilon_n \vee \upsilon_n = \pi_n$. *A* `SVNS` π_n *is called single valued neutrosophic approximation connected (briefly,* `SVNA-connected`*) if it is not* `SVNA-disconnected`.
3. $(\tilde{\mathcal{F}}, \delta)$ *is called single valued neutrosophic approximation disconnected space (briefly, SVNA-disconnected space) if there exist SVNA-separated sets* $\varepsilon_n, \upsilon_n \in \zeta^{\tilde{\mathcal{F}}}$, *such that* $\varepsilon_n \vee \upsilon_n = \langle 1, 0, 0\rangle$. *A* `SVNAS` $(\tilde{\mathcal{F}}, \delta)$ *is called* `SVNA-connected` *if it is not* `SVNA-disconnected`

Definition 4.2. *Let* $(\tilde{\mathcal{F}}, \delta, \hbar)$ *be a* `SVNIAS` *and* $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$. *Then,*

1. *The* `SVNSs` $\varepsilon_n, \upsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ *are called single valued neutrosophic ideal approximation separated (briefly,* `SVNIA-separated`*) if*

$$\left(\mathrm{Cl}_\delta^{\alpha_n}\right)^\star_{\alpha_n}(\varepsilon_n) \wedge \upsilon_n = \varepsilon_n \wedge \left(\mathrm{Cl}_\delta^{\alpha_n}\right)^\star_{\alpha_n}(\upsilon_n) = \langle 0, 1, 1\rangle.$$

2. *A* `SVNS` $\pi_n \in \zeta^{\tilde{\mathcal{F}}}$ *is called single valued neutrosophic ideal approximation disconnected set (briefly,* `SVNIA-disconnected`*) if there exist* `SVNIA-separated` *sets* $\varepsilon_n, \upsilon_n \in \zeta^{\tilde{\mathcal{F}}}$, *such that*

$\varepsilon_n \vee \upsilon_n = \pi_n$. *A* `SVNS` π_n *is called single valued neutrosophic ideal approximation connected (briefly,* `SVNIA-connected`*) if it is not* `SVNIA-disconnected`*.*

3. $(\tilde{\mathcal{F}}, \delta, \hbar)$ *is called single valued neutrosophic ideal approximation disconnected space (briefly, SVNIA-disconnected space) if there exist* `SVNIA-separated` *sets* $\varepsilon_n, \upsilon_n \in \zeta^{\tilde{\mathcal{F}}}$, *such that* $\varepsilon_n \vee \upsilon_n = \langle 1, 0, 0\rangle$. *A* `SVNIAS` $(\tilde{\mathcal{F}}, \delta, \hbar)$ *is called* `SVNIA-connected` *if it is not* `SVNIA-disconnected`*.*

Remark 4.1. *Any two* `SVNIA-separated` *sets* $\varepsilon_n, \upsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ *are* `SVNA-separated` *as well (from that* $\mathrm{Cl}_{\delta}^{\alpha_n}(\pi_n) \leq \left(\mathrm{Cl}_{\delta}^{\alpha_n}\right)^{\star}_{\alpha_n}(\varepsilon_n)\ \forall\ \pi_n \in \zeta^{\tilde{\mathcal{F}}}$*). That is, neutrosophic ideal approximation disconnectedness implies single-valued neutrosophic approximation disconnectedness and thus, single-valued neutrosophic approximation connectedness implies single-valued neutrosophic ideal approximation connectedness.*

Example 4.1. *This is an example that shows that* $[\varepsilon_n]^{\star}_{\alpha_n} \leq \mathrm{Cl}_{\delta}^{\alpha_n}(\varepsilon_n)$.

Let δ *be* `SVNR` *on a set* $\tilde{\mathcal{F}} = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ *as shown down.*

δ	ω_1	ω_2	ω_3	ω_4	ω_5
ω_1	(1, 0, 0)	(1, 0, 0)	(0, 1, 1)	(0, 1, 1)	(0, 1, 1)
ω_2	(1, 0, 0)	(1, 0, 0)	(1, 0, 0)	(0, 1, 1)	(0, 1, 1)
ω_3	(0, 1, 1)	(1, 0, 0)	(1, 0, 0)	(0, 1, 1)	(0, 1, 1)
ω_4	(0, 1, 1)	(0, 1, 1)	(0, 1, 1)	(1, 0, 0)	(1, 0, 0)
ω_5	(0, 1, 1)	(0, 1, 1)	(0, 1, 1)	(1, 0, 0)	(1, 0, 0)

TABLE 2. SVNRof δ

Assume that $\alpha_n = \langle (0,1,1), (0,1,1), (0.2, 0.2, 0.2), (1,0,0), (1,0,0)\rangle$. *Then,*

$$\tilde{\varrho}_{(\alpha_n)^{\delta}}(\omega_1) = \tilde{\varrho}_{\alpha_n}(\omega_1) \wedge \bigvee_{\alpha_n(\mu)>0, \mu \neq \omega_1} \tilde{\varrho}_{[\omega]}(\mu) = 0,$$

$$\tilde{\sigma}_{(\alpha_n)^{\delta}}(\omega_1) = \tilde{\sigma}_{\alpha_n}(\omega_1) \vee \bigwedge_{\alpha_n(\mu)>0, \mu \neq \omega_1} (\tilde{\sigma}_{[\omega]})(\mu) = 1,$$

$$\tilde{\varsigma}_{(\alpha_n)^{\delta}}(\omega_1) = \tilde{\varrho}_{\alpha_n}(\omega_1) \vee \bigwedge_{\alpha_n(\mu)>0, \mu \neq \omega_1} \tilde{\varsigma}_{[\omega]}(\mu) = 1,$$

Hence, $(\alpha_n)^{\delta}(\omega_1) = (0,1,1)$. *Similarly, we can obtain* $(\alpha_n)^{\delta}(\omega_2) = (0,1,1)$ *and*

$$\tilde{\varrho}_{(\alpha_n)^{\delta}}(\omega_3) = \tilde{\varrho}_{\alpha_n}(\omega_3) \wedge \bigvee_{\alpha_n(\mu)>0, \mu \neq \omega_3} \tilde{\varrho}_{[\omega]}(\mu) = 0.2,$$

$$\tilde{\sigma}_{(\alpha_n)^{\delta}}(\omega_3) = \tilde{\sigma}_{\alpha_n}(\omega_3) \vee \bigwedge_{\alpha_n(\mu)>0, \mu \neq \omega_3} (\tilde{\sigma}_{[\omega]})(\mu) = 0.2,$$

$$\tilde{\varsigma}_{(\alpha_n)^{\delta}}(\omega_3) = \tilde{\varrho}_{\alpha_n}(\omega_3) \vee \bigwedge_{\alpha_n(\mu)>0, \mu \neq \omega_3} \tilde{\varsigma}_{[\omega]}(\mu) = 0.2,$$

Hence, $(\alpha_n)^{\delta}(\omega_3) = (0.2, 0.2, 0.2)$ *and*

$$\tilde{\varrho}_{(\alpha_n)^{\delta}}(\omega_4) = \tilde{\varrho}_{\alpha_n}(\omega_4) \wedge \bigvee_{\alpha_n(\mu)>0, \mu\neq\omega_4} \tilde{\varrho}_{[\omega]}(\mu) = 1,$$

$$\tilde{\sigma}_{(\alpha_n)^{\delta}}(\omega_4) = \tilde{\sigma}_{\alpha_n}(\omega_4) \vee \bigwedge_{\alpha_n(\mu)>0, \mu\neq\omega_4} (\tilde{\sigma}_{[\omega]})(\mu) = 0,$$

$$\tilde{\varsigma}_{(\alpha_n)^{\delta}}(\omega_4) = \tilde{\varrho}_{\alpha_n}(\omega_4) \vee \bigwedge_{\alpha_n(\mu)>0, \mu\neq\omega_4} \tilde{\varsigma}_{[\omega]}(\mu) = 0,$$

Hence, $(\alpha_n)^{\delta}(\omega_4) = (1, 0, 0)$*. Similarly,* $(\alpha_n)^{\delta}(\omega_5) = (1, 0, 0)$*. Thus, by equations (3) and (4) we get than*

$$(\alpha_n)^{\delta} = \langle (0,1,1), (0,1,1), (0.2, 0.2, 0.2), (1,0,0), (1,0,0) \rangle = (\alpha_n)_{\delta},$$

$$[(\alpha_n)_{\delta}]^{c} = \langle (1,0,0), (1,0,0), (0.2, 0.8, 0.2), (0,1,1), (0,1,1) \rangle.$$

Suppose that

$$\varepsilon_n = \langle (0,1,1), (0,1,1), (0,1,1), (0.6, 0.6, 0.6), (0,1,1) \rangle,$$
$$\upsilon_n = \langle (0,1,1), (0,1,1), (0,1,1), (0,1,1), (0.6, 06, 0.6) \rangle,$$

then

$$(\varepsilon_n)^{\delta} = \langle (0,1,1), (0,1,1), (0,1,1), (0.6, 0.6, 0.6), (0,1,1) \rangle,$$
$$(\upsilon_n)^{\delta} = \langle (0,1,1), (0,1,1), (0,1,1), (0,1,1), (0.6, 06, 0.6) \rangle,$$

and thus from Definitions $\mathrm{Cl}_{\delta}^{\alpha_n}, \mathrm{int}_{\delta}^{\alpha_n}$ *in [14], we get than*

$$\mathrm{Cl}_{\delta}^{\alpha_n}(\varepsilon_n) = [(\alpha_n)_{\delta}]^{c} \vee [\varepsilon_n]^{\delta} = \langle (1,0,0), (1,0,0), (0.2, 08, 0.2), (0.6, 0.6, 0.6), (0,1,1) \rangle$$

, and also, $\mathrm{Cl}_{\delta}^{\alpha_n}(\upsilon_n) = [(\alpha_n)_{\delta}]^{c} \vee [\upsilon_n]^{\delta} = \langle (1,0,0), (1,0,0), (0.2, 08, 0.2), (0,1,1), (0.6, 0.6, 0.6) \rangle$*, which means that*

$$\mathrm{Cl}_{\delta}^{\alpha_n}(\varepsilon_n) \wedge \upsilon_n = \varepsilon_n \wedge \mathrm{Cl}_{\delta}^{\alpha_n}(\upsilon_n) = \langle 0,1,1 \rangle.$$

Thus, $\varepsilon_n, \upsilon_n$ *are* `SVNA-separated` *sets, and moreover the* `SVNS`

$$(\varepsilon_n \vee \upsilon_n) = \langle (1,0,0), (1,0,0), (0,1,1), (0.6, 0.6, 0.6), (0.6, 0.6, 0.6) \rangle$$

is `SVNA-disconnected` *set.*

Now, suppose

$$\hbar = \{\eta_n \in \zeta^{\tilde{\mathcal{F}}} : \eta_n \leq \langle (0.7, 0.7, 0.7), (0.7, 0.7, 0.7), (0.7, 0.7, 0.7), (0.7, 0.7, 0.7), (0.7, 0.7, 0.7) \rangle\}.$$

Then

$$((\alpha_n)^{\delta})_{\delta}^{\star} = \langle (1,0,0), (1,0,0), (0.2, 0.2, 0.2), (0.3, 0.3, 0.3), (0.3, 0.3, 0.3) \rangle.$$

Hence,

$$\left(\mathrm{Cl}_{\delta}^{\alpha_n}\right)_{\alpha_n}^{\star}(\varepsilon_n) = \langle (1,0,0), (1,0,0), (0.2, 0.2, 0.2), (0.6, 0.3, 0.3), (0.3, 0.3, 0.3) \rangle,$$

which means that

$$(\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\varepsilon_n) \wedge \upsilon_n = \langle (0,1,1),(0,1,1),(0,1,1),(0,1,1),(0.3,06,0.6)\rangle \neq \langle 0,1,1\rangle.$$

Hence, not every `SVNA-separated` *sets are* `SVNIA-separated` *sets, and moreover, the* `SVNS` *set* $(\varepsilon_n \vee \upsilon_n)$ *will be* `SVNIA-connected` *set whenever* $\hbar \neq \zeta^{\tilde{\mathcal{F}}}$ *and* $\hbar \neq \langle 0,1,1\rangle$*, that is, whenever* $\hbar$ *is a proper* `SVNI` *on* $\tilde{\mathcal{F}}$*.*

Theorem 4.1. *Let* $(\tilde{\mathcal{F}},\delta,\hbar)$ *be a* `SVNIAS` *and* $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$*. Then, the following are equivalent.*

1. $(\tilde{\mathcal{F}},\delta,\hbar)$ *is* `SVNIA-connected`*.*
2. $\varepsilon_n \wedge \upsilon_n = \langle 0,1,1\rangle, \left(\mathrm{int}_{\delta}^{\alpha_n}\right)^{\star}_{\alpha_n}(\varepsilon_n) = \varepsilon_n, \left(\mathrm{int}_{\delta}^{\alpha_n}\right)^{\star}_{\alpha_n}(\upsilon_n) = \upsilon_n$ *and* $\varepsilon_n \vee \upsilon_n = \langle 1,0,0\rangle$ *imply* $\varepsilon_n = \langle 0,1,1\rangle$ *or* $\upsilon_n = \langle 0,1,1\rangle$*.*
3. $\varepsilon_n \wedge \upsilon_n = \langle 0,1,1\rangle, \left(\mathrm{Cl}_{\delta}^{\alpha_n}\right)^{\star}_{\alpha_n}(\varepsilon_n) = \varepsilon_n, \left(\mathrm{Cl}_{\delta}^{\alpha_n}\right)^{\star}_{\alpha_n}(\upsilon_n) = \upsilon_n$ *and* $\varepsilon_n \vee \upsilon_n = \langle 1,0,0\rangle$ *imply* $\varepsilon_n = \langle 0,1,1\rangle$ *or* $\upsilon_n = \langle 0,1,1\rangle$*.*

Proof. $(1) \Rightarrow (2)$: Let $\varepsilon_n, \upsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ with $\left(\mathrm{int}_{\delta}^{\alpha_n}\right)^{\star}_{\alpha_n}(\varepsilon_n) = \varepsilon_n, \left(\mathrm{int}_{\delta}^{\alpha_n}\right)^{\star}_{\alpha_n}(\upsilon_n) = \upsilon_n$ such that $\varepsilon_n \wedge \upsilon_n = \langle 0,1,1\rangle$ and $\varepsilon_n \vee \upsilon_n = \langle 1,0,0\rangle$. Then, from (2) in Proposition 3.2., we get that

$$(\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\varepsilon_n) = (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}((\upsilon_n)^c) = ((\mathrm{int}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\upsilon_n))^c = (\upsilon_n)^c = \varepsilon_n,$$

$$(\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\upsilon_n) = (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}((\varepsilon_n)^c) = ((\mathrm{int}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\varepsilon_n))^c = (\varepsilon_n)^c = \upsilon_n,$$

Hence, $\left(\mathrm{Cl}_{\delta}^{\alpha_n}\right)^{\star}_{\alpha_n}(\varepsilon_n) \wedge \upsilon_n = \varepsilon_n \wedge \left(\mathrm{Cl}_{\delta}^{\alpha_n}\right)^{\star}_{\alpha_n}(\upsilon_n) = \varepsilon_n \wedge \upsilon_n = \langle 0,1,1\rangle$. That is, $\varepsilon_n, \upsilon_n$ are `SVNIA-separated` sets so that $\varepsilon_n \vee \upsilon_n = \langle 1,0,0\rangle$. But $(\tilde{\mathcal{F}},\delta,\hbar)$ is `SVNIA-connected` implies that $\varepsilon_n = \langle 0,1,1\rangle$ or $\upsilon_n = \langle 0,1,1\rangle$.

$(2) \Rightarrow (3)$:, $(3) \Rightarrow (1)$: Clear. □

Theorem 4.2. *Let* $(\tilde{\mathcal{F}},\delta,\hbar)$ *be a* `SVNIAS` *and* $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$*. Then, the following are equivalent.*

1. ε_n *is* `SVNIA-connected` *set.*
2. *If* υ_n, π_n *are* `SVNIA-separated` *sets with* $\varepsilon_n \leq (\upsilon_n \vee \pi_n)$*, then* $\varepsilon_n \wedge \upsilon_n = \langle 0,1,1\rangle$ *or* $\varepsilon_n \wedge \pi_n = \langle 0,1,1\rangle$*.*
3. *If* υ_n, π_n *are* `SVNIA-separated` *sets with* $\varepsilon_n \leq (\upsilon_n \vee \pi_n)$*, then* $\varepsilon_n \leq \upsilon_n$ *or* $\varepsilon_n \leq \pi_n$*.*

Proof. $(1) \Rightarrow (2)$: Let υ_n, π_n are `SVNIA-separated` sets with $\varepsilon_n \leq (\upsilon_n \vee \pi_n)$. That is,

$$\left(\mathrm{Cl}_{\delta}^{\alpha_n}\right)^{\star}_{\alpha_n}(\upsilon_n) \wedge \pi_n = \upsilon_n \wedge \left(\mathrm{Cl}_{\delta}^{\alpha_n}\right)^{\star}_{\alpha_n}(\pi_n) = \langle 0,1,1\rangle.$$

So that $\varepsilon_n \leq (\upsilon_n \vee \pi_n)$. Since

$$\begin{aligned}(\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\varepsilon_n \wedge \upsilon_n) \wedge (\varepsilon_n \wedge \pi_n) &= (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\varepsilon_n) \wedge (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\upsilon_n) \wedge (\varepsilon_n \wedge \pi_n)\\ &= (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\varepsilon_n) \wedge \varepsilon_n \wedge (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\upsilon_n) \wedge \pi_n\\ &= \langle 0,1,1\rangle.\end{aligned}$$

$$\begin{aligned}(\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\varepsilon_n \wedge \pi_n) \wedge (\varepsilon_n \wedge \upsilon_n) &= (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\varepsilon_n) \wedge (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\pi_n) \wedge (\varepsilon_n \wedge \upsilon_n)\\ &= (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\varepsilon_n) \wedge \varepsilon_n \wedge (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\pi_n) \wedge \upsilon_n\\ &= \langle 0,1,1\rangle.\end{aligned}$$

Then, $(\mu \wedge v)$ and $(\mu \wedge \rho)$ are `SVNIA-separated` sets with $\varepsilon_n = (\varepsilon_n \wedge \upsilon_n) \vee (\varepsilon_n \wedge \pi_n)$. But ε_n is `SVNIA-connected` means that $\varepsilon_n \wedge \upsilon_n = \langle 0,1,1\rangle$ or $\varepsilon_n \wedge \pi_n = \langle 0,1,1\rangle$.
(2) $\Rightarrow$ (3) : If $\varepsilon_n \wedge \upsilon_n = \langle 0,1,1\rangle$, $\varepsilon_n \leq (\upsilon_n \vee \pi_n)$ means that $\varepsilon_n = \varepsilon_n \wedge (\upsilon_n \vee \pi_n) = (\varepsilon_n \wedge \upsilon_n) \vee (\varepsilon_n \wedge \pi_n) = (\varepsilon_n \wedge \pi_n)$ and thus $(\varepsilon_n \leq \pi_n)$. Also, if $\varepsilon_n \wedge \pi_n = \langle 0,1,1\rangle$, then $\varepsilon_n \leq \upsilon_n$.
(3) $\Rightarrow$ (1): Let υ_n, π_n be `SVNIA-separated` sets so that $\varepsilon_n = \upsilon_n \vee \pi_n$. Then, from (3), $\varepsilon_n \leq \upsilon_n$ or $\varepsilon_n \leq \pi_n$. If $\varepsilon_n \leq \upsilon_n$, then

$$\pi_n = (\upsilon_n \vee \pi_n) \wedge \pi_n = \varepsilon_n \wedge \pi_n \leq \upsilon_n \wedge \pi_n \leq (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\upsilon_n) \wedge \pi_n = \langle 0,1,1\rangle.$$

Also, if $\mu \leq \rho$, then

$$\upsilon_n = (\upsilon_n \vee \pi_n) \wedge \upsilon_n = \varepsilon_n \wedge \upsilon_n \leq \pi_n \wedge \upsilon_n \leq (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(\pi_n) \wedge \upsilon_n = \langle 0,1,1\rangle.$$

Hence, ε_n is `SVNIA-connected` set. □

Definition 4.3. *Let $(\tilde{\mathcal{F}}, \delta)$, $(\tilde{\mathcal{G}}, \delta^{\star})$ be two `SVNAS` and $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$, $\varepsilon_n \in \zeta^{\tilde{\mathcal{G}}}$ are `SVNSs`. Then, the mapping $f : (\tilde{\mathcal{F}}, \delta) \to (\tilde{\mathcal{G}}, \delta^{\star})$ is called single valued neutrosophic approximation continuous (briefly, `SVNA-continuous`) if $f^{-1}(\mathrm{int}_{\delta^{\star}}^{\varepsilon_n}(\upsilon_n)) \leq \mathrm{int}_{\delta}^{\alpha_n}(f^{-1}(\upsilon_n))\ \ \forall\ \ \upsilon_n \in \zeta^{\tilde{\mathcal{G}}}$.*
Equivalently. f is called `SVNA-continuous` if $f^{-1}(\mathrm{Cl}_{\delta^{\star}}^{\varepsilon_n}(\upsilon_n)) \geq \mathrm{Cl}_{\delta}^{\alpha_n}(f^{-1}(\upsilon_n))\ \ \forall\ \ \upsilon_n \in \zeta^{\tilde{\mathcal{G}}}$.

Definition 4.4. *A mapping $f : (\tilde{\mathcal{F}}, \delta, \hbar) \to (\tilde{\mathcal{G}}, \delta^{\star})$ is called single valued neutrosophic ideal approximation continuous (briefly, `SVNIA-continuous`) if $f^{-1}(\mathrm{int}_{\delta^{\star}}^{\varepsilon_n}(\upsilon_n)) \leq (\mathrm{int}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}\left(f^{-1}(\upsilon_n)\right) \forall \upsilon_n \in \zeta^{\tilde{\mathcal{G}}}$.*
Equivalently. f is called `SVNIA-continuous` if $f^{-1}(\mathrm{Cl}_{\delta^{\star}}^{\varepsilon_n})(\upsilon_n) \geq (\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(f^{-1}(\upsilon_n)) \forall \upsilon_n \in \zeta^{\tilde{\mathcal{G}}}$.

Every `SVNIA-continuous` mapping will be `SVNA-continuous` as well (from (1) in Proposition 3.2.) but not converse.

Remark 4.2. *Since $\hbar$ and $\hbar^{\star}$ are independent `SVNIs` on $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ respectively, then the mapping $f : (\tilde{\mathcal{F}}, \delta, \hbar) \to (\tilde{\mathcal{G}}, \delta^{\star}, \hbar^{\star})$ still not `SVNIA-continuous` in general even if we have taken f is a bijective map with respect to $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ and $f(\alpha_n) \in \zeta^{\tilde{\mathcal{G}}}$ and the `SVNR` δ an $\tilde{\mathcal{F}}$ and $\delta^{\star}$ on $\tilde{\mathcal{G}}$ wrere $\delta^{\star} = \delta \circ \left(f^{-1} \times f^{-1}\right) = (f \times f)(\delta)$. This special case itself could be as an example of a `SVNA-continuous` mapping but not `SVNIA-continuous` in general*

Theorem 4.3. *Let $(\tilde{\mathcal{F}}, \delta, \hbar), (\tilde{\mathcal{G}}, \delta^{\star}, \hbar^{\star})$, associated with $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$, $\varepsilon_n \in \zeta^{\tilde{\mathcal{G}}}$ respectively, be `SVNIAS` and $f : (\tilde{\mathcal{F}}, \delta, \hbar) \to (\tilde{\mathcal{G}}, \delta^{\star}, \hbar^{\star})$ is a `SVNIA-continuous` mapping. Then, $f(\eta_n) \in \zeta^{\tilde{\mathcal{G}}}$ is a `SVNIA-connected` set if η_n is a `SVNIA-connected` set in $\tilde{\mathcal{F}}$.*

Proof. Let $\upsilon_n, \pi_n \in \zeta^{\tilde{\mathcal{G}}}$ be `SVNIA-separated` sets with $f(\eta_n) = \upsilon_n \vee \pi_n$. That is, $(\mathrm{Cl}_{\delta^{\star}}^{\varepsilon_n})^{\star}_{\varepsilon_n}(\upsilon_n) \wedge \pi_n = \mathrm{Cl}_{\delta^{\star}}^{\varepsilon_n})^{\star}_{\varepsilon_n}(\pi_n) \wedge \upsilon_n = \langle 0,1,1\rangle$. Then, $\eta_n \leq \left(f^{-1}(\upsilon_n) \vee f^{-1}(\pi_n)\right)$, and from f is `SVNIA-continuous`, we

get that

$$\begin{aligned}(\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(f^{-1}(\upsilon_n)) \wedge f^{-1}(\pi_n) &\leq f^{-1}(\mathrm{Cl}_{\delta^{\star}}^{\varepsilon_n}(\upsilon_n)) \wedge f^{-1}(\pi_n)\\ &= f^{-1}(\mathrm{Cl}_{\delta^{\star}}^{\varepsilon_n}(\upsilon_n) \wedge \pi_n) = f^{-1}(\langle 0,1,1\rangle)\\ &= \langle 0,1,1\rangle.\end{aligned}$$

and similarly, we have

$$\begin{aligned}(\mathrm{Cl}_{\delta}^{\alpha_n})^{\star}_{\alpha_n}(f^{-1}(\pi_n)) \wedge f^{-1}(\upsilon_n) &\leq f^{-1}(\mathrm{Cl}_{\delta^{\star}}^{\varepsilon_n}(\pi_n)) \wedge f^{-1}(\upsilon_n)\\ &= f^{-1}(\mathrm{Cl}_{\delta^{\star}}^{\varepsilon_n}(\pi_n) \wedge \upsilon_n) = f^{-1}(\langle 0,1,1\rangle)\\ &= \langle 0,1,1\rangle.\end{aligned}$$

Hence, $f^{-1}(\upsilon_n)$ and $f^{-1}(\pi_n)$ are SVNIA-separated sets in $\tilde{\mathcal{F}}$ so that $\eta_n \leq (f^{-1}(\upsilon_n) \vee f^{-1}(\pi_n))$. But by (3) in Theorem (4.2), we get that $\eta_n \leq f^{-1}(\upsilon_n)$ or $\eta_n \leq f^{-1}(\pi_n)$ which means that $f(\eta_n) \leq \upsilon_n$ or $f(\eta_n) \leq \pi_n$. Thus, from that η_n is SVNIA-connected set in $\tilde{\mathcal{F}}$, and again from (3) in Theorem (4.2), we get that $f(\eta_n)$ is SVNIA-connected in $\tilde{\mathcal{G}}$. □

The implications in the following diagram are satisfied whenever f is SVNIA-continuous.

$$\begin{array}{ccc}\eta_n \textit{ is } \texttt{SVNA-connected} & \longrightarrow & \eta_n \textit{ is } \texttt{SVNIA-connected}\\ \downarrow & & \downarrow\\ f(\eta_n) \textit{ is } \texttt{SVNA-connected} & \longrightarrow & f(\eta_n) \textit{ is } \texttt{SVNIA-connected}\\ \uparrow & & \uparrow\\ \eta_n \textit{ is } \texttt{SVNIA-connected} & & \eta_n \textit{ is } \texttt{SVNA-connected}\end{array}$$

Only the implications in the following diagram are satisfied whenever f is SVNA-continuous.

$$\begin{array}{ccc}\eta_n \textit{ is } \texttt{SVNA-connected} & \longrightarrow & \eta_n \textit{ is } \texttt{SVNIA-connected}\\ \downarrow & & \downarrow\\ f(\eta_n) \textit{ is } \texttt{SVNA-connected} & \longrightarrow & f(\eta_n) \textit{ is } \texttt{SVNIA-connected}\\ \uparrow & & \\ \eta_n \textit{ is } \texttt{SVNIA-connected} & & \end{array}$$

Example 4.2. *Let* $\tilde{\mathcal{F}} = \tilde{\mathcal{G}} = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ *and* $f : \tilde{\mathcal{F}} \to \tilde{\mathcal{G}}$ *such that*

$$f(\omega_1) = f(\omega_2) = \omega_1, \;\; f(\omega_3) = \omega_2, \;\; f(\omega_4) = \omega_4,$$

δ *and* δ^* *are* SVNR *on* $\tilde{\mathcal{F}}$, $\tilde{\mathcal{G}}$ *respectively as follows:*

Assume that

$$\alpha_n = \langle (1,0,0), (1,0,0), (1,0,0), (0.2,0.2,0.2)\rangle \in \zeta^{\tilde{\mathcal{F}}},$$

and

$$\eta_n = \langle (0,1,1), (0.3,0.3,0.3), (0.5,0.5,0.5), (0.2,0.2,0.2)\rangle \in \zeta^{\tilde{\mathcal{G}}}$$

δ	ω_1	ω_2	ω_3	ω_4
ω_1	(1, 0, 0)	(0, 1, 1)	(0, 1, 1)	(0, 1, 1)
ω_2	(0, 1, 1)	(1, 0, 0)	(0, 1, 1)	(0, 1, 1)
ω_3	(0, 1, 1)	(0, 1, 1)	(1, 0, 0)	(0.3, 0.3, 0.3)
ω_4	(0, 1, 1)	(0, 1, 1)	(0.3, 0.3, 0.3)	(1, 0, 0)

TABLE 3. SVNRof δ

δ^*	ω_1	ω_2	ω_3	ω_4
ω_1	(1, 0, 0)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0, 1, 1)
ω_2	(0.5, 0.5, 0.5)	(1, 0, 0)	(0.5, 0.5, 0.5)	(0, 1, 1)
ω_3	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(1, 0, 0)	(0, 1, 1)
ω_4	(0, 1, 1)	(0, 1, 1)	(0, 1, 1)	(1, 0, 0)

TABLE 4. SVNRof δ

Then,

$$\tilde{\varrho}_{(\alpha_n)^\delta}(\omega_1) = \tilde{\varrho}_{\alpha_n}(\omega_1) \wedge \bigvee_{\alpha_n(\mu)>0, \mu\neq\omega_1} \tilde{\varrho}_{[\omega]}(\mu) = 0,$$

$$\tilde{\sigma}_{(\alpha_n)^\delta}(\omega_1) = \tilde{\sigma}_{\alpha_n}(\omega_1) \vee \bigwedge_{\alpha_n(\mu)>0, \mu\neq\omega_1} (\tilde{\sigma}_{[\omega]})(\mu) = 1,$$

$$\tilde{\varsigma}_{(\alpha_n)^\delta}(\omega_1) = \tilde{\varrho}_{\alpha_n}(\omega_1) \vee \bigwedge_{\alpha_n(\mu)>0, \mu\neq\omega_1} \tilde{\varsigma}_{[\omega]}(\mu) = 1,$$

Hence, $(\alpha_n)^\delta(\omega_1) = (\alpha_n)^\delta(\omega_2) = (\alpha_n)^\delta(\omega_3) = (0, 1, 1)$. *Also,*

$$\tilde{\varrho}_{(\alpha_n)^\delta}(\omega_4) = \tilde{\varrho}_{\alpha_n}(\omega_4) \wedge \bigvee_{\alpha_n(\mu)>0, \mu\neq\omega_4} \tilde{\varrho}_{[\omega]}(\mu) = 0.2,$$

$$\tilde{\sigma}_{(\alpha_n)^\delta}(\omega_4) = \tilde{\sigma}_{\alpha_n}(\omega_4) \vee \bigwedge_{\alpha_n(\mu)>0, \mu\neq\omega_4} (\tilde{\sigma}_{[\omega]})(\mu) = 0.3,$$

$$\tilde{\varsigma}_{(\alpha_n)^\delta}(\omega_4) = \tilde{\varrho}_{\alpha_n}(\omega_4) \vee \bigwedge_{\alpha_n(\mu)>0, \mu\neq\omega_4} \tilde{\varsigma}_{[\omega]}(\mu) = 0.3,$$

Thus, $(\alpha_n)^\delta(\omega_4) = (0.2, 0.3, 0.3)$ *and than,*

$$(\alpha_n)^\delta = \langle (0, 1, 1), (0, 1, 1), (0, 1, 1), (0.2, 0.3, 03) \rangle,$$

$$(\alpha_n)_\delta = \langle (1, 0, 0), (1, 0, 0), (1, 0, 0), (0.3, 0.2, 0.2) \rangle,$$

$$[(\alpha_n)_\delta]^c = \langle (0, 1, 1), (0, 1, 1), (0, 1, 1), (0.2, 0.8, 0.3) \rangle.$$

For $\beta_n = \langle (0,1,1),(0,1,1),,(0.2,0.8,0.2),(0,1,1)\rangle \in \zeta^{\tilde{\mathcal{G}}}$*, we get that*

$$f^{-1}(\beta_n) = \langle 0,1,1\rangle$$

and then $\mathrm{Cl}_{\delta}^{\alpha_n}(f^{-1}(\beta_n)) = \langle 0,1,1\rangle$*. Since*

$$(\eta_n)_{\delta^*} = \langle (0.5,0.5,0.5),(0.5,0.3,0.3),(0.5,0.5,0.5),(1,0,0\rangle,$$

$$(\beta_n)^{\delta^*} = \langle (0,1,1),(0,1,1),(0.2,0.8,0.5),(0,1,1)\rangle,$$

$$[(\eta_n)_{\delta^*}]^c = \langle (0.5,0.5,0.5),(0.3,0.7,0.5),(0.5,0.5,0.5),(0,1,1)\rangle.$$

Then, $\mathrm{Cl}_{\delta^*}^{\eta_n}(\beta_n) = [(\eta_n)_{\delta^*}]^c \vee (\beta_n)^{\delta^*} = \langle (0.5,0.5,0.5),(0.3,0.7,0.5),(0.5,0.5,0.5),(0,1,1)\rangle$*. Thus,*

$$f^{-1}(\mathrm{Cl}_{\delta^*}^{\eta_n}(\beta_n)) = \langle (0.5,0.5,0.5),(0.5,0.5,0.5),(0.3,0.7,0.5),(0,1,1)\rangle \geq \langle 0,1,1\rangle = \mathrm{Cl}_{\delta}^{\alpha_n}(f^{-1}(\beta_n))$$

. Hence, there is a fuzzy set $\beta_n \in \zeta^{\tilde{\mathcal{G}}}$ *satisfying the condition of single valued neutrosophic approximation continuity. Next, we will show that* β_n *itself will not satisfy the condition of fuzzy ideal approximation continuity.*

Since, $\mathrm{Cl}_{\delta}^{\eta_n}(f^{-1}(\beta_n)) = \langle 0,1,1\rangle$*, then*

$$(\mathrm{Cl}_{\delta}^{\alpha_n})_{\alpha_n}^{\star}\left(f^{-1}(\beta_n)\right) = \mathrm{Cl}_{\delta}^{\alpha_n}\left(f^{-1}(\beta_n)\right) \vee ((\alpha_n)^{\delta})_{\alpha_n}^{\star} = ((\alpha_n)^{\delta})_{\alpha_n}^{\star},$$

that is,

$$(\mathrm{Cl}_{\delta}^{\alpha_n})_{\alpha_n}^{\star}\left(f^{-1}(\beta_n)\right) = \{\langle (0,1,1),(0,1,1),(0,1,1),(0.2,0.3,03)\rangle\}_{\alpha_n}^{\star}.$$

Now, define a single-valued neutrosophic ideal $\hbar$ *over* $\tilde{\mathcal{F}}$ *as next*

$$\varphi \in \hbar \Leftrightarrow \varphi \leq \{\langle (1,0,0),(1,0,0),(0.2,0.2,0.2),(0.2,0.2,0.2)\rangle\}$$

. Then, from being $\mathrm{Cl}_{\delta}^{\eta_n}(\mu_n) = (\alpha_n)_{\delta}^{c} \vee (\mu_n)^{\delta} \geq \langle (0,1,1),(0,1,1),(0,1,1),(0.2,0.8,0.3)\rangle$*, we get that,*

$$(\mathrm{Cl}_{\delta}^{\alpha_n})_{\alpha_n}^{\star}\left(f^{-1}(\beta_n)\right) = \{\langle (0,1,1),(0,1,1),(0.2,0.8,0.2),(0.2,0.8,0.2)\rangle\}.$$

according to the definition of $\hbar$ *and the definition of*

$$[\mu_n]_{\alpha_n}^{\star}(\delta,\hbar) = \bigwedge\{\nu_n \in \zeta^{\tilde{\mathcal{F}}} : \mu_n \overline{\wedge} \nu_n = \langle \tilde{\varrho}_{\mu_n} \overline{\wedge} \tilde{\varrho}_{\nu_n}(\omega), \tilde{\sigma}_{\mu_n} \overline{\vee} \tilde{\sigma}_{\nu_n}(\omega), \tilde{\xi}_{\mu_n} \overline{\vee} \tilde{\xi}_{\nu_n}(\omega)\rangle \in \hbar, \mathrm{Cl}_{\delta}^{\alpha_n}(\nu_n) = \nu_n\}$$

. Hence, we obtain

$$f^{-1}(\mathrm{Cl}_{\delta^*}^{\eta_n}(\beta_n)) = \langle (0.5,0.5,0.5),(0.5,0.5,0.5),(0.3,0.7,0.5),(0,1,1)\rangle \not\geq (\mathrm{Cl}_{\delta}^{\alpha_n})_{\alpha_n}^{\star}\left(f^{-1}(\beta_n)\right)$$

$$= \{\langle (0,1,1),(0,1,1),(0.2,0.8,0.2),(0.2,0.8,0.2)\rangle\}$$

Therefore, not any single-valued neutrosophic approximation continuous map must be a single-valued neutrosophic ideal approximation continuous but the converse is a must.

5. Conclusion

In this paper, we introduced the notion of single-valued neutrosophic approximation space. Joining a single-valued neutrosophic ideal to the single-valued neutrosophic approximation space, we got a single-valued neutrosophic ideal approximation space with other properties different from those of single-valued neutrosophic approximation spaces. In future work, we will define single-valued neutrosophic approximation rough groups and single-valued neutrosophic approximation rough rings as applications of this paper.

Discussion for further works: The theories that were used in this article could be extended to study some similar notions in the neutrosophic metric topological spaces.

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References

[1] Z. Pawlak, Rough Sets, Int. J. Comput. Inform. Sci. 11 (1982), 341–356. https://doi.org/10.1007/bf01001956.

[2] Z. Pei, D. Pei, L. Zheng, Topology vs Generalized Rough Sets, Int. J. Approx. Reason. 52 (2011), 231–239. https://doi.org/10.1016/j.ijar.2010.07.010.

[3] X. Chen, Q. Li, Construction of Rough Approximations in Fuzzy Setting, Fuzzy Sets Syst. 158 (2007), 2641–2653. https://doi.org/10.1016/j.fss.2007.05.016.

[4] D. Dubois, H. Prade, Rough Fuzzy Sets and Fuzzy Rough Sets, Int. J. Gen. Syst. 17 (1990), 191–209. https://doi.org/10.1080/03081079008935107.

[5] Z. Li, T. Xie, Roughness of Fuzzy Soft Sets and Related Results, Int. J. Comput. Intell. Syst. 8 (2015), 278–296. https://doi.org/10.1080/18756891.2015.1001951.

[6] G. Liu, Generalized Rough Sets Over Fuzzy Lattices, Inform. Sci. 178 (2008), 1651–1662. https://doi.org/10.1016/j.ins.2007.11.010.

[7] W. Wu, Generalized Fuzzy Rough Sets, Inform. Sci. 151 (2003), 263–282. https://doi.org/10.1016/s0020-0255(02)00379-1.

[8] M.I. Ali, A Note on Soft Sets, Rough Soft Sets and Fuzzy Soft Sets, Appl. Soft Comput. 11 (2011), 3329–3332. https://doi.org/10.1016/j.asoc.2011.01.003.

[9] D. Molodtsov, Soft Set Theory–First Results, Comput. Math. Appl. 37 (1999), 19–31. https://doi.org/10.1016/s0898-1221(99)00056-5.

[10] D. Boixader, J. Jacas, J. Recasens, Upper and Lower Approximations of Fuzzy Sets, Int. J. Gen. Syst. 29 (2000), 555–568. https://doi.org/10.1080/03081070008960961.

[11] W.Z. Wu, A Study on Relationship Between Fuzzy Rough Approximation Operators and Fuzzy Topological Spaces, Springer-Verlag, Berlin, Heidelberg, (2005).

[12] A.M. Abd El-Latif, Generalized Soft Rough Sets and Generated Soft Ideal Rough Topological Spaces, J. Intell. Fuzzy Syst. 34 (2018), 517–524. https://doi.org/10.3233/jifs-17610.

[13] J. Mahanta, P.K. Das, Fuzzy Soft Topological Spaces, J. Intell. Fuzzy Syst. 32 (2017), 443–450. https://doi.org/10.3233/jifs-152165.

[14] C.L. Chang, Fuzzy Topological Spaces, J. Math. Anal. Appl. 24 (1968), 182–190. https://doi.org/10.1016/0022-247x(68)90057-7.

[15] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics, 6th ed. InfoLearnQuest, Ann Arbor, MI, USA. (2007).

[16] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Single Valued Neutrosophic Sets, Multispace Multistruct. 4 (2010), 410–413.

[17] H.L. Yang, Z.L. Guo, Y. She, X. Liao, On Single Valued Neutrosophic Relations, J. Intell. Fuzzy Syst. 30 (2016), 1045–1056. https://doi.org/10.3233/ifs-151827.

[18] Q. Jin, K. Hu, C. Bo, L. Li, A New Single-Valued Neutrosophic Rough Sets and Related Topology, J. Math. 2021 (2021), 5522021. https://doi.org/10.1155/2021/5522021.

[19] F. Alsharari, Y.M. Saber, $\mathcal{G}\Theta_{\tau_i}^{\star\tau_j}$-Fuzzy Closure Operator, New Math. Nat. Comput. 16 (2020), 123–141. https://doi.org/10.1142/s1793005720500088.

[20] F. Alsharari, Y.M. Saber, F. Smarandache, Compactness on Single-Valued Neutrosophic Ideal Topological Spaces, Neutrosophic Sets Syst. 41 (2021), 127–145.

[21] Y.M. Saber, M.A. Abdel-Sattar, Ideals on Fuzzy Topological Spaces, Appl. Math. Sci. 8 (2014), 1667–1691. https://doi.org/10.12988/ams.2014.33194.

[22] Y. Saber, F. Alsharari, F. Smarandache, An Introduction to Single-Valued Neutrosophic Soft Topological Structure, Soft Comput. 26 (2022), 7107–7122. https://doi.org/10.1007/s00500-022-07150-4.

[23] Y.M. Saber, F. Alsharari, Generalized Fuzzy Ideal Closed Sets on Fuzzy Topological Spaces in Sostak Sense, Int. J. Fuzzy Log. Intell. Syst. 18 (2018), 161–166. https://doi.org/10.5391/ijfis.2018.18.3.161.

[24] Y. Saber, F. Alsharari, F. Smarandache, On Single-Valued Neutrosophic Ideals in Sostak Sense, Symmetry. 12 (2020), 193. https://doi.org/10.3390/sym12020193.

[25] Y. Saber, F. Alsharari, F. Smarandache, M. Abdel-Sattar, Connectedness and Stratification of Single-Valued Neutrosophic Topological Spaces, Symmetry. 12 (2020), 1464. https://doi.org/10.3390/sym12091464.

[26] Y. Saber, F. Alsharari, F. Smarandache, M. Abdel-Sattar, On Single Valued Neutrosophic Regularity Spaces, Comput. Model. Eng. Sci. 130 (2022), 1625–1648. https://doi.org/10.32604/cmes.2022.017782.

[27] Y. Saber, Connectedness in Single-Valued Neutrosophic Soft Grill Topological Spaces, Int. J. Anal. Appl. 21 (2023), 137. https://doi.org/10.28924/2291-8639-21-2023-137.

[28] Y. Saber, H. Alohali, T. Elmasry, F. Smarandache, On Single-Valued Neutrosophic Soft Uniform Spaces, AIMS Math. 9 (2024), 412–439. https://doi.org/10.3934/math.2024023.

[29] A.M. Zahran, S.A.A. El-Baki, Y.M. Saber, Decomposition of Fuzzy Ideal Continuity via Fuzzy Idealization, Int. J. Fuzzy Log. Intell. Syst. 9 (2009), 83–93. https://doi.org/10.5391/ijfis.2009.9.2.083.

[30] A.M. Zahran, S.E. Abbas, S.A. Abd El-baki, Y.M. Saber, Decomposition of Fuzzy Continuity and Fuzzy Ideal Continuity via Fuzzy Idealization, Chaos Solitons Fractals. 42 (2009), 3064–3077. https://doi.org/10.1016/j.chaos.2009.04.010.

The Meaning of Independence in Local Institutions from Neutrosophic Perspective

Alaa A. Mohammed[1], Sabah S. Abdulrahman[2], Huda E. Khalid[3*], Florentin Smarandache[4], A. A. Salama[5]

[1]University of Telafer / Director of Legal Affairs Department, Telafer University, Iraq

Email:*alaa a. mohammed@sci.psu.edu.eg*

[2]University of Telafer / Director of Legal Affairs Department, Telafer University, Iraq

Email:*sabah s. abdulrahma@sci.psu.edu.eg*

[3]University of Telafer, The Administration Assistant for the President of the Telafer University, Telafer, Iraq; Email: https://orcid.org/0000-0002-0968-5611 , dr.huda-ismael@uotelafer.edu.iq

[4]Emeritus Professor; Mathematics, Physics, and Natural Science Division; University of New Mexico; 705 Gurley Ave., Gallup, NM 87301, USA Email: smarand@unm.edu

[5] Dept. of Math and Computer Sci., Faculty of Science, Port Said University, Egypt

Email: *ahmed_salama_2000@sci.psu.edu.eg*

*Correspondence: dr.huda-ismael@uotelafer.edu.iq

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Abstract: It is well known that the term vagueness is spread in all aspects of our lives, this manuscript will clarify the meaning of independence (strict meaning of independence, the illusory meaning of independence, Oscillating between the true meaning of independence, and the fallacy of claiming independence and not applying it), this triplet actually refers to the three components that the Neutrosophic theory standing on which is (truthiness, falsity, indeterminacy), the strategy of applying the notion of independency in the local institutions, enlighten on the allegation of applying the independency notion in some governments' institutions without a real impact of this term, and studying the meaning of independence from the perspective of a country legal system, all these issues and some comparisons in applying the term (independency) between some countries such as Iraq, France, Algeria, USA, etc. have been enclosed in this article.

Keywords: Neutrosophic Notion of Independency, Neutrosophic Theory, Administrative and Financial Independence, Local Institutions, Iraqi Law, British Law, French Law.

1. Introduction

It is well known that there are three kinds of authority in quite several countries which are: legislative authority, executive authority, and judicial authority, each of them consists of some additional internal institutions, such as, in Iraq the Federal Judicial Authority consists of:

1- The Supreme Judicial Council,

2- The Federal Supreme Court,

3- The Court of Cassation,
4- The Public Prosecution Service,
5- The Judicial Supervision Authority,
6- Other federal courts regulated by law,
7- Central Criminal Court.

The first phrase that we will see in what the English jurist Howard Machin [1] expressed that contradicts applying the term (independence) in the local institute is (According to logic and reason, it is difficult to accept the existence of disorganized bodies in the legal system of the state, and accordingly, the subjection of local institutions to parliamentary and judicial oversight does not affect their independence) .
The text mentioned above (in our opinion) is somehow regarded as a limitation for the notion of – independence- especially for those Independent Bodies and Institutions in Iraq such as:

1- The High Commission for Human Rights,
2- The Independent High Electoral Commission,
3- The Integrity Commission

All the above bodies and institutions are independent commissions subject to the oversight of the Council of Representatives.
Our question that should be raised is:
How could we call these commissions independent commissions but there is a subjective oversight on them by the Council of Representatives? This means that not all local institutions have the same degree of independence.

The following bodies are also independent

4- The Central Bank of Iraq,
5- The Board of Financial Supervision,
6- The Communications and Media Commission,
7- The Sunni Endowment Office,
8- The Shiite Endowment Office,
9- The Christian Endowment Office

All the above bodies (from 4 to 9) are administratively and financially independent.

10- The Martyrs' Foundation is affiliated with the Council of Ministers. / So how can it be independent with belonging to the Council of Ministries?!
11- The Federal Public Service Council regulates the affairs of the federal civil service, including appointment and promotion.

In the literatures, the notion of independence of local bodies is defined as

Definition 1: Independence means that local institutions are not subject to presidential control or guardianship, and do not receive orders or instructions from any other party. The executive authority also does not have the freedom to dismiss its members. However, independence does not mean that these bodies operate separately from the goals of the government, but rather are considered supportive bodies for government policies.

Article 102 of the Iraqi Constitution states: The High Commission for Human Rights, the Independent High Electoral Commission, and the Integrity Commission are independent bodies subject to the oversight of the Council of Representatives, and their work is regulated by law.

If a comparison between the above definition of independence and Article 102 of the Iraqi Constitution states has been made. We can conclude from this that the concept of independence is a concept restricted by conditions and determinants, consequently, we can say that the independence notion needs to be independence, the contradiction here is clear, how could these bodies be completely independent such as the Integrity Commission wherein this body is subjected to the oversight of the Council of Representatives?! Eventually, we can say, that there is no local institution that is completely independent, but it is either partially independent or There is a claim that it is independent, but this independence is merely apparent.

The importance of this research lies in considering the administrative and financial independence of local institutions as a vital topic that contributes to enhancing the effectiveness and efficiency of institutions' performance, which is positively reflected in the provision of public services and the achievement of sustainable development. This topic can be addressed by conducting a comparative study between Iraqi law and the laws of some countries such as British law, French law, Algerian law, etc. to review some contradictions and some similarities and differences in this context. The comparative study between laws allows understanding how different countries organize the independence of their local institutions, which helps in identifying strengths and weaknesses and providing recommendations to improve legal and regulatory frameworks.

Definition 2: [2] The neutrosophy considers every idea < A > together with its opposite or negation < antiA > and with their spectrum of neutralities < neutA > in between them (i.e. notions or ideas supporting neither < A > nor < antiA >). The < neutA > and < antiA > ideas together are referred to as < nonA >. Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic).

2. Organizational Independence Features

In order to provide aspects of organizational independence for local institutions, we need several basic requirements, the most important of which are:

1- Diversity of members of these institutions in addition to their multiple experiences to enhance their ability to make decisions optimally.
2- The existence of a law that guarantees that no member can combine more than one job at the same time, which may affect the independence of these institutions.
3- The existence of protection for these members from any external interference that may affect their official decision-making.
4- The period of membership of these members in these institutions may affect the neutrality and independence of their official decisions. For example, if the membership period exceeds the duration of the parliamentary session and the duration of the government, this will enhance the independence of these institutions from the political influences of the government and parties.

2.1. Comparison Study for the first Requirement (Diversity of Institutional Members)

The legal texts regulating the formation of local institutions, in general, have been reviewed, and the diversity of the composition of the members can be observed through them. For example, in Algeria, the bodies of the independence of the authorities are characterized by the diversity of the composition of their members, as the Competition Council consists of nine members, the Banking Commission consists of six members, the Stock Exchange Regulation and Supervision Committee consists of seven members and the Monetary and Credit Council consists of nine members. The members of these bodies are appointed by a decree issued by the President of the Republic, except the Chairman of the Stock Exchange Regulation and Supervision Committee who is appointed by the government, in addition to the need for diversity in terms of experience and specialization to achieve the objectives of local institutions; these bodies were established to meet sensitive and specialized activities, and therefore their composition of professionals and specialists is essential [3].

In France, the law on the Ombudsman of the Republic stipulates that the appointment of the mediator is made by decree of the Council of Ministers for six years [4], and the mediator chooses his assistants and assistants from among civil and military employees [5], as is the case for the Competition Council, which consists of seventeen members appointed by decree for six years. The Council is composed of eight members representing the judicial bodies (Council of State, Court of Cassation, Court of Auditors, or other ordinary and administrative courts), four members chosen based on their specialization in the field of competition and consumption, and five others appointed based on their activity in the production and domestic trade sector. In addition, a government commissioner is appointed within the Council by decision of the Minister of Economy to represent the State's point of view [6], as is the case with the law of the Supreme Council for Audiovisual Communication, which provides for the Council to be composed of nine members [7].

In the United States, the matter is no different from what we mentioned above, with regard to independent regulatory commissions, the American legislator uses the term "independent regulatory agencies or commissions" to refer to local institutions [8]. For example, the Federal Trade Commission consists of five members, the Equal Employment Opportunity Commission consists of fourteen members and the Federal Savings Board consists of seven members. These members are appointed by the President with the approval of the Senate [9].

In Britain, a common characteristic of semi-independent NGOs is that British jurist and legislators use the term "semi-independent NGOs" to refer to local institutions in general, which are characterized by their collective management [10]. For example, the British Broadcasting Corporation (BBC) consists of a chairman, a deputy chairman, and several members who form the board of directors. The Minister appoints these members on behalf of the Crown and in consultation with the Prime Minister for five years. The chairman of the institution enjoys a strong executive position [11]. While in Iraq, members of local institutions are appointed upon the recommendation of the Council of Ministers and the approval of the Council of Representatives, according to the text of the Constitution, which states: "The Council of Ministers shall exercise the following powers: "Recommending to the Council of Representatives the approval of the appointment of holders of special ranks" [12]. In another context, the Constitution stipulates the powers of the Council of Representatives, including: "Approving the appointment of ambassadors and holders of special ranks based on a proposal from the Council of Ministers."[13]

A collective and diverse nature characterizes the formation of all independent bodies, and this is clearly evident in the legal texts regulating their establishment. For example, the Law of the Independent High Electoral Commission stipulates that the Board of Commissioners shall consist of nine members, at least two of whom shall be chosen from among the legal experts by a majority in the Council of Representatives after their nomination by the Council of Representatives Committee. They must be qualified and experienced, and must have a good reputation for competence, integrity, and independence, taking into account the representation of women [14].

The Law of the High Commission for Human Rights in Iraq also stipulates the formation of a committee of experts consisting of representatives of the Council of Representatives, the Council of Ministers, the Supreme Judicial Council, and organizations Civil society and the United Nations Human Rights Office in Iraq, whose number does not exceed fifteen members [15], as the Council consists of eleven original members and three reserve members, who are chosen after being nominated by the committee, and their selection is approved by an absolute majority of the number of members present from the Council of Representatives [16].

From the Neutrosophic perspective, we can conclude that despite of claiming of all above governments councils and their representatives that the diversity of members in the forementioned bodies will maintain a kind of independency, but there are still a kind of vague or suspicious when we will take the statement of the English jurist Howerd Machin in consideration who has pointed out that "the impartiality of the members may be questionable, since the power of appointment lies

with the minister" [1], that is the heads of these organizations are usually chosen by the government, represented by the prime minister or the relevant minister.

2.2. Not to Combine Two or More Jobs for any Member of Independent Bodies

To ensure the full independence of members of independent bodies, it is required that they not combine their work in the body with any job of an executive or legislative nature. This guarantee is reinforced by general constitutional principles, where the combination of executive and legislative functions is prohibited. In France, for example, its constitution stipulates that the functions of members of the government are incompatible with any legislative activity, as well as any representative task of a national nature or any job in the public sector or other professional activity [17]. In the United States of America, the constitution stipulates that no member of the Senate or House of Representatives may combine his membership in Congress with any civil office under the authority of the federal government during his term of election. It is also prohibited for anyone holding a position in the United States government to become a member of either house of Congress while continuing in his government position [18]. In Iraq, the constitution clearly stipulates that it is not permissible to combine membership in the House of Representatives with any other official job or position [19]. It should be noted that the matter did not stop at the limits of general constitutional principles; the laws specific to independent bodies were keen to confirm this fact in particular. For example, in France, the Ombudsman loses his position if he is elected to parliament, as he may not run for parliament or municipal councils during his term as Ombudsman. Similarly, the decree of 4th February on the French High Audiovisual Council, as well as the law on the French Competition Council, confirm this policy [20]. American law further prohibits directors of specialized agencies from holding positions in companies or institutions that were previously subject to their control. In addition, Congress requires that agency members not belong to any political party after their appointment [6]. In Iraq, this rule has been repeated in the texts of the laws of some independent bodies, such as the Law of the Financial Control Board, where the head of the board, his deputies, heads of departments, or those in charge of control may not practice any other profession or work outside official working hours, except for publishing books and research, giving lectures, and translating, as well as membership in professional and civil society councils and organizations.

From the Neutrosophic concept, what if X of the agency members were not affiliated with any political party after their appointment, but previously and before the appointment they belonged to certain political parties and then resigned from their parties after their appointment and their loyalties remained to their parties and some of their actions within the government agencies lean in one way or another towards the parties to which they belonged previously, here we will face a kind of ambiguity and indeterminacy in the way these members perform. So how do we guarantee the independence of the local institution?! Sometimes it is possible to choose completely independent members who have never belonged to certain parties to ensure polishing the image of the independent institution, but the number of these actually independents will be small so that it does

not affect decision-making within the local institutions whose decisions are supposed to be independent, thus we return to the same neutrosophic opinion that confirms that mixing the concept of apparent independence (hidden dependency) with the concept of real independence creates confusion in the real meaning of the concept of independence.

2.3. Protecting Members from External Interference

Members of independent bodies enjoy a guarantee against dismissal from office, which ensures their independence in performing their duties. For example, in the Arab Republic of Egypt, this guarantee was explicitly stated in the suspended 2012 Constitution, which states: (The President of the Republic appoints the heads of independent bodies and oversight bodies after the approval of the Shura Council. They may not be dismissed except with the approval of the majority of the members of the Council, and they are prohibited from doing what is prohibited for ministers). In Algeria, it is noted that the heads of independent administrative authorities are appointed by a decree issued by the President of the Republic and are not dismissed except in the event of committing a serious professional error or in exceptional circumstances [1].

In France, the texts regulating the Republican Mediator clearly included this guarantee, as the mediator can only be dismissed from his position if he is unable to perform his duties, and this is done based on a request from the President of the Republic addressed to a committee consisting of the Vice-President of the State Council , the President of the Cassation Court, and the President of the Accounts Court , and the decision to dismiss the mediator must be taken unanimously by the members of the committee [20], while the President of the Competition Council and his deputies can't be dismissed before the end of their six-year term [6]. Thus, the French Council of State confirmed in its ruling issued on August 27, 1989, that members of independent bodies in France enjoy guarantees that protect their independence, and the ruling ended this system by canceling the negative decision resulting from the Prime Minister's silence regarding the request of Mr. Ordonneau, who was dismissed from his position as President of the Competition Council before the end of the six-year term specified for his term, as well as the obligation to pay compensation as a result of this cancellation based on Mr. Ordonneau's request. This confirms the legal protection of the independence of members of independent bodies in France [21].

In the United States, the political and legal situation confirms the independence of independent bodies in dealing with the powers of the US President to dismiss their members. This is clearly shown in the case of the executor of the will of (Humphrey) v. the United States of America in 1935. In this case, US President Roosevelt dismissed Mr. (Humphrey), who was a member of the Federal Trade Commission, during the period of the program called (New Deal). President Roosevelt's dismissal of Mr. (Humphrey) sparked a legal controversy due to its repercussions on the independence of the Federal Trade Commission and interference in its independent work. The case was discussed before the Supreme Court, where the court ruled that the president does not have the absolute authority to dismiss members of independent agencies without a legitimate legal reason, highlighting the keenness of the American legal system to protect the independence of independent

bodies in performing their functions without illegal interference, believing that he was seeking to thwart policies related to this program. Mr. (Humphrey) died shortly after his dismissal. His wife filed a lawsuit, claiming that the president does not have the right to dismiss members of independent regulatory bodies for political reasons. Accordingly, the Supreme Court has ruled that independent regulatory commissions and agencies are not part of the executive branch, but rather perform quasi-legislative and quasi-judicial functions. The President's use of his power to dismiss could certainly lead to the politicization of these commissions and agencies. Based on this ruling, the Supreme Court has confirmed that independent regulatory agencies are not subordinate to the executive branch and are not affected by current policies. Therefore, the President's use of his power to dismiss members of these agencies could jeopardize their independence and functions and expose them to inappropriate political influences, thus undermining their quasi-legislative and quasi-judicial role in the government system. In the same vein, although the US Constitution grants presidents the right to make appointments during the period of the Senate's recess without the need for its approval, the Senate imposed a limit to this power, which was represented by the passage of a law prohibiting the payment of salaries to officials who served for more than one year without the approval of the Senate. This law has greatly restricted the use of this authority, as it has only been used once since its issuance during the Roosevelt era, which has continued until the present time, as it was used for the only time during the Bush era (W. Bush) in 2002 [22].

In Iraq, independent bodies enjoy this guarantee, as their officials are held accountable under the constitution according to the procedures for questioning and dismissing ministers. For example, the Council of Representatives can hold discussions and examinations of the performance of the heads and members of independent bodies, and in the event of serious violations, the parliament can decide to dismiss them by the required majority of votes, which represents an application of the principle of democratic accountability in the management of public affairs [23]. Some laws specify cases for terminating the membership of the head and members, as these cases are determined based on specific criteria that include exceeding the legal age, committing serious violations related to the performance of the job, or the inability to perform the duties stipulated in the law, or submitting an official resignation [24]. In the Federal Public Service Law, the Iraqi legislator did not preferably use the term dismissal, but rather adopted the term exemption. Exemption from the job according to the provisions of the Law on Discipline for State and Public Sector Employees means the permanent removal of the employee from the job, without the possibility of re-employment in state departments and the public sector [25].

2.4. Term of Membership (Term of Office)

The organic guarantees relate to the term of office, where the term of office of members of independent bodies usually exceeds the term of the parliamentary session and the term of office of the government. This arrangement enhances the independence of local institutions from the political influences of the government and parties. For example, in France, the term of office of the

Competition Council and the Ombudsman of the Republic is six years, as is the case for the High Council for Audiovisual Communication [4].

In the United States, the term of office of members of independent bodies varies. For example, the term of office of the Federal Trade Commission is seven years, while the term of office of the Equal Employment Opportunity Commission is five years [26]. In Britain, members of the BBC Foundation are appointed for five years [11].

In Iraq, the term of office of members of independent bodies varies in a variable and irregular manner, with the term of office of the Independent High Electoral Commission and the Integrity Commission being five years, while the term of office of the Board of Financial Supervision and the High Commission for Human Rights is four years [27].

3. Features of Financial and Administrative Independence

For local institutions to enjoy the features of financial and administrative independence, they must have the following:

1- They must have an independent budget that enables them to carry out their functions without external interference.
2- They must have the ability to determine and manage their financial resources without being subject to external authorities.
3- They must be able to make administrative and executive decisions that serve their goals without external interference or influence.
4- Their structure and management must be multi-party to enhance transparency and accountability.
5- They must have the ability to organize and implement their tasks in accordance with the specified laws and regulations without external interference or pressure.
6- They must follow internal accountability procedures that preserve their independence and enhance their integrity and efficiency.
7- The existence of legal protection that enables them to protect members and the institution from any interference or attempt to influence their work in illegal ways

The existence of the above features guarantees the independence of the body or organization so that it can perform its tasks efficiently and independently without external influences that threaten its independence.

The aspects of independence are represented in two basic pillars:

1- Financial independence of local institutions.

2- Administrative independence of local institutions.

In the next two paragraphs, we will focus on explaining each of the aforementioned pillars in detail.

3.1. Financial independence of local institutions.

Financial independence of local institutions generally means the ability of local institutions to form their financial resources effectively and then choose policies and methods that reflect their actual independence. It is not enough to have financial responsibility and authority to manage resources, as independence must be realistic rather than merely formal theoretical. The French jurist Fidel explains this point by saying that the independence of public institutions, which depend heavily on financial grants provided by the central authority, can be an apparent, ineffective independence. In other words, real independence is achieved more within the framework of the law rather than in practical reality [28].

One of the manifestations of the financial independence of local institutions is the existence of separate financial accounts for them [29]. Local institutions are distinguished by their financial accounts that are separate from the accounts of the state or the bodies to which they belong organically, in addition to the existence of an independent budget that differs from the general budget, which is an exception to the principle of budget unity. This exception is justified by the need to achieve the independence enjoyed by local institutions and their right to legal personality [30].

In France, since independent administrative authorities do not have legal personality, their budget is either linked to the Prime Minister, as is the case with the Superior Council for Audiovisual Communication and the Commission for Access to Administrative Documents, or to a minister, such as the Minister of Economy in the case of the Competition Council [31]. From neutrosophic perspective, this is not truly independence, but it is partially independent mixing with partially dependent which supports the notion of neutrosophic philosophical aspects that offered in def. (2).

In Iraq, all local institutions enjoy financial and administrative independence thanks to their legal personality, as stipulated in Article (2) of the Integrity Commission Law, which describes it as an independent body subject to the oversight of the House of Representatives, and with financial and administrative independence. Likewise, Article (5) of the Federal Financial Control Board Law stipulates that the Board is a financially and administratively independent body with legal personality.

3.2. Administrative Independence of Local Institutions.

Local institutions enjoy administrative independence, and we have previously mentioned that this independence is distinct in nature from the independence enjoyed by public institutions, as they are not subject to presidential and supervisory control.

Among the elements of administrative independence is the ability of local institutions to formulate their internal regulations independently [32]. In this context, the ability of local institutions to choose the rules that regulate their work without the need for ratification by the executive authority is evident, in addition to their freedom to choose their members and set rules for ending their duties. The laws specific to local institutions have confirmed this aspect, as stated in France in the decree of 26 July 1989 on the organization and functioning of the High Council for Audiovisual Communication [33]. Similarly, in Britain, NGOs have the authority to appoint a number of their employees without the need to follow the procedures in force for selecting public employees [34].

In Iraq, the Integrity Commission Law No. 30 of 2011 referred to this, stating: "The Commission has the right to issue internal regulations to organize its departments" and allows the Commission to determine the formations of its departments under this law, which facilitates the organization of its activities and management of its internal affairs in an independent manner according to the standards and needs it determines [35], as well as the Independent High Electoral Commission Law No. 11 of 2007, which stipulated: (The Board of Commissioners shall establish internal regulations for the Commission) [36]. However, the subordination of this authority in some cases and legal systems, such as in France, affects the regulatory authority granted to the goverment's president or the Prime Minister [37].

The subjection of the regulatory jurisdiction of some independent bodies to the regulatory authority granted to the Prime Minister is nothing more than a distribution of jurisdiction or a regulatory process to determine the jurisdiction of each of them, similar to the distribution of jurisdiction between the legislator and the government. It is worth noting that the rules issued by independent bodies must be consistent with the decisions issued by the Prime Minister or the Prime Minister. Independence appears from an administrative perspective in the inability to cancel or amend the decisions of independent bodies by a higher authority [38], but this

does not prevent the legislative authority from interfering in the systems established by local institutions; independence here is in the context of resisting governmental influence, which means that the legislator reserves the right to amend the rules related to these bodies, whether in terms of their formation, appointment of their members, or determination of their powers [39]. In the United States of America, Congress in its two chambers (the House of Representatives and the Senate) has the right to use the legislative path to amend the regulations of independent agencies, when either chamber deems those regulations inappropriate or illegal [33]. In addition to the manifestations of administrative independence mentioned above, it includes diversity in the composition of members in decision-making [39], and some legal texts related to independent bodies have revealed that reporting this nature does not require a text, as this can be inferred from their collective formation or taking the form of a committee or body. Administrative independence is also represented by its acquisition of legal personality, although there is a trend in jurisprudence that considers it not to be a decisive factor in measuring the degree of independence [1]. However, the arbitration of reality indicates otherwise, given the important results associated with granting legal personality as we mentioned. It is also mentioned that the appointment of a government commissioner in the independent administrative authorities in France to express its point of view does not affect functional independence [40]. On the other hand, it must be emphasized that administrative independence does not mean not being subject to oversight, as confirmed by the French State Council in its 2001 report, where it confirmed that independence does not exempt from responsibility.

The report on the administrative independence of local institutions, according to the aforementioned meaning, aims to strengthen the citizen's confidence in the political and administrative authority, and to ensure the fairness of state interventions [41]. From this standpoint, the administration was prevented from changing or canceling decisions issued by independent bodies except by virtue of a judicial decision [42].

4. Results

This section dedicated to four results have been concluded from this article:

1. Diversity of composition of members and comprehensive participation of the legislative and executive authorities in their appointment.
2. Guarantees against dismissal and protection from illegal interference.
3. Financial and administrative independence of local institutions.

4. Relative independence that appears in the subjection of bodies to oversight in its various forms.

These represent the basic points that distinguish the manifestations of independence of local institutions in many legal systems around the world.

5. Recommendations

We have reached a set of proposals that we think as capable of supporting the legal organization of independent bodies in Iraq, which are as follows:

1. We propose amending the constitution to ensure that the authority to appoint members of independent bodies is nominated by the Supreme Judicial Council and approved by the House of Representatives, in order to achieve the required independence in the face of the government.
2. We call on the Iraqi legislator to amend the term of office of members of independent bodies to six years, thus exceeding the term of office of the government and parliament, with no possibility of renewal.
3. We propose amending Article (61) Paragraph (e) of the eighth item of the Constitution so that the vote on the dismissal of officials of independent bodies shall be by a two-thirds majority.
4. We propose setting explicit and clear conditions for terminating the membership of members of independent bodies, based on what was applied in the Federal Public Service Law, to avoid any unspecified interpretation or explanation.

With regard to Articles (102), (103), and (104) of the Constitution of the Republic of Iraq, the terms liaison, responsibility, and oversight have raised wide controversy. We point out the following:

A- The terms oversight and responsibility refer to one meaning, which is political oversight organized by Parliament based on the texts of the Constitution, embodying the principle of legitimacy. The Iraqi legislator must be satisfied with the term oversight only, as mentioning responsibility implies direct oversight.

B- The term liaison does not raise any problem related to the independence of independent bodies, as the existence of loose bodies in the legal system cannot be considered logical. The connection of independent bodies to different authorities does not mean submission to presidential or guardianship oversight, but rather aims to embody legal and political unity. It must be emphasized that determining the connection of independent bodies to different authorities depends on the criterion of function; if the function is executive, a report on its connection to the Council of Ministers must be submitted, and if it is supervisory, its connection to the House of Representatives must be specified.

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References

1. Hadri Samir: Independence of the separating authorities in economic and financial matters, Master's thesis, Faculty of Law and Commercial Sciences, University of Oumed Boubakra, Boumerdes, 2006.
2. V. Kandasamy & F. Smarandache, Fuzzy Relational Maps and Neutrosophic Relational Maps, Rehoboth: American Research Press, 2004.
3. Here is a reformulated version of your text: "The State Council, in its 2001 report on jurisprudence and its 2000 advisory opinion, abolishes independent administrative authorities."
4. Law No. 73-6 of January 3, 1973. http://www.legifrance.gouv.fr. Article2.
5. Al-Hindawi Adnan, "The Oversight of Independent Bodies to Guarantee Individual Rights", Comparative Study, PhD Thesis, Faculty of Law, University of Baghdad, 1999, p. 86.
6. Ordinance No. 86-1243 of December 1, 1986. http://www.legifrance.gouv.fr artc (2,3).
7. Law No. 86-1067 of September 30, 1986. http://www.legifrance.gouv.fr.
8. Independence of Authorities: Evaluation of an Unspecified Legal Entity (Volume 2: Appendices). http://www.senat.fr/rap/r05-404-2/r05-404-249.html.
9. Maurice B. Fiorina et al.: The New American Democracy, First Arabic Edition, Translated by Lamis Fouad Al-Yahya, Al-Ahliya for Publishing and Distribution, Amman, 2008, p. 379. , and in the same sense:

   - Stephen v. Monsma: The American Political Approach, System,

   - Dryden Press, USA, 2nd, Ed, 1973, p.67.
10. Michael Cole: Quasi-Government in Britain: Origins and Consequences of the Persistence of the Term "Kwangju". Public Policy and Administration. 1998. p. 75.
11. Majeed Al-Anbaki: The British Constitution, Part One, Dar Al-Kutub and Al-Watha'iq, Baghdad, 2003, p. 86.
12. Paragraph Five, Legal Article 80 of the Constitution of the Republic of Iraq for the year 2005, published in the Iraqi Gazette, Issue 4012 on December 28, 2005.
13. Paragraph B of Clause Five, Legal Article (61) of the Constitution of the Republic of Iraq for the year 2005.

14. Clause Two, Legal Article 3 of the Independent High Electoral Commission Law No. 11 of 2007, as amended.
15. Article Seven, Law of the High Commission for Human Rights for the year No. 53 of 2008 AD.
16. Clause (First) Legal Article (8) of the Law of the High Commission for Human Rights for the year No. 53 of 2008 as amended.
17. Article 23, Constitution of the French Republic for the year 1958 as amended
18. Paragraph Six Second, Article (1) of the Constitution of the United States of America for the year 1787 as amended.
19. Clause Six, Article (49) Constitution of the Republic of Iraq for the year 2005.
20. Badir Ali, Protecting Individuals: The Mediator's Mission in the French System, Legal Journal, Issue Two, College of Law - University of Baghdad, 1996, p. 94.
21. Advisory Opinion of the Federal Supreme Court No. 228/T/2006 dated 10/9/2006. (Unpublished).
22. David K. Nichols: The Myth of the Modern American Presidency, translated by Sadiq Ibrahim Awda, Dar Al-Ahliya for Publishing and Distribution, Amman, 1997, pp. 154-156.
23. Paragraph (e) of item (eighth) of Article (61) of the Constitution of the Republic of Iraq for the year 2005, and this was confirmed by the laws related to local institutions; such as the text of item (first) of legal article (seventh) of the law related to the Integrity Commission No. 30 of 2011. Article (24) of the Federal Board of Financial Supervision No. 31 of 2011 as amended.
24. Article (6) of the Independent High Electoral Commission Law No. 11 of 2007 as amended. And Article (15) of the High Commission for Human Rights Law No. 53 of 2008 as amended.
25. Item (Eighth) Legal Article (8) of the Civil Service and Public Sector Employees Discipline Law No. (14) of 1991 as amended, published in the Iraqi Gazette, Issue 3356 on 6/3/1991.
26. Details of the term of membership of independent agencies in the United States of America: Morris B. Fiorina and others: The New American Democracy, previous source, p. 379.
27. Item (First) Legal Article (7) of the Independent High Electoral Commission Law No. 11 of 2007 as amended, Item (Third) Legal Article (4) of the Integrity Commission Law No. 30 of 2011, Item (Second) Legal Article (22) of the Federal Financial Supervision Board Law No. 31 of 2011 as amended, and Item (Sixth) Legal Article (8) of the High Commission for Human Rights Law No. 53 of 2008 as amended.
28. Saad Al-Aloush: The Theory of Public Institution and Its Application in Iraqi Legislation - A Comparative Study, Dar Al-Nahda Al-Arabiya, Cairo, 1968, p. 85.
29. The text of clause (fourth) of legal article (9) of the Independent High Electoral Commission Law No. 11 of 2007, as amended, states that the Commission shall have an independent annual budget prepared in accordance with the recognized principles and rules proposed by the Board of Commissioners in consultation with the Ministry of Finance, approved by the Council of Representatives and included in the general budget of the state), as stated in clause (fifth) of legal article (12) of the High Commission

for Human Rights Law No. 53 of 2008, as amended: (Proposing the financial budget of the Commission and submitting it to the Council of Representatives for approval), and also the text of clause (second) of legal article (6) of the Integrity Commission Law No.

30. Les autorités administratives indépendantes face aux autres institutions politiques et administratives http://www.vie-publique.fr/decouverte-institutions/institutions/approfondissements/autorites-administratives-independantes-face-aux-autres-institutions-politiques-administratives.html .
31. Item (First) Legal Article (2) of the Law of the High Commission for Human Rights No. 53 of 2008 as amended, and Article (2) of the Federal Public Service Council Law No. 4 of 2009.
32. Decree No. 89-518 dated July 26, 1989. Article 4. http://www.legifrance.gouv.fr
33. Abdullah Hanafi: Independence of Authorities, p. 168.
34. Item Six Article (6) "Integrity Commission Law" No. 30 of 2011.
35. Article (4) of the French Competition Council Law: Decree No. 86-1243 dated December 1, 1986 regarding freedom of prices and competition. http://www.legifrance.gouv.fr.
36. Decree No. 86-1243 of December 1, 1986 on freedom of prices and competition. http://www.legifrance.gouv.fr. Art. 15 and Article (32) of Decree 93-10 on the Algerian Stock Exchange for the year 1993. Published in the Algerian Official Gazette, No. (34) of May 23, 1993.
37. Independence of the authorities, on the website: http://www.djelfa.info/vb/archive/index.php/t-519361.html.p30
38. Issam Ibrahim Khalil: Legal organization and control of radio and television stations, previous source, pp. 265-266.
39. Item Five, Article (Third), Law of the Independent High Electoral Commission No. 11 of 2007 as amended, Item (First) Legal Article (10) of the Law of the High Commission for Human Rights No. 53 of 2008 as amended, Item (Second) Legal Article (8) of the Federal Public Service Council Law No. 4 of 2009.
40. Ibrahim Taha Al-Fayyadh: Procedures and Drafting of Rulings in the Administrative Judiciary, Bayt Al-Hikma, Baghdad, 1999, p. 85.
41. https://fr.wikipedia.org/wiki/Autorit%C3%A9
42. Shalabi Nasr and others: Independent Administrative Authorities.

Generating Neutrosophic Random Variables Based on Gamma Distribution

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

Khalifa Alshaqsi
Department of Mathematics, College of Arts and Sciences, University of Nizwa, Nizwa, OMAN

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Abstract

In practical life, we encounter many systems that cannot be studied directly, either due to their high cost or because some of these systems cannot be studied directly. Therefore, we resort to the simulation method, which depends on applying the study to systems similar to real ones and then projecting these results if they are suitable for the real system. The simulation process requires a good understanding of probability distributions and the methods used to transform random numbers that follow a regular distribution in the field [0,1] into random variables that follow them, so that we can achieve the greatest possible benefit from the simulation process and obtain more accurate and appropriate results for all conditions that arise. In previous research, we presented a neutrosophical vision of the process of generating random numbers that follow a regular distribution in the field [0, 1] and some techniques used to generate random variables, such as the inverse transformation technique that was used to generate random variables that follow a uniform distribution in the domain [a, b] and the exponential distribution, the rejection and acceptance technique, which was used to generate random variables that follow the beta distribution, and the mixed technique, which was used to generate random variables that follow the Poisson distribution. In this research, we present a neutrosophic study to generate neutrosophic random variables that follow the gamma distribution, a distribution that is frequently used in engineering applications.

Keywords: Simulation, Generating Random Numbers, Neutrosophic Logic, Generating Neutrosophic Random Numbers, Rejection and Acceptance Technique.

1 | Introduction

Operations research is the applied side of mathematics, and since its inception, it has helped improve the performance of many systems that use its methods in their workflow. Among the methods of operations research is the simulation method. The importance of the simulation process in all branches of science comes from the great difficulty that we can face when studying the operation of any system. From real systems, due to the high cost or the fact that some systems cannot be studied directly, the simulation process depends on generating a series of random numbers subject to a regular probability distribution in the range [0,1], then converting these random numbers into random variables subject to the law of probability distribution

according to which the system to be simulated operates, the attention of researchers and scholars interested in the simulation method has focused on providing scientific methods and methods that help in obtaining the transformation of random numbers into random variables that follow the probability distributions most used in practical applications, and in the face of the great revolution brought about by neutrosophic logic in All fields of science, which are confirmed by the research presented in this field [1-17], are a continuation of the research we presented previously, the aim of which is to present a neutrosophical vision of Monte Carlo simulation [18-22]. In this research, we present a neutrosophic study to generate neutrosophic random variables that follow the gamma distribution, a distribution that is frequently used in engineering applications.

2 | Discussion

In this paper we have founded a new type of hybrid algebra, called Neutrosophic TwoFold Algebra and its corresponding Neutrosophic TwoFold Law, by combining two algebras:

2.1 | The Principle Behind the Simulation Process

It is the generation of a series of random numbers that follow a uniform distribution over the field $[0,1]$, and then converting these random numbers into random variables that follow the distribution according to which the system to be simulated operates. There are many techniques used to generate these random numbers, such as the mean square, the product mean., the Fibonacci method and others, and to generate neutrosophic random numbers we presented a study in the research [18] and we used the mean square method and we reached the following results:

- The mean square method for generating a series of random numbers is defined by the following relation:

$$R_{i+1} = Mid\left[R_i^2\right]; i = 0,1,2,3, \dots$$

Where Mid symbolizes the middle four ranks of R_i^2, and R_i is chosen, i.e., a fractional random number consisting of four ranks (called a seed) that does not contain a zero in any of its four ranks.

- We convert these random numbers into neutrosophic random numbers and here we distinguish three states of the field $[0,1]$ with a margin of indeterminacy, in the three states we have $\varepsilon \in [0, n]$ and $0 < n < 1$

First case: $[0 + \varepsilon, 1]$ indeterminacy at the minimum of the range. We substitute in the relation:

$$R_{iN} = \frac{R_i - \varepsilon}{1 - \varepsilon}$$

Second case: $[0,1 + \varepsilon]$ indeterminacy at the upper limit of the range. We substitute in the relation:

$$R_{iN} = \frac{R_i}{1 + \varepsilon}$$

Third case: $[0 + \varepsilon, 1 + \varepsilon]$ Indeterminacy in the upper and lower limits of the field. We substitute in the relation:

$$R_{iN} = R_i - \varepsilon$$

2.2 | Generating Neutrosophic Random Variables

To generate neutrosophic random variables that follow the probability distribution according to which the system to be simulated operates, we distinguish three cases:

First case: Neutrosophic random numbers and probability distribution in the classical form.

Second case: Classical random numbers and the probability distribution is in the neutrosophic form.

Third case: Neutrosophic random numbers and probability distribution in neutrosophic form.

2.3 | Generating Neutrosophic Random Variables Based Gamma Distribution

The gamma distribution is one of the important distributions used in engineering applications. Given this importance and so that we can provide more accurate simulation results suitable for all conditions, we present in this research a neutrosophic study of the process of generating random variables that follow the gamma distribution, based on the classical and neutrosophic studies presented in the references [18- 25].

2.3.1 | Classic formula for gamma distribution [23, 24]

The probability density function is given by the following classical formula:

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad ; \quad x \geq 0 \, , \alpha > 0, \beta > 0$$

$$\Gamma(x) = \int_0^{\infty} x^{c-1} \, e^{-x} dx$$

where $x \geq 0 \, , \alpha > 0, \beta > 0$ are non-negative values,

Special cases of gamma distribution:

- As the value of α increases, the gamma distribution approaches the normal distribution.
- When $\alpha = 1$, the gamma distribution matches the exponential distribution.

2.3.2 | Neutrosophic Formula for Gamma Distribution [25]

If we take $\alpha \; or \; \beta$ or both, a neutrosophic number, where the neutrosophic number is written in the following standard form, $N = a + bI$, where $a \; and \; b$ are real or complex coefficients. a represents the definite part and bI represents the indefinite part (indeterminacy) of the number N, and it is possible that $[\lambda_1 \, , \, \lambda_2]$ or $\{\lambda_1 \, , \, \lambda_1\}$ or...otherwise it is any set close to the real value a,we obtain the neutrosophic gamma distribution. Here we distinguish the following cases:

1. We take α a neutrosophic value, β classic value, then the probability density function is written in the following form:

$$f_N(x) = \frac{\beta^{\alpha_N}}{\Gamma(\alpha_N)} x^{\alpha_N - 1} e^{-\beta . x}$$

$$x \geq 0 \, , \alpha_N > 0, \beta > 0$$

2. We take β a neutrosophic value, α classic value, then the probability density function is written in the following form:

$$f_N(x) = \frac{{\beta_N}^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta_N . x}$$

$$x \geq 0 \, , \alpha > 0, \beta_N > 0$$

Thus, we obtain the following special cases of the neutrosophic gamma distribution:

- As the value of α increases, the gamma distribution approaches the neutrosophic normal distribution.
- When $\alpha = 1$, the gamma distribution matches the neutrosophic exponential distribution.

3. We take α and β are neutrosophic values, that is, we take them in the form. Then the probability density function is written in the following form:

$$f_N(x) = \frac{{\beta_N}^{\alpha_N}}{\Gamma(\alpha_N)} x^{\alpha_N - 1} e^{-\beta_N . x}$$

$$x \geq 0\ , \alpha_N > 0, \beta_N > 0$$

Where $\beta_N = \beta \pm \gamma$ and $\alpha_N = \alpha \pm \delta$ and γ, δ is the indeterminacy of the mediators β, α and it can be any set close to the real values β, α.

2.3.3 | Generating Neutrosophic Random Variables Based Gamma Distribution

From the classical study presented in references [23,24] and the neutrosophic study presented in the references [18,21,22], the general formulation of the technique used to generate neutrosophic random variables following a gamma distribution is as follows:

1. We search for the largest value of the function $f_N(x)$ on the domain of its definition by calculating the first derivative and making it equal to zero. We obtain the following value:

$$x_N = \frac{\alpha_N - 1}{\beta_N} = mod$$

2. We replace this value with the function $f_N(x)$. We obtain M, the largest value of the function, which corresponds to the mode. Then we follow the following steps:
   a. We generate two random numbers from the range [0,1], such as $R_1\ , R_2$.
   b. We convert these two numbers into neutrosophic random numbers $R_{N1}\ , R_{N2}$.
   c. We form a new variable from one of the two numbers, for example $x_N = \frac{1}{R_{N1}}$, then the transformation field of x becomes $[0, \infty[$, which is identical to the distribution field $f_N(x)$.
   d. We compare R_{N2} with $\frac{f_N\left(\frac{1}{R_{N1}}\right)}{M}$.
   e. If $R_{N2} > \frac{f_N\left(\frac{1}{RN_1}\right)}{M}$, we consider the variable $x_N = \frac{1}{R_{N1}}$ to be a neutrosophic random variable subject to the gamma distribution.
   f. If $R_2 > x = \frac{1}{R_{N1}}$, we reject the numbers $R_{N1}\ , R_{N2}$ and return to step (a).

3 | Example

We explain the above through an example that corresponds to the third case of neutrosophic random numbers, and the gamma function is a neutrosophic function.

Find a random variable that follows the gamma distribution defined by the following probability density function:

$$f_N(x) = \frac{{\beta_N}^{\alpha_N}}{\Gamma(\alpha_N)} x^{\alpha_N - 1} e^{-\beta_N . x}$$

Where $x \geq 0\ , \beta_N \in [0.2.0.25], \alpha_N \in \{3,5\}$

Based on the data in the example, the study is divided into two parts:

3.1 | First Part

Data $x \geq 0\ , \alpha_N = 3\ , \beta_N \in [0.02,0.025]$

First, we calculate M from the following relation:

$$M_N = \frac{\alpha_N - 1}{\beta_N}$$

$$M_N = \frac{\alpha_N - 1}{\beta_N} = \frac{3-1}{[0.2,0.25]} = \frac{2}{[0.2,0.25]} \in [8,10]$$

a. We generate two random numbers that follow a uniform distribution in the range $[0,1]$ using the mean square method given by the following relation:

$$R_{i+1} = Mid\left[R_i^2\right]; i = 0,1,2,3, \dots$$

We take the seed $R_0 = 0.3176$ and from it we get $R_1 = 0.0869$, $R_2 = 0.7551$.

b. We convert the two numbers into two neutrosophic random numbers on the range $[0+\varepsilon, 1+\varepsilon]$ using the relation:

$$R_{iN} = R_i - \varepsilon$$

We take in this example $\varepsilon \in [0,0.03]$, we get the following two neutrosophic random numbers:

$$R_{1N} = 0.0869 - [0,0.03] \in [0.0569,0.0869]$$

$$R_{2N} = 0.7551 - [0,0.03] \in [0.7251,0.7551]$$

c. We form a new variable from one of the two numbers, for example

$$x_N = \frac{1}{R_{N1}} = \frac{1}{[0.0569,0.0869]} \in [11,5075,17.5746]$$

d. To compare $R_{2N} \in [0.7251,0.7551]$ with $\frac{f_N\left(\frac{1}{R_{N1}}\right)}{M_N}$.

- We calculate $f_N\left(\frac{1}{R_{N1}}\right)$ we find: $f_N(x) = \frac{{\beta_N}^{\alpha_N}}{\Gamma(\alpha_N)} x^{\alpha_N - 1} e^{-\beta_N . x}$

$$f_N([11,5075,17.5746]) = \frac{([0.2,0.25])^3}{\Gamma(3)} ([11.5075,17.5746])^2 e^{-[0.2,0.25].[11.5075,17.5746]}$$
$$\in [0.1766,0.8031]$$

- We calculate $\frac{f_N\left(\frac{1}{RN_1}\right)}{M_N}$ we find:

$$\frac{f_N\left(\frac{1}{RN_1}\right)}{M_N} = \frac{[0.1766,0.8031]}{[8,10]} \in [0.0221,0.0803]$$

We have, $R_{2N} \in [0.7251,0.7551]$ and $\frac{f_N\left(\frac{1}{R_{N1}}\right)}{M_N} \in [0.0221,0.0803]$

We note that: $[0.7251,0.7551] > [0.0221,0.0803] \Longrightarrow R_{2N} > \frac{f_N\left(\frac{1}{R_{N1}}\right)}{M_N}$

Check the acceptance criterion, so we consider $x_N = \frac{1}{R_{N1}} \in [11,5075,17.5746]$

To be a neutrosophic random variable subject to the gamma distribution.

3.2 | Second Part

Based on the data $x \geq 0$, $\alpha_N = 5$, $\beta_N \in [0.02,0.025]$

We calculate M from the following relation:

$$M_N = \frac{\alpha_N - 1}{\beta_N}$$

$$M_N = \frac{\alpha_N - 1}{\beta_N} = \frac{5-1}{[0.2,0.25]} = \frac{4}{[0.2,0.25]} \in [16,20]$$

a. We generate two random numbers that follow a uniform distribution in the range $[0,1]$ using the mean square method given by the following relation:

$$R_{i+1} = Mid\left[R_i^2\right]; i = 0,1,2,3, \dots$$

We take the seed $R_0 = 0.3176$ and from it we get $R_1 = 0.0869$, $R_2 = 0.7551$.

b. We convert the two numbers into two neutrosophic random numbers on the range $[0+\varepsilon, 1+\varepsilon]$ using the relation:

$$R_{iN} = R_i - \varepsilon$$

We take in this example $\varepsilon \in [0,0.03]$, we get the following two neutrosophic random numbers:

$$R_{1N} = 0.0869 - [0,0.03] \in [0.0569,0.0869]$$

$$R_{2N} = 0.7551 - [0,0.03] \in [0.7251,0.7551]$$

c. We form a new variable from one of the two numbers, for example:

$$x_N = \frac{1}{R_{N1}} = \frac{1}{[0.0569,0.0869]} \in [11,5075,17.5746]$$

d. To compare $R_{2N} \in [0.7251,0.7551]$ with $\frac{f_N\left(\frac{1}{R_{N1}}\right)}{M_N}$.

- We calculate $f_N\left(\frac{1}{R_{N1}}\right)$ we find: $f_N(x) = \frac{{\beta_N}^{\alpha_N}}{\Gamma(\alpha_N)} x^{\alpha_N - 1} e^{-\beta_N . x}$

$$f_N([11,5075,17.5746]) = \frac{([0.2,0.25])^5}{\Gamma(5)} ([11.5075,17.5746])^4 e^{-[0.2,0.25].[11.5075,17.5746]}$$
$$\in [0.1766,0.8031]$$

- We calculate $\frac{f_N\left(\frac{1}{R_{N1}}\right)}{M_N}$ we find: $\frac{f_N\left(\frac{1}{R_{N1}}\right)}{M_N} = \frac{[0.1766,0.8031]}{[16,20]} \in [0.011,0.0401]$

We have, $R_{2N} \in [0.7251,0.7551]$ and $\frac{f_N\left(\frac{1}{R_{N1}}\right)}{M_N} \in [0.011,0.0401]$

We note that: $[0.7251,0.7551] > [0.011,0.0401] \Longrightarrow R_{2N} > \frac{f_N\left(\frac{1}{R_{N1}}\right)}{M_N}$

Check the acceptance criterion, so we consider $x_N = \frac{1}{R_{N1}} \in [11,5075,17.5746]$

To be a neutrosophic random variable subject to the gamma distribution. From the results of the study in the first and second sections, we note that: $x_N = \frac{1}{R_{N1}} \in [11,5075,17.5746]$

Can be considered a neutrosophic random variable that follows a gamma distribution defined by the following probability density function:

$f_N(x) = \frac{{\beta_N}^{\alpha_N}}{\Gamma(\alpha_N)} x^{\alpha_N - 1} e^{-\beta_N . x}$, where $x \geq 0$, $\alpha_N \in \{3,5\}$, $\beta_N \in [0.2,0.25]$.

3.3 | Classic Wording of the Example

Find a random variable that follows the gamma distribution defined by the following probability density function:

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

Where $x \geq 0$, $\alpha = 3, \beta = 0.2$

Data $x \geq 0$, $\alpha = 3, \beta = 0.2$

First, we calculate M from the following relation:

$$M = \frac{\alpha - 1}{\beta}$$

$$M = \frac{\alpha - 1}{\beta} = \frac{3-1}{0.2} = \frac{2}{0.2} = 8$$

a. We generate two random numbers that follow a uniform distribution in the range $[0,1]$ using the mean square method given by the following relation:

$$R_{i+1} = Mid\left[R_i^2\right]; i = 0,1,2,3, \ldots$$

We take the seed $R_0 = 0.3176$ and from it we get $R_1 = 0.0869$, $R_2 = 0.7551$.

b. We form a new variable from one of the two numbers, for example R_1, we find:

$$x = \frac{1}{R_1} = \frac{1}{0.0869} = 11.5075$$

c. To compare R_2 with $\frac{f\left(\frac{1}{R_1}\right)}{M}$.

- We calculate $f\left(\frac{1}{R_1}\right)$ we find: $f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

$$f_N(11.5075) = \frac{(0.2)^3}{\Gamma(3)} (11.5075)^2 e^{-0.2.11.5075} = 0.1766$$

- We calculate $\frac{f\left(\frac{1}{R_1}\right)}{M}$ we find: $\frac{f\left(\frac{1}{R_1}\right)}{M} = \frac{0.1766}{8} = 0.0221$

We have, $R_2 = 0.7551$and $\frac{f\left(\frac{1}{R_1}\right)}{M} = 0.0221$

We note that: $0.7551 > 0.0221 \Longrightarrow R_2 > \frac{f\left(\frac{1}{R_1}\right)}{M}$

Check the acceptance criterion, so we consider: $x_N = \frac{1}{R_1} = 11.5075$

To be a neutrosophic random variable subject to the gamma distribution.

From the results of the study in the first and second sections, we note that:

$$x_N = \frac{1}{R_1} = 11.5075$$

Can be considered a neutrosophic random variable that follows a gamma distribution defined by the following probability density function:

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

Where $x \geq 0$, $\alpha = 3, \beta = 0.2$.

The classic example is treated in the same way as before if the data are:

Find a random variable that follows the gamma distribution defined by the following probability density function:

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad ; \quad x \geq 0 \, , \alpha > 0, \beta > 0,$$ where $x \geq 0$, $\alpha = 5, \beta = 0.2$.

4 | Conclusion and Results

In this research, we presented a neutrosophic study, the purpose of which is to obtain neutrosophic random variables that follow the gamma distribution, a distribution that has many uses in engineering applications. We found through this study that the neutrosophic values of the required random variables have a margin of freedom and take into account all the conditions that the system's operating environment passes through. What is intended to be simulated works according to the gamma distribution, while the classical study gives a specific value for the random variable that we need for the simulation process, a value appropriate to conditions similar to the conditions in which such data were adopted, and the results that we obtained through the neutrosophic presentation of the example and the classical presentation confirm these results, and therefore to obtain more accurate simulation results that take into account all conditions, we recommend using the neutrosophic study.

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Author Contributaion

All authors contributed equally to this work.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Doi :https://doi.org/10.54216/PAMDA.020102

[2] G. Dhanalakshmi, S. Sandhiya, Florentin Smarandache, Selection of the best process for desalination under a Treesoft set environment using the multi-criteria decision-making method,Doi:https://doi.org/10.54216/IJNS.230312, Vol,23, No,3

[3] Maissam Jdid, NEUTROSOPHIC TRANSPORT AND ASSIGNMENT ISSUES, Publisher: Global Knowledge's, ISBN, 978_1 _59973_770_6, (Arabic version). USA,2023. https://fs.unm.edu/NeutrosophicOptimization-ar.pdf

[4] Mountajab Alhasan, Malath F Alaswad. A Study of the Movement of a Neutrosophic Material Point in the Neutrosophic Plane by Using a Neutrosophic AH-Isometry, Journal Prospects for Applied Mathematics and Data Analysis, Doi:https://doi.org/10.54216/PAMDA.020101, Vol 2, No 1, 2023.

[5] Maissam Jdid, Florentin Smarandache, Graphical Method for Solving Neutrosophical Nonlinear Programming Models, International Journal of Data Science and Big Data Analytics, Vol,3, No,2. https://dx.doi.org/10.51483/IJDSBDA.3.2.2023.66-72

[6] Mona Gharib, Florentin Smarandache, Mona Mohamed, CSsEv: Modelling QoS Metrics in Tree Soft Toward Cloud Services Evaluator based on Uncertainty Environment, Doi:https://doi.org/10.54216/IJNS.230204, Vol,23, No,2

[7] Fatma Taher, Neutrosophic Multi-Criteria Decision-Making Methodology to Identify Key Barriers in Education, https://doi.org/10.54216/IJNS.220410

[8] Maissam Jdid, Neutrosophic linear models and algorithms to find their optimal solution, Biblio Publishing, ISBN, 978_1 _59973_778_2, (Arabic version). USA,2023, https://fs.unm.edu/NeutrosophicLinearModels-ar.pdf

[9] Antonios Paraskevas, Extended Event Calculus using Neutrosophic Logic: Method, Implementation, Analysis, Recent Progress and Future Directions, https://doi.org/10.61356/j.nswa.2024.116

[10] Marwah Yahya Mustafa , Zakariya Yahya Algamal, Neutrosophic inverse power Lindley distribution: A modeling and application for bladder cancer patients, Doi :https://doi.org/10.54216/IJNS.210218

[11] Maissam Jdid, Florentin Smarandache Converting Some Zero-One Neutrosophic Nonlinear Programming Problems into Zero-One Neutrosophic Linear Programming Problems, Neutrosophic Optimization and Intelligent systems, Vol. 1 (2024) 39-45, DOI: https://doi.org/10.61356/j.nois.2024.17489

[12] Renee Miriam M., Nivetha Martin , Aleeswari A. , Said Broumi, Rework Warehouse Inventory Model for Product Distribution with Quality Conservation in Neutrosophic Environment, Doi :https://doi.org/10.54216/IJNS.210215

[13] Maissam Jdid, Florentin Smarandache, A Study of Systems of Neutrosophic Linear Equations, Doi :https://doi.org/10.54216/IJNS.230202, Volume 23, Issue 2

[14] Nabil Khuder Salman. On the Symbolic 3-Plithogenic Real Functions by Using Special AH-Isometry. Vol. 59, 2023, pp. 139-152. DOI: zenodo.10031188/10.5281

[15] Mahmoud M. Ismail, Mahmoud M. Ibrahim and Shereen Zaki, A Neutrosophic Approach for Multi-Factor Analysis of Uncertainty and Sustainability of Supply Chain Performance, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 263-277. DOI: 10.5281/zenodo.8404468

[16] Said Broumi, S. krishna Prabha, Fermatean Neutrosophic Matrices and Their Basic Operations, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 572-595. DOI: 10.5281/zenodo.8404536

[17] Mohamed Abdel-Basset, Uncertainty-infused Representation Learning Using Neutrosophic-based Transformer Network, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 644-657. DOI: 10.5281/zenodo.10028758

[18] Maissam Jdid, Rafif Alhabib and A. A. Salama, Fundamentals of Neutrosophical Simulation for Generating Random Numbers Associated with Uniform Probability Distribution, Neutrosophic Sets and Systems, Vol. 49, 2022, pp. 92-102. DOI: 10.5281/zenodo.6426375

[19] Maissam Jdid, A. Salama, Using the Inverse Transformation Method to Generate Random Variables that follow the Neutrosophic Uniform Probability Distribution, Doi :https://doi.org/10.54216/JNFS.060202

[20] Maissam Jdid, Rafif Alhabib and A. A. Salama, The Basics of Neutrosophic Simulation for Converting Random Numbers Associated with a Uniform Probability Distribution into Random Variables Follow an Exponential Distribution, Neutrosophic Sets and Systems, Vol. 53, 2023, pp. 358-366. DOI: 10.5281/zenodo.7536049

[21] Maissam Jdid, Said Broumi, Neutrosophical Rejection and Acceptance Method for the Generation of Random Variables, Neutrosophic Sets and Systems, Vol. 56, 2023, pp. 153-166. DOI: 10.5281/zenodo.8194749

[22] Maissam Jdid and Nada A. Nabeeh, Generating Random Variables that follow the Beta Distribution Using the Neutrosophic Acceptance-Rejection Method, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 139-148. DOI: 10.5281/zenodo.8404445

[23] Alali. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004. (Arabic version).

[24] Bukajh JS -. Mualla, W... and others-Operations Research Book translated into Arabic - The Arab Center for Arabization, Translation, Authoring and Publishing -Damascus -1998. (Arabic version).

[25] F. Smarandache. Introduction to Neutrosophic statistics, Sitech & Education Publishing, 2014.

Generalized Neutrosophic Sampling Strategy for Elevated estimation of Population Mean

Subhash Kumar Yadav[1], Florentin Smarandache[2*]
1 Department of Statistics, Babasaheb Bhimrao Ambedkar University, Lucknow, INDIA, drskystats@gmail.com
2 Department of Mathematics, University of New Mexico, 705 Gurley Ave, Gallup, USA, fsmarandache@gmail.com
*Corresponding: fsmarandache@gmail.com

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Abstract: One of the disadvantages of the point estimate in survey sampling is that it fluctuates from sample to sample due to sampling error, as the estimator only provides a point value for the parameter under discussion. The neutrosophic approach, pioneered by Florentin Smarandache, is an excellent tool for estimating the parameters under consideration in sampling theory since it yields interval estimates in which the parameter lies with a very high probability. As a result, the neutrosophic technique, which is a generalization of classical approach, is used to deal with ambiguous, indeterminate, and uncertain data. In this investigation, we suggest a new general family of ratio and exponential ratio type estimators for the elevated estimation of neutrosophic population mean of the primary variable utilizing known neutrosophic auxiliary parameters. For the first degree approximation, the bias and Mean Squared Error (MSE) of the suggested estimators are computed. The neutrosophic optimum values of the characterizing constants are determined, as well as the minimum value of the neutrosophic MSE of the suggested estimator is obtained for these optimum values of the characterizing scalars. Because the minimum MSE of the classical estimators of population mean lies inside the estimated interval of the neutrosophic estimators, the neutrosophic estimators are better than the equivalent classical estimators. The empirical investigation, which used both real and simulated data sets, backs up the theoretical findings. For practical utility in various areas of applications, the estimator with the lowest MSE or highest Percentage Relative Efficiency (PRE) is recommended.

Keywords: Classical Ratio Estimators, Neutrosophic Estimators, Bias, MSE, PRE, Simulation.

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1. Introduction

Due to time and financial constraints, sampling becomes unavoidable when the population is big. The most apt estimator for the parameter under consideration is the corresponding statistic and so is the sample mean ($\bar{y}$) for the population mean ($\bar{Y}$) of main variable Y . Although $\bar{y}$ is an unbiased estimator of $\bar{Y}$, its sampling variance is rather high, hence the sampling distribution of $\bar{y}$ will not be very close to the genuine $\bar{Y}$. Therefore, we look for a population mean estimator that is even biased yet has a sampling distribution closer to the true $\bar{Y}$. The employment of an auxiliary variable (X) having a high degree of positive or negative association with $\bar{Y}$ achieves the goal of finding efficient estimators. The use of supplementary information to elevate the effectiveness of the estimators of the parameters under consideration is well established in sampling theory. For elevated estimation of $\bar{Y}$ using positively and negatively correlated auxiliary information with main variable, respectively, ratio and product technique of estimation processes are utilized with the condition that the line of regression pass through origin. If the line does not cross through the origin, the regression method of estimation is favored above the ratio and product approaches. The ratio method is preferred in real-world applications due to its broad

applicability; for example, area and production in crop yield applications, income and investment in business and economics, hospital infrastructure and health are some examples of applications, where ratio estimators are used to estimate . As a result, the current research focuses on estimating using known positively associated auxiliary variable.

1.2. Estimation under Classical Sampling Theory

In estimation methods of classical sampling theory, the data utilized for elevated estimation of $\bar{Y}$ using ratio, product, or regression type estimators are known and produced by crisp numbers. In classical statistics, various authors worked on numerous estimators of in the presence of known and suggested various ratio type estimators. In classical sampling theory, [1] introduced the conventional ratio estimator of using the positively correlated . As an auxiliary parameter, he utilized the known population mean ($\bar{X}$) of . Various authors later used well-known auxiliary parameters like coefficient of variation (CV), coefficient of skewness, coefficient of kurtosis, standard deviation, quartiles, and so on to improve the estimation of . [2] worked on a modified ratio estimator of utilizing the known CV of . For the elevated estimation of , [3] proposed the exponential ratio estimator employing a known . [4] proposed two ratio estimators for more efficient estimation of , utilizing known coefficient of kurtosis and the CV of . [5] focused on improving estimate utilizing known population correlation coefficient between and , and their results outperformed rival estimators. For increased estimate of , [6] suggested the modifications on ratio estimator of , that makes the use of known coefficient of kurtosis of . [7] proposed several modified ratio estimators of based on known information on some well-known auxiliary parameters. [8] suggested two ratio type estimators of utilizing known skewness and kurtosis of , which outperformed rival estimators. [9] presented an increased estimation approach for population mean using auxiliary parameters on characteristic. [10] worked in the direction of improving a family of ratio and product estimators of with known parameters of and [11] worked on a generic family of estimators of using transformed . [12] proposed a generalized family of dual to ratio-cum-product estimators with known auxiliary parameters. [13] developed a new ratio estimator for utilizing linear transformation of as minimum and maximum values. Using auxiliary parameters, [14] provided several efficient estimators for . [15] introduced a new family of estimators based on the main variable's known population median and shown improvement over the estimators in competition. [16] proposed a new modified ratio type estimator based on an auxiliary variable's exponential parameter. [17] proposed an improved family of $\bar{Y}$ estimators utilizing known parameters of and for improving the efficiency of the estimators, [18] used some well-known traditional and non-traditional auxiliary parameters. Many more authors have attempted to improve $\bar{Y}$ estimation using known data on traditional and non-traditional, robust and non-robust auxiliary parameters in classical sampling theory.

1.3. Estimation under Neutrosophic Sampling Theory

The data in classical sampling theory is mostly deterministic with no uncertainty in the measurements of the observations for the characteristics under investigation, however, we frequently encounter difficulties in everyday life where the data for the attributes under examination are not determined, for instance the measurement of temperature at any place along with other applications including information technology, information systems, decision support systems, financial data set detection, new economy growth, decline analysis, and more. In such cases, we seek alternate ways for dealing with undetermined

data, and the fuzzy logic pioneered by Prof. Lofti A. Zadeh in 1965 gives a solution for dealing with such data when exact measurements of the variable under examination are unavailable. Although fuzzy statistics deals with ambiguous, unclear, or imprecise data, it does not take into account the indeterminacy measurements. Neutosophic logic, further, is a generalized fuzzy logic that measures indeterminacy together with the determinate component of the observations and is utilized to analyze when the observations are imprecise or ambiguous, [19, 20]. [21] utilized the fuzzy logic in decision making for more precise decisions. Later different procedures using fuzzy logic have been developed and utilized extensively for making decisions in different areas of applications, [22-26]. [27] mentioned that the complex fuzzy sets are the advanced fuzzy sets and its generalization is the complex neutrosophic set. [28] suggested a diagram of fuzzy sets along with the generalizations of the sets and utilized the interval-valued neutrosophic sets for making decisions.

According to [29], Neutosophic statistics are used when data has some indeterminacy. Neutrosophic statistics is the extended form of classical statistics and are applied when the observations in the population or sample are imprecise, indeterminate, or vague. Further he mentioned that the methods of Neutrosophic statistics are utilized to analyze Neutrosophic data, which is indeterminate to some degree and the sample size may not be an exact number. In their works, [30] and [19] argue that neutrosophic statistics are particularly useful and acceptable for use in the system with the uncertainty. [31] used neutrosophic statistics to analyze the effect on scale and anisotropy for neutrosophic numbers of rock joint roughness coefficient. [20] focused on a Neutrosophic analysis of variance for university student data. [32] used a neutrosophic soft matrix (NSM) and relative weights of experts to develop an algorithmic strategy for group decision making (GDM) challenges. [33] used neutrosophic statistics to examine data from diabetes patients who had undergone a new diagnosis test. [34] worked on the estimation of the ratio of a crisp variable and a neutrosophic variable and shown improvement over the classical ratio method of estimation. [35] employed NEWMA chart and recurrent sampling to monitor road traffic crashes using neutrosophic statistics and in his research, [36] used neutrosophic statistics to develop a new goodness of fit test utilizing unclear parameters. In a study of skewness and kurtosis estimators of wind speed distributions under indeterminacy, [37] employed neutrosophic statistics. [38] devised a decision-making approach for determining the best fit of those damages in a neutrosophic environment, with the badly damaged machine receiving preference. [39] developed several new single-valued neutrosophic graph (SVNG) concepts, stating that the fuzzy set and the neutrosophic set are two effective instruments for dealing with the uncertainties and ambiguity of any real-world scenario.

When dealing with the uncertainties of a real-life scenario, the neutrosophic set outperforms the fuzzy set. [40] used neutrosophic parameterized hypersoft set theory to develop a decision-making application. They first conceptualized the neutrosophic parameterized hypersoft set, as well as some of its basic features and operations, and then used this theory to construct a decision-making-based method. For both one and two sample hypothesis testing situations, [41] suggested a modified Sign test that takes into account the indeterminate condition and true data form. They evaluated the suggested improved Sign test using two real data sets: covid-19 reproduction rate and covid-positive daily cases in ICU in Pakistan, and found that the suggested methodologies are appropriate for the problems of nonparametric in decision-making involving interval-valued data. To handle medical diagnostics and decision-making difficulties, [42] worked on algorithms for a generalization of multipolar neutrosophic soft set with measures of information. They proposed a general multipolar neutrosophic soft set, complete with operations and fundamental features. Later, they extended it to tackle decision-making problems by introducing various information measures for the generalization of multipolar neutrosophic soft set, such as distance, similarity, and correlation coefficient. [43] mentioned that in traditional survey sample studies where data is definite, certain, and unambiguous, the estimates are a single valued crisp results

that may be incorrect, overestimated, or understated, which might be a disadvantage. There are a variety of scenarios where data is neutrosophic in nature, and this is when Neutrosophic statistics is used instead of traditional approaches.

Uncertain and ambiguous values of the variables, non-clear contentions, and imprecise interval values are examples of neutrosophic data. As a result, data from trials or populations may be interval-valued neutrosophic numbers. The factual observation, that was ambiguous at the time of collecting, was thought to be a value within that range. There are more indeterminate data than definite data available in real life. As a result, more statistical techniques that are neutrosophic are needed. In real life, there are so many research variables that gathering information is quite costly, especially when the information is confusing. Thus, using traditional methods for indeterminate data to determine the unknown real value of the parameter will be dangerous and costly. After a thorough review of the literature, no study in sample surveys for ratio method of estimation for utilizing known under neutrosophic data has been found. There are not enough promising articles in this subject of statistics yet. There was no available solution to tackle the issue using ratio estimation when and were neutrosophic in nature. As a result, [43] presented a neutrosophic ratio-type estimation approach as the initial step in this direction. Further [43] mentioned that Neutrosophic Statistical analysis aids in the study of data with a degree of indeterminacy or insufficient knowledge, as well as conflicting beliefs. For the problem of indeterminacy, traditional statistics unsucceeded to analyze the data since certain observations were presented in a range of unknown values with the possibility of including a factual measurement within that range. As a result, in an uncertain environment, neutrosophic statistics is used, which is a more flexible alternative to and generalization of classical statistics. There have been numerous studies in the field of sample surveys under the Neutrosophy, where the method of ratio estimation is still new and necessitates a great deal of attention to the uncertain data system. For instance, the measurements of a machine product such as nuts or bolts may have slight measurement or manufacturing errors, and we may accept such product if it falls within the specified measurement range. Marks in grade system and health parameters through different testing procedures may be the areas of applications where neutrosophic statistics may be a better choice than the traditional one. Thus it is clear that in many situations, discussed above, the Neutrosophic estimators are used for improved estimation of population mean over the classical estimators where the observations of the study variable are not deterministic rather these are nondeterministic.

In this investigative work, we suggest a novel generalized neutrosophic ratio estimator for enhanced estimation of utilizing the known parameters of X . The sampling properties of the suggested estimator are studied for the first degree of approximation. The complete manuscript is being presented in different sections from introduction to the references.

1.4. Observations in Neutrosophic Environment and Notations

Quantitative neutrosophic data, where a number may lie in an uncertain interval [a, b], is one sort of observation in the neutrosophic environment, [30]. Neutosophic numbers' interval value can be represented in a variety of ways. [43] have defined neutrosophic interval values as $Z_N = Z_L + Z_U I_N$, where, $I_N \in [I_L, I_U]$. We also use the same notations of [43] for the considered neutrosophic data, which are in the interval form as $Z_N \in [Z_L, Z_U]$, where Z_L and Z_L are the lower and upper values of the neutrosophic variable Z_N. Let the neutrosophic population consists of N distinct units $(P_1, P_2, ..., P_N)$ and a neutrosophic random sample of size $n_N \in [n_L, n_U]$ is taken from the above population using simple random sampling without replacement (srswor) technique. Let $y_N(i)$ be the observation on the ith unit of

the sample for the neutrosophic data under consideration for the main variable y_N, of the form $y_N(i) \in [y_L, y_U]$ and by the same way for the auxiliary variable $x_N(i) \in [x_L, x_U]$. Let $\bar{y}_N(i) \in [\bar{y}_L, \bar{y}_U]$ be the sample mean for the neutrosophic study variable y_N and $\bar{x}_N(i) \in [\bar{x}_L, \bar{x}_U]$ be sample mean for the neutrosophic x_N which is correlated with y_N. Further let $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$ and $\bar{X}_N \in [\bar{X}_L, \bar{X}_U]$ be the population means for the neutrosophic variables y_N and x_N respectively, which are the overall averages of the neutrosophic data set. The neutrosophic coefficients of variation of y_N and x_N are given as $C_{yN} \in [C_{yNL}, C_{yNU}]$ and $C_{xN} \in [C_{xNL}, C_{xNU}]$ respectively. The correlation coefficient between the neutrosophic variables y_N and x_N is represented as $\rho_{yxN} \in [\rho_{yxNL}, \rho_{yxNU}]$. The neutrosophic coefficients of skewness and kurtosis for x_N are given by $\beta_{1(x)N} \in [\beta_{1(x)NL}, \beta_{1(x)NU}]$ and $\beta_{2(x)N} \in [\beta_{2(x)NL}, \beta_{2(x)NU}]$ respectively. The neutrosophic quartiles of x_N are given by $Q_{iN} \in [Q_{iNL}, Q_{iNU}]$, $i = 1, 3$ and the neutrosophic median of auxiliary variable as $M_{dN} \in [M_{dNL}, M_{dNU}]$.

1.5. Flow Chart of the Study

The graph given below represents the flow chart of the suggested study using neutrosophic numbers. The following chart is a recreated flow chart suggested by [43].

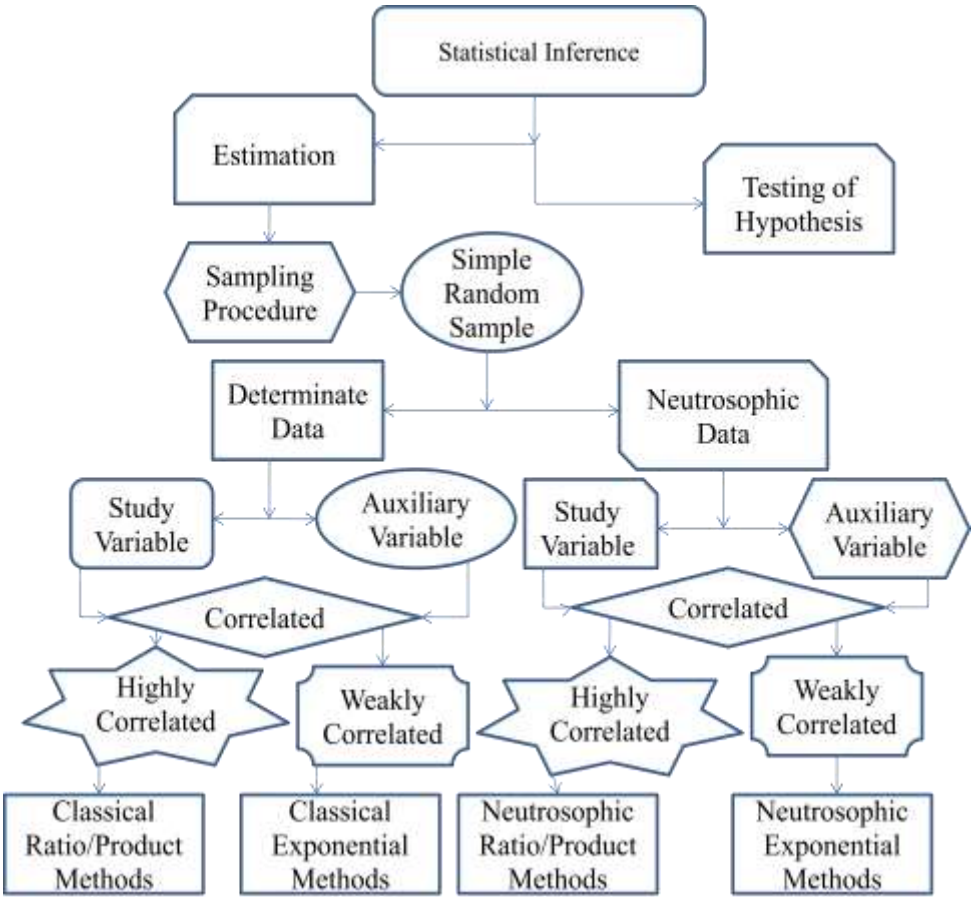

Figure-1: Flow chart of the study

1.6. Standard Approximations

Following are some standard approximations used for the sampling properties of the neutrosophic estimators, suggested by [43] as,

Let $\bar{e}_{yN} \in [\bar{e}_{yL}, \bar{e}_{yU}]$ and $\bar{e}_{xN} \in [\bar{e}_{xL}, \bar{e}_{xU}]$ be the mean errors for the study and the auxiliary neutrosophic variables along with $\bar{e}_{yN}(i) = \bar{y}_N(i) - \bar{Y}_N$ and $\bar{e}_{xN}(i) = \bar{x}_N(i) - \bar{X}_N$ respectively. The expectations of these errors for different orders are defined as;

$E(\bar{e}_{yN}) = E(\bar{e}_{xN}) = 0$ and,

$E(\bar{e}_{yN}^2) = \theta_N \bar{Y}_N^2 C_{yN}^2$, $E(\bar{e}_{xN}^2) = \theta_N \bar{X}_N^2 C_{xN}^2$, $E(\bar{e}_{yN}\bar{e}_{xN}) = \theta_N \bar{X}_N \bar{Y}_N \rho_{yxN} C_{yN} C_{xN}$

Where,

$\bar{e}_{yN} \in [\bar{e}_{yL}, \bar{e}_{yU}]$, $\bar{e}_{xN} \in [\bar{e}_{xL}, \bar{e}_{xU}]$, $\bar{e}_{yN}\bar{e}_{xN} \in [\bar{e}_{yL}\bar{e}_{xL}, \bar{e}_{yU}\bar{e}_{xU}]$, $\bar{e}_{yN}^2 \in [\bar{e}_{yL}^2, \bar{e}_{yU}^2]$, $\bar{e}_{xN}^2 \in [\bar{e}_{xL}^2, \bar{e}_{xU}^2]$,

$C_{xN}^2 = \frac{\sigma_{xN}^2}{\bar{X}_N^2}$, $C_{yN}^2 = \frac{\sigma_{yN}^2}{\bar{Y}_N^2}$, $C_{xN}^2 \in [C_{xL}^2, C_{xU}^2]$, $C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]$, $\rho_{yxN} = \frac{\sigma_{yxN}}{\sigma_{yN}\sigma_{xN}}$, $\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}]$,

$\theta_N = \frac{1 - f_N}{n_N}$, $\theta_N \in [\theta_L, \theta_U]$, $n_N \in [n_L, n_U]$, $\sigma_{xN}^2 \in [\sigma_{xL}^2, \sigma_{xU}^2]$, $\sigma_{yN}^2 \in [\sigma_{yL}^2, \sigma_{yU}^2]$, $\sigma_{yxN} \in [\sigma_{yxL}, \sigma_{yxU}]$

On the basis of the errors of the neutrosophic variables, bias and the Mean Squared Error (MSE) of the introduced and the competing estimators are obtained for an approximation of order one. The Bias and the MSE in neutrosophic environment are defined as, $Bias(\bar{y}_N) \in [Bias_L, Bias_U]$ and $MSE(\bar{y}_N) \in [MSE_L, MSE_U]$. Further the correlated auxiliary variables are used for the elevated estimation of $\bar{Y}_N$ and neutrosophic ratio type estimators are applied when there is indeterminacy in the data.

1.7. Review of Neutrosophic Estimators

The most appropriate neutrosophic estimator for the neutrosophic $\bar{Y}_N$ of is the corresponding neutrosophic sample mean and is given by,

$t_0 = \bar{y}_N$

The variance of the neutrosophic sample mean for the first degree of approximation is,

$V(t_0) = \theta_N \bar{Y}_N^2 C_{yN}^2$ (1)

Where, $t_{0N} \in [t_{0L}, t_{0U}]$

Using [1], [43] suggested the usual neutrosophic ratio estimator of $\bar{Y}_N$ using the known neutrosophic population mean of as,

$t_{RN} = \bar{y}_N \left(\frac{\bar{X}_N}{\bar{x}_N} \right)$

The bias and MSE of the neutrosophic ratio estimator t_{RN}, for an approximation of degree one respectively are,

$\text{Bias}(t_{RN}) = \theta_N \bar{Y}_N [C_{xN}^2 - C_{yxN}]$, where, $C_{yxN} = \rho_{yxN} C_{yN} C_{xN}$

$\text{MSE}(t_{RN}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + C_{xN}^2 - 2C_{yxN}]$ (2)

Where, $t_{RN} \in [t_{RL}, t_{RU}]$

Motivated by [2], [43] suggested the following neutrosophic ratio estimator using CV of neutrosophic variable as,

$$t_{1N} = \bar{y}_N \left(\frac{\bar{X}_N + C_{xN}}{\bar{x}_N + C_{xN}} \right)$$

The bias and MSE of the neutrosophic ratio estimator t_{1N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{1N}) = \theta_N \bar{Y}_N [\lambda_{1N}^2 C_{xN}^2 - \lambda_{1N} C_{yxN}]$$

$$\text{MSE}(t_{1N}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + \lambda_{1N}^2 C_{xN}^2 - 2\lambda_{1N} C_{yxN}] \quad (3)$$

Where, $\lambda_{1N} = \frac{\bar{X}_N}{\bar{X}_N + C_{xN}}$ and $t_{1N} \in [t_{1L}, t_{1U}]$

Based on [3], [43] proposed the following neutrosophic exponential ratio estimator as,

$$t_{2N} = \bar{y}_N \exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N} \right)$$

The bias and MSE of the neutrosophic exponential ratio estimator t_{2N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{2N}) = \theta_N \bar{Y}_N \left[\frac{3}{8} C_{xN}^2 - \frac{1}{2} C_{yxN} \right]$$

$$\text{MSE}(t_{2N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{C_{xN}^2}{4} - C_{yxN} \right] \quad (4)$$

Where, $t_{2N} \in [t_{2L}, t_{2U}]$

Motivated by [4], the two neutrosophic ratio estimators using CV and coefficient of kurtosis of may be given as,

$$t_{3N} = \bar{y}_N \left(\frac{C_{xN} \bar{X}_N + \beta_{2(x)N}}{C_{xN} \bar{x}_N + \beta_{2(x)N}} \right)$$

X

$$t_{4N} = \bar{y}_N \left(\frac{\beta_{2(x)N} \bar{X}_N + C_{xN}}{\beta_{2(x)N} \bar{x}_N + C_{xN}} \right)$$

The biases and MSEs of the neutrosophic ratio estimators t_{3N} and t_{4N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{3N}) = \theta_N \bar{Y}_N [\lambda_{3N}^2 C_{xN}^2 - \lambda_{3N} C_{yxN}]$$

$$\text{Bias}(t_{4N}) = \theta_N \bar{Y}_N [\lambda_{4N}^2 C_{xN}^2 - \lambda_{4N} C_{yxN}]$$

$$\text{MSE}(t_{3N}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + \lambda_{3N}^2 C_{xN}^2 - 2\lambda_{3N} C_{yxN}] \quad (5)$$

$$\text{MSE}(t_{4N}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + \lambda_{4N}^2 C_{xN}^2 - 2\lambda_{4N} C_{yxN}] \quad (6)$$

Where, $\lambda_{3N} = \frac{C_{xN} \bar{X}_N}{C_{xN} \bar{X}_N + \beta_{2(x)N}}$, $\lambda_{4N} = \frac{\beta_{2(x)N} \bar{X}_N}{\beta_{2(x)N} \bar{X}_N + C_{xN}}$ and $t_{3N} \in [t_{3L}, t_{3U}]$, $t_{4N} \in [t_{4L}, t_{4U}]$

Motivated by [5], the neutrosophic ratio estimator t_{5N}, using known population coefficient of correlation may be given as,

$$t_{5N} = \bar{y}_N \left(\frac{\bar{X}_N + \rho_{yxN}}{\bar{x}_N + \rho_{yxN}} \right)$$

The bias and MSE of the neutrosophic ratio estimator t_{5N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{5N}) = \theta_N \bar{Y}_N [\lambda_{5N}^2 C_{xN}^2 - \lambda_{5N} C_{yxN}]$$

$$\text{MSE}(t_{5N}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + \lambda_{5N}^2 C_{xN}^2 - 2\lambda_{5N} C_{yxN}] \quad (7)$$

Where, $\lambda_{5N} = \dfrac{\bar{X}_N}{\bar{X}_N + \rho_{yxN}}$ and $t_{5N} \in [t_{5L}, t_{5U}]$

[43] suggested the following neutrosophic ratio estimator by adapting the estimator by [6], using coefficient of kurtosis of as,

$$t_{6N} = \bar{y}_N \left(\frac{\bar{X}_N + \beta_{2(x)N}}{\bar{x}_N + \beta_{2(x)N}} \right)$$

The bias and MSE of the neutrosophic ratio estimator t_{6N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{6N}) = \theta_N \bar{Y}_N [\lambda_{6N}^2 C_{xN}^2 - \lambda_{6N} C_{yxN}]$$

$$\text{MSE}(t_{6N}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + \lambda_{6N}^2 C_{xN}^2 - 2\lambda_{6N} C_{yxN}] \quad (8)$$

Where, $\lambda_{6N} = \dfrac{\bar{X}_N}{\bar{X}_N + \beta_{2(x)N}}$ and $t_{6N} \in [t_{6L}, t_{6U}]$

Motivated by [44], the two neutrosophic ratio estimators using first and third quartiles of may be given as,

$$t_{7N} = \bar{y}_N \left(\frac{\bar{X}_N + Q_{1N}}{\bar{x}_N + Q_{1N}} \right)$$

$$t_{8N} = \bar{y}_N \left(\frac{\bar{X}_N + Q_{3N}}{\bar{x}_N + Q_{3N}} \right)$$

The biases and MSEs of the neutrosophic ratio estimators t_{7N} and t_{8N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{7N}) = \theta_N \bar{Y}_N [\lambda_{7N}^2 C_{xN}^2 - \lambda_{7N} C_{yxN}]$$

$$\text{Bias}(t_{8N}) = \theta_N \bar{Y}_N [\lambda_{8N}^2 C_{xN}^2 - \lambda_{8N} C_{yxN}]$$

$$\text{MSE}(t_{7N}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + \lambda_{7N}^2 C_{xN}^2 - 2\lambda_{7N} C_{yxN}] \quad (9)$$

$$\text{MSE}(t_{8N}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + \lambda_{8N}^2 C_{xN}^2 - 2\lambda_{8N} C_{yxN}] \quad (10)$$

Where, $\lambda_{7N} = \dfrac{\bar{X}_N}{\bar{X}_N + Q_{1N}}$, $\lambda_{8N} = \dfrac{\bar{X}_N}{\bar{X}_N + Q_{3N}}$ and $t_{7N} \in [t_{7L}, t_{7U}]$, $t_{8N} \in [t_{8L}, t_{8U}]$

Motivated by [8], the two neutrosophic ratio estimators using coefficients of skewness and kurtosis of may be represented as,

$$t_{9N} = \bar{y}_N \left(\frac{\overline{X}_N + \beta_{1(x)N}}{\bar{x}_N + \beta_{1(x)N}} \right)$$

$$t_{10N} = \bar{y}_N \left(\frac{\beta_{1(x)N}\overline{X}_N + \beta_{2(x)N}}{\beta_{1(x)N}\bar{x}_N + \beta_{2(x)N}} \right)$$

X

The biases and MSEs of the neutrosophic ratio estimators t_{9N} and t_{10N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{9N}) = \theta_N \overline{Y}_N [\lambda_{9N}^2 C_{xN}^2 - \lambda_{9N} C_{yxN}]$$

$$\text{Bias}(t_{10N}) = \theta_N \overline{Y}_N [\lambda_{10N}^2 C_{xN}^2 - \lambda_{10N} C_{yxN}]$$

$$\text{MSE}(t_{9N}) = \theta_N \overline{Y}_N^2 [C_{yN}^2 + \lambda_{9N}^2 C_{xN}^2 - 2\lambda_{9N} C_{yxN}] \quad (11)$$

$$\text{MSE}(t_{10N}) = \theta_N \overline{Y}_N^2 [C_{yN}^2 + \lambda_{10N}^2 C_{xN}^2 - 2\lambda_{10N} C_{yxN}] \quad (12)$$

Where, $\lambda_{9N} = \frac{\overline{X}_N}{\overline{X}_N + \beta_{1(x)N}}$, $\lambda_{10N} = \frac{\beta_{1(x)N}\overline{X}_N}{\beta_{1(x)N}\overline{X}_N + \beta_{2(x)N}}$ and $t_{9N} \in [t_{9L}, t_{9U}]$, $t_{10N} \in [t_{10L}, t_{10U}]$

Motivated by [45], the two neutrosophic ratio estimators using median and coefficients of variation of , we may define as,

$$t_{11N} = \bar{y}_N \left(\frac{\overline{X}_N + M_{d(x)N}}{\bar{x}_N + M_{d(x)N}} \right)$$

$$t_{12N} = \bar{y}_N \left(\frac{C_{xN}\overline{X}_N + M_{d(x)N}}{C_{xN}\bar{x}_N + M_{d(x)N}} \right)$$

X

The biases and MSEs of the neutrosophic ratio estimators t_{11N} and t_{12N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{11N}) = \theta_N \overline{Y}_N [\lambda_{11N}^2 C_{xN}^2 - \lambda_{11N} C_{yxN}]$$

$$\text{Bias}(t_{12N}) = \theta_N \overline{Y}_N [\lambda_{12N}^2 C_{xN}^2 - \lambda_{12N} C_{yxN}]$$

$$\text{MSE}(t_{11N}) = \theta_N \overline{Y}_N^2 [C_{yN}^2 + \lambda_{11N}^2 C_{xN}^2 - 2\lambda_{11N} C_{yxN}] \quad (13)$$

$$\text{MSE}(t_{12N}) = \theta_N \overline{Y}_N^2 [C_{yN}^2 + \lambda_{12N}^2 C_{xN}^2 - 2\lambda_{12N} C_{yxN}] \quad (14)$$

Where, $\lambda_{11N} = \frac{\overline{X}_N}{\overline{X}_N + M_{d(x)N}}$, $\lambda_{12N} = \frac{C_{xN}\overline{X}_N}{C_{xN}\overline{X}_N + M_{d(x)N}}$ and $t_{1N} \in [t_{11L}, t_{11U}]$, $t_{12N} \in [t_{12L}, t_{12U}]$

Motivated by [46], the neutrosophic ratio estimator t_{13N}, using known population coefficient of correlation may be given as,

$$t_{13N} = \bar{y}_N \left(\frac{\overline{X}_N + n_N}{\bar{x}_N + n_N} \right)$$

The bias and MSE of the neutrosophic ratio estimator t_{13N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{13N}) = \theta_N \bar{Y}_N [\lambda_{13N}^2 C_{xN}^2 - \lambda_{13N} C_{yxN}]$$
$$\text{MSE}(t_{13N}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + \lambda_{13N}^2 C_{xN}^2 - 2\lambda_{13N} C_{yxN}] \quad (15)$$

Where, $\lambda_{13N} = \dfrac{\bar{X}_N}{\bar{X}_N + n_N}$ and $t_{13N} \in [t_{13L}, t_{13U}]$

Motivated by [47], [43] suggested the following neutrosophic modified exponential ratio estimator as,

$$t_{14N} = \bar{y}_N \exp\left(\frac{(a\bar{X}_N + b) - (a\bar{x}_N + b)}{(a\bar{X}_N + b) + (a\bar{x}_N + b)} \right)$$

where, a and b are the neutrosophic auxiliary parameters.

The bias and MSE of the neutrosophic exponential ratio estimator t_{14N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{14N}) = \theta_N \bar{Y}_N \left[\frac{3}{8} \lambda_{14N}^2 C_{xN}^2 - \frac{1}{2} \lambda_{13N} C_{yxN} \right]$$
$$\text{MSE}(t_{14N}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + \lambda_{14N}^2 C_{xN}^2 - 2\lambda_{14N} C_{yxN}] \quad (16)$$

Where, $\lambda_{14N} = \dfrac{a\bar{X}_N}{2(a\bar{X}_N + b)}$ and $t_{14N} \in [t_{14L}, t_{14U}]$

Motivated by [48], [43] proposed the following generalized neutrosophic exponential ratio estimator as,

$$t_{15N} = \bar{y}_N \exp\left[\alpha \left(\frac{\bar{X}_N^{\frac{1}{h}} - \bar{x}_N^{\frac{1}{h}}}{\bar{X}_N^{\frac{1}{h}} + (a-1)\bar{x}_N^{\frac{1}{h}}} \right) \right]$$

where, α and h are the real known constants with $-\infty < \alpha < \infty$ and $h > 0$. The characterizing scalar a ($a \neq 0$) is determined so that the MSE of t_{15N} is minimum.

The bias and MSE of the neutrosophic generalized exponential ratio estimator t_{15N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{15N}) = \theta_N \bar{Y}_N \left[\frac{\alpha C_{xN}^2}{ah^2} - \frac{\alpha C_{xN}^2}{a^2 h^2} + \frac{\alpha^2 C_{xN}^2}{2a^2 h^2} - \frac{\alpha C_{yxN}}{ah} \right]$$
$$\text{MSE}(t_{15N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{\alpha^2 C_{xN}^2}{a^2 h^2} - \frac{2\alpha C_{yxN}}{ah} \right] \quad (17)$$

The optimum value of the characterizing constant a is obtained by minimizing $\text{MSE}(t_{15N})$ and the optimum value is,

$$a_{opt} = \frac{\alpha C_{xN}^2}{h C_{yxN}} \quad (18)$$

The minimum value of the $\text{MSE}(t_{15N})$ for the optimum value of a_{opt} is,

$$\text{MSE}_{\min}(t_{15N}) = \theta_N \bar{Y}_N^2 C_{yN}^2 (1 - \rho_{yxN}^2) \quad (19)$$

2. Material and Methods

Motivated by [49], we suggest a ratio cum exponential ratio class of neutrosophic main variable using the neutrosophic auxiliary parameters as,

$$t_{pN} = \kappa_1 \bar{y}_N \left(\frac{a\bar{X}_N + b}{a\bar{x}_N + b} \right) + \kappa_2 \bar{y}_N \exp\left(\frac{(a\bar{X}_N + b) - (a\bar{x}_N + b)}{(a\bar{X}_N + b) + (a\bar{x}_N + b)} \right)$$

Where, κ_1 and κ_2 are the characterizing scalars to be determine such that the MSE of t_{pN} is minimum. It is worth notable that,

(i) If $\kappa_2 = 0$, then the introduced estimator t_{pN} reduces to [49] ratio type estimators having different estimators by different authors as its special cases.

(ii) If $\kappa_2 = 0$ and $\kappa_1 = 1$, the introduced estimator t_{pN} reduces to ratio type estimators having different estimators by different authors as its special cases.

(iii) If $\kappa_1 = 0$, then the suggested class of estimators t_{pN} reduces to [49] exponential ratio type estimators having different estimators by different authors as its special cases.

(iv) If $\kappa_1 = 0$ and $\kappa_2 = 1$, the suggested family of estimators t_{pN} reduces to exponential ratio type estimators having different estimators by different authors as its special cases.

Expressing the introduced estimator in terms of $\bar{e}_{yN}$ and $\bar{e}_{xN}$, we have

$$t_{pN} = \kappa_1 \bar{Y}_N (1 + \bar{e}_{yN}) \left(\frac{a\bar{X}_N + b}{a\bar{X}_N (1 + \bar{e}_{xN}) + b} \right) + \kappa_2 \bar{Y}_N (1 + \bar{e}_{yN}) \exp\left(\frac{(a\bar{X}_N + b) - (a\bar{X}_N (1 + \bar{e}_{xN}) + b)}{(a\bar{X}_N + b) + (a\bar{X}_N (1 + \bar{e}_{xN}) + b)} \right)$$

$$= \bar{Y}_N (1 + \bar{e}_{yN}) \left[\kappa_1 \left(\frac{a\bar{X}_N + b}{a\bar{X}_N (1 + \bar{e}_{xN}) + b} \right) + \kappa_2 \exp\left(\frac{(a\bar{X}_N + b) - (a\bar{X}_N (1 + \bar{e}_{xN}) + b)}{(a\bar{X}_N + b) + (a\bar{X}_N (1 + \bar{e}_{xN}) + b)} \right) \right]$$

$$= \bar{Y}_N (1 + \bar{e}_{yN}) \left[\kappa_1 (1 + \lambda \bar{e}_{xN})^{-1} + \kappa_2 \exp\left(-\frac{\lambda \bar{e}_{xN}}{2} (1 + \frac{\lambda \bar{e}_{xN}}{2})^{-1} \right) \right]$$

Expanding the terms on the right hand side and simplifying and retaining the terms for the first degree of approximation, we get

$$t_{pN} = \bar{Y}_N [\kappa_1 (1 + \bar{e}_{yN} - \lambda \bar{e}_{xN} - \lambda \bar{e}_{yN} \bar{e}_{xN} + \lambda^2 \bar{e}_{xN}^2) + \kappa_2 (1 + \bar{e}_{yN} - \frac{\lambda \bar{e}_{xN}}{2} - \frac{\lambda \bar{e}_{yN} \bar{e}_{xN}}{2} + \frac{3}{8} \lambda^2 \bar{e}_{xN}^2)]$$

Subtracting $\bar{Y}_N$ on both sides of the above equation, we have

$$t_{pN} - \bar{Y}_N = \bar{Y}_N [\kappa_1 (1 + \bar{e}_{yN} - \lambda \bar{e}_{xN} - \lambda \bar{e}_{yN} \bar{e}_{xN} + \lambda^2 \bar{e}_{xN}^2) + \kappa_2 (1 + \bar{e}_{yN} - \frac{\lambda \bar{e}_{xN}}{2} - \frac{\lambda \bar{e}_{yN} \bar{e}_{xN}}{2} + \frac{3}{8} \lambda^2 \bar{e}_{xN}^2) - 1] \quad (20)$$

Taking expectations on both sides of (20) and putting values of different expectations, we get the bias of t_{pN} as,

$$Bias(t_{pN}) = \bar{Y}_N [\kappa_1 (1 - \lambda \theta_N \bar{Y}_N \bar{X}_N C_{yxN} + \lambda^2 \theta_N \bar{X}_N^2 C_{xN}^2) + \kappa_2 (1 - \frac{\lambda}{2} \theta_N \bar{Y}_N \bar{X}_N C_{yxN} + \frac{3}{8} \lambda^2 \theta_N \bar{X}_N^2 C_{xN}^2) - 1] \quad (21)$$

Squaring on both sides of (20), simplifying for the first degree of approximation, we get

$$(t_{pN} - \bar{Y}_N)^2 = \bar{Y}_N^2 \left\{ \begin{array}{l} 1 + \kappa_1^2 (1 + \bar{e}_{yN}^2 + 3\lambda^2 \bar{e}_{xN}^2 - 4\lambda \bar{e}_{yN} \bar{e}_{xN}) + \kappa_2^2 (1 + \bar{e}_{yN}^2 + \lambda^2 \bar{e}_{xN}^2 - 2\lambda \bar{e}_{yN} \bar{e}_{xN}) \\ - 2\kappa_1 (1 + \lambda^2 \bar{e}_{xN}^2 - \lambda \bar{e}_{yN} \bar{e}_{xN}) - 2\kappa_2 (1 + \frac{3}{8} \lambda^2 \bar{e}_{xN}^2 - \frac{\lambda}{2} \bar{e}_{yN} \bar{e}_{xN}) \\ + 2\kappa_1 \kappa_2 (1 + \bar{e}_{yN}^2 + \frac{15}{8} \lambda^2 \bar{e}_{xN}^2 - 3\lambda \bar{e}_{yN} \bar{e}_{xN}) \end{array} \right\}$$

Putting values of different expectations after taking expectation on both sides, we get the MSE of t_{pN} as,

$$MSE(t_{pN}) = \bar{Y}_N^2 \left\{ \begin{array}{l} 1+\kappa_1^2(1+\theta_N \bar{Y}_N^2 C_{yN}^2 + 3\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 4\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN}) \\ +\kappa_2^2(1+\theta_N \bar{Y}_N^2 C_{yN}^2 + \lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 2\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN}) \\ -2\kappa_1(1+\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - \lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN}) \\ -2\kappa_2(1+\frac{3}{8}\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - \frac{\lambda}{2}\theta_N \bar{Y}_N \bar{X}_N C_{yxN}) \\ +2\kappa_1\kappa_2(1+\theta_N \bar{Y}_N^2 C_{yN}^2 + \frac{15}{8}\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 3\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN}) \end{array} \right\} \quad (22)$$

$$MSE(t_{pN}) = \bar{Y}_N^2[1 + A\kappa_1^2 + B\kappa_2^2 - 2C\kappa_1 - 2D\kappa_2 + 2F\kappa_1\kappa_2] \quad (23)$$

Where,

$$A = (1+\theta_N \bar{Y}_N^2 C_{yN}^2 + 3\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 4\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN})$$

$$B = (1+\theta_N \bar{Y}_N^2 C_{yN}^2 + \lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 2\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN})$$

$$C = (1+\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - \lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN})$$

$$D = (1+\frac{3}{8}\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - \frac{\lambda}{2}\theta_N \bar{Y}_N \bar{X}_N C_{yxN})$$

$$F = (1+\theta_N \bar{Y}_N^2 C_{yN}^2 + \frac{15}{8}\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 3\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN})$$

The optimum values of the characterizing constants κ_1 and κ_2 which minimizes the MSE of the suggested estimator t_{pN} respectively are,

$$\kappa_{1(opt)} = \frac{(DF - BC)}{(F^2 - AB)} \text{ and } \kappa_{2(opt)} = \frac{(CF - AD)}{(F^2 - AB)}$$

The minimum value of the MSE of t_{pN} for these optimum values of κ_1 and κ_2 is,

$$MSE_{\min}(t_{pN}) = \bar{Y}_N^2 \left\{ 1 - \frac{\left\lfloor \begin{array}{l} 2C(DF - BC)(F^2 - AB) + 2D(CF - AD)(F^2 - AB) \\ -2F(DF - BC)(CF - AD) - (DF - BC) - (CF - AD) \end{array} \right\rfloor}{(F^2 - AB)^2} \right\} \quad (24)$$

$$MSE_{\min}(t_{pN}) = \bar{Y}_N^2 \left\{ 1 - \frac{P}{Q} \right\} \quad (25)$$

Where,

$$P = \left\lfloor \begin{array}{l} 2C(DF - BC)(F^2 - AB) + 2D(CF - AD)(F^2 - AB) \\ -2F(DF - BC)(CF - AD) - (DF - BC) - (CF - AD) \end{array} \right\rfloor$$

$$Q = (F^2 - AB)^2$$

3. Theoretical Efficiency Comparison

Under this section, we have compared the introduced neutrosophic estimator with the competing neutrosophic estimators of $\bar{Y}$ using the neutrosophic auxiliary parameters. The efficiency of the introduced estimator has been compared in terms of MSEs and the efficiency condition of the introduced estimator to be more efficient than the competing one is obtained.

The suggested estimator t_{pN} is more efficient than t_0 for the condition if,

$V(t_0) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N C_{yN}^2 - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} has lesser MSE than estimator t_{RN} for the following condition.

$MSE(t_{RN}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N [C_{yN}^2 + C_{xN}^2 - 2C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The suggested estimator t_{pN} is better than the estimator t_{1N} by [43] under the restriction if,

$MSE(t_{1N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N [C_{yN}^2 + \lambda_{1N}^2 C_{xN}^2 - 2\lambda_{1N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The suggested estimator t_{pN} is better than the exponential ratio type estimator t_{2N} by [43] for the condition if,

$MSE(t_{2N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N \left[C_{yN}^2 + \frac{C_{xN}^2}{4} - C_{yxN} \right] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} performs better than the estimator t_{3N} if,

$MSE(t_{3N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N [C_{yN}^2 + \lambda_{3N}^2 C_{xN}^2 - 2\lambda_{3N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} has lesser MSE than the estimator t_{4N} if it satisfies,

$MSE(t_{4N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N [C_{yN}^2 + \lambda_{4N}^2 C_{xN}^2 - 2\lambda_{4N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The proposed estimator t_{pN} is better than the estimator t_{5N} if,

$MSE(t_{5N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N [C_{yN}^2 + \lambda_{5N}^2 C_{xN}^2 - 2\lambda_{5N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The suggested estimator t_{pN} performs better than the ratio estimator t_{6N} by [43] if,

$MSE(t_{6N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N[C_{yN}^2 + \lambda_{6N}^2 C_{xN}^2 - 2\lambda_{6N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} is better than the ratio estimator t_{7N} under the condition if,

$MSE(t_{7N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N[C_{yN}^2 + \lambda_{7N}^2 C_{xN}^2 - 2\lambda_{7N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The proposed estimator t_{pN} has lesser MSE than the ratio estimator t_{8N} for the condition if,

$MSE(t_{8N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N[C_{yN}^2 + \lambda_{8N}^2 C_{xN}^2 - 2\lambda_{8N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} performs better than the estimator t_{9N} if,

$MSE(t_{9N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N[C_{yN}^2 + \lambda_{9N}^2 C_{xN}^2 - 2\lambda_{9N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} has lesser MSE than the estimator t_{10N} if,

$MSE(t_{10N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N[C_{yN}^2 + \lambda_{10N}^2 C_{xN}^2 - 2\lambda_{10N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The proposed estimator t_{pN} has lesser MSE in comparison to the ratio estimator t_{11N} under the condition if,

$MSE(t_{11N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N[C_{yN}^2 + \lambda_{11N}^2 C_{xN}^2 - 2\lambda_{11N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} is better than the ratio estimator t_{12N} for the condition if,

$MSE(t_{12N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N[C_{yN}^2 + \lambda_{12N}^2 C_{xN}^2 - 2\lambda_{12N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The proposed estimator t_{pN} perform better the estimator t_{13N} if,

$MSE(t_{13N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N[C_{yN}^2 + \lambda_{13N}^2 C_{xN}^2 - 2\lambda_{13N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The suggested estimator t_{pN} has lesser MSE with that of the ratio estimator t_{14N} under the condition if,

$MSE(t_{14N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N[C_{yN}^2 + \lambda_{14N}^2 C_{xN}^2 - 2\lambda_{14N} C_{yxN}] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} is better than that of the estimator t_{15N} if,

$\mathrm{MSE}_{\min}(t_{15N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N C_{yN}^2 (1 - \rho_{yxN}^2) - \left\{1 - \frac{P}{Q}\right\} > 0$$

4. Simulation Study

To verify the theoretical efficiency conditions and evaluate the efficiencies of the suggested and competing neutrosophic estimator of $\overline{Y}$ utilizing known auxiliary parameters, we have simulated a neutrosophic data set using the same parameters of [43]. To generate the neutrosophic data, we have considered that the neutrosophic main and auxiliary random variables Y_N and X_N follow the neutrosophic normal distributions. Thus $Y_N \sim NN(\mu_{yN}, \sigma_{yN}^2)$; $Y_N \in (Y_L, Y_U)$, $\mu_{yN} \in (\mu_{yL}, \mu_{yU})$, $\sigma_{yN}^2 \in (\sigma_{yL}^2, \sigma_{yU}^2)$ and $X_N \sim NN(\mu_{xN}, \sigma_{xN}^2)$; $X_N \in (X_L, X_U)$, $\mu_{xN} \in (\mu_{xL}, \mu_{xU})$, $\sigma_{xN}^2 \in (\sigma_{xL}^2, \sigma_{xU}^2)$. For the numerical illustration, we have taken $Y_N \sim NN([76.0, 84.9], [(12.9)^2, (17.2)^2])$, where, $\mu_{yN} \in (76.0, 84.9)$, $\sigma_{yN} \in (12.9, 17.2)$ and $X_N \sim NN([171.2, 180.4], [(5.8)^2, (6.7)^2])$, where, $\mu_{xN} \in (171.2, 180.4)$, $\sigma_{xN} \in (5.8, 6.7)$ and generated 1000 normal random observation for both the variables. The descriptive statistics for the simulated data is presented in Table 1.

Table 1. Descriptive statistics of the simulated data for the neutrosophic data

Parameter	Neutrosophic	Value Parameter	Neutrosophic Value
N_N	[1000,1000]	C_{xN}	[0.0332, 0.0369]
n_N	[20, 20]	$\beta_{1(x)N}$	[0.0020, 0.0051]
μ_{yN}	[76.20, 85.63]	$\beta_{2(x)N}$	[3.0227, 2.9539]
μ_{xN}	[171.08,180.34]	$Q_{1(x)N}$	[167.3941,176.1144]
σ_{yN}	[12.79,17.37]	$M_{d(x)N}$	[170.9067,180.3451]
σ_{xN}	[5.67, 6.65]	$Q_{3(x)N}$	[174.9269,184.7586]
C_{yN}	[0.1679, 0.2028]	ρ_{yxN}	[0.01933,0.00703]

The Table 2 is representing the neutrosophic MSEs of different competing along with the suggested estimator of population mean.

Table 2. Neutrosophic MSEs of different competing and suggested estimator

SR. No.	Estimators	MSE
1.	t_0	[8.019213, 14.77799]
2.	t_{RN}	[17.39673, 27.98680]
3.	t_{1N}	[17.39674, 27.98681]
4.	t_{2N}	[8.066852, 14.8812]
5.	t_{3N}	[17.39709, 27.98701]

6.	t_{4N}	[17.39674, 27.98681]
7.	t_{5N}	[17.39674, 27.98681]
8.	t_{6N}	[17.3978, 27.98741]
9.	t_{7N}	[17.42703, 28.00546]
10.	t_{8N}	[17.4277, 28.00592]
11.	t_{9N}	[17.39673, 27.98681]
12.	t_{10N}	[17.4517, 28.01563]
13.	t_{11N}	[17.42734, 28.00569]
14.	t_{12N}	[17.45602, 28.02323]
15.	t_{13N}	[17.40314, 27.99058]
16.	t_{14N}(a=1,b=0)	[17.42736, 28.00569]
17.	t_{14N}(a=1,b=1)	[17.42754, 28.00579]
18.	t_{15N}	[8.016216, 14.77726]
19.	t_{pN}	**[7.864525, 13.821846]**

5. Results and Discussion

From Table 2, it may clearly be observed that the estimator t_0 of $\overline{Y}_N$ has its neutrosophic sampling variance as [8.019213, 14.77799] and the neutrosophic MSE of the exponential ratio estimator t_{2N} is [8.066852, 14.8812] while the neutrosophic MSEs of all the mentioned ratio type estimators lie in the interval [17.45602, 28.02323]. The neutrosophic ratio type estimators have high MSEs than the neutrosophic estimator t_0 because of the low neutrosophic correlation between neutrosophic y and x. The neutrosophic MSE of the suggested class of estimators is [7.864525, 13.821846], which is the minimum among the group of all neutrosophic estimators of $\overline{Y}_N$ in competition.

6. Conclusion

In this scripture, we have suggested a novel family of neutrosophic estimators of $\overline{Y}_N$ for the elevated estimation of neutrosophic $\overline{Y}_N$ using the known neutrosophic auxiliary parameters. We studied the neutrosophic sampling properties mainly bias and MSE of the proposed family of estimators for the approximation of degree one. The neutrosophic optimum values of the characterizing scalars of the introduced estimator are obtained and the neutrosophic minimum MSE of the suggested estimator has also been obtained for these neutrosophic optimum values of the characterizing scalars. The introduced estimator has been compared with the neutrosophic competing estimators theoretically and the efficiency condition over the competing estimators have been obtained. These efficiency conditions are verified using a neutrosophic simulated data set. The results in Table-2 are showing that the suggested estimator is most efficient among the class of all neutrosophic competing estimators of $\overline{Y}_N$. Thus the introduced class of estimators may be recommended for elevated estimation of neutrosophic $\overline{Y}_N$ in different areas of applications. It is to be mentioned here that the neutrosophic estimators are most suitable for improved estimation of population mean for the situations where the observations of the study variable are nondeterministic but for the situation where its observations are deterministic, it may be inferior to the classical estimators.

References

[1] Cochran, W. G. The Estimation of the Yields of the Cereal Experiments by Sampling for the Ratio of Grain to Total Produce. *The Journal of Agric. Science* **1940**, 30, 262-275.

[2] Sisodia, B.V.S. and Dwivedi, V.K. A modified ratio estimator using coefficient of variation of auxiliary variable, *Journal of the Indian Society of Agricultural Statistics* **1981**, 33, 13-18.

[3] Bahl, S. and Tuteja, R.K. Ratio and product type exponential estimator, Information and Optimization Sciences **1991**, XII (I), 159-163.

[4] Upadhyaya, L. N. and Singh, H. P. Use of transformed auxiliary variable in estimating the finite population mean, *Biometrical Journal* **1999**, 41, 627-636.

[5] Singh, H.P. and Tailor, R. Use of known correlation coefficient in estimating the finite population mean, *Statistics in Transition* **2003**, 6, 4, 555-560.

[6] Singh, H.P., Tailor, R., Tailor, R. and Kakran, M.S. An Improved Estimator of population means using Power transformation. Journal of the Indian Society of Agricultural Statistics **2004**, 58(2), 223-230.

[7] Kadilar, C. and Cingi, H. Improvement in estimating the population mean in simple random sampling, *Applied Mathematics Letters* **2006**, 19, 75-79.

[8] Yan, Z. and Tian, B. Ratio method to the mean estimation using coefficient of skewness of auxiliary variable. *ICICA, Part II, CCIS* **2010**, 106, 103-110.

[9] Singh, H.P. and Solanki, R.S. Improved estimation of population mean in simple random sampling using information on auxiliary attribute, *Applied Mathematics and Computation* **2012**, 218, 15, 7798-7812.

[10] Yadav, S.K. and Kadilar, C. Improved class of ratio and product estimators. *Applied Mathematics and Computation*, **2013**, 219 (2)2, 10726-10731.

[11] Grover, L.K. and Kaur, P. A generalized class of ratio type exponential estimators of population mean under linear transformation of auxiliary variable. *Communications in Statistics-Simulation and Computation* **2014**, 43, 1552-1574.

[12] Vishwakarma, G.K. and Kumar, M. A general family of dual to ratio-cum-product estimators of population mean in simple random sampling, *Chilean Journal of Statistics* **2015**, 6 (2), 69-79.

[13] Cekim, H.O. and Cingi, H. Some estimator types for population mean using linear transformation with the help of the minimum and maximum values of the auxiliary variable, *Hacettepe Journal of Mathematics and Statistics* **2017**, 46 (4), 685-694.

[14] Subzar, M. Maqbool, S. Raja, T.A. Pal, S.K. and Sharma, P. Efficient Estimators of Population Mean Using Auxiliary Information under Simple Random Sampling, *Statistics in Transition new series* **2018**, 19 (2), 219-238.

[15] Yadav, S.K. Sharma, D.K. and Brown, K. New class of estimators of the population mean using the known population median of the study variable, *International Journal of Mathematics in Operational Research* **2020**, 16 (2), 179-201.

[16] Zaman, T. and Dunder, E. Proposing Novel Modified Ratio Estimators by Adding an Exponential Parameter, *Lobachevskii Journal of Mathematics* **2020**, 41 (3), 451-458.

[17] Yadav, S.K. Sharma, D.K. and Baghel, S. Upgraded family of estimators of population mean using known parameters of auxiliary and study variables. *International Journal of Mathematical Modelling and Numerical Optimisation* **2021**, 11(3), 252-274.

[18] Yadav, S.K. and Zaman, T. Use of some conventional and non-conventional parameters for improving the efficiency of ratio-type estimators, Journal of Statistics and Management Systems **2021**, DOI: 10.1080/09720510.2020.1864939.

[19] Aslam, M. A new sampling plan using neutrosophic process loss consideration. *Symmetry* **2018**, 10(5), 132. https://doi.org/10.3390/sym10050132

[20] Aslam, M. Neutrosophic analysis of variance: application to university students. *Complex Intell Syst* **2019**, 5, 403-407. https://doi.org/10.1007/s40747-019-0107-2

[21] Bellman R.E. and Zadeh L.A. Decision making in a fuzzy environment, *Management Science* **1970**, 17, 140-164.

[22] Jan, N. Zedam, L. Mahmood, T. Ullah, K. Ali, Z. Multiple attribute decision making method under linguistic cubic informa tion. *Journal of Intell Fuzzy Syst* **2019**, 36, 253-269. https://doi.org/10.3233/JIFS-181253

[23] Ali, Z. and Mahmood, T. Complex neutrosophic generalized dice similarity measures and their application to decision making. *CAAI Trans Intelligence Technol* **2020**, 5, https://doi.org/10.1049/trit.2019.0084

[24] Xue, Y. and Deng, Y. Decision making under measure-based granular uncertainty with intuitionistic fuzzy sets. *Appl Intell* **2021**, 51**,** 6224-6233. DOI: 10.1007/s10489-021-02216-6

[25] Chang, J-F., Lai, C-J., Wang, C-N., Hsueh, M-H and Nguyen, V.T. Fuzzy Optimization Model for Decision-Making in Supply Chain Management, *Mathematics* **2021**, 9, 312. DOI: 10.3390/math9040312

[26] Riaz, M. Hamid, M.T. Afzal, D. Pamucar, D. Chu, Y-M. Multi-criteria decision making in robotic agri-farming with q-rung orthopair m-polar fuzzy sets. *PLoS ONE* **2021**, 16(2), e0246485. https://doi.org/10.1371/journal.pone.0246485

[27] Liu, P. Ali, Z. Mahmood, T. The distance measures and cross-entropy based on complex fuzzy sets and their application in decision making. J Intell Fuzzy Syst **2020**, 39, 1-24. https://doi.org/10.3233/JIFS-191718

[28] Li, D-F., Mahmood, T., Ali, Z., Dong Y. Decision making based on interval-valued complex single-valued neutrosophic hesitant fuzzy generalized hybrid weighted averaging operators. J Intell Fuzzy Syst **2020**, 38, 4359-4401. https://doi.org/10.3233/JIFS-191005

[29] Smarandache, F. Neutrosophy: *neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis.*: American Research Press, **1998**.

[30] Smarandache, F. *Introduction to neutrosophic statistics*: Sitech & Education Publishing, **2014**.

[31] Chen, J. Ye, J. Du, S. Scale effect and anisotropy analyzed for neutrosophic numbers of rock joint roughness coefficient based on neutrosophic statistics. *Symmetry* **2017**, 9(10), 208. https://doi.org/10.3390/sym9100208

[32] Das, S. Kumar, S. Kar, S. Pal, T. Group decision making using neutrosophic soft matrix: An algorithmic approach, *Journal of King Saud University-Computer and Information Sciences* **2019**, 31, 459-468.

[33] Aslam, M. Al-Shareef, A. Khan, K. Monitoring the temperature through moving average control under uncertainty environment, *Scientific Reports* **2020**, 10, 12182, Doi.org/10.1038/s41598-020-69192-8

[34] Bouza-Herrera, C.N. Subzar, M. Estimating the Ratio of a Crisp Variable and a Neutrosophic Variable, *International Journal of Neutrosophic Science* **2020**, 11(1), 9-21.

[35] Aslam, M. Monitoring the road traffic crashes using NEWMA chart and repetitive sampling, *Int J Inj Contr Saf Promot* **2021a**, 28(1), 39-45. https://doi.org/10.1080/17457300.2020.1835990

[36] Aslam, M. A new goodness of fit test in the presence of uncertain parameters, *Complex Intell Syst* **2021b**, 7(1), 359-365. https://doi.org/10.1007/s40747-020-00214-8

[37] Aslam, M. A study on skewness and kurtosis estimators of wind speed distribution under indeterminacy, *Theoret Appl Climatol* **2021c**, 143(3), 1227-1234. https://doi.org/10.1007/s00704-020-03509-5

[38] Chinnadurai, V. Sindhu, M.P. Bharathivelan, K. An Introduction to Neutro-Prime Topology and Decision Making Problem, *Neutrosophic Sets and Systems* **2021**, 41, 146-167.

[39] Mohanta, K. Dey, A. Pal, A. A note on different types of product of neutrosophic graphs, *Complex & Intelligent Systems* **2021**, 7, 857-871. Doi.org/10.1007/s40747-020-00238-0

[40] Rahman, A.U. Saeed, M. Dhital, A. Decision Making Application Based on Neutrosophic Parameterized Hypersoft Set Theory, *Neutrosophic Sets and Systems* **2021**, 41, 1-14.

[41] Sherwani, R.A.K. Shakeel, H. Saleem, M. Awan, W.B. Aslam, M. Farooq, M. A new neutrosophic sign test: An application to COVID-19 data, *PLoS ONE* **2021**, 16(8), e0255671. https://doi.org/10.1371/journal.pone.0255671

[42] Zulqarnain, R.M. Garg, H. Siddique, I. Ali, R. Alsubie, A. Hamadneh, N.N Khan, I. Algorithms for a Generalized Multipolar Neutrosophic Soft Set with Information Measures to Solve Medical Diagnoses and Decision-Making Problems, *Journal of Mathematics* **2021**, https://doi.org/10.1155/2021/6654657

[43] Tahir, Z. Khan, H. Aslam, M. Shabbir, J. Mahmood, Smarandache, F. Neutrosophic ratio-type estimators for estimating the population mean, Complex & Intelligent Systems, **2021**, Doi.org/10.1007/s40747-021-00439-1

[44] Al-Omari, A.I. Ratio estimation of the population mean using auxiliary information in simple random sampling and median ranked set sampling, *Statistics and Probability Letters* **2012**, 82, 1883-18890.

[45] Subramani, J. and Kumarapandiyan, G. Estimation of Population Mean Using Coefficient of Variation and Median of an Auxiliary Variable, *International Journal of Probability and Statistics* **2012**, 1(4), 111-118.

[46] Jerajuddin, M. and Kishun, J. Modified Ratio Estimators for Population Mean Using Size of the Sample, Selected From Population, *IJSRSET* **2016**, 2(2), 10-16.

[47] Singh, R. Chauhan, P. Sawan, N. Smarandache, F. *Auxiliary information and a priori values in construction of improved estimators*: Infinite Study, **2007**.

[48] Khan, H. Sanaullah, A. Amjad, M. Siddiqi, A. Improved exponential ratio type estimators for estimating population mean regarding full information in survey sampling, *World Applied Science Journal* **2014**, 26(5), 1897-1902.

[49] Searls, D. T. The Utilization of a Known Coefficient of Variation in the Estimation Procedure, *Journal of the American Statistical Association* **1964**, 59, 1225-1226.

Enhanced Neutrosophic Set and Machine Learning Approach for Breast Cancer Prediction

Ashika T[1], Hannah Grace[2],* Nivetha Martin[3], and Florentin Smarandache[4]

[1]Vellore Institute of Technology, Department of Mathematics, School of Advanced Sciences, Chennai, India.
E-mail: ashika.t2023@vitstudent.ac.in
[2]Vellore Institute of Technology, Department of Mathematics, School of Advanced Sciences, Chennai, India.
E-mail: hannahgrace.g@vit.ac.in
[3]Arul Anandar College (Autonomous), Karumathur, Madurai, India. E-mail: nivetha.martin710@gmail.com
[4] Emeritus Professor, University of New Mexico, Gallup, United States. E-mail: fsmarandache@gmail.com
* Correspondence: hannahgrace.g@vit.ac.in

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Abstract: Breast cancer is the most prevalent type of cancer that affects women worldwide and poses a serious risk to female mortality. In order to lower death rates and enhance treatment results, early detection is critical. Neutrosophic Set Theory (NST) and machine learning (ML) approaches are integrated in this study to provide a novel hybrid methodology (NS-ML) that improves breast cancer diagnosis. Using the Wisconsin Diagnostic Breast Cancer (WDBC) dataset, the research transforms these data into Neutrosophic (N) representations to effectively capture uncertainties. When trained on the N-dataset instead of traditional datasets, ML algorithms such as Decision Tree (DT), Random Forest (RF), and Adaptive Boosting (AdaBoost) perform better. Notably, N-AdaBoost models achieve outstanding results with 99.12% accuracy and 100% precision, highlighting the efficacy of NS in enhancing diagnostic reliability.

Keywords: Neutrosophic sets; Machine Learning; Uncertainity handling; Breast cancer; Classification.

1. Introduction

Women globally have the risk of breast cancer which as a risk factor of cancer due to abnormal growth of cells in breast tissue [1]. Countries like Belgium and the Netherlands reported high rates of incidence, affecting thousands of women, while Barbados and Fiji had notable mortality rates due to the disease [2]. In 2020, it caused over 2 million new cases and led to 6,85,000 deaths globally [3]. By 2030, global cases are expected to rise significantly, impacting approximately 27 million people [4].

Breast cancer involves various tumor types, categorized as malignant or benign, with malignant tumors posing a higher risk due to their rapid spread. Factors contributing to its increasing incidence include lack of awareness, economic disparities, inadequate healthcare access, and screening challenges [5]. Developing independent apps for accurate cancer diagnosis is crucial to overcome these challenges. By analyzing variables and pinpointing the most important elements for a precise diagnosis across several models, ML has shown to be extremely successful in the diagnosis of breast cancer. Several models have been presented in earlier research, using various methods and techniques to identify cancers. ML, particularly integrated with NS to handle data uncertainty, presents an effective approach for enhancing breast cancer models for forecasting.

The main contributions of this study are:

a) Transforming the WDBC into an N-dataset.

b) Model training employing ML classification techniques, such as AdaBoost, DT, and RF.

c) Evaluating the performance of these models.

d) Comparing the performance between the original dataset and the N-dataset.

The remaining sections of the paper are organized as follows: The section 2 summarizes the literature on breast cancer prognosis. The study's materials and procedures are described in depth in Section 3. The study's findings are presented in Section 4. The methodology and results are covered in Section 5, and the paper's conclusions are given in Section 6.

2. Related work

Breast cancer is a serious worldwide health issue, and increasing survival rates requires early identification. Advanced ML techniques, such as Eagle Strategy Optimization (ESO), Gravitational Search Optimization (GSO), and their combined approach, are used to improve classification accuracy on the WDBC dataset. By prioritizing informative features and reducing computational complexity, the approach shows promising results in improving diagnostic precision and efficiency [6]. Breast cancer, characterized by complex development involving various cell types, remains a significant challenge worldwide. Advances in understanding pathogenesis and genetic factors have led to improved prevention and treatment strategies. Effective screening and research into drug-resistant mechanisms have enhanced patient outcomes and quality of life [7]. Over recent decades, significant advancements in breast cancer research have revolutionized treatment approaches, leading to better outcomes. Early detection through improved awareness and screening methods has enabled curative treatments such as surgery and radiation therapy. Ongoing research aims to further enhance diagnostic and therapeutic strategies [8].

Breast cancer is a multifactorial illness with a wide range of subtypes and symptoms. Understanding this diversity is crucial for developing targeted treatment approaches. Research focuses on genetic mutations, micro environmental factors, and epigenetic changes to improve personalized treatment strategies [9]. Technological advancements in mammographic screening and therapeutic interventions have transformed breast cancer management. Innovations in surgical techniques and radiotherapy have improved disease control and cosmetic outcomes. Clinical trials exploring combination therapies and gene-expression profiling aim to enhance treatment selection and patient outcomes [10]. Developing accurate prediction models tailored to specific populations, such as Cuban women, is crucial for effective breast cancer management. ML-based model that achieves high accuracy in estimating breast cancer risk for Cuban women, outperforms existing models. The potential for early diagnosis to enhance patient outcomes and save healthcare expenditures is highlighted by this approach [11]. To select and classify features in multidimensional breast cancer datasets, a new approach called the Rat Swarm Optimizer (RSO) hybridization with Levy Flight-based Cuckoo Search Optimization Algorithm (H-RS-LVCSO) was presented. By merging hybrid adaptive LVCSO with moment invariant wavelet feature extraction, this method greatly improves accuracy, precision, and execution speed. The research contributes to advancing breast cancer classification through innovative Feature Selection (FS) and classification techniques [12].

Evaluating various ML algorithms for breast cancer diagnosis, highlights RF's robustness in handling high-dimensional data and nonlinear decision boundaries. The approach demonstrates high accuracy in distinguishing between healthy individuals and those with breast cancer, showcasing its potential for accurate early detection [13]. The application of ensemble data mining techniques enhances the precision of breast cancer diagnosis by combining Rotation Forest with feature selection based on genetic algorithms. The approach optimizes input variable selection and employs robust classification methods, achieving high accuracy rates and demonstrating the effectiveness of ensemble methods in medical diagnostics [14]. Particle swarm optimization (PSO) was used for FS in data mining techniques to create a predictive model for breast cancer recurrence. The study demonstrates how Particle Swarm Optimization (PSO) enhances classification performance by assessing classifiers such as Naive Bayes (NB), K-Nearest Neighbor (KNN), and the rapid Decision Tree learner (REPTree). The results highlight the effectiveness of feature selection (FS) techniques in optimizing predictive models for breast cancer recurrence [15].

Cardiotocography (CTG) data uncertainty is crucial for classifying fetal heart rate in the biomedical field. The proposed Interval Neutrosophic Rough Neural Network (IN-RNN) framework, utilizing the backpropagation algorithm, enhances RNN's performance through NST. The experimental results indicate exceptional performance, with scores around 95%. Using WEKA application, the framework was compared with algorithms like Neural Network (NN), decision tables, and nearest neighbors, confirming its efficiency. The Receiver Operating Characteristic (ROC) curve displays high and acceptable area-under-curve values for the pathologic, normal, and suspicious states. The IN-RNN framework estimates uncertainty boundaries based on membership, truth, and indeterminacy values, with performance metrics indicating its effectiveness in classifying CTG data [16]. Differentiating COVID-19 from other lung illnesses, like bacterial and viral pneumonia, has become more difficult due to the COVID-19 pandemic. To differentiate between these diseases, a neutrosophic method was put forth, which involved grouping data into sets labeled True (T), False (F), and Indeterminacy (I) to improve feature extraction. Alpha-mean and beta-enhancement preprocessing is applied to chest X-ray pictures in order to decrease indeterminacy and boost opacity detection. Then, in a transfer learning setup, these improved images are examined using ResNet-50, VGG-16, and XGBoost, yielding an accuracy of 97.33% [17].

Decision-making (DM) is naturally challenging because of the uncertain and ambiguous nature of environments, particularly when multiple attributes are considered. The Multi-Polar Interval-Valued Neutrosophic Set (MPIVNS) and the Hypersoft Set (HS) framework were combined to address these issues. New aggregate operators, distance metrics, and similarity measures created especially for MPIVN-HSs are presented. These tools are essential for resolving complex attribute-based decision-making problems. The research utilizes the KNN algorithm to improve decision processes, showcasing practical applications in areas like site selection and beyond. The study significantly advances fields relying on language-based DM, including Artificial Intelligence (AI) and sentiment analysis [18]. The KNN algorithm is a popular non-parametric supervised classifier that assigns class labels to unknown samples based on their nearest neighbors in a training set using distance metrics. While effective, efforts have extended KNN to enhance its accuracy. Neutrosophic KNN, which integrates NST to improve classification. NST computes a final membership $U = T + I - F$ for class labeling, similar to fuzzy KNN, and assigns T, I, and F memberships using a supervised Neutrosophic C-Means (NCM) algorithm. Extensive experiments on synthetic and real-world datasets validate the method's efficacy compared to traditional KNN, fuzzy KNN, and weighted KNN approaches [19].

NST, especially single-valued NS (SVNSs), improves handling of imprecision and uncertainty in medical applications. By integrating NST with fuzzy techniques, more effective solutions for medical image processing, DM, and information fusion are achieved. These methods have shown efficacy in de-noising, clustering, and segmenting medical images. Neutrosophic logic (NL) offers a framework for modeling vagueness and uncertainty, making it ideal for dealing with incomplete or inconsistent information. The importance of NS is emphasized in various medical applications and proposes a framework for leveraging NS to enhance medical image processing and diagnosis [20].

Developing decision support tools for healthcare facility maintenance and asset renewal is challenging due to uncertainties and subjectivity in DM. In order to reduce subjectivity, Neutrosophic Logic (NL), Multi-Attribute Utility Theory (MAUT), and the Analytic Network Process (ANP) were integrated to assess hospital building assets according to their criticality and performance inequalities. ML algorithms, such as DT, KNN, and NB, automate and standardize the prioritization process. Applying the model to healthcare institutions in Canada showed a notable improvement in prediction performance, outperforming the previous model by about 11%. With the help of this framework, hospital asset renewal will be prioritized in a way that is impartial, automated, and consistent, guaranteeing effective resource allocation [21].

A hybrid fuzzy Multi-Criteria Decision-Making (MCDM) methodology utilizing Single-Valued Neutrosophic Fuzzy Sets (SVNFS), Best-Worst Method (BWM), and VIKOR is proposed for assessing cybersecurity risks targeting Connected and Autonomous Vehicles (CAVs). Expert opinions on cyber-attack likelihood and severity are integrated to rank threat-agent categories, identifying insider attackers as posing the greatest risk. This approach addresses subjectivity in opinions and incorporates criteria weights based on the consequences of cyber-attacks, offering a flexible framework applicable beyond CAV cybersecurity to other complex decision contexts with uncertain data [22]. The decision support system (DSS) utilizes the CRITIC and CRADIS models within SVNS to prioritize hydrogen technologies for decarbonizing Iran's oil refining industry. It assesses blue, green, and low-carbon hydrogen technologies across environmental, economic, social, and reliability criteria, identifying solar renewable energy as optimal due to its clean energy conversion and geographical suitability. This study enhances DM under uncertainty, suggesting future research explore broader qualitative factors and stakeholder perspectives [23]. The research focuses on advancing strategic DM in the planning of historic pedestrian bridge remediation through an innovative algorithm based on Rough NS (RNS). This novel approach integrates Rough Sets (RS) and NS theories within a MCDM model. A key contribution is the introduction of a new RN symmetric cross entropy measure and its weighted variant, specifically designed to address uncertainties and the challenge of unknown criteria weights inherent in complex DM processes. By incorporating the VIKOR method, the model enables effective prioritization of bridge remediation efforts by providing robust and reliable rankings. Case studies validate the model's efficacy, demonstrating its practical utility compared to traditional methods in real-world scenarios [24].

A novel approach to Multi-Attribute Decision-making (MADM) was introduced by integrating RS, NS, and Grey System Theory (GST). The RN Grey Relational Analysis (RNGRA) method addresses indeterminate and inconsistent data using RNS, characterized by T, I, and F-membership degrees. Attribute weights are partially known and determined via an information entropy method. The Accumulated Geometric Operator (AGO) converts RN numbers into SVN numbers. The method employs the Hamming distance to calculate the NGR coefficient for assessing reliability and unreliability. Finally, a RN relational degree is established to rank alternatives, with a numerical example provided to demonstrate the method's effectiveness and applicability [25].

A RN TOPSIS method was presented for Multi-Attribute Group DM (MAGDM), effectively handling uncertainty, indeterminacy, and inconsistency in data. By evaluating alternatives and features using RNS, which are distinguished by T, I, and F-membership degrees, the method enhances the conventional TOPSIS technique. Individual opinions are aggregated into a group consensus using the RN weighted averaging operator. The distance between each alternative and the positive and negative Rough Neutrosophic (RN) ideal solutions is estimated using the Euclidean distance. A numerical example demonstrates the method's practicality and efficiency, making it applicable in pattern recognition, AI, and medical diagnosis [26].

3. Materials and Methods

The materials and methods used in the study are described in detail in this section.

3.1. Proposed methodology

The objective is to advance breast cancer prediction by integrating NS with ML algorithms. The WDBC dataset (I_D) was initially retrieved from the UCI ML Repository and then carefully preprocessed (O_{Dpp}) to ensure its quality and consistency.

$$I_D = WDBC \quad (1)$$

$$O_{Dpp} = f_{pp}(I_D) \quad (2)$$

The dataset was then transformed into an N-representation (O_N), where each data point was characterized not only by its specific attributes but also by degrees of T, I, and F. This approach offers a more nuanced depiction of uncertainty and variability inherent in medical datasets.

$$O_N = f_{T,I,F}(O_{Dpp}) \quad (3)$$

Following the transformation, the N-dataset was split ($O_{N_{(s)}}$) into training and testing subsets.

$$O_{N_{(s)}} = s(f_{O_N}) = s\ (X_{train}, X_{test}, y_{train}, y_{test}) \quad (4)$$

The N- dataset was normalized ($O_{N_{(Nor)}}$) to a range of 0 to 1 using Min-Max Scaler.

$$O_{N_{(Nor)}} = f_{Nor}\ (O_{N_{(s)}}) \quad (5)$$

The normalized N - training dataset was employed to train ML classifiers ($O_{N_{(ML)}}$) such as DT, RF, and AdaBoost. These classifiers were selected based on their capacity to handle intricate feature interactions and identify subtle patterns necessary for an accurate diagnosis of breast cancer.

$$O_{N_{(ML)}} = f_{ML}\ (O_{N_{(Nor)}}) = O_{ML}\ (DT_{O_{N_{(Nor)}}}, RF_{O_{N_{(Nor)}}}, AB_{O_{N_{(Nor)}}}) \quad (6)$$

The main performance metrics ($O_{N_{Metrics}}$) for the classifier, which include accuracy, precision, recall, and F1 score, were used to evaluate its performance and provide a thorough examination of its ability to predict.

$$\boldsymbol{O_{N_{Metrics}} = O_{N_{(ML)}}(Metrics_{A_{cc},P_r,R_c,F_1})} \quad (7)$$

Finally, the study conducted a Comparative Analysis (O_{CA}) between the N-dataset and the original dataset to evaluate how integrating NS enhances the accuracy and reliability of breast cancer prediction models.

$$O_{CA} = CA\,(I_D;\, O_N) \quad (8)$$

The workflow is depicted in Figure 1.

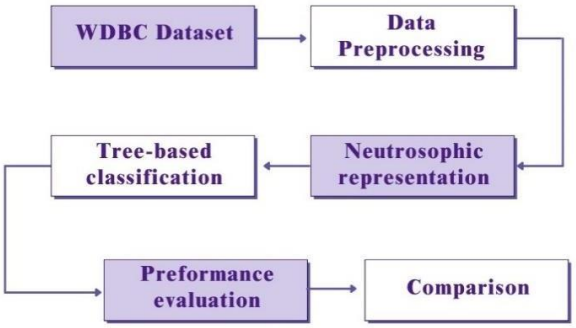

Fig. 1: NS-ML framework for Breast Cancer Prediction

3.2. Dataset description

A popular dataset for ML and statistical analysis, the WDBC dataset was collected by the University of Wisconsin Hospitals in Madison and is particularly useful for predicting and classifying breast cancer. Derived from digital photographs of breast tumors, it has attributes including radius, texture, area, perimeter, smoothness, compactness, concavity, concave spots, symmetry, and fractal dimension. With 212 instances classified as malignant (indicating presence of cancer) and 357 as benign (indicating absence of cancer), this dataset serves as a robust resource for developing accurate models that distinguish between malignant and benign breast cancer cases based on these comprehensive tumor characteristics.

3.3. Preprocessing of data

The breast cancer dataset was preprocessed for effective ML analysis. First, it was divided into features attributes describing tumor and a categorical target variable that distinguishes between benign and malignant cases. This categorical target was encoded into numerical values to make it easier for the models to understand. All features were normalized using Min-Max scaling in order to guarantee accurate predictions. This preprocessing process improved the machine learning models' ability to predict breast cancer diagnosis and maintained data consistency.

3.4. Neutrosophic Sets

Let X be a point or object-containing space, and let x be any element belongs to X. Three unique membership functions define a NS, A within X: T, I and F membership often referred as $T_A(X)$, $I_A(X)$, and $F_A(X)$. These functions assign real values within the interval [0, 1] indicating the degree to which x pertains to the subsets of T, I, and F, respectively:

$$T_A\, maps\, X\, to\, the\, interval\,]\,0^-, 1^+[$$
$$I_A\, maps\, X\, to\, the\, interval\,]\,0^-, 1^+[$$
$$F_A\, maps\, X\, to\, the\, interval\,]\,0^-, 1^+[$$

The sum of $T_A(X)$, $I_A(X)$, $F_A(X)$ for every X, the falls ranges from 0 to 3. This flexibility enables NS to effectively represent and manage uncertainty, ambiguity, and contradiction within sets,

making them valuable in contexts where handling incomplete or uncertain information is crucial [26].

3.5. Neutrosophic dataset formation

To address the uncertainties inherent in the WDBC dataset for binary classification, An N- dataset was introduced as an inclusive and generalized solution. This dataset goes beyond conventional and high-risk categories by incorporating a degree of neutrality. Thus, the N-dataset is defined as $< T_A, I_A, F_A >$, where each element of the set $X = \{x_1, x_2, \dots, x_n\}$ is specified as follows:

$$\forall x\,(t, i, f) \in < T_A, I_A, F_A >$$

where, *t, i, and f*, respectively, are the real numbers for T, I, and F.

In order to better capture the uncertainty in the data, the model is being developed by adding N-components to the original dataset. The first step in the method is to compute the mean vectors for the training set as a whole (ρ^{all}), the positive class (ρ^{+}), and the negative class (ρ^{-}) which is provided in Eq. (1).

$$\rho^{all} = \sum_{k=1}^{n^{all}} x \;;\; \rho^{+} = \sum_{k=1}^{n^{+}} x \;;\; \rho^{-} = \sum_{k=1}^{n^{-}} x \tag{1}$$

The following Eqs. (2), (3) & (4) are u sed to compute the T, I, and F components for a given sample (x):

$$T = 1 - \frac{||x - \rho^{+}||}{\max(||X_{train} - \rho^{+})} \tag{2}$$

$$I = 1 - \frac{||x - \rho^{all}||}{\max(||X_{train} - \rho^{all})} \tag{3}$$

$$F = 1 - \frac{||x - \rho^{-}||}{\max(||X_{train} - \rho^{-})} \tag{4}$$

These formulas are applied to each sample in the training and testing datasets, yielding features that measure the degrees of T, I, and F. Consequently, an N-dataset is produced, which enhances the ML algorithm's capacity to identify data with inherent uncertainty. This strategy considerably improves the classifier's performance in processing ambiguous and complex biomedical data by utilizing the capabilities of NST. The algorithm for the formation of O_N is provided below.

INPUT: The WDBC dataset (I_D)
OUTPUT: Neutrosophic dataset (O_N)

Step 1: Mean Vector Computation
Step 1.1: Positive class (ρ^{+})

$$\rho^{+} = \sum_{k=1}^{n^{+}} x_k$$

where $y_k = 1$ and n^{+} is the no. of positive samples in X_{train}
Step 1.2: Negative class (ρ^{-})

$$\rho^{-} = \sum_{k=1}^{n^{-}} x_k$$

where $y_k = 0$ and n^{-} is the no. of negative samples in X_{train}
Step 1.3: Overall mean (ρ^{+})

$$\rho^{+} = \sum_{k=1}^{n^{+}} x_k$$

where $y_k = 1$ and n^+ is the no. of positive samples in X_{train}

Step 2: Calculate the N-components (t, i, f) for each sample.

For each sample x_i in both X_{train} and X_{test}

$$T_i = 1 - \frac{||x_i - \rho^+||}{\max(||X_{train} - \rho^+)}$$

$$I_i = 1 - \frac{||x_i - \rho^{all}||}{\max(||X_{train} - \rho^{all})}$$

$$F_i = 1 - \frac{||x_i - \rho^-||}{\max(||X_{train} - \rho^-)}$$

Step 3: Group the N-components

$$f_{T_i, I_i, F_i} = O_N$$

3.6. Classification algorithms

3.6.1. Decision Tree

A hierarchical supervised learning model called a DT makes predictions by gradually dividing the data into groups based on the values of its features. It creates a tree-like structure with internal nodes representing decision points, branches indicating possible outcomes, and leaf nodes delivering final predictions. The tree is built recursively, starting from the root and progressing downwards. At each internal node, the algorithm chooses a feature and threshold that best separates the data, typically using metrics like Gini impurity or entropy. This process continues, refining predictions at each level, until reaching leaf nodes. In order to classify new data, branches are followed depending on feature values as they go from the root of the tree to the leaf. This approach effectively breaks complex decisions into simpler steps, making DT both powerful and interpretable for various prediction and classification tasks [28]. The Gini Impurity and Entropy is calculated using the following Eqs. (5) and (6)

Gini Impurity

$$G = 1 - \sum_{j=1}^{C} {p_j}^2 \quad (5)$$

Entropy:

$$H = -\sum^{C} p_j \log(p_j) \quad (6)$$

where p_j represents the probability of class j in the node, and C is the number of classes

3.6.2. Random Forest

An ensemble learning method called RF combines several DTs to reduce overfitting and improve prediction accuracy. Using a bootstrap sampling of the original dataset, it creates a large number of trees. In order to reduce correlations between the different trees and introduce variety through feature bagging, a random selection of features is chosen at each node for splitting. For classification tasks, the model aggregates predictions by majority voting, while for regression, it uses averaging. By leveraging the collective wisdom of many diverse trees, RF effectively reduces variance and enhances generalization. This method excels in handling complex, high-dimensional datasets and is

widely adopted in ML for its robust performance, ability to capture non-linear relationships, and resilience against overfitting. Additionally, the model's capacity to provide feature importance rankings and handle missing values further contributes to its popularity across various domains [29].

3.6.3. Adaptive Boosting

Boosting is a ML technique that combines multiple weak learners to form a strong predictive model. AdaBoost, developed by Freund and Schapire [30], exemplifies this approach and remains widely used in various fields. In AdaBoost, weak learners are trained iteratively on weighted distributions of training data, with weights adjusted based on their performance. Each weak hypothesis receives a weight αt proportional to its accuracy, thereby minimizing errors. By assigning greater weight to more accurate learners, the final model aggregates these weak learners into a robust overall predictor.

3.7. Performance Evaluation

The proposed methods were compared using metrics including recall, accuracy, precision, and F1-score. The percentage of correctly identified subjects is called accuracy. The precision measure shows the percentage of successfully diagnosed positive subjects out of all predicted positive subjects. The recall metric assesses how well the model can detect positive samples. A fair assessment of the model's performance is provided by the F1-score, which is the harmonic mean of precision and recall.

4. Results

4.1. Experimental setup

Three distinct ML tree-based classifiers were utilized to predict breast cancer using the WDBC dataset. Prior to training, the dataset was transformed into an N-dataset to enhance the models' robustness in handling uncertainties. Several tree-based algorithms were trained on this N-dataset, and standard metrics were employed to evaluate the predictive performance of the methods. Furthermore, comparisons were made between results obtained from the N-dataset and the original dataset. All these experiments were efficiently executed on Google Colab, leveraging GPU acceleration to manage the computational complexity of ML tasks effectively.

4.2. Experimental results

The table provides a comparative analysis of ML algorithm performance using both the N-dataset and the original dataset. N-DT, N-RF, and N-AdaBoost denote models trained with the N-dataset using DT, RF, and AdaBoost algorithms respectively.

Table 2. Comparative evaluation of N-DT and DT model

Metrics	**N-DT**	**DT**
Accuracy	**93.86**	90.35
Precision	**91.66**	84.61
Recall	**93.62**	93.62
F1 score	**92.63**	88.88

Table 3. Comparative evaluation of N-RF and RF model

Metrics	N-RF	RF
Accuracy	**96.49**	96.49
Precision	**95.74**	97.77
Recall	**95.74**	93.61
F1 score	**95.74**	95.65

Table 4. Comparative evaluation of N-AdaBoost and AdaBoost model

Metrics	N-AdaBoost	AdaBoost
Accuracy	**99.12**	98.24
Precision	**100.00**	100.00
Recall	**97.87**	95.74
F1 score	**98.92**	97.82

These metrics provide a detailed comparison of ML algorithms using both the N-dataset and the original dataset. According to Table 2, N-DT demonstrate improvements in accuracy, precision, and F1 score, but slightly lower than RF and AdaBoost in precision and F1 score. Table 3 shows that N-RF exhibits notable enhancements in precision and F1 score, indicating improved capability to accurately classify positive instances while maintaining overall performance metrics. Table 4 highlights performance of N-AdaBoost, achieving 99.12% accuracy, perfect precision of 100.00%, 97.87% recall, and a F1 score of 98.92%. These results underscore AdaBoost's effectiveness in handling uncertainties inherent in biomedical data. In contrast, when using the original dataset, DT, RF, and AdaBoost shows lower precision and F1 scores compared to their performance with the N-dataset. The comparison of the results is displayed in Figure 2. Overall, integrating NS theory enhances the predictive capabilities of these algorithms for breast cancer diagnosis.

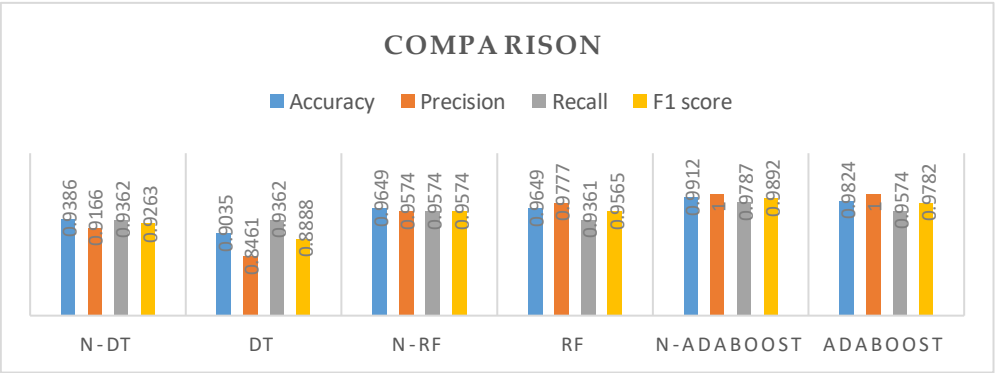

Figure 2. Comparison of N-tree based and conventional ML classifiers

5. Discussion

The proposed hybrid approach, NS-ML aimed at improving the prediction of breast cancer diagnosis. The research findings highlight that N-AdaBoost models, achieve superior accuracy and precision in detecting breast cancer. This underscores the effectiveness of integrating NST into ML models for biomedical applications, particularly in enhancing the reliability and accuracy of breast cancer diagnosis.

Conclusion

This research aims at advancing breast cancer prediction by integrating NS with ML techniques. Dataset was collected from WDBC dataset from the UCI ML Repository. The dataset was transformed into an N-representation, enriching each data point with degrees of T, I, and F to capture the complexities of medical datasets. Subsequently, the N-dataset was partitioned into training and testing subsets for training tree-based classifiers such as DT, RF, and AdaBoost. Predictive performance was measured using evaluation metrics, which demonstrated the models' capacity to recognize trends in breast cancer. Comparative analysis between the N-datasets and original datasets demonstrated improved performance metrics for DT, RF, and AdaBoost with the N-dataset. Notably, N-AdaBoost models demonstrated enhanced reliability of breast cancer diagnosis utilizing NS with scores of 99.12% accuracy, 100.00% precision, 97.87% recall, and an F1 score of 98.92%.

As a future work, the NS-ML approach can be expanded to incorporate Deep Learning (DL) and Neural Network (NN) architectures, aiming to enhance their capability in handling complex biomedical data. This integration will leverage NST to effectively model uncertainties and variability, thereby improving accuracy in tasks like image-based diagnostics and genomic analysis. Developing new neural network structures that integrate N-elements will be crucial for capturing intricate patterns in biomedical data.

References

[1] Panigrahi, L.; Verma, K.; Singh, B. K. Ultrasound image segmentation using a novel multi-scale Gaussian kernel fuzzy clustering and multi-scale vector field convolution. *Expert Syst. Appl.* 2019, *115*, 486–498.

[2] World Cancer Research Fund International, London, breast cancer statistics. Available online: https://www.wcrf.org/cancer-trends/breast-cancer-statistics/.

[3] World Health Organization, Switzerland, statistics on breast cancer. Available online: https://www.who.int/news-room/fact-sheets/detail/breast-cancer (accessed on 9 August 2024).

[4] Kahya, M. A. Classification enhancement of breast cancer histopathological image using penalized logistic regression. *Indones. J. Electr. Eng. Comput. Sci.* 2019, *13*, 1.

[5] Begum, S.A.; Mahmud, T.; Rahman, T.; Zannat, J.; Khatun, F.; Nahar, K.; Towhida, M.; Joarder, M.; Harun, A.; Sharmin, F. Knowledge, attitude and practice of Bangladeshi women towards breast cancer: a cross-sectional study. *Mymensingh Med. J.* 2019, *28*, 96–104.

[6] Singh, L.K.; Khanna, M.; Singh, R. Artificial intelligence-based medical decision support system for early and accurate breast cancer prediction. *Adv. Eng. Softw.* 2023, *175*, 103338.

[7] Sun, Y.S.; Zhao, Z.; Yang, Z.N.; Xu, F.; Lu, H.J.; Zhu, Z.Y.; Zhu, H.P. Risk factors and preventions of breast cancer. *Int. J. Biol. Sci.* 2017, *13*, 1387–1397.

[8] Sharma, G.N.; Dave, R.; Sanadya, J.; Sharma, P.; Sharma, K. Various types and management of breast cancer: an overview. *J. Adv. Pharm. Technol. Res.* 2010, *1*, 109–126.

[9] Polyak, K. Breast cancer: origins and evolution. *J. Clin. Investig.* 2007, *117*, 3155–3163.

[10] Benson, J.R.; Jatoi, I.; Keisch, M.; Esteva, F.J.; Makris, A.; Jordan, V.C. Early breast cancer. *Lancet* 2009,

373, 1463–1479.

[11] Valencia-Moreno, J.M.; Gonzalez-Fraga, J.A.; Gutierrez-Lopez, E.; Estrada-Senti, V.; Cantero-Ronquillo, H.A.; Kober, V. Breast cancer risk estimation with intelligent algorithms and risk factors for Cuban women. *Comput. Biol. Med.* 2024, *179*, 108818.

[12] Rekha, K.S.; Divya, D.; Amali, M.J.; Yuvaraj, N. Hybrid ML-MDKL feature subset selection and classification technique accompanied with rat swarm optimizer to classify the multidimensional breast cancer mammogram image. *Optik* 2024, *297*, 171574.

[13] Malakouti, S.M.; Menhaj, M.B.; Suratgar, A.A. ML: Early Breast Cancer Diagnosis. *Curr. Probl. Cancer Case Rep.* 2024, *13*, 100278.

[14] Aličković, E.; Subasi, A. Breast cancer diagnosis using GA feature selection and Rotation Forest. *Neural Comput. Appl.* 2017, *28*, 753–763.

[15] Sakri, S.B.; Rashid, N.B.A.; Zain, Z.M. Particle swarm optimization feature selection for breast cancer recurrence prediction. *IEEE Access* 2018, *6*, 29637–29647.

[16] Amin, B.; Salama, A.A.; El-Henawy, I.M.; Mahfouz, K.; Gafar, M.G. Intelligent neutrosophic diagnostic system for cardiotocography data. *Comput. Intell. Neurosci.* 2021, *2021*, 6656770.

[17] Jennifer, J.S.; Sharmila, T.S. A neutrosophic set approach on chest X-rays for automatic lung infection detection. *Inf. Technol. Control* 2023, *52*, 37–52.

[18] Saqlain, M.; Garg, H.; Kumam, P.; Kumam, W. Uncertainty and decision-making with multi-polar interval-valued neutrosophic hypersoft set: A distance, similarity measure and machine learning approach. *Alex. Eng. J.* 2023, *84*, 323–332.

[19] Akbulut, Y.; Sengur, A.; Guo, Y.; Smarandache, F. NS-k-NN: Neutrosophic set-based k-nearest neighbors classifier. *Symmetry* 2017, *9*, 179.

[20] Nguyen, G.N.; Son, L.H.; Ashour, A.S.; Dey, N. A survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses. *Int. J. Mach. Learn. Cybern.* 2019, *10*, 1–13.

[21] Ahmed, R.; Nasiri, F.; Zayed, T. A novel Neutrosophic-based machine learning approach for maintenance prioritization in healthcare facilities. *J. Build. Eng.* 2021, *42*, 102480.

[22] Tanaji, B.A.; Roychowdhury, S. BWM Integrated VIKOR method using Neutrosophic fuzzy sets for cybersecurity risk assessment of connected and autonomous vehicles. *Appl. Soft Comput.* 2024, *159*, 111628.

[23] Fetanat, A.; Tayebi, M. Sustainability and reliability-based hydrogen technologies prioritization for decarbonization in the oil refining industry: A decision support system under single-valued neutrosophic set. *Int. J. Hydrog. Energy* 2024, *52*, 765–786.

[24] Rogulj, K.; Kilić Pamuković, J.; Ivić, M. Hybrid MCDM based on VIKOR and cross entropy under rough

neutrosophic set theory. *Mathematics* 2021, *9*, 1334.

[25] Mondal, K.; Pramanik, S. Rough neutrosophic multi-attribute decision-making based on rough accuracy score function. *Neutrosophic Sets Syst.* 2015, *8*, 14–21.

[26] Wang, H.; Smarandache, F.; Zhang, Y.; Sunderraman, R. Single valued neutrosophic sets. *Infinite Study*

2010.

[27] Alshikho, M.; Jdid, M.; Broumi, S. Artificial Intelligence and Neutrosophic Machine learning in the Diagnosis and Detection of COVID 19. *J. Prospect. Appl. Math. Data Anal.* 2023, *1*, 2.

[28] Mitchell, T. Decision tree learning. *Mach. Learn.* 1997, *414*, 52–78.

[29] Breiman, L. Random forests. *Mach. Learn.* 2001, *45*, 5–32.

[30] Schapire, R.E. Explaining adaboost. In *Empirical Inference: Festschrift in Honor of Vladimir N. Vapnik*; Springer Berlin Heidelberg: Berlin, Heidelberg, 2013; pp. 37–52.

Multi Attribute Neutrosophic Optimization Technique for Optimal Crop Selection in Ariyalur District

S. Angammal,[1] Hannah G. Grace,[2] Nivetha Martin,[3] Florentin Smarandache[4]

[1,2]Vellore Institute of Technology Chennai, School of Advanced Sciences, Department of Mathematics. India

[3] Arul Anandar College (Autonomous), Karumathur. India

[4] The University of New Mexico, United States

*Correspondence: hannahgrace.g@vit.ac.in

Angammal, S., Hannah G. Grace, Nivetha Martin, and Florentin Smarandache. "Multi-Attribute Neutrosophic Optimization Technique for Optimal Crop Selection in Ariyalur District." *Neutrosophic Sets and Systems*, vol. 73, 2024, pp. 153–168.

Abstract. Ranking crops is a vital part of sustainable farming practices. A deliberate strategy that incorporates multi-criteria decision-making is essential for achieving sustainability in agriculture. The process of choosing crops involves a lot of unknowns and unpredictable elements. The neutrosophic set, characterized by the three independent degrees of truth (T), falsity (F), and indeterminacy (I), is more adept at handling incomplete data. The current study combines single-valued multi-criteria neutrosophic programming with the TOPSIS approach to examine the crop selection process in the Ariyalur district. This study incorporated expert advice and a thorough literature analysis to identify the fundamental criteria for developing sustainable crop planning for important crops in the Ariyalur area. In order to rate the crops and achieve sustainability for agricultural production, eleven significant criteria were selected based on environmental, social, economic, and soil nutrient concerns. In this study, the relative importance of criteria and alternatives is assessed by consolidating the views of different decision-makers into a unified opinion through a single-valued neutrosophic set-based weighted averaging operator. The current approach will greatly enhance the self-sufficiency of the agricultural sector and boost the country's GDP. Additionally, it will support the Ministry of Agriculture and other stakeholders in formulating regulations related to crop harvesting methods.

Keywords: Crop selection; TOPSIS approach; Neutrosophic set; Multi criteria optimization; Group Decision.

1. Introduction

Agriculture is an essential part of the global economy, providing food, feed, and fibre for an expanding population. Undoubtedly, the primary source of livelihood in India is agriculture and its related sectors, especially in the country's vast rural areas. It also makes a substantial contribution to the GDP. Some countries around the world, especially India, are encountering heightened difficulties in meeting their food demands due to population growth. Food

consumption often rises annually in tandem with a country's population growth; as a result, sufficient crop production that meets sustainability standards and yield rates is necessary. Sustainable agriculture is crucial for comprehensive rural development because it promotes rural employment, ensures food security, and supports environmentally friendly practices such as sustainable management of natural resources, soil conservation, and biodiversity preservation. Crop selection is important in agriculture because it influences the sustainability, profitability and productivity of farming operations. Because of fluctuating socioeconomic conditions and limited resources that differ from state to state and region to region in India, choosing crop patterns is even more difficult. The minimum support price (MSP) of crops is just insufficient to support farmers' economically viable growth, which presents another challenge for them in realising crop value. This study employs a multi-attribute decision-making (MADM) approach to examine the crop selection system in Ariyalur district, Tamil Nadu, India, with the aim of achieving sustainable and profitable agriculture. Numerous fields, including operation research, urban planning, natural science, and management science, have used MADM technique with either numerical or descriptive attribute values. When it comes to crop selection, it entails assessing and choosing the optimal crops to plant based on a variety of attributes, including yield, cost, environmental effect, climatic resilience, market demand, etc.

Classical MADM techniques, such as TOPSIS [1], VIKOR [2], PROMETHEE [3], and ELECTRE [4], use crisp numbers to represent the weights of each attribute and alternative ratings. However, due to attribute complexity and ambiguity, MADM problems' attribute values are not always described with precise numerical values. To overcome such an issue, Zadeh [5] introduced fuzzy set theory. It is highly effective for decision-making in MADM scenarios where information is imprecise. To determine the most suitable crop for the land, F. Sari and F. Koyunch [6] integrated AHP and TOPSIS with GIS (geographical information systems). AHP and fuzzy TOPSIS techniques were employed by Weilun Huanga and Qi Zhang [7] to determine the best economic crop in the minority region. The fuzzy TOPSIS approach was proposed by Singh, R.K., and Mallick, J. [8] as a means of selecting the vegetable cash crop for sustainable agriculture within the green chamber. To select the optimal biomass crop type for bio-energy production, Cobuloglu H. I. and Buyuktahtak I. E. [9] introduced a new stochastic analytical hierarchy process (AHP) capable of managing ambiguous data and discovering the relative importance of criteria in the MCDM process. Various authors have created fuzzy-MCDM techniques for addressing the plant location selection (PLS) problem, drawing on principles from fuzzy set theory. For the PLS problem in a linguistic environment, Yong [10] suggested the TOPSIS method. The location of the facility was chosen by Ertugrul and Karakasoglu [11] using the fuzzy TOPSIS and fuzzy AHP approaches. The degree of belonging constitutes the only element in the fuzzy set. The most effective way to solve the

MADM problem requires keeping in mind that membership and non-membership functions are equally important. Therefore, Atanassov [12] proposed the intuitionistic fuzzy set (IFS), defined by belongings and non-belongings degrees simultaneously. To solve the PLS problem, Pankaj Gupta et al. [13] suggested an expanded VIKOR technique using intuitionistic trapezoidal fuzzy parameters. To address the challenge of choosing R & D projects, Wan et al. [14] created a novel approach for handling multi-attribute group decision-making (MAGDM) problems, including incomplete attribute weight information and Atanassov's interval-valued intuitionistic fuzzy values. Abbas Mardani et al. [15] introduced an innovative combined approach that applies the step-wise weight assessment ratio analysis (SWARA) and the complex proportional assessment (COPRAS) method within the context of IFSs to determine the optimal biomass crop type for sustainable production of bio-fuel. In IFSs, the sum of a vague parameter's degrees of belonging and non-belonging does not amount to one. Consequently, an intuitionistic fuzzy set has some degree of incompleteness or indeterminacy. It is unable to effectively handle every kind of uncertainty in various real-world physical challenges, including those with indeterminate data.

To address ambiguous or indeterminate data often encountered in practical situations, Smarandache [16] introduced the idea of a neutrosophic set (NS) within a philosophical context. However, applying NS directly in scientific and engineering fields proves to be challenging. Wang et al. [17] created the single-valued neutrosophic set (SVNS), a subclass of NS, to address challenges. SVNSs have been utilised by neumorous academics to create decision-making models. [18–21]. Using the weighted correlation coefficient of SVNSs, Ye [22] investigated the MCDM problem. MCDM problem under interval neutrosophic set information were examined by Zhang et al. [23]. Chi and Liu [24] developed an enhanced TOPSIS method for addressing multi-attribute decision-making (MADM) problems involving interval neutrosophic sets. A TOPSIS model was introduced by Nancy and Grag [25] to evaluate the MCDM when there was insufficient weight data available for SVNSs. An innovative weighted aggregated sum product assessment (WASPAS) framework using SVNS was developed by Arunodaya Raj Misra et al. [26] to identify the biomass crop option that is most optimal for producing biofuel. For professional selection, Abdel-Basset M. et al. [27] introduced a novel hybrid neutrosophic MCDM framework that combines neutrosophic analytical network processes (ANP) and TOPSIS using bipolar neutrosophic numbers. To manage uncertainty in SVNS data, Shahzaib et al. [28] created a hybrid averaging and geometric aggregation operator utilising a sine trignometric function. Additionally, the approach is used for agricultural land selection in order to demonstrate its efficacy. In order to define the available water resources in the agricultural sector based on possibility measurements in a generalised single-valued non-linear bipolar neutrosophic environment, Garai T., and Garg H. [29] devised a multi-criterion water resource

management technique. To address multi-attribute decision-making issues in a multi-valued neutrosophic environment, Dongsheng Xu and Lijuan Peng [30] suggested a novel approach based on TOPSIS and TODIM. The benefits of TOPSIS include being simpler, easier to understand, and more computationally efficient, according to an assessment of the literature. Consequently, choosing agricultural crops may benefit from the current study's combination of the TOPSIS technique with SVNSs. The primary goal of this research is to evaluate and rank the most suitable crops grown in the Ariyalur district using a TOPSIS-based (MAGDM) approach.

This study is structured as follows: Section 2 reviews preliminary research on SVNSs; Section 3 covers the study area; Section 4 details the methodology employed; Section 5 illustrates the application of the methodology; and Section 6 summarizes the conclusions drawn from the study.

2. **Preliminary**

To advance the paper, certain fundamental definitions of a single valued neutrosophic set are given in this section.

2.1. *Single Valued Neutrosophic Set*

A single valued neutrosophic set (SVNS) P, over the universe of discourse U is represented as

$$P = \{(u, T_P(u), I_P(u), F_P(u))/u \in U\}$$

where $T_P(u), I_P(u), F_P(u)$ are values in the range [0,1], and the sum $T_P(u) + I_P(u) + F_P(u)$ satisfies $0 \leq T_P(u) + I_P(u) + F_P(u) \leq 3$.

2.2. *Complement*

The complement of a SVNS P, represented as c(P), is defined by: $T_{c(P)}(u) = F_P(u)$, $I_{c(P)}(u) = 1–I_P(u)$, $F_{c(P)}(u) = T_P(u)$, for all u in U.

2.3. *Neutrosophic Operators*

If P and Q be two SVNSs then

(i) $P \oplus Q = (T_P(u) + T_Q(u) - T_P(u).T_Q(u), I_P(u) + I_Q(u), F_P(u) + F_Q(u))$

(ii) $P \otimes B = (T_P(u).T_Q(u), I_P(u) + I_Q(u) - I_P(u).I_Q(u), F_P(u) + F_Q(u) - F_P(u).F_Q(u))$

(iii) $P \cup Q = (\max(T_P(u), T_Q(u)), \max(I_P(u), I_Q(u)), \min(F_P(u), F_Q(u)))$ or

$P \cup Q = (\max(T_P(u), T_Q(u)), \min(I_P(u), I_Q(u)), \min(F_P(u), F_Q(u)))$

(iv) $P \cap B = (\min(T_P(u), T_Q(u)), \min(I_P(u), I_Q(u)), \max(F_P(u), F_Q(u)))$ or

$P \cap Q = (\min(T_P(u), T_Q(u)), \max(I_P(u), I_Q(u)), \max(F_P(u), F_Q(u)))$

Study Area

Appropriate crop selection is crucial to establishing sustainable agriculture and increasing agricultural profitability. This paper presents a TOPSIS-based neutrosophic programming approach to crop planning in the Ariyalur district, one of the Cauvery Delta regions. The Ariyalur District covers a total area of 193,338 hectares. 94,725 hectares are the net area that has been planted. Of which, 58441 ha are rainfed and 36284 ha are under irrigation. Rainfall averages 954 mm per year. This district grows a variety of crops, and the majority of its residents work in agriculture. Since 70% of the population makes their living from agriculture and related industries, agriculture remains the most important component of the district's economy. In order to increase production, the Agriculture Department has implemented several development schemes and disseminated pertinent technology, stepping up its efforts to attain a higher growth rate in the sector. The department's objectives and policies focus on sustaining agricultural production stability and promoting sustainable growth to meet the food demands of a growing population, while also providing raw materials for agro-based industries, thereby generating employment for the rural population. The Ariyalur district's soil is made up of ferruginous red clay and limestone. The colour typically varies from red at the top to yellow at the lower horizon, with a loamy texture. These soils are of medium depth with good drainage, characterized by low levels of organic content, nitrogen, and phosphorus, yet generally containing ample amounts of potash and lime. The pH range is 6.5 to 8.0, and they are free from salt and calcium carbonate formation. This district is used to develop sugarcane, groundnuts, maize, cotton, and paddy crops. The primary irrigation sources in this district include open wells, canals, tanks, and tube wells. Bore wells and tube wells contribute significantly to irrigation. The alternatives that were examined in this study were sugarcane, maize, groundnut, cotton, brinjal, tomato, chilli, and onion.

4. Multi Attribute TOPSIS Based Neutrosophic Technique

Hwang and Yoon introduced the TOPSIS method in 1981. This technique determines the optimal solution as the one that is closest to the positive ideal solution and farthest from the negative ideal solution. In terms of determining the ideal response, the negative approach maximizes the cost criterion while decreasing the benefit criteria, and the positive approach maximises the benefit criteria while minimising the cost criteria. Numerous studies have used TOPSIS to address MCDM problems in research. The next set of steps presents the computing procedures for the TOPSIS-based neutrosophic technique.

Step 1: Assume the alternatives A_i and the criteria C_j, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Additionally, assume that the decision makers have provided $w = \{w_1, w_2, \ldots, w_n\}$ as the weight vector for the criteria $C_1, C_2, \ldots, C_n$. Also, create the decision matrix that

follows, which shows the values corresponding to the alternatives and criteria for MADM problems.

$$D = [d_{ij}]_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{bmatrix} \end{array}$$

Step 2: Use professional advice or another technically sound method to determine which criteria are most important.

Step 3: Convert each value assigned to the alternatives according to the attributes in the decision matrix, as defined in step one, into a single-valued neutrosophic number. Following such conversion, the resulting matrix is defined as follows:

$$(d_{ij})_{m\times n} = (T_{ij}, I_{ij}, F_{ij})_{m\times n}$$

$$D = [d_{ij}]_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} (T_{11}, I_{11}, F_{11}) & (T_{12}, I_{12}, F_{12}) & \cdots & (T_{1n}, I_{1n}, F_{1n}) \\ (T_{21}, I_{21}, F_{21}) & (T_{22}, I_{22}, F_{22}) & \cdots & (T_{2n}, I_{2n}, F_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{m1}, I_{m1}, F_{m1}) & (T_{m2}, I_{m2}, F_{m2}) & \cdots & (T_{mn}, I_{mn}, F_{mn}) \end{bmatrix} \end{array}$$

Step 4: Convert the decision matrix mentioned in step 3 into the normalized single valued neutrosophic decision matrix $\left(\widetilde{d}_{ij}\right)_{m\times n}$, where $\left(\widetilde{d}_{ij}\right) = d_{ij}$ for benefit criteria C_j and $\left(\widetilde{d}_{ij}\right) = d_{ij}^{c}$ for cost criteria.

Step 5: In a collective decision-making scenario, decision-makers are not of equal importance. Therefore, at this stage, the weight of the decision-maker is established. Assume that there exist r decision makers and that the linguistic word mentions their importance and expresses it in terms of neutrosophic numbers. Then, the weight of t^{th} decision-maker is specified by:

$$\varphi_t = \frac{1 - \sqrt{\frac{\{(1-T_s(u))^2 + (I_s(u))^2 + (F_s(u))^2\}}{3}}}{\sum_{t=1}^{r}\left(1 - \sqrt{\frac{\{(1-T_s(u))^2 + (I_s(u))^2 + (F_s(u))^2\}}{3}}\right)}, \text{and} \sum_{t=1}^{r} \varphi_t = 1 \tag{1}$$

Step 6: Create an aggregated neutrosophic decision matrix by applying the single-value neutrosophic weighted averaging (SVNWA) operator as described:

$$[d_{ij}]_{m\times n}=\begin{array}{c} \\ A_1\\ A_2\\ \vdots\\ A_m\end{array}\begin{array}{c}\begin{array}{cccc} C_1 & C_2 & \cdots & C_n\end{array}\\ \begin{bmatrix} (AT_{11},AI_{11},AF_{11}) & (AT_{12},AI_{12},AF_{12}) & \cdots & (AT_{1n},AI_{1n},AF_{1n})\\ (AT_{21},AI_{21},AF_{21}) & (AT_{22},AI_{22},AF_{22}) & \cdots & (AT_{2n},AI_{2n},AF_{2n})\\ \vdots & \vdots & \ddots & \vdots\\ (AT_{m1},AI_{m1},AF_{m1}) & (AT_{m2},AI_{m2},AF_{m2}) & \cdots & (AT_{mn},AI_{mn},AF_{mn})\end{bmatrix}\end{array} \tag{2}$$

where $AT_{ij}=1-\prod_{t=1}^{r}\left(1-T_{ij}^{(t)}\right)^{\varphi_t}$, $AI_{ij}=\prod_{t=1}^{r}\left(I_{ij}^{(t)}\right)^{\varphi_t}$ and $AF_{ij}=\prod_{t=1}^{r}\left(F_{ij}^{(t)}\right)^{\varphi_t}$, $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$.

Step 7: Every decision-maker in a group setting has a different viewpoint on every criterion. Gathering a thorough assessment of each attribute's significance from all decision-makers is necessary to obtain the grouped opinion on the chosen attribute. Consequently, the SVNWA operator is used to determine the aggregated weights for the criteria at this phase as follows: $w_j=\left(1-\prod_{t=1}^{r}\left(1-T_j^{(t)}\right)^{\varphi_t},\prod_{t=1}^{r}\left(I_j^{(t)}\right)^{\varphi_t},\prod_{t=1}^{r}\left(F_j^{(t)}\right)^{\varphi_t}\right)$, where $j=1,2,\ldots,n$.

Step 8: Compute the weighted aggregated neutrosophic decision matrix using the two neutrosophic sets multiplication approach as follows:

$$\begin{array}{ccccc}(d_{ij})_{m\times n} & \times & w_j & = & (AT_{ij}\cdot w_j, AI_{ij}\cdot w_j, AF_{ij}\cdot w_j)\\ & & C_1 & & C_2 \quad \cdots \quad C_n\end{array}$$

$$=\begin{array}{c} \\ A_1\\ A_2\\ \vdots\\ A_m\end{array}\begin{array}{c}\\ \begin{bmatrix} (AT_{11}\cdot w_1, AI_{11}\cdot w_1, AF_{11}\cdot w_1) & (AT_{12}\cdot w_2, AI_{12}\cdot w_2, AF_{12}\cdot w_2) & \cdots & (AT_{1n}\cdot w_n, AI_{1n}\cdot w_n, AF_{1n}\cdot w_n)\\ (AT_{21}\cdot w_1, AI_{21}\cdot w_1, AF_{21}\cdot w_1) & (AT_{22}\cdot w_2, AI_{22}\cdot w_2, AF_{22}\cdot w_2) & \cdots & (AT_{2n}\cdot w_n, AI_{2n}\cdot w_n, AF_{2n}\cdot w_n)\\ \vdots & \vdots & \ddots & \vdots\\ (AT_{m1}\cdot w_1, AI_{m1}\cdot w_1, AF_{m1}\cdot w_1) & (AT_{m2}\cdot w_2, AI_{m2}\cdot w_2, AF_{m2}\cdot w_2) & \cdots & (AT_{mn}\cdot w_n, AI_{mn}\cdot w_n, AF_{mn}\cdot w_n)\end{bmatrix}\end{array} \tag{3}$$

where $AT_{ij}\cdot w_j=\left(1-\prod_{t=1}^{r}\left(1-T_{ij}^{(t)}\right)^{\varphi_t}\right)\times\left(1-\prod_{t=1}^{r}\left(1-T_j^{(t)}\right)^{\varphi_t}\right)$

$AI_{ij}\cdot w_j=\prod_{t=1}^{r}\left(I_{ij}^{(t)}\right)^{\varphi_t}+\prod_{t=1}^{r}\left(I_j^{(t)}\right)^{\varphi_t}-\prod_{t=1}^{r}\left(I_{ij}^{(t)}\right)^{\varphi_t}\cdot\prod_{t=1}^{r}\left(I_j^{(t)}\right)^{\varphi_t}$

$AF_{ij}\cdot w_j=\prod_{t=1}^{r}\left(F_{ij}^{(t)}\right)^{\varphi_t}+\prod_{t=1}^{r}\left(F_j^{(t)}\right)^{\varphi_t}-\prod_{t=1}^{r}\left(F_{ij}^{(t)}\right)^{\varphi_t}\cdot\prod_{t=1}^{r}\left(F_j^{(t)}\right)^{\varphi_t}\ \forall\ i=1,2,\ldots,m$ and $j=1,2,\ldots,n$.

Step 9: Determine the relative neutrosophic positive ideal solution A_N^{+} and negative ideal solution A_N^{-} for the attributes of benefit and cost as described below:

$$A_N^{+}=\left(d_{1.w}^{+},d_{2.w}^{+},\ldots,d_{j.w}^{+}\right);\quad A_N^{-}=\left(d_{1.w}^{-},d_{2.w}^{-},\ldots,d_{j.w}^{-}\right)$$

where $d_{j.w}^{+}=\left((AT_j.w)^{+},(AI_j.w)^{+},(AF_j.w)^{+}\right);\quad d_{j.w}^{-}=\left((AT_j.w)^{-},(AI_j.w)^{-},(AF_j.w)^{-}\right)$

$(AT_j.w)^{+}=\left\{\left(\max_i\left\{(AT_{ij}.w_j)\mid j\in j_1\right\}\right),\left(\min_i\left\{(AT_{ij}.w_j)\mid j\in j_2\right\}\right)\right\}$

$(AI_j.w)^{+}=\left\{\left(\min_i\left\{(AI_{ij}.w_j)\mid j\in j_1\right\}\right),\left(\max_i\left\{(AI_{ij}.w_j)\mid j\in j_2\right\}\right)\right\}$

$$(AF_j.w)^+ = \left\{ \left(\min_i \{ (AF_{ij}.w_j) \mid j \in j_1 \} \right), \left(\max_i \{ (AF_{ij}.w_j) \mid j \in j_2 \} \right) \right\}$$

$$(AT_j.w)^- = \left\{ \left(\min_i \{ (AT_{ij}.w_j) \mid j \in j_1 \} \right), \left(\max_i \{ (AT_{ij}.w_j) \mid j \in j_2 \} \right) \right\}$$

$$(AI_{j.w})^- = \left\{ \left(\max_i \left\{ \left(AI_{ij.w_j}\right) \mid j \in j_1 \right\} \right), \left(\min_i \left\{ \left(AI_{ij.w_j}\right) \mid j \in j_2 \right\} \right) \right\}$$

$$(AF_{j.w})^- = \left\{ \left(\max_i \left\{ \left(AF_{ij.w_j}\right) \mid j \in j_1 \right\} \right), \left(\min_i \left\{ \left(AF_{ij.w_j}\right) \mid j \in j_2 \right\} \right) \right\}$$

Step 10: Calculate the closeness coefficient for each alternative based on its distance from the relative neutrosophic positive and negative ideal solutions.

$$C_i^* = \frac{D_i^-}{D_i^+ + D_i^-}$$

where,

$$D_i^+ = \sqrt{\frac{1}{3n} \left\{ \sum_{j=1}^{n} \left[\left(AT_{ij}.w_j - (AT_j.w)^+\right)^2 + \left(AI_{ij}.w_j - (AI_j.w)^+\right)^2 + \left(AF_{ij}.w_j - (AF_j.w)^+\right)^2 \right] \right\}}$$

$$D_i^- = \sqrt{\frac{1}{3n} \left\{ \sum_{j=1}^{n} \left[\left(AT_{ij}.w_j - (AT_j.w)^-\right)^2 + \left(AI_{ij}.w_j - (AI_j.w)^-\right)^2 + \left(AF_{ij}.w_j - (AF_j.w)^-\right)^2 \right] \right\}}$$

Step 11: Rank the alternatives according to their closeness coefficients from highest to lowest.

Step 12: Identify the best alternatives in accordance with the closeness coefficients. The best alternative is the one with the maximum closeness coefficient.

5. **Model Implementation**

Crop selection has a significant role in creating sustainable agriculture. In this study, to improve the sustainable agriculture of Ariyalur district, which plays a major role in the agriculture sector, the above-mentioned method is used to rank the crops (sugarcane, paddy, cotton, groundnut, maize, brinjal, tomato, chilli, onion) cultivated there on the basis of criteria (production, profitability, water availability, seed growth, soil texture, precipitation, irrigation, crop demand, price of crop, expenditure and fertilizer). Based on the advice of agriculture field specialists and a comprehensive assessment of the literature, these criteria were developed. An alternative and attribute-based questionnaire is used to gather data for the current study from the agriculture experts in the Ariyalur district. The agriculture experts provided the complete data in the predetermined format. The weight of the attributes and the weight of the alternatives based on the attributes are collected from the four-decision maker in the form of linguistic terms. The linguistic terms' rates are expressed as a single-valued neutrosophic set. The linguistic terms for attributes and importance of decision-makers are shown in Table 1. Also, the linguistic terms of alternatives are mentioned in Table 2 in SVNS rating

TABLE 1. Linguistic terms of decision maker and Attributes.

Linguistics Term	**SVNN**
Very important (VI)	(0.95, 0.05, 0.05)
Important (I)	(0.85, 0.15, 0.15)
Medium (M)	(0.50, 0.40, 0.45)
Unimportant (UI)	(0.30, 0.60, 0.70)
Very unimportant (VUI)	(0.05, 0.85, 0.95)

TABLE 2. Alternatives' Linguistic Terms.

Linguistics Term	**SVNN**
Extremely high (EH)	(1, 0, 0)
Very high (VH)	(0.95, 0.10, 0.05)
High (H)	(0.85, 0.15, 0.15)
Medium (M)	(0.65, 0.40, 0.35)
Low (L)	(0.20, 0.75, 0.80)
Very low (VL)	(0.10, 0.85, 0.90)
Extremely low (EL)	(0.05, 0.90, 0.95)

format. In this present study, the linguistic term for (DM_1, DM_2, DM_3, DM_4) is (VI, I, VI, M) respectively.

Using Eq. (1), the decision makers' weights are ascertained as follows:

$$\varphi_1 = \frac{1 - \sqrt{\frac{\{(1-T_1(x))^2+(I_1(x))^2+(F_1(x))^2\}}{3}}}{\sum_{t=1}^{4}\left(1 - \sqrt{\frac{\{(1-T_s(x))^2+(I_s(x))^2+(F_s(x))^2\}}{3}}\right)}$$

$$= \frac{1 - \sqrt{\frac{\{(0.05^2+0.05^2+0.05^2\}}{3}}}{4 - \sqrt{0.0025} - \sqrt{0.0025} - \sqrt{0.0025} - \sqrt{0.2042}}$$

$$\varphi_1 = 0.2888$$

Similarly, $\varphi_2 = 0.2577$, $\varphi_3 = 0.288$, $\varphi_4 = 0.1662$. Therefore (0.288, 0.2577, 0.288, 0.1662) is the four decision makers' weight vectors.

The linguistic term of every alternative $A_i(i = 1, 2, \ldots, 9)$ in relation to every criterion $C_j(j = 1, 2, \ldots, 11)$, as well as the linguistic terms for the weights of the criteria provided by the four decision makers (DM_1, DM_2, DM_3, DM_4) are displayed in Table 3. Using Table 1 & Table 2, convert the linguistic terms of the alternatives and attributes in Table 3 into SVNNs. Then, as stated in Step 3, generate the single-valued neutrosophic decision matrix. Additionally, apply Step 4 to normalize the decision matrix. After these conversions, Step 6 is used to define the aggregated neutrosophic decision matrix in the following manner:

$$d_{11} = \left(1 - \prod_{t=1}^{4} \left(1 - T_{11}^{(t)}\right)^{\varphi_t}, \prod_{t=1}^{4} \left(I_{11}^{(t)}\right)^{\varphi_t}, \prod_{t=1}^{4} \left(F_{11}^{(t)}\right)^{\varphi_t}\right)$$

TABLE 3. Weight for alternatives and attributes in linguistic terms.

Alter	D.M	Prod	Prof	W.A	S.G	S.T	P	I.T	C.D	P.C	E	F
Sugarcane	1	H	H	H	H	VH	L	VH	VH	H	M	VH
	2	H	M	M	H	M	L	H	H	L	H	H
	3	M	M	M	H	VH	L	VH	M	L	H	H
	4	M	M	M	M	H	M	H	M	M	M	VH
Paddy	1	H	M	VH	H	VH	L	VH	VH	H	H	VH
	2	M	M	M	M	M	L	M	M	L	M	H
	3	M	M	L	L	VH	L	M	L	M	VH	M
	4	VL	VL	VL	H	M	VL	H	VL	VL	VH	VH
Cotton	1	H	H	M	H	VH	M	VH	VH	H	M	H
	2	M	M	VL	H	H	L	H	M	L	H	H
	3	VH	L	VL	H	H	VL	H	M	L	M	H
	4	L	M	M	M	L	VL	VH	VH	VL	VH	M
Groundnut	1	M	M	M	H	VH	M	VH	H	H	H	VH
	2	M	M	M	H	H	L	H	H	M	M	H
	3	M	M	M	M	M	VL	H	H	M	H	H
	4	M	M	M	M	H	L	VH	H	M	H	VH
Maize	1	M	H	M	H	VH	M	VH	M	H	M	H
	2	M	M	M	H	VH	M	H	M	M	M	H
	3	H	M	L	H	H	VL	H	H	L	M	H
	4	M	M	VL	M	M	L	VH	VH	M	H	H
Brinjal	1	M	H	M	H	H	M	H	H	H	M	G
	2	M	M	M	M	H	M	H	M	M	M	H
	3	H	L	M	M	H	VL	H	H	H	H	H
	4	VL	L	L	M	M	VL	H	M	M	VH	M
Tomato	1	M	H	M	H	H	M	M	VH	H	M	H
	2	L	M	M	H	H	L	H	H	H	M	H
	3	H	M	M	H	H	VL	M	H	M	M	H
	4	M	M	M	M	H	VL	H	M	VL	M	VH
Chilli	1	VH	VH	M	H	M	M	M	VH	H	M	H
	2	H	H	H	H	VH	L	H	H	H	M	H
	3	M	M	M	M	H	VL	H	M	M	H	H
	4	M	M	L	M	M	VL	VH	M	M	H	M
Onion	1	M	H	M	M	H	M	H	VH	H	H	H
	2	M	H	M	H	H	H	H	H	M	M	H
	3	H	M	M	M	H	VL	M	H	H	M	H
	4	L	L	VL	L	M	VL	VH	L	M	M	H
weight	1	VI	I	VI	VI	VI	VI	VI	VI	I	I	VI
	2	I	VI	I	I	I	I	I	I	VI	I	I
	3	VI	VI	VI	VI	VI	VI	VI	VI	VI	I	I
	4	VI	I	I	VI	VI	VI	VI	VI	I	I	VI

$$AT_{11} = 1 - \prod_{t=1}^{4} \left(1 - T_{11}^{(t)}\right)^{\varphi_t}$$
$$= 1 - \left((1-0.85)^{0.288} \times (1-0.85)^{0.2577} \times (1-0.65)^{0.288} \times (1-0.65)^{0.1662}\right) = 0.7796$$

$$AI_{11} = \prod_{t=1}^{4} \left(I_{11}^{(t)}\right)^{\varphi_t} = (0.15)^{0.288} \times (0.15)^{0.2577} \times (0.4)^{0.288} \times (0.4)^{0.1662} = 0.2342$$

$$AF_{11} = \prod_{t=1}^{4} \left(F_{11}^{(t)}\right)^{\varphi_t} = (0.15)^{0.288} \times (0.15)^{0.2577} \times (0.35)^{0.288} \times (0.35)^{0.1662} = 0.2204$$

Similarly, all the values of $(AT_{ij}, AI_{ij}, AF_{ij})$ will be determined. Consequently, the following is the final aggregated neutrosophic decision matrix:

Now the aggregated weight of the attributes is defined by using Step7 as follows:

TABLE 4. Aggregated neutrosophic decision matrix.

Alter	Prod	Prof	W.A	S.G	S.T	P	I.T	C.D	P.C	E	F
S	(0.7796, 0.2342, 0.2204)	(0.7257, 0.3016, 0.2742)	(0.7257, 0.3016, 0.2742)	(0.8273, 0.1766, 0.1727)	(0.9009, 0.1529, 0.0991)	(0.3027, 0.6756, 0.6973)	(0.9203, 0.1188, 0.0.0797)	(0.8393, 0.2084, 0.1607)	(0.5694, 0.4250, 0.4306)	(0.4504, 0.5637, 0.5495)	(0.9089, 0.1247, 0.0911)
P	(0.6791, 0.3418, 0.3209)	(0.5905, 0.4534, 0.4095)	(0.7033, 0.3645, 0.2967)	(0.6977, 0.3071, 0.3023)	(0.8859, 0.18, 0.1141)	(0.1842, 0.7658, 0.8158)	(0.9203, 0.228, 0.1736)	(0.8393, 0.3645, 0.2967)	(0.5694, 0.402, 0.397	(0.4504, 0.6613 0.6688	0.9089 0.1655 0.1162
Co	(0.8204, 0.2246, 0.1796)	(0.652, 0.3614, 0.348)	(0.4139, 0.6036, 0.5861)	(0.8273, 0.1766, 0.1727)	(0.8556, 0.1744, 0.1444)	(0.3348, 0.6624, 0.6652)	(0.9089, 0.1248, 0.0911)	(0.8554, 0.2131, 0.1446)	(0.4962, 0.4817, 0.5038)	(0.4932, 0.5332, 0.5068)	(0.8273, 0.1766, 0.1727)
GN	(0.65, 0.4, 0.35)	(0.65, 0.4, 0.35)	(0.65, 0.4, 0.35)	(0.7796, 0.2342, 0.2204)	(0.8604, 0.1771, 0.1396)	(0.3477, 0.6488, 0.6523)	(0.9089, 0.1248, 0.0911)	(0.85, 0.15, 0.15)	(0.7258, 0.3016, 0.2742)	(0.3535, 0.6379, 0.6465)	(0.9089, 0.1248, 0.0911)
M	(0.7258, 0.3016, 0.2742)	(0.7258, 0.3016, 0.2742)	(0.4803, 0.5434, 0.5095)	(0.8273, 0.1766, 0.1727)	(0.9052, 0.1415, 0.0948)	(0.4729, 0.5518, 0.5271)	(0.9089, 0.1248, 0.0911)	(0.8015, 0.2395, 0.1985)	(0.652, 0.3614, 0.348)	(0.5984, 0.4441, 0.4016)	(0.85, 0.15, 0.15)
B	(0.6791, 0.3418, 0.3209)	(0.6008, 0.4013, 0.3992)	(0.5984, 0.4441, 0.4016)	(0.7258, 0.3016, 0.2742)	(0.8273, 0.1766, 0.1727)	(0.4625, 0.5634, 0.5375)	(0.85, 0.15, 0.15)	(0.7851, 0.2274, 0.2149)	(0.7851, 0.2274, 0.2149)	(0.4804, 0.5434, 0.5196)	(0.8273, 0.1766, 0.1727)
T	(0.6606, 0.3546, 0.3394)	(0.7258, 0.3016, 0.2742)	(0.65, 0.4, 0.35)	(0.8273, 0.1766, 0.1727)	(0.85, 0.15, 0.15)	(0.3348, 0.6624, 0.6652)	(0.7556, 0.264, 0.2444)	(0.8741, 0.1571, 0.1259)	(0.7421, 0.2655, 0.2579)	(0.65, 0.4, 0.35)	(0.875, 0.1403, 0.125)
Ch	(0.8393, 0.2084, 0.1607)	(0.8393, 0.2084, 0.1607)	(0.6772, 0.3449, 0.3228)	(0.78, 0.2342, 0.2204)	(0.8339, 0.211, 0.1661)	(0.3348, 0.6624, 0.6652)	(0.8405, 0.186, 0.1595)	(0.8393, 0.2084, 0.1607)	(0.78, 0.2342, 0.2204)	(0.4905, 0.5322, 0.5095)	(0.8273, 0.1766, 0.1727)
O	(0.6854, 0.3348, 0.3146)	(0.7471, 0.26, 0.2529)	(0.5905, 0.4534, 0.4095)	(0.6772, 0.3449, 0.3228)	(0.8273, 0.1766, 0.1727)	(0.5679, 0.4375, 0.4321)	(0.8405, 0.186, 0.1595)	(0.8556, 0.1744, 0.1444)	(0.7851, 0.2274, 0.2149)	(0.5559, 0.4794, 0.4441)	(0.85, 0.15, 0.15)

$$w_1 = \left(1 - \prod_{t=1}^{4} \left(1 - T_1^{(t)}\right)^{\varphi_t}, \prod_{t=1}^{4} \left(I_1^{(t)}\right)^{\varphi_t}, \prod_{t=1}^{4} \left(F_1^{(t)}\right)^{\varphi_t}\right)$$

where

$$1 - \prod_{t=1}^{4} \left(1 - T_1^{(t)}\right)^{\varphi_t} = 1 - (1 - 0.95)^{0.288} \times (1 - 0.85)^{0.2577} \times (1 - 0.95)^{0.288} \times (1 - 0.95)^{0.1662} = 0.9336$$

$$\prod_{t=1}^{4} \left(I_1^{(t)}\right)^{\varphi_t} = 0.05^{0.288} \times 0.15^{0.2577} \times 0.05^{0.288} \times 0.05^{0.1662} = 0.0664$$

$\prod_{t=1}^{4}\left(F_1^{(t)}\right)^{\varphi_t} = 0.05^{0.288} \times 0.15^{0.2577} \times 0.05^{0.288} \times 0.05^{0.1662} = 0.0664$

Thus, $w_1 = (0.9336, 0.0664, 0.0664)$. Similarly, the weights of the criteria (profitability, water availability, seed growth, soil texture, precipitation, irrigation, crop demand, price of crop, expenditure and fertilize) will be determined, and the final aggregated weight of the criteria is as follows:

$$W = \begin{bmatrix} (0.9336, 0.0664, 0.0664); & (0.9176, 0.0824, 0.0824) \\ (0.9203, 0.0797, 0.0797); & (0.9336, 0.0664, 0.0664) \\ (0.9336, 0.0664, 0.0664); & (0.9336, 0.0664, 0.0664) \\ (0.9336, 0.0664, 0.0664); & (0.9336, 0.0664, 0.0664) \\ (0.9176, 0.0824, 0.0824); & (0.85, 0.15, 0.15) \\ (0.9089, 0.0911, 0.0911) & \end{bmatrix}$$

The weighted aggregated neutrosophic decision matrix is computed using Step 8 in the following manner after locating the aggregated neutrosophic decision matrix and the attributes' aggregated weights:

$d_{11} \times w_1 = (\mathrm{AT}_{11} \cdot w_1,\ \mathrm{AI}_{11} \cdot w_1,\ \mathrm{AF}_{11} \cdot w_1)$

where,

$$\begin{aligned} \mathrm{AT}_{11} \cdot w_1 &= \left(1 - \prod_{t=1}^{4}\left(1 - T_{11}^{(t)}\right)^{\varphi_t}\right) \times \left(1 - \prod_{t=1}^{4}\left(1 - T_1^{(t)}\right)^{\varphi_t}\right) \\ &= 0.7796 \times 0.9336 = 0.7278 \\ \mathrm{AI}_{11} \cdot w_1 &= \prod_{t=1}^{4}\left(I_{11}^{(t)}\right) + \prod_{t=1}^{4}\left(I_1^{(t)}\right)^{\varphi_t} - \prod_{t=1}^{4}\left(I_{11}^{(t)}\right) \times \prod_{t=1}^{4}\left(I_1^{(t)}\right)^{\varphi_t} \\ &= 0.2342 + 0.0664 - 0.2342 \times 0.0664 = 0.2851 \\ \mathrm{AF}_{11} \cdot w_1 &= \prod_{t=1}^{4}\left(F_{11}^{(t)}\right) + \prod_{t=1}^{4}\left(F_1^{(t)}\right)^{\varphi_t} - \prod_{t=1}^{4}\left(F_{11}^{(t)}\right) \times \prod_{t=1}^{4}\left(F_1^{(t)}\right)^{\varphi_t} \\ &= 0.2204 + 0.0664 - 0.2204 \times 0.0664 = 0.2722 \end{aligned}$$

Similarly, all the values of $(\mathrm{AT}_{ij} \cdot w_j, \mathrm{AI}_{ij} \cdot w_j, \mathrm{AF}_{ij} \cdot w_j)$ will be determined, and the final weighted aggregated neutrosophic decision matrix is presented in Table 5.

Using Step 9, the relative positive and negative ideal solutions are generated from Table 5 as follows:

$$A_N^+ = \begin{bmatrix} (0.7836, 0.261, 0.2164); & (0.7702, 0.2736, 0.2298) \\ 0.6679, 0.3573, 0.3321); & (0.7724, 0.2313, 0.2276) \\ (0.8451, 0.1985, 0.1549); & (0.5302, 0.4749, 0.4698) \\ (0.8592, 0.1773, 0.1408); & (0.8161, 0.2065, 0.1839) \\ (0.7204, 0.2910, 0.2796); & (0.1408, 0.8279, 0.8592) \\ (0.8261, 0.2045, 0.1739) & \end{bmatrix} \quad A_N^- = \begin{bmatrix} (0.6068, 0.4399, 0.3932); & (0.5418, 0.4985, 0.4582) \\ 0.3809, 0.6352, 0.6191); & (0.6322, 0.3884, 0.3678) \\ (0.7724, 0.2634, 0.2276); & (0.1719, 0.7813, 0.8281) \\ (0.7054, 0.3128, 0.2946); & (0.6566, 0.4067, 0.3434) \\ (0.4553, 0.5244, 0.5447); & (0.2975, 0.66, 0.7025) \\ (0.7519, 0.2516, 0.2481) & \end{bmatrix}$$

Using the above positive and negative ideal solutions, and Step 10 the distance measures are determined as follows:

$$D_1^+ = \sqrt{\frac{1}{33}\left(\sum_{j=1}^{n}\left[\left(\mathrm{AT}_{1j} \cdot w_j - (\mathrm{AT}_j \cdot w)^+\right)^2 + \left(\mathrm{AI}_{1j} \cdot w_j - (\mathrm{AI}_j \cdot w)^+\right)^2 + \left(\mathrm{AF}_{1j} \cdot w_j - (\mathrm{AF}_j \cdot w)^+\right)^2\right]\right)} = 0.1012$$

TABLE 5. Weighted aggregated neutrosophic decision matrix.

Alter	Prod	Prof	W.A	S.G	S.T	P	I.T	C.D	P.C	E	F
S	(0.7278, 0.2851, 0.2722)	(0.666, 0.3591, 0.334)	(0.6679, 0.3573, 0.3321)	(0.7724, 0.2313, 0.2276)	(0.8411, 0.2092, 0.1589)	(0.2826, 0.6972, 0.7174)	(0.8592, 0.1773, 0.1408)	(0.7836, 0.261, 0.2164)	(0.5225, 0.4724, 0.4775)	(0.2104, 0.7668, 0.7896)	(0.8261, 0.2045, 0.1739)
P	(0.6341, 0.3855, 0.3659)	(0.5418, 0.4985, 0.4582)	(0.6473, 0.4152, 0.3527)	(0.6514, 0.3531, 0.3486)	(0.8271, 0.2345, 0.1729)	(0.1719, 0.7813, 0.8281)	(0.7715, 0.2792, 0.2285)	(0.6566, 0.4067, 0.3434)	(0.5533, 0.4512, 0.4467)	(0.1408, 0.8279, 0.8592)	(0.8032, 0.2415, 0.1968)
Co	(0.7659, 0.2761, 0.2341)	(0.5983, 0.4141, 0.4017)	(0.3809, 0.6352, 0.619)	(0.7724, 0.2313, 0.2276)	(0.7988, 0.2293, 0.2012)	(0.3126, 0.6848, 0.6874)	(0.8486, 0.1829, 0.1514)	(0.7986, 0.2654, 0.2014)	(0.4553, 0.5244, 0.5447)	(0.2309, 0.7468, 0.7806)	(0.7519, 0.2516, 0.2481)
GN	(0.6068, 0.4399, 0.3932)	(0.5964, 0.4495, 0.4036)	(0.5982, 0.4479, 0.4018)	(0.7278, 0.2851, 0.2722)	(0.8033, 0.2317, 0.1967)	(0.3246, 0.6721, 0.6754)	(0.8486, 0.1829, 0.1514)	(0.7935, 0.2065, 0.2065)	(0.666, 0.3591, 0.3340)	(0.1757, 0.8105, 0.8243)	(0.8261, 0.2045, 0.1739)
M	(0.6776, 0.348, 0.3224)	(0.666, 0.3591, 0.3340)	(0.4421, 0.5798, 0.5486)	(0.7724, 0.2313, 0.2276)	(0.8451, 0.1985, 0.1549)	(0.4415, 0.5815, 0.5585)	(0.8486, 0.1829, 0.1514)	(0.7483, 0.29, 0.2517)	(0.5983, 0.4141, 0.4017)	(0.2723, 0.6904, 0.7277)	(0.7723, 0.2275, 0.2275)
B	(0.6341, 0.3855, 0.3659)	(0.5513, 0.4506, 0.4487)	(0.5507, 0.4884, 0.4493)	(0.6776, 0.348, 0.3224)	(0.7724, 0.2313, 0.2276)	(0.4317, 0.5924, 0.5683)	(0.7935, 0.2065, 0.2065)	(0.733, 0.2787, 0.267)	(0.7204, 0.2910, 0.2796)	(0.2142, 0.7531, 0.7858)	(0.7519, 0.2516, 0.2481)
T	(0.6168, 0.3975, 0.3832)	(0.666, 0.3591, 0.3340)	(0.5982, 0.4479, 0.4018)	(0.7724, 0.2313, 0.2276)	(0.7935, 0.2065, 0.2065)	(0.3126, 0.6848, 0.6874)	(0.7054, 0.3128, 0.2946)	(0.8161, 0.2131, 0.1839)	(0.6809, 0.3260, 0.3191)	(0.2975, 0.66, 0.7025)	(0.7953, 0.2186, 0.2047)
Ch	(0.7836, 0.261, 0.2164)	(0.7702, 0.2736, 0.2298)	(0.6232, 0.3971, 0.3768)	(0.7278, 0.2851, 0.2722)	(0.7785, 0.2634, 0.2215)	(0.3126, 0.6848, 0.6874)	(0.7847, 0.2401, 0.2153)	(0.7836, 0.261, 0.2164)	(0.7153, 0.2973, 0.2847)	(0.2259, 0.7474, 0.7741)	(0.7519, 0.2516, 0.2481)
O	(0.6399, 0.379, 0.3601)	(0.6855, 0.321, 0.3145)	(0.5434, 0.497, 0.4566)	(0.6322, 0.3884, 0.3678)	(0.7724, 0.2313, 0.2276)	(0.5302, 0.4749, 0.4698)	(0.7847, 0.2401, 0.2153)	(0.7988, 0.2293, 0.2012)	(0.7204, 0.2910, 0.2796)	(0.2531, 0.7138, 0.7469)	(0.7725, 0.2275, 0.2275)

$$D_1^- = \sqrt{\frac{1}{33}\left(\sum_{j=1}^{n}\left[\left(\mathrm{AT}_{1j}\cdot w_j-(\mathrm{AT}_j\cdot w)^-\right)^2+\left(\mathrm{AI}_{1j}\cdot w_j-(\mathrm{AI}_j\cdot w)^-\right)^2+\left(\mathrm{AF}_{1j}\cdot w_j-(\mathrm{AF}_j\cdot w)^-\right)^2\right]\right)} = 0.1375$$

Similarly, all the relative neutrosophic distance measures will be calculated. After calculating all D_i^+, D_i^-, the closeness coefficient is determined by using the formula mentioned in step 10 as follows:

$$C_1^* = \frac{D_1^-}{D_1^+ + D_1^-} = \frac{0.1375}{0.1012+0.1375} = 0.576$$

The distance measures and the closeness coefficients of all the alternatives are mentioned in the Table 6. Additionally, Figure 1 illustrates the graphical representation of alternatives versus the closeness coefficient. From Table 6, and Figure 1, it is clear that the closeness coefficient of the alternative chilli is greater than the other alternatives. Therefore, chilli is the most desirable crop. Also, the order preference of all the alternatives is Chilli > Onion > Sugarcane > Groundnut > Brinjal > Maize > Tomato > Cotton > Paddy.

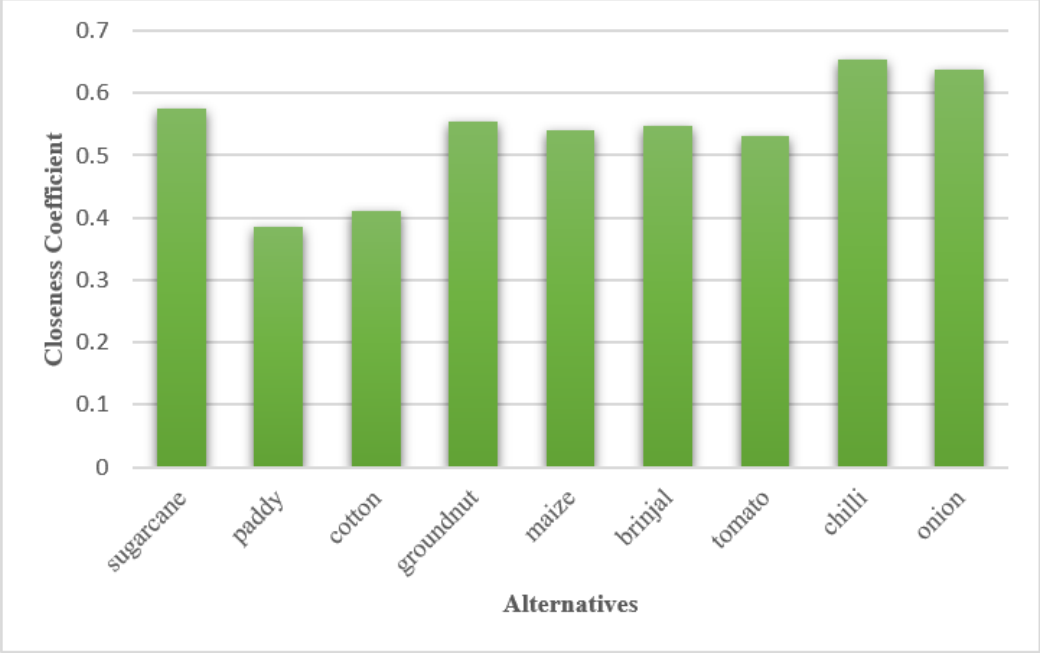

FIGURE 1. Alternatives vs Closeness coefficients

TABLE 6. Closeness Coefficients of the Alternatives.

Alternatives	D^+	D^-	C^*
Sugarcane	0.1012	0.1375	0.576
Paddy	0.1573	0.0986	0.3853
Cotton	0.1456	0.1019	0.4117
Groundnut	0.1035	0.1286	0.5541
Maize	0.1037	0.1218	0.5401
Brinjal	0.1053	0.1276	0.5479
Tomato	0.1121	0.1271	0.5314
Chilli	0.0814	0.1535	0.6535
Onion	0.0885	0.1553	0.637

6. **Conclusion**

Crop selection for sustainable agriculture is a complicated procedure. It presents a number of difficulties, particularly when it is accomplished while taking into consideration a wide range of sustainability-influencing criteria. The TOPSIS-based SVNS is used in this study for Ariyalur district farmers to rank the nine crops—sugarcane, paddy, cotton, groundnut, maize, brinjal, tomato, chilli, and onion—based on the criteria of production, profitability, water availability, seed growth, soil texture, precipitation, irrigation, crop demand, crop price, expenditure, and fertilizer. From the closeness coefficient of this study, it is concluded that

chilli is the most optimal crop to cultivate in Ariyalur district. A neutrosophic-based multi-criteria approach is crucial because multiple uncertainties and instabilities frequently impact the crop selection process. This strategy can assist policymakers and farmers in creating all-encompassing policies that support sustainable farming methods, which the world desperately needs. Subsequent studies could look into ways to mitigate and adapt to the consequences of climate change, which would eventually lead to more sustainable farming methods and better decision-making.

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References

1. Hwang, C.L.; Yoon, K. Multiple Attribute Decision Making: Methods and Applications.; Springer-Verlag, New York, 1981.
2. Opricovic, S.; Tzeng, G.H. Compromise solution by MCDM methods: a comparative analysis of VIKOR and TOPSIS. European Journal of Operational Research 2004, 156, 445–455.
3. Brans, J.P.; Vincke, P.; Mareschal, B. How to select and how to rank projects: the PROMETHEE method. European Journal of Operational Research 1986, 24, 228–238.
4. Roy, B. The outranking approach and the foundations of ELECTRE methods. Theory and Decision 1991, 31, 49–73.
5. Zadeh, L.A. Fuzzy sets. Information and Control 1965, 8, 338–353.
6. Sarı, F.; Koyuncu, F. Multi-criteria decision analysis to determine the suitability of agricultural crops for land consolidation areas. International Journal of Engineering and Geosciences 2021, 6(2), 64–73.
7. Huang, W.; Zhang, Q. Selecting the optimal economic crop in minority regions with the criteria about soil and water conservation. Agricultural Water Management 2020, 241, 106295.
8. Singh, R.K.; Mallick, J. Fuzzy-based multi-criteria method for sustainable green chamber farming practices. Indian Journal of Agricultural Sciences 2019, 89(10), 1732–1736.
9. Cobuloglu, H.I.; Buyuktahtakın, I.E. A stochastic multi-criteria decision analysis for sustainable biomass crop selection. Expert Systems with Applications 2015, 42(15-16), 6065–6074.
10. Yong, D. Plant location selection based on fuzzy TOPSIS. The International Journal of Advanced Manufacturing Technology 2006, 28, 839–844.
11. Ertugrul, I.; Karakasoglu, N. Comparison of fuzzy AHP and fuzzy TOPSIS methods for facility location selection. The International Journal of Advanced Manufacturing Technology 2008, 39, 783–795.
12. Atanassov, K.T. Intuitionistic fuzzy sets. Fuzzy Sets and Systems 1986, 20, 87–96.
13. Gupta, P.; Mehlawat, M.K.; Grover, N. Intuitionistic fuzzy multi-attribute group decision-making with an application to plant location selection based on a new extended VIKOR method. Information Sciences 2016, 370, 184–203.
14. Wan, S.; Dong, J. A new method for Atanassov's interval-valued intuitionistic fuzzy MAGDM with incomplete attribute weight information. In Decision Making Theories and Methods Based on Interval-Valued Intuitionistic Fuzzy Sets; 2020; 37–69.

15. Mardani, A.; Devi, S.; Alrasheedi, M.; Arya, L.; Singh, M.P.; Pandey, K. Hybrid intuitionistic fuzzy entropy-SWARA-COPRAS method for multi-criteria sustainable biomass crop type selection. Sustainability 2023, 15(10), 7765.
16. Smarandache, F. A unifying field in Logics: Neutrosophic Logic, in Philosophy (American Research Press, 1999) pp. 1–141.
17. Wang, H.; Smarandache, F.; Sunderraman, R.; Zhang, Y.Q. Single valued neutrosophic sets. Multispace and Multistructure 2010, 4, 410–413.
18. Garai, T.; Garg, H.; Roy, T.K. A ranking method based on possibility mean for multi-attribute decision making with single valued neutrosophic numbers. Journal of Ambient Intelligence and Humanized Computing 2020, 11, 5245–5258.
19. Xu, D.; Wei, X.; Ding, H.; Bin, H. A new method based on PROMETHEE and TODIM for multi-attribute decision-making with single-valued neutrosophic sets. Mathematics 2020, 8(10), 1816.
20. Karaaslan, F. Gaussian single-valued neutrosophic numbers and its application in multi-attribute decision making. Neutrosophic Sets and Systems 2018, 22(1), 101–117.
21. Alzahrani, F.A.; Ghorui, N.; Gazi, K.H.; Giri, B.C.; Ghosh, A.; Mondal, S.P. Optimal site selection for women university using neutrosophic multi-criteria decision-making approach. Buildings 2023, 13(1), 152.
22. Ye, J. Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. International Journal of General Systems 2013, 42(4), 386–394.
23. Zhang, H.Y.; Wang, J.Q.; Chen, X.H. Interval neutrosophic sets and their application in multicriteria decision making problems. The Scientific World Journal 2014, 2014(1), 645953.
24. Chi, P.; Liu, P. An extended TOPSIS method for the multiple attribute decision making problems based on interval neutrosophic set. Neutrosophic Sets and Systems 2013, 1(1), 63–70.
25. Garg, H. A novel divergence measure and its based TOPSIS method for multi criteria decision-making under single-valued neutrosophic environment. Journal of Intelligent & Fuzzy Systems 2019, 36(1), 101–115.
26. Mishra, A.R.; Rani, P.; Prajapati, R.S. Multi-criteria weighted aggregated sum product assessment method for sustainable biomass crop selection problem using single-valued neutrosophic sets. Applied Soft Computing 2021, 113, 108038.
27. Abdel-Basset, M.; Gamal, A.; Son, L.H.; Smarandache, F. A bipolar neutrosophic multi criteria decision making framework for professional selection. Applied Sciences 2020, 10(4), 1202.
28. Ashraf, S.; Abdullah, S. Decision support modeling for agriculture land selection based on sine trigonometric single valued neutrosophic information. International Journal of Neutrosophic Science (IJNS) 2020, 9(2), 60–73.
29. Garai, T.; Garg, H. Multi-criteria decision making of water resource management problem (in Agriculture field, Purulia district) based on possibility measures under generalized single valued non-linear bipolar neutrosophic environment. Expert Systems with Applications 2022, 205, 117715.
30. Xu, D.; Peng, L. An Improved Method Based on TODIM and TOPSIS for Multi-Attribute Decision-Making with Multi-Valued Neutrosophic Sets. CMES-Computer Modeling in Engineering & Sciences 2021, 129(2).

An Optimization Approach with Single-Valued Neutrosophic Hesitant Fuzzy Dombi Aggregation Operators

Sania Batool,[1] Masooma Raza Hashmi,[1] Muhammad Riaz,[1] Florentin Smarandache,[2] Dragan Pamucar,[3] Dejan Spasic[4]

[1] Department of mathematics, University of the Punjab, Lahore 54590, Pakistan
[2] Mathematics, Physical and Natural Sciences Division, University of New Mexico, Gallup Campus, Gallup, NM 87301, USA
[3] Department of Operations Research and Statistics, Faculty of Organizational Sciences, University of Belgrade, 11000 Belgrade, Serbia
[4] The Faculty of Civil Aviation, Megatrend University, 11000 Belgrade, Serbia

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Abstract: Using the strength of a single-valued neutrosophic set (SVNS) with the flexibility of a hesitant fuzzy set (HFS) yields a robust model named the single-valued neutrosophic hesitant fuzzy set (SVNHFS). Due to the ability to utilize three independent indexes (truthness, indeterminacy, and falsity), an SVNHFS is an efficient model for optimization and computational intelligence (CI) as well as an intelligent decision support system (IDSS). Taking advantage of the flexibility of operational parameters in Dombi's t-norm and t-conorm operations, new aggregation operators (AOs) are proposed, which are named the SVN fuzzy Dombi weighted averaging (SVNHFDWA) operator, SVN hesitant fuzzy Dombi ordered weighted averaging (SVNHFDOWA) operator, SVN hesitant fuzzy Dombi hybrid averaging (SVNHFDHWA) operator, SVN hesitant fuzzy Dombi weighted geometric (SVNHFDWG) operator, SVN hesitant fuzzy Dombi ordered weighted geometric (SVNHFDOWG) operator as well as SVN hesitant fuzzy Dombi hybrid weighted geometric (SVNHFDHWG) operator. The efficiency of these AOs is investigated in order to determine the best option using SVN hesitant fuzzy numbers (SVNHFNs) in an IDSS. Additionally, a practical application of SVNHFDWA and SVNHFDWG is also presented to examine symmetrical analysis in the selection of wireless charging station for vehicles.

Keywords: SVNHFS; Dombi operators; intelligent decision support system

1. Introduction

The researchers have struggled for centuries to comprehend the complex relationship between precision and ambiguity. Zadeh [1] was attempting to solve the problem of dealing with uncertain and imprecise information in the framework of computational intelligence when the novel notion of a fuzzy set (FS) developed. In this framework, each object's membership degree (MD) is composed of a single value which lies in $[0, 1]$. One of its disadvantages is that it only supports MD and does not allow non-membership degree (NMD) to be expressed. Atanassov [2] invented an intuitionistic fuzzy set (IFS) to circumvent the constraints imposed by FS. The MD and NMD of an IFS, both of which have values from 0 and 1, can be used to evaluate the structure of objects. A wide range of information aggregation have been developed in order to consolidate knowledge that is intuitively ambiguous. These aggregation methods have been created to make the information more reasonable and accurate.

Xu [3] was responsible for the creation of multiple kinds of aggregation operators (AOs), including IF weighted, and other operators. Xu is [3] credited with the invention of multiple types of averaging operators, including the IF weighted, ordered weighted, and hybrid averaging operators, amongst other types of averaging operators. These operators are still very useful today. An interval-valued intuitionistic fuzzy set (IVIFS) was presented in [4] as a further developments of an IFS, which is defined by its interval MF and interval NMF in $[0, 1]$.

The authors in [4] suggested the further aspects of an interval-valued intuitionistic fuzzy set (IVIFS), which is distinguished by MD and NMD as fuzzy intervals. This was completed in an effort to further generalize IFSs. Since IFSs and IVIFSs cannot effectively describe uncertain and vague data, Smarandache [5] designed a neutrosophic set (NS) from a mathematical perspective as a technique of expressing imprecise information. The study of an SVNS [6] is more realistic due to its components in compact set $[0, 1]$. An interval neutrosophic set (INS) [7] is another interesting extension of the NS model. A hesitant fuzzy set [8,9] has the ability to seek a solution to the problem of determining the extent to which an individual component belongs to a set, which is notoriously difficult to do. HFS can be seen as a suitable way of permitting multiple different degrees for an element to be added to a set, which is suitable for addressing hesitancy, which is the most prevalent difficulty that arises while making decisions. Because of the circumstance that there is just few truth-membership hesitant grades in an HFS, the challenge cannot be characterized utilizing truth-MD, indeterminacy, or falsity-MD. This is due to the hesitation of decision makers to make suitable choices. On the other hand, an SVNS enables a chance and opportunity to capture vague knowledge that arises from the real-world circumstances. It is quite preferable to handle information that is vague as well as uncertain if possible. It is very useful to describe the degree of truth-MD, indeterminacy, and falsity-MD for IDSS and CI. In order to deal with this scenario, Ye [10] was the one who gave the idea of the SVN hesitant fuzzy set.

One perspective on decision making describes it as a process of problem solving that culminates in the selection of a solution that is considered to be either the optimal or at least an acceptable and reasonable alternative among a set of feasible alternatives. MCDM, which stands for multi-criteria decision making, is a branch of operations research that focuses on the method of determining the best option for a specific set of criteria by thoroughly and rigorously analyzing all of the options that are available. This is accomplished by comparing and contrasting all of the potential choices.

On a continual basis, MCDM challenges and their associated solutions are confronted in multiple fields including the social sciences, economics, management, and medicine. Among the most complex obstacles that one faces when encountering complexities that require MCDM is scrambling to figure out how to incorporate uncertain information pieces that have been provided by a diverse range of sources in the process of arriving at a judgement or conclusions. This is one of the most significant challenges that one faces when encountering challenges that require MCDM.

Researchers utilized a variety of methods, such as rules, aggregation-specific procedures, and various extensions of FS theory, in order to obtain an outstanding aggregation. All of these methodologies are founded on various quantitative aggregation operations. The aggregation operators are remarkably beneficial tools to use when it involves the process of consolidating a variety of information into a single information. A substantial amount of study is currently being directed into comprehending the information aggregation techniques, which are an important tool in their own right. Ye [10] proposed SVNHF averaging operator and SVNHF geometric operator (SVNHFWG). In [11] SVNHF ordered weighted averging (SVNHFOWA) operator and SVNHF hybrid averaging (SVNHFHA) operator were studied. Both The Dombi t-norm and the t-conorm operations, which show how essential it is to have a high degree of flexibility with regard to the operational parameter, were hypothesized of and produced by Dombi [12]. Some operators for IFS, IVIFS, PFS, SVNS, HFS and SVNHFSs are listed in Table 1.

Table 1. Operators for fuzzy sets.

Fuzzy Set	Operator	Reference
IFS	Dombi aggregation operator	[13]
	Dombi Bonferroni mean operators	[14]
	Einstein aggregation operators	[15]
	Prioritized aggregation operators	[16]
	Prioritized AO with priority degrees	[17]
	Choquet integral operator	[18]
	Power AAO and GAO	[19]
	Quasi AO	[20]
IVIFS	Dombi Heronian mean AO	[21]
	Dombi Hamy mean operators	[22]
	Einstein Geometric Choquet Integral operator	[23]
	Hybrid WAO based on Einstein operation	[24]
	Hamacher AO	[25]
PFS	Dombi aggregation AO	[26,27]
	Einstein hybrid averaging aggregation operator	[28]
	Prioritized aggregation operators	[29]
	Prioritized aggregation operators based on priority degrees	[30]
	Interaction power Bonferroni mean AO	[31]
	Bonferroni mean AO	[32]
SVNS	Dombi weighted aggregation operators	[33]
	Dombi power aggregation operators	[34]
	Dombi prioritized weighted aggregation operators	[35]
	Schweizer–Sklar prioritized aggregation operator	[36]
	Einstein prioritized aggregation operators	[37]
HFS	Dombi aggregation operators	[38]
	Dombi–Archimedean weighted aggregation operators	[39,40]
	Einstein aggregation operators	[41]
	Hamacher Aggregation operators	[42]
SVNHFS	Choquet aggregation operators	[43]
	Prioritized aggregation operators	[44]
	Normalized geometric aggregation operators	[45]

Hanif et al. [46] developed linear Diophontine fuzzy (LDF) graphs with a fresh take on how decisions are made in regard to medical diagnosis. Prakash et al. [47] came up with the innovative ideas of LDF-graph and examined the elements of extending the lifespan of a wireless charging network by employing a mobile robot. Mohamed et al. [48] He developed a study on the assessment of the autonomous charger for electric vehicles (EV) and hybrid electric vehicles (HEV). Tang et al. [49] proposed a symmetric implicational algorithm derived from the intuitionistic fuzzy entropy method. Wang [50] proposed a robust intuitionistic fuzzy analytic hierarchy process. Fuzzy optimization techniques have been utilized by various researchers [51–58]. Shi and Ye [59] proposed Dombi AO for aggregation of neutrosophic cubic sets.

This article provides a number of aggregation operators for the SVNHF framework. Some key features are also demonstrated, including idempotency, monotonicity, boundedness, and commutativity. In addition, a technique for handling IDSS problems is given by utilizing the SVNHFDWA operator and SVNHFDWG operator. In IDSS, an application of SVNHFDWA operator and SVNHFDWG operator is provided. A practical application of SVNHFDWA and SVNHFDWG is also presented to examine symmetrical analysis in the selection of a feasible mobile robot (mobile charger) for electric vehicles (EV) as well as hybrid electric vehicles (HEV).

In Section 2, we review NS, SVNS, HFS, SVNHFS, and their fundamental operations. In Section 3, we study some operational laws of SVNHFSs, and a series of robust AOs are developed in Section 4 that include SVNHFDWA, SVNHFDWG, SVNHFDOWA, SVNHFDOWG, SVNHFDHWA, and SVNHFDHWG. In Section 5, an illustration is established to utilize SVNHFDWA and SVNHFDWG that are helpful in IDSS that might be applied in real-world situations particularly in the selection of feasible mobile robots (mobile charger).

2. Preliminaries

Definition 1 ([5]). *Let X be any set. A neutrosophic set (NS) $\mathfrak{A}$ in X can be identified by the mappings $T_{\mathfrak{A}}, I_{\mathfrak{A}}, N_{\mathfrak{A}} : X \longrightarrow (0^-, 1^+)$ such that $0^- \leq \sup T_{\mathfrak{A}}(x) + \sup I_{\mathfrak{A}}(x) + \sup N_{\mathfrak{A}}(x) \leq 3^+$, $x \in X$.*

Example 1. *Let $X = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a universal set. Then, the NS will be*

$$\mathfrak{A} = \Big\{ \langle \rho_1, \{0.325, 0.746, 0.267\}\rangle, \langle \rho_2, \{0.313, 0.435, 0.918\}\rangle, \langle \rho_3, \{0.142, 0.278, 0.538\}\rangle, \langle \rho_4, \{0.434, 0.787, 0.167\}\rangle, \langle \rho_5, \{0.934, 0.487, 0.067\}\rangle \Big\}$$

Definition 2 ([6]). *An SVN $\mathfrak{S}$ in X is identified by truth-MD $\mathcal{T}_{\mathfrak{S}}(x)$, indeterminacy $\mathcal{I}_{\mathfrak{S}}(x)$, and falsity-MD $\mathcal{N}_{\mathfrak{S}}(x)$ such that for each point x in X, $\mathcal{T}_{\mathfrak{S}}(x), \mathcal{I}_{\mathfrak{S}}(x), \mathcal{N}_{\mathfrak{S}}(x) \in [0,1]$ as well as $0 \leq \mathcal{T}_{\mathfrak{S}}(x) + \mathcal{I}_{\mathfrak{S}}(x) + \mathcal{N}_{\mathfrak{S}}(x) \leq 3$. Then, an SVNS $\mathfrak{S}$ becomes of the form*

$$\mathfrak{S} = \{\langle x, \mathcal{T}_{\mathfrak{S}}(x), \mathcal{I}_{\mathfrak{S}}(x), \mathcal{N}_{\mathfrak{S}}(x)\rangle | x \in X\}$$

Definition 3 ([9]). *An HFS $\mathfrak{H}$ in X associates a finite subset of $[0,1]$ to each object $\varsigma \in X$. It can be expressed as*

$$\mathfrak{H} = \{\langle \varsigma, h(\varsigma)\rangle \, \varsigma \in X\}.$$

Example 2. *Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be a set. Then, the HFS will be*

$$\mathfrak{H} = \Big\{ \langle x_1, \{0.125, 0.3450.867\}\rangle, \langle x_2, \{0.333, 0.445, 0.509, 0.689\}\rangle, \langle x_3, \{0.378, 0.432\}\rangle, \langle x_4, \{0.004, 0.243, 0.754\}\rangle, \langle x_5, \{0.669, 0.775\}\rangle \Big\}$$

where

$$h(x_1) = \{0.125, 0.3450.867\}$$
$$h(x_2) = \{0.333, 0.445, 0.509, 0.689\}$$
$$h(x_3) = \{0.378, 0.432\}$$
$$h(x_4) = \{0.004, 0.243, 0.754\}$$
$$h(x_5) = \{0.669, 0.775\}$$

are HF-elements corresponding to x_1, x_2, x_3, x_4 and x_5, respectively.

Definition 4 ([10]). *Let X be a universal set; an SVNHFS in X is expressed as*

$$T = \{\langle x, \mathbb{P}(x), \mathbb{I}(x), \mathbb{N}(x)\rangle | \, x \in X\}$$

such that the possible truth-membership hesitant degrees (THD) are represented by $\mathbb{P}(x)$, indeterminacy-membership hesitant degrees (IHD) are represented by $\mathbb{I}(x)$, and falsity-membership hesitant degrees (FHD) are represented by $\mathbb{N}(x)$, respectively.

Example 3. *Let $X = \{x_1, x_2, x_3, x_4\}$ be a universal set. Then, the SVNHFS will be*

$$T = \Big\{ \langle x_1, \{0.645, 0.125, 0.667\}, \{0.211\}, \{0.111, 0.200\}\rangle, \\ \langle x_2, \{0.133, 0.345\}, \{0.988, 0.259\}, \{0.335, 0.121\}\rangle, \\ \langle x_3, \{0.511\}, \{0.124, 0.654\}, \{0.652, 0.783, 0.284\}\rangle, \\ \langle x_4, \{0.481, 0.872, 0.100, 0.321\}, \{0.865\}, \{0.768\}\rangle \Big\}$$

Definition 5 ([10]). *If $\mathfrak{x}_1 = \langle \mathbb{P}_1(x), \mathbb{I}_1(x), \mathbb{N}_1(x)\rangle, \mathfrak{x}_2 = \langle \mathbb{P}_2(x), \mathbb{I}_2(x), \mathbb{N}_2(x)\rangle$ are two SNVHFNs in X, then*

(i) $\mathfrak{x}_1^c = \{\mathbb{N}_1(x), 1 - \mathbb{I}_1(x), \mathbb{P}_1(x)\}$

(ii) $\mathfrak{x}_1 \subseteq \mathfrak{x}_2 \Longleftrightarrow \mathbb{P}_1(x) \leq \mathbb{P}_2(x), \mathbb{I}_1(x) \geq \mathbb{I}_2(x)$ *and* $\mathbb{N}_1(x) \geq \mathbb{N}_2(x)$ *for any* $x \in \mathbb{N}$.

(iii) $\mathfrak{x}_1 \cup \mathfrak{x}_2 = \Big\{ \mathbb{P} \in (\mathbb{P}_1 \cup \mathbb{P}_2) | \mathbb{P} \geq \max(\mathbb{P}_1^- \mathbb{P}_2^-), \mathbb{I} \in (\mathbb{I}_1 \cap \mathbb{I}_2) | \mathbb{I} \leq \min(\mathbb{I}_1^+ \mathbb{I}_2^+), \mathbb{N} \in (\mathbb{N}_1 \cap \mathbb{N}_2) | \mathbb{N} \leq \min(\mathbb{N}_1^+ \mathbb{N}_2^+) \Big\}$

(iv) $\mathfrak{x}_1 \cap \mathfrak{x}_2 = \Big\{ \mathbb{P} \in (\mathbb{P}_1 \cap \mathbb{P}_2) | \mathbb{P} \leq \min(\mathbb{P}_1^+ \mathbb{P}_2^+), \mathbb{I} \in (\mathbb{I}_1 \cup \mathbb{I}_2) | \mathbb{I} \geq \max(\mathbb{I}_1^- \mathbb{I}_2^-), \mathbb{N} \in (\mathbb{N}_1 \cup \mathbb{N}_2) | \mathbb{N} \geq \max(\mathbb{N}_1^- \mathbb{N}_2^-) \Big\}$

(v) $\mathfrak{x}_1 \oplus \mathfrak{x}_2 = \{\mathbb{P}_1 \oplus \mathbb{P}_2, \mathbb{I}_1 \oplus \mathbb{I}_2, \mathbb{N}_1 \oplus \mathbb{N}_2\}$
$= \bigcup_{\phi_1 \in \mathbb{P}_1 \varphi_1 \in \mathbb{I}_1 \psi_1 \in \mathbb{N}_1 \phi_2 \in \mathbb{P}_2 \varphi_2 \in \mathbb{I}_2 \psi_2 \in \mathbb{N}_2} \Big\{ \{\phi_1 + \phi_2 - \phi_1\phi_2\}, \{\varphi_1\varphi_2\}, \{\psi_1\psi_2\} \Big\}$

(vi) $\mathfrak{x}_1 \otimes \mathfrak{x}_2 = \{\mathbb{P}_1 \otimes \mathbb{P}_2, \mathbb{I}_1 \otimes \mathbb{I}_2, \mathbb{N}_1 \otimes \mathbb{N}_2\}$
$= \bigcup_{\phi_1 \in \mathbb{P}_1 \varphi_1 \in \mathbb{I}_1 \psi_1 \in \mathbb{N}_1 \phi_2 \in \mathbb{P}_2 \varphi_2 \in \mathbb{I}_2 \psi_2 \in \mathbb{N}_2} \Big\{ \{\phi_1\phi_2\}, \{\varphi_1 + \varphi_2 - \varphi_1\varphi_2\}, \{\psi_1 + \psi_2 - \psi_1\psi_2\} \Big\}$

(vii) $\theta\mathfrak{x}_1 = \bigcup_{\phi_1 \in \mathbb{P}_1 \varphi_1 \in \mathbb{I}_1 \psi_1 \in \mathbb{N}_1} \Big\{ \{1 - (1 - \phi_1)^\theta\}, \{\varphi_1)^\theta\}, \{{\psi_1}^\theta\} \Big\}, \theta > 0$

(viii) $\mathfrak{x}_1^\theta = \bigcup_{\phi_1 \in \mathbb{P}_1 \varphi_1 \in \mathbb{I}_1 \psi_1 \in \mathbb{N}_1} \Big\{ \{(\phi_1)^\theta\}, \{1 - (1 - \varphi_1)^\theta\}, \{1 - (1 - \psi_1)^\theta\} \Big\}, \theta > 0$

Definition 6 ([60]). *Let $\mathfrak{x} = \langle \mathbb{P}, \mathbb{I}, \mathbb{N}\rangle$ be a SVNHFN. The score function $S(\mathfrak{x})$ of $\mathfrak{x}$ becomes*

$$S(\mathfrak{x}) = \frac{1}{3}\left[2 + \frac{1}{\mathfrak{l}_{\mathbb{P}}} \sum_{\phi \in \mathbb{P}} \phi - \frac{1}{\mathfrak{l}_{\mathbb{I}}} \sum_{\varphi \in \mathbb{I}} \varphi - \frac{1}{\mathfrak{l}_{\mathbb{N}}} \sum_{\psi \in \mathbb{N}} \psi\right] \tag{1}$$

Definition 7 ([60]). *Let $\mathfrak{x} = \langle \mathbb{P}, \mathbb{I}, \mathbb{N}\rangle$ be a SVNHFN. The accuracy function $A(\mathfrak{x})$ of $\mathfrak{x}$ becomes*

$$A(\mathfrak{x}) = \frac{1}{\mathfrak{l}_{\mathbb{P}}} \sum_{\phi \in \mathbb{P}} \phi - \frac{1}{\mathfrak{l}_{\mathbb{N}}} \sum_{\psi \in \mathbb{N}} \psi \tag{2}$$

Definition 8 ([60]). *Let $\mathfrak{x} = \langle \mathbb{P}, \mathbb{I}, \mathbb{N}\rangle$ be a SVNHFN. The certain function $C(\mathfrak{x})$ of $\mathfrak{x}$ becomes*

$$C(\mathfrak{x}) = \frac{1}{\mathfrak{l}_{\mathbb{P}}} \sum_{\phi \in \mathbb{P}} \phi \tag{3}$$

Let $\mathfrak{x}_1$ and $\mathfrak{x}_2$ be two SVNHFNs. Then:

1. As $S(\mathfrak{x}_1) > S(\mathfrak{x}_2)$, then $\mathfrak{x}_1$ is superior to $\mathfrak{x}_2$, designated by $\mathfrak{x}_1 \succ \mathfrak{x}_2$
2. As $S(\mathfrak{x}_1) = S(\mathfrak{x}_2)$ and $A(\mathfrak{x}_1) > A(\mathfrak{x}_2)$, then $\mathfrak{x}_1$ is superior to $\mathfrak{x}_2$, which is designated by $\mathfrak{x}_1 \succ \mathfrak{x}_2$.
3. As $S(\mathfrak{x}_1) = S(\mathfrak{x}_2), A(\mathfrak{x}_1) = A(\mathfrak{x}_2)$ and $C(\mathfrak{x}_1) > C(\mathfrak{x}_2)$, then $\mathfrak{x}_1$ is superior to $\mathfrak{x}_2$, which is designated by $\mathfrak{x}_1 \succ \mathfrak{x}_2$

4. As $S(\mathfrak{x}_1) = S(\mathfrak{x}_2), A(\mathfrak{x}_1) = A(\mathfrak{x}_2)$ and $C(\mathfrak{x}_1) = C(\mathfrak{x}_2)$, then $\mathfrak{x}_1$ is equal to $\mathfrak{x}_2$, which is designated by $\mathfrak{x}_1 \sim \mathfrak{x}_2$

Definition 9 ([12]). *Consider a and b any two real numbers. Dombi t-norm and Dombi t-conorm can be elaborated as follows.*

$$\mathrm{Dom}(a,b) = \frac{1}{1+\left\{\left(\frac{1-a}{a}\right)^{\lambda}+\left(\frac{1-b}{b}\right)^{\lambda}\right\}^{1/\lambda}} \tag{4}$$

$$\mathrm{Dom}^{c}(a,b) = 1-\frac{1}{1+\left\{\left(\frac{a}{1-a}\right)^{\lambda}+\left(\frac{b}{1-b}\right)^{\lambda}\right\}^{1/\lambda}} \tag{5}$$

where $\lambda \geq 1$ *and* $(a,b) \in [0,1]^2$.

3. Dombi Operations for SVNHFNs

Let $\mathfrak{x}_1 = \langle \mathbb{P}_1(x), \mathbb{I}_1(x), \mathbb{N}_1(x)\rangle$, $\mathfrak{x}_2 = \langle \mathbb{P}_2(x), \mathbb{I}_2(x), \mathbb{N}_2(x)\rangle$ and $\mathfrak{x} = \langle \mathbb{P}(x), \mathbb{I}(x), \mathbb{N}(x)\rangle$ be three SNVHFNs in X; then, the Dombi operations can be defined as:

1. $\mathfrak{x}_1 \oplus \mathfrak{x}_2 = \bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2}$
$$\left\{1-\frac{1}{1+\left\{\left(\frac{\phi_1}{1-\phi_1}\right)^{\lambda}+\left(\frac{\phi_2}{1-\phi_2}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\left(\frac{1-\varphi_1}{\varphi_1}\right)^{\lambda}+\left(\frac{1-\varphi_2}{\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\left(\frac{1-\psi_1}{\psi_1}\right)^{\lambda}+\left(\frac{1-\psi_2}{\psi_2}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$
2. $\mathfrak{x}_1 \otimes \mathfrak{x}_2 = \bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2}$
$$\left\{\frac{1}{1+\left\{\left(\frac{1-\phi_1}{\phi_1}\right)^{\lambda}+\left(\frac{1-\phi_2}{\phi_2}\right)^{\lambda}\right\}^{1/\lambda}}, 1-\frac{1}{1+\left\{\left(\frac{\varphi_1}{1-\varphi_1}\right)^{\lambda}+\left(\frac{\varphi_2}{1-\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}}, 1-\frac{1}{1+\left\{\left(\frac{\psi_1}{1-\psi_1}\right)^{\lambda}+\left(\frac{\psi_2}{1-\psi_2}\right)^{\lambda}\right\}^{1/\lambda}},\right\}$$
3. $\rho\mathfrak{x} = \bigcup_{\phi\in\mathbb{P},\varphi\in\mathbb{I},\psi\in\mathbb{N}} \left\{1-\frac{1}{1+\left\{\rho\left(\frac{\phi}{1-\phi}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\rho\left(\frac{1-\varphi}{\varphi}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\rho\left(\frac{1-\psi}{\psi}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$
4. $\mathfrak{x}^{\rho} = \bigcup_{\phi\in\mathbb{P},\varphi\in\mathbb{I},\psi\in\mathbb{N}} \left\{\frac{1}{1+\left\{\rho\left(\frac{1-\phi}{\phi}\right)^{\lambda}\right\}^{1/\lambda}}, 1-\frac{1}{1+\left\{\rho\left(\frac{\varphi}{1-\varphi}\right)^{\lambda}\right\}^{1/\lambda}}, 1-\frac{1}{1+\left\{\rho\left(\frac{\psi}{1-\psi}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$

Theorem 1. *Let* $\mathfrak{x}_1 = \langle \mathbb{P}_1(x), \mathbb{I}_1(x), \mathbb{N}_1(x)\rangle$ *and* $\mathfrak{x}_2 = \langle \mathbb{P}_2(x), \mathbb{I}_2(x), \mathbb{N}_2(x)\rangle$ *be two SVNHFNs and* $\rho, \rho_1, \rho_2 \geq 0$ *be any real numbers; then,*

(i) $\mathfrak{x}_1 \oplus \mathfrak{x}_2 = \mathfrak{x}_2 \oplus \mathfrak{x}_1$.

(ii) $\mathfrak{x}_1 \otimes \mathfrak{x}_2 = \mathfrak{x}_2 \otimes \mathfrak{x}_1$.

(iii) $(\mathfrak{x}_1 \otimes \mathfrak{x}_2)^{\rho} = \mathfrak{x}_1^{\rho} \otimes \mathfrak{x}_2^{\rho}$.

(iv) $\rho(\mathfrak{x}_1 \oplus \mathfrak{x}_2) = \rho\mathfrak{x}_1 \oplus \rho\mathfrak{x}_2$.

(v) $\mathfrak{x}_1^{\rho_1} \otimes \mathfrak{x}_1^{\rho_2} = \mathfrak{x}_1^{\rho_1+\rho_2}$.

(vi) $\rho_1(\rho_2\mathfrak{x}_1) = (\rho_1\rho_2)\mathfrak{x}_1$.

(vii) $(\mathfrak{x}_1^{\rho_1})^{\rho_2} = (\mathfrak{x}_1)^{\rho_1\rho_2}$.

(viii) $\rho_1\mathfrak{x}_1 \oplus \rho_2 = (\rho_1+\rho_2)\mathfrak{x}_1$.

(ix) $\mathfrak{x}_1^{c} \oplus \mathfrak{x}_2^{c} = (\mathfrak{x}_1 \otimes \mathfrak{x}_2)^{c}$.

(x) $\mathfrak{x}_1^c \otimes \mathfrak{x}_2^c = (\mathfrak{x}_1 \oplus \mathfrak{x}_2)^c$.

Proof. Proof is straight forward. □

4. Dombi Operators for SVNHF Information

Now, we look at the Dombi SVNHF aggregation operators using SVNHF Dombi operations which are discussed in the previous section. This section introduces a number of aggregation operators, including the SVNHFDWA, SVNHFDWG, SVNHFDOWA, SVNHFDOWG, SVNHFDHA, SVNHFDHG.

4.1. SVNHFDWA Operator

Definition 10. *Consider a collection of SVNHFNs $\mathfrak{x}_i (i = 1, 2, \cdots, n)$; then, the SVNHFDWA operator is defined as follows:*

$$SVNHFDWA(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \overset{n}{\underset{i=1}{\oplus}}(w_i \mathfrak{x}_i) = w_1\mathfrak{x}_1 \oplus w_2\mathfrak{x}_2 \oplus \cdots \oplus w_n\mathfrak{x}_n \tag{6}$$

Equation (6) gives $w = (w_1, w_2, \cdots, w_n)^T$, which is a weight vector (WV) of the SVNHFNs $\mathfrak{x}_i (i = 1, 2, \cdots, n)$, $w_i \in [0, 1]$ with $\sum_{i=1}^{n} w_i = 1$.

Theorem 2. *Consider a collection of SVNHFNs $\mathfrak{x}_i (i = 1, 2, \cdots, n)$ with WVs $w = (w_1, w_2, \cdots, w_n)^T$; then, the aggregated SVNHFN is SVNHFDWA* $(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \oplus_{i=1}^{n}(w_i \mathfrak{x}_i) = \underset{\phi_1 \in \mathbb{P}_1, \phi_2 \in \mathbb{P}_2, \cdots, \phi_n \in \mathbb{P}_n, \varphi_1 \in \mathbb{I}_1, \varphi_2 \in \mathbb{I}_2, \cdots, \varphi_n \in \mathbb{I}_n, \psi_1 \in \mathbb{N}_1, \psi_2 \in \mathbb{N}_2, \cdots, \psi_n \in \mathbb{N}_n}{\cup}$

$$\left\{1 - \frac{1}{1 + \left\{\sum_{i=1}^{n} w_i \left(\frac{\phi_i}{1-\phi_i}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1 + \left\{\sum_{i=1}^{n} w_i \left(\frac{1-\varphi_i}{\varphi_i}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1 + \left\{\sum_{i=1}^{n} w_i \left(\frac{1-\psi_i}{\psi_i}\right)^{\lambda}\right\}^{1/\lambda}}\right\} \tag{7}$$

Proof. Equation (7) can be proved by using induction on n:

For $n = 2$, as

$$w_1\mathfrak{x}_1 = \underset{\phi_1 \in \mathbb{P}_1, \varphi_1 \in \mathbb{I}_1, \psi_1 \in \mathbb{N}_1}{\cup} \left\{1 - \frac{1}{1+\left\{w_1\left(\frac{\phi_1}{1-\phi_1}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{w_1\left(\frac{1-\varphi_1}{\varphi_1}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{w_1\left(\frac{1-\psi_1}{\psi_1}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

$$w_2\mathfrak{x}_2 = \underset{\phi_2 \in \mathbb{P}_2, \varphi_2 \in \mathbb{I}_2, \psi_2 \in \mathbb{N}_2}{\cup} \left\{1 - \frac{1}{1+\left\{w_2\left(\frac{\phi_2}{1-\phi_2}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{w_2\left(\frac{1-\varphi_2}{\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{w_2\left(\frac{1-\psi_2}{\psi_2}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

then, taking union for $\phi_1 \in \mathbb{P}_1, \phi_2 \in \mathbb{P}_2, \varphi_1 \in \mathbb{I}_1, \varphi_2 \in \mathbb{I}_2, \psi_1 \in \mathbb{N}_1, \psi_2 \in \mathbb{N}_2$

$$w_1\mathfrak{x}_1 \oplus w_2\mathfrak{x}_2 = \cup\left\{1 - \frac{1}{1+\left(\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{\phi_1}{1-\phi_1}\right)^{\lambda}\right\}^{1/\lambda}}}{1-1+\frac{1}{1+\left\{w_1\left(\frac{\phi_1}{1-\phi_1}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda} + \left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{\phi_2}{1-\phi_2}\right)^{\lambda}\right\}^{1/\lambda}}}{1-1+\frac{1}{1+\left\{w_2\left(\frac{\phi_2}{1-\phi_2}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda}\right)^{1/\lambda}}, \right.$$

$$\frac{1}{1+\left(\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{1-\varphi_1}{\varphi_1}\right)^{\lambda}\right\}^{1/\lambda}}}{1-1+\frac{1}{1+\left\{w_1\left(\frac{1-\varphi_1}{\varphi_1}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda} + \left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{1-\varphi_2}{\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}}}{1-1+\frac{1}{1+\left\{w_2\left(\frac{1-\varphi_2}{\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda}\right)^{1/\lambda}},$$

$$\left. \frac{1}{1+\left(\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{1-\psi_1}{\psi_1}\right)^{\lambda}\right\}^{1/\lambda}}}{1-1+\frac{1}{1+\left\{w_1\left(\frac{1-\psi_1}{\psi_1}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda} + \left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{1-\psi_2}{\psi_2}\right)^{\lambda}\right\}^{1/\lambda}}}{1-1+\frac{1}{1+\left\{w_2\left(\frac{1-\psi_2}{\psi_2}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda}\right)^{1/\lambda}} \right\}$$

$$= \cup \left\{ 1 - \frac{1}{1+\left(\left(\frac{\left\{ w_1 \left(\frac{\phi_1}{1-\phi_1} \right)^{\lambda} \right\}^{1/\lambda} - 1}{1+\left\{ w_1 \left(\frac{\phi_1}{1-\phi_1} \right)^{\lambda} \right\}^{1/\lambda}} \Big/ \frac{1}{1+\left\{ w_1 \left(\frac{\phi_1}{1-\phi_1} \right)^{\lambda} \right\}^{1/\lambda}} \right)^{\lambda} + \left(\frac{\left\{ w_2 \left(\frac{\phi_2}{1-\phi_2} \right)^{\lambda} \right\}^{1/\lambda} - 1}{1+\left\{ w_2 \left(\frac{\phi_2}{1-\phi_2} \right)^{\lambda} \right\}^{1/\lambda}} \Big/ \frac{1}{1+\left\{ w_2 \left(\frac{\phi_2}{1-\phi_2} \right)^{\lambda} \right\}^{1/\lambda}} \right)^{\lambda} \right)^{1/\lambda}}, \right.$$

$$\frac{1}{1+\left(\left(\frac{\left\{ w_1 \left(\frac{1-\varphi_1}{\varphi_1} \right)^{\lambda} \right\}^{1/\lambda} - 1}{1+\left\{ w_1 \left(\frac{1-\varphi_1}{\varphi_1} \right)^{\lambda} \right\}^{1/\lambda}} \Big/ \frac{1}{1+\left\{ w_1 \left(\frac{1-\varphi_1}{\varphi_1} \right)^{\lambda} \right\}^{1/\lambda}} \right)^{\lambda} + \left(\frac{\left\{ w_2 \left(\frac{1-\varphi_2}{\varphi_2} \right)^{\lambda} \right\}^{1/\lambda} - 1}{1+\left\{ w_2 \left(\frac{1-\varphi_2}{\varphi_2} \right)^{\lambda} \right\}^{1/\lambda}} \Big/ \frac{1}{1+\left\{ w_2 \left(\frac{1-\varphi_2}{\varphi_2} \right)^{\lambda} \right\}^{1/\lambda}} \right)^{\lambda} \right)^{1/\lambda}},$$

$$\left. \frac{1}{1+\left(\left(\frac{\left\{ w_1 \left(\frac{1-\psi_1}{\psi_1} \right)^{\lambda} \right\}^{1/\lambda} - 1}{1+\left\{ w_1 \left(\frac{1-\psi_1}{\psi_1} \right)^{\lambda} \right\}^{1/\lambda}} \Big/ \frac{1}{1+\left\{ w_1 \left(\frac{1-\psi_1}{\psi_1} \right)^{\lambda} \right\}^{1/\lambda}} \right)^{\lambda} + \left(\frac{\left\{ w_2 \left(\frac{1-\psi_2}{\psi_2} \right)^{\lambda} \right\}^{1/\lambda} - 1}{1+\left\{ w_2 \left(\frac{1-\psi_2}{\psi_2} \right)^{\lambda} \right\}^{1/\lambda}} \Big/ \frac{1}{1+\left\{ w_2 \left(\frac{1-\psi_2}{\psi_2} \right)^{\lambda} \right\}^{1/\lambda}} \right)^{\lambda} \right)^{1/\lambda}} \right\}$$

$$= \bigcup_{\phi_1 \in \mathbb{P}_1, \phi_2 \in \mathbb{P}_2, \varphi_1 \in \mathbb{I}_1, \varphi_2 \in \mathbb{I}_2, \psi_1 \in \mathbb{N}_1, \psi_2 \in \mathbb{N}_2}$$

$$\left\{ 1 - \frac{1}{1+\left\{ w_1 \left(\frac{\phi_1}{1-\phi_1} \right)^{\lambda} + w_2 \left(\frac{\phi_2}{1-\phi_2} \right)^{\lambda} \right\}^{1/\lambda}}, \frac{1}{1+\left\{ w_1 \left(\frac{1-\varphi_1}{\varphi_1} \right)^{\lambda} + w_2 \left(\frac{1-\varphi_2}{\varphi_2} \right)^{\lambda} \right\}^{1/\lambda}}, \right.$$

$$\left. \frac{1}{1+\left\{ w_1 \left(\frac{1-\psi_1}{\psi_1} \right)^{\lambda} + w_2 \left(\frac{1-\psi_2}{\psi_2} \right)^{\lambda} \right\}^{1/\lambda}} \right\}$$

As Equation (7) is satisfied for $n = k$, that is

$$\oplus_{i=1}^{k} (w_i \mathfrak{x}_i) = \bigcup_{\phi_1 \in \mathbb{P}_1, \phi_2 \in \mathbb{P}_2, \cdots, \phi_k \in \mathbb{P}_k, \varphi_1 \in \mathbb{I}_1, \varphi_2 \in \mathbb{I}_2, \cdots, \varphi_k \in \mathbb{I}_k, \psi_1 \in \mathbb{N}_1, \psi_2 \in \mathbb{N}_2, \cdots, \psi_k \in \mathbb{N}_k}$$

$$\left\{ 1 - \frac{1}{1+\left\{ \sum_{i=1}^{k} w_i \left(\frac{\phi_i}{1-\phi_i} \right)^{\lambda} \right\}^{1/\lambda}}, \right.$$

$$\left. \frac{1}{1+\left\{ \sum_{i=1}^{k} w_i \left(\frac{1-\varphi_i}{\varphi_i} \right)^{\lambda} \right\}^{1/\lambda}}, \frac{1}{1+\left\{ \sum_{i=1}^{k} w_i \left(\frac{1-\psi_i}{\psi_i} \right)^{\lambda} \right\}^{1/\lambda}} \right\}$$

then, when $n = k+1$, by the Dombi addition operation, we have

$$\oplus_{i=1}^{k+1} (w_i \mathfrak{x}_i) = \oplus_{i=1}^{k} (w_i \mathfrak{x}_i) \oplus (w_{k+1} \mathfrak{x}_{k+1}) =$$

$$\bigcup_{\phi_1 \in \mathbb{P}_1, \phi_2 \in \mathbb{P}_2, \cdots, \phi_k \in \mathbb{P}_k, \varphi_1 \in \mathbb{I}_1, \varphi_2 \in \mathbb{I}_2, \cdots, \varphi_k \in \mathbb{I}_k, \psi_1 \in \mathbb{N}_1, \psi_2 \in \mathbb{N}_2, \cdots, \psi_k \in \mathbb{N}_k}$$

$$\left\{ 1 - \frac{1}{1+\left\{ \sum_{i=1}^{k} w_i \left(\frac{\phi_i}{1-\phi_i} \right)^{\lambda} \right\}^{1/\lambda}}, \frac{1}{1+\left\{ \sum_{i=1}^{k} w_i \left(\frac{1-\varphi_i}{\varphi_i} \right)^{\lambda} \right\}^{1/\lambda}}, \frac{1}{1+\left\{ \sum_{i=1}^{k} w_i \left(\frac{1-\psi_i}{\psi_i} \right)^{\lambda} \right\}^{1/\lambda}} \right\} \oplus$$

$$\bigcup_{\phi_{k+1} \in \mathbb{P}_{k+1}, \varphi_{k+1} \in \mathbb{I}_{k+1}, \psi_{k+1} \in \mathbb{N}_{k+1}}$$

$$\left\{ 1 - \frac{1}{1+\left\{ w_{k+1} \left(\frac{\phi_{k+1}}{1-\phi_{k+1}} \right)^{\lambda} \right\}^{1/\lambda}}, \frac{1}{1+\left\{ w_{k+1} \left(\frac{1-\varphi_{k+1}}{\varphi_{k+1}} \right)^{\lambda} \right\}^{1/\lambda}}, \frac{1}{1+\left\{ w_{k+1} \left(\frac{1-\psi_{k+1}}{\psi_{k+1}} \right)^{\lambda} \right\}^{1/\lambda}} \right\}$$

$$= \bigcup_{\phi_1 \in \mathbb{P}_1, \phi_2 \in \mathbb{P}_2, \cdots, \phi_{k+1} \in \mathbb{P}_{k+1}, \varphi_1 \in \mathbb{I}_1, \varphi_2 \in \mathbb{I}_2, \cdots, \varphi_{k+1} \in \mathbb{I}_{k+1}, \psi_1 \in \mathbb{N}_1, \psi_2 \in \mathbb{N}_2, \cdots, \psi_{k+1} \in \mathbb{N}_{k+1}}$$

$$\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{k+1} w_i\left(\frac{\phi_i}{1-\phi_i}\right)^{\lambda}\right\}^{1/\lambda}},\right.$$
$$\left.\frac{1}{1+\left\{\sum_{i=1}^{k+1} w_i\left(\frac{1-\varphi_i}{\varphi_i}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\sum_{i=1}^{k+1} w_i\left(\frac{1-\psi_i}{\psi_i}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

i.e., Equation (7) is satisfied for $n = k+1$. Thus, Equation (7) is satisfied for all n. Then, $\text{SVNHFDWA}(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \oplus_{i=1}^{n}(w_i\mathfrak{x}_i) =$

$$\bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\cdots,\phi_n\in\mathbb{P}_n,\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\cdots,\varphi_n\in\mathbb{I}_n,\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2,\cdots,\psi_n\in\mathbb{N}_n}$$
$$\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\phi_i}{1-\phi_i}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\varphi_i}{\varphi_i}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\psi_i}{\psi_i}\right)^{\lambda}\right\}^{1/\lambda}}\right\} \quad \square$$

Example 4. *Let us consider three SVNHFNs given in Table 2 and* $w = \{0.25, 0.35, 0.40\}^T$, *which is the WV of SVNHFNs. Then, the aggregated SVNHFN with* $\lambda = 2$ *is*

Table 2. SVNHFNs.

$\mathfrak{x}_1$	$\langle\{0.432, 0.567\}, \{0.111\}, \{0.334\}\rangle$
$\mathfrak{x}_2$	$\langle\{0.519, 0.678, 0.800\}, \{0.200, 0.350\}, \{0.156, 0.259\}\rangle$
$\mathfrak{x}_3$	$\langle\{0.750\}, \{0.290, 0.381, 0.476\}, \{0.500\}\rangle$

$$SVNHFDWA(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3) = \oplus_{i=1}^{3}(w_i\mathfrak{x}_i) = \bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\phi_3\in\mathbb{P}_3,\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\varphi_3\in\mathbb{I}_3,\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2,\psi_3\in\mathbb{N}_3}$$
$$\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{3} w_i\left(\frac{\phi_i}{1-\phi_i}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\sum_{i=1}^{3} w_i\left(\frac{1-\varphi_i}{\varphi_i}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\sum_{i=1}^{3} w_i\left(\frac{1-\psi_i}{\psi_i}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$
$$= \langle\{0.671, 0.678, 0.697, 0.703, 0.754, 0.756\}, \{0.169, 0.174, 0.175, 0.184, 0.189, 0.192\}, \{0.227, 0.326\}\rangle$$

Now, we examine the nature of λ, which is an operational parameter, and Table 3 shows some of the SVNHFN's aggregated values.

Table 3. Aggregated SVNHFNs based on SVNHFDWA operator.

Parameter	Aggregated SVNHFNs
$\lambda = 1$	$\langle\{0.639, 0.680, 0.736, 0.656, 0.694, 0.745\}, \{0.186, 0.198, 0.207, 0.216, 0.232, 0.244\}, \{0.264, 0.345\}\rangle$
$\lambda = 2$	$\langle\{0.671, 0.678, 0.697, 0.703, 0.754, 0.756\}, \{0.169, 0.174, 0.175, 0.184, 0.189, 0.192\}, \{0.227, 0.326\}\rangle$
$\lambda = 4$	$\langle\{0.706, 0.715, 0.769, 0.707, 0.716, 0.769\}, \{0.147, 0.147, 0.147, 0.150, 0.150, 0.150\}, \{0.193, 0.303\}\rangle$
$\lambda = 5$	$\langle\{0.714, 0.720, 0.773, 0.715, 0.720, 0.773\}, \{0.140, 0.140, 0.140, 0.141, 0.141, 0.141\}, \{0.186, 0.296\}\rangle$
$\lambda = 10$	$\langle\{0.732, 0.733, 0.784, 0.732, 0.733, 0.784\}, \{0.125, 0.125, 0.125, 0.125, 0.125, 0.125\}, \{0.170, 0.279\}\rangle$
$\lambda = 20$	$\langle\{0.741, 0.741, 0.791, 0.741, 0.741, 0.791\}, \{0.118, 0.118, 0.118, 0.118, 0.118, 0.118\}, \{0.163, 0.269\}\rangle$
$\lambda = 50$	$\langle\{0.747, 0.747, 0.797, 0.747, 0.747, 0.797\}, \{0.114, 0.114, 0.114, 0.114, 0.114, 0.114\}, \{0.158, 0.263\}\rangle$
$\lambda = 100$	$\langle\{0.748, 0.748, 0.798, 0.748, 0.748, 0.798\}, \{0.112, 0.112, 0.112, 0.112, 0.112, 0.112\}, \{0.158, 0.261\}\rangle$

The SVNHFDWA operator satisfies the following properties:

Theorem 3 (Commutative property). *Let* $\mathfrak{x}_i = \langle\mathbb{P}_i, \mathbb{I}_i, \mathbb{N}_i\rangle$ *be a collection of SVNHFNs and* $\check{\mathfrak{x}}_i = \langle\check{\mathbb{P}}_i, \check{\mathbb{I}}_i, \check{\mathbb{N}}_i\rangle$ *be any permutation of* $\mathfrak{x}_i = \langle\mathbb{P}_i, \mathbb{I}_i, \mathbb{N}_i\rangle$. *Then,*

$$SVNHFDWA(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = SVNHFDWA(\check{\mathfrak{x}}_1, \check{\mathfrak{x}}_2, \cdots, \check{\mathfrak{x}}_n) \tag{8}$$

Proof. Proof is obvious. $\square$

Theorem 4 (Idempotency property). *Let $\mathfrak{x}_i = \langle \mathbb{P}_i, \mathbb{I}_i, \mathbb{N}_i \rangle$ be a collection of SVNHFNs. If all $\mathfrak{x}_i$ are equal, i.e., $\mathfrak{x}_i = \mathfrak{x}$ $\forall i$, then*

$$SVNHFDWA(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \mathfrak{x} \tag{9}$$

Proof. Since $\mathfrak{x}_i = \mathfrak{x}$ $\forall i$, then by Equation (7), we have

$$\text{SVNHFDWA}(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \oplus_{i=1}^{n}(w_i\mathfrak{x}_i) =$$

$$\bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\cdots,\phi_n\in\mathbb{P}_n,\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\cdots,\varphi_n\in\mathbb{I}_n,\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2,\cdots,\psi_n\in\mathbb{N}_n}$$

$$\left\{1 - \frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\phi_i}{1-\phi_i}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\varphi_i}{\varphi_i}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\psi_i}{\psi_i}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

$$= \bigcup_{\phi\in\mathbb{P},\varphi\in\mathbb{I},\psi\in\mathbb{N}} \left\{1 - \frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\phi}{1-\phi}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\varphi}{\varphi}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\psi}{\psi}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

$$= \bigcup_{\phi\in\mathbb{P},\varphi\in\mathbb{I},\psi\in\mathbb{N}} \left\{1 - \frac{1}{1+\left\{\left(\sum_{i=1}^{n} w_i\right)^{\lambda}\left(\frac{\phi}{1-\phi}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\left(\sum_{i=1}^{n} w_i\right)^{\lambda}\left(\frac{1-\varphi}{\varphi}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\left(\sum_{i=1}^{n} w_i\right)^{\lambda}\left(\frac{1-\psi}{\psi}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

$$= \bigcup_{\phi\in\mathbb{P},\varphi\in\mathbb{I},\psi\in\mathbb{N}} \left\{1 - \frac{1}{1+\left\{\left(\frac{\phi}{1-\phi}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\left(\frac{1-\varphi}{\varphi}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1+\left\{\left(\frac{1-\psi}{\psi}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

$$= \bigcup_{\phi\in\mathbb{P},\varphi\in\mathbb{I},\psi\in\mathbb{N}} \left\{1 - \frac{1}{1+\left\{\left(\frac{\phi}{1-\phi}\right)\right\}}, \frac{1}{1+\left\{\left(\frac{1-\varphi}{\varphi}\right)\right\}}, \frac{1}{1+\left\{\left(\frac{1-\psi}{\psi}\right)\right\}}\right\}$$

$$= \bigcup_{\phi\in\mathbb{P},\varphi\in\mathbb{I},\psi\in\mathbb{N}} \{\phi, \varphi, \psi\} = \langle \mathbb{P}, \mathbb{I}, \mathbb{N} \rangle = \mathfrak{x}$$

□

Theorem 5 (Monotonicity property). *Let $\mathfrak{x}_i = \langle \mathbb{P}_i, \mathbb{I}_i, \mathbb{N}_i \rangle$ and $\breve{\mathfrak{x}}_i = \langle \breve{\mathbb{P}}_i, \breve{\mathbb{I}}_i, \breve{\mathbb{N}}_i \rangle$ be the families of two distinct SVNHFNs. If $\breve{\mathfrak{x}}_i \geq \mathfrak{x}_i$ $\forall i$, then*

$$SVNHFDWA(\breve{\mathfrak{x}}_1, \breve{\mathfrak{x}}_2, \cdots, \breve{\mathfrak{x}}_n) \geq SVNHFDWA(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n)$$

Proof. Here, $\breve{\mathbb{P}}_i \geq \mathbb{P}_i, \breve{\mathbb{I}}_i \leq \mathbb{I}_i, \breve{\mathbb{N}}_i \leq \mathbb{N}_i$ $\forall i$. If $\breve{\mathbb{P}}_i \geq \mathbb{P}_i$,

$\Leftrightarrow \breve{\phi}_i \geq \phi_i \Leftrightarrow -\breve{\phi}_i \leq -\phi_i \Leftrightarrow 1-\breve{\phi}_i \leq 1-\phi_i \Leftrightarrow \frac{\breve{\phi}_i}{1-\breve{\phi}_i} \leq \frac{\phi_i}{1-\phi_i} \Leftrightarrow (\frac{\breve{\phi}_i}{1-\breve{\phi}_i})^{\lambda} \leq (\frac{\phi_i}{1-\phi_i})^{\lambda}$

$\Leftrightarrow \sum_{i=1}^{n} w_i(\frac{\breve{\phi}_i}{1-\breve{\phi}_i})^{\lambda} \leq \sum_{i=1}^{n} w_i(\frac{\phi_i}{1-\phi_i})^{\lambda}$

$\Leftrightarrow \{\sum_{i=1}^{n} w_i(\frac{\breve{\phi}_i}{1-\breve{\phi}_i})^{\lambda}\}^{\frac{1}{\lambda}} \leq \{\sum_{i=1}^{n} w_i(\frac{\phi_i}{1-\phi_i})^{\lambda}\}^{\frac{1}{\lambda}}$

$\Leftrightarrow 1 + \{\sum_{i=1}^{n} w_i(\frac{\breve{\phi}_i}{1-\breve{\phi}_i})^{\lambda}\}^{\frac{1}{\lambda}} \leq 1 + \{\sum_{i=1}^{n} w_i(\frac{\phi_i}{1-\phi_i})^{\lambda}\}^{\frac{1}{\lambda}}$

$\Leftrightarrow \frac{1}{1+\{\sum_{i=1}^{n} w_i(\frac{\breve{\phi}_i}{1-\breve{\phi}_i})^{\lambda}\}^{\frac{1}{\lambda}}} \geq \frac{1}{1+\{\sum_{i=1}^{n} w_i(\frac{\phi_i}{1-\phi_i})^{\lambda}\}^{\frac{1}{\lambda}}}$

$\Leftrightarrow \bigcup_{\breve{\phi}_1\in\breve{\mathbb{P}}_1,\breve{\phi}_2\in\breve{\mathbb{P}}_2,\cdots,\breve{\phi}_n\in\breve{\mathbb{P}}_n} \left\{1 - \frac{1}{1+\{\sum_{i=1}^{n} w_i(\frac{\breve{\phi}_i}{1-\breve{\phi}_i})^{\lambda}\}^{\frac{1}{\lambda}}}\right\} \geq \bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\cdots,\phi_n\in\mathbb{P}_n} \left\{1 - \frac{1}{1+\{\sum_{i=1}^{n} w_i(\frac{\phi_i}{1-\phi_i})^{\lambda}\}^{\frac{1}{\lambda}}}\right\}$

$\breve{\mathbb{I}}_i \leq \mathbb{I}_i$,

$\Leftrightarrow \breve{\varphi}_i \leq \varphi_i \Leftrightarrow -\breve{\varphi}_i \geq -\varphi_i \Leftrightarrow 1-\breve{\varphi}_i \geq 1-\varphi_i \Leftrightarrow \frac{1-\breve{\varphi}_i}{\breve{\varphi}_i} \geq \frac{1-\varphi_i}{\varphi_i} \Leftrightarrow (\frac{1-\breve{\varphi}_i}{\breve{\varphi}_i})^\lambda \geq (\frac{1-\varphi_i}{\varphi_i})^\lambda$
$\Leftrightarrow \sum_{i=1}^n w_i(\frac{1-\breve{\varphi}_i}{\breve{\varphi}_i})^\lambda \geq \sum_{i=1}^n w_i(\frac{1-\varphi_i}{\varphi_i})^\lambda$
$\Leftrightarrow \{\sum_{i=1}^n w_i(\frac{1-\breve{\varphi}_i}{\breve{\varphi}_i})^\lambda\}^{\frac{1}{\lambda}} \geq \{\sum_{i=1}^n w_i(\frac{1-\varphi_i}{\varphi_i})^\lambda\}^{\frac{1}{\lambda}}$
$\Leftrightarrow 1+\{\sum_{i=1}^n w_i(\frac{1-\breve{\varphi}_i}{\breve{\varphi}_i})^\lambda\}^{\frac{1}{\lambda}} \geq 1+\{\sum_{i=1}^n w_i(\frac{1-\varphi_i}{\varphi_i})^\lambda\}^{\frac{1}{\lambda}}$
$\Leftrightarrow \bigcup_{\breve{\varphi}_1\in\breve{\mathbb{I}}_1,\breve{\varphi}_2\in\breve{\mathbb{I}}_2,\cdots,\breve{\varphi}_n\in\breve{\mathbb{I}}_n}\left\{\frac{1}{1+\{\sum_{i=1}^n w_i(\frac{1-\breve{\varphi}_i}{\breve{\varphi}_i})^\lambda\}^{\frac{1}{\lambda}}}\right\} \leq \bigcup_{\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\cdots,\varphi_n\in\mathbb{I}_n}\left\{\frac{1}{1+\{\sum_{i=1}^n w_i(\frac{1-\varphi_i}{\varphi_i})^\lambda\}^{\frac{1}{\lambda}}}\right\}$
and
$\breve{\mathbb{N}}_i \leq \mathbb{N}_i$,
$\Leftrightarrow \breve{\psi}_i \leq \psi_i \Leftrightarrow -\breve{\psi}_i \geq -\psi_i \Leftrightarrow 1-\breve{\psi}_i \geq 1-\psi_i \Leftrightarrow \frac{1-\breve{\psi}_i}{\breve{\psi}_i} \geq \frac{1-\psi_i}{\psi_i} \Leftrightarrow (\frac{1-\breve{\psi}_i}{\breve{\psi}_i})^\lambda \geq (\frac{1-\psi_i}{\psi_i})^\lambda$
$\Leftrightarrow \sum_{i=1}^n w_i(\frac{1-\breve{\psi}_i}{\breve{\psi}_i})^\lambda \geq \sum_{i=1}^n w_i(\frac{1-\psi_i}{\psi_i})^\lambda$
$\Leftrightarrow \{\sum_{i=1}^n w_i(\frac{1-\breve{\psi}_i}{\breve{\psi}_i})^\lambda\}^{\frac{1}{\lambda}} \geq \{\sum_{i=1}^n w_i(\frac{1-\psi_i}{\psi_i})^\lambda\}^{\frac{1}{\lambda}}$
$\Leftrightarrow 1+\{\sum_{i=1}^n w_i(\frac{1-\breve{\psi}_i}{\breve{\psi}_i})^\lambda\}^{\frac{1}{\lambda}} \geq 1+\{\sum_{i=1}^n w_i(\frac{1-\psi_i}{\psi_i})^\lambda\}^{\frac{1}{\lambda}}$
$\Leftrightarrow \bigcup_{\breve{\psi}_1\in\breve{\mathbb{N}}_1,\breve{\psi}_2\in\breve{\mathbb{N}}_2,\cdots,\breve{\psi}_n\in\breve{\mathbb{N}}_n}\left\{\frac{1}{1+\{\sum_{i=1}^n w_i(\frac{1-\breve{\psi}_i}{\breve{\psi}_i})^\lambda\}^{\frac{1}{\lambda}}}\right\} \leq \bigcup_{\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2,\cdots,\psi_n\in\mathbb{N}_n}\left\{\frac{1}{1+\{\sum_{i=1}^n w_i(\frac{1-\psi_i}{\psi_i})^\lambda\}^{\frac{1}{\lambda}}}\right\}$
we obtain that $\breve{\mathbb{P}}_i \geq \mathbb{P}_i, \breve{\mathbb{I}}_i \leq \mathbb{I}_i, \breve{\mathbb{N}}_i \leq \mathbb{N}_i\ \forall i$. Therefore,

$$SVNHFDWA(\breve{\mathfrak{x}}_1,\breve{\mathfrak{x}}_2,\cdots,\breve{\mathfrak{x}}_n) \geq SVNHFDWA(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n)$$

□

Theorem 6 (Boundary). *Let $\mathfrak{x}_i = \langle \mathbb{P}_i, \mathbb{I}_i, \mathbb{N}_i\rangle$ be a collection of SVNHFNs, then*

$$\mathfrak{x}^- \leq SVNHFDWA(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n) \leq \mathfrak{x}^+ \tag{10}$$

where $\mathfrak{x}^- = \langle \mathbb{P}^-, \mathbb{I}^-, \mathbb{N}^-\rangle$, $\mathfrak{x}^+ = \langle \mathbb{P}^+, \mathbb{I}^+, \mathbb{N}^+\rangle$

$$\mathbb{P}^- = \bigcup_{\phi_i\in\mathbb{P}_i} \min_i(\phi_i), \mathbb{P}^+ = \bigcup_{\phi_i\in\mathbb{P}_i} \max_i(\phi_i)$$
$$\mathbb{I}^- = \bigcup_{\varphi_i\in\mathbb{I}_i} \max_i(\varphi_i), \mathbb{I}^+ = \bigcup_{\varphi_i\in\mathbb{I}_i} \min_i(\varphi_i)$$
$$\mathbb{N}^- = \bigcup_{\psi_i\in\mathbb{N}_i} \max_i(\psi_i), \mathbb{N}^+ = \bigcup_{\psi_i\in\mathbb{N}_i} \min_i(\psi_i)$$

Proof. As

$$\bigcup_{\phi_i\in\mathbb{P}_i} \min_i(\phi_i) \leq \bigcup_{\phi_i\in\mathbb{P}_i} (\phi_i) \leq \bigcup_{\phi_i\in\mathbb{P}_i} \max_i(\phi_i)$$
$$\bigcup_{\phi_i\in\mathbb{P}_i} (1-\max_i\phi_i) \leq \bigcup_{\phi_i\in\mathbb{P}_i} (1-\phi_i) \leq \bigcup_{\phi_i\in\mathbb{P}_i} (1-\min_i\phi_i)$$
$$\bigcup_{\phi_i\in\mathbb{P}_i}\left(\frac{\max_i\phi_i}{1-\max_i\phi_i}\right) \geq \bigcup_{\phi_i\in\mathbb{P}_i}\left(\frac{\phi_i}{1-\phi_i}\right) \geq \bigcup_{\phi_i\in\mathbb{P}_i}\left(\frac{\min_i\phi_i}{1-\min_i\phi_i}\right)$$
$$\bigcup_{\phi_i\in\mathbb{P}_i}\left(\frac{\max_i\phi_i}{1-\max_i\phi_i}\right)^\lambda \geq \bigcup_{\phi_i\in\mathbb{P}_i}\left(\frac{\phi_i}{1-\phi_i}\right)^\lambda \geq \bigcup_{\phi_i\in\mathbb{P}_i}\left(\frac{\min_i\phi_i}{1-\min_i\phi_i}\right)^\lambda$$

$$\bigcup_{\phi_i\in\mathbb{P}_i}\sum_{i=1}^n w_i\left(\frac{\max_i\phi_i}{1-\max_i\phi_i}\right)^\lambda \geq \bigcup_{\phi_i\in\mathbb{P}_i}\sum_{i=1}^n w_i\left(\frac{\phi_i}{1-\phi_i}\right)^\lambda \geq \bigcup_{\phi_i\in\mathbb{P}_i}\sum_{i=1}^n w_i\left(\frac{\min_i\phi_i}{1-\min_i\phi_i}\right)^\lambda$$

$$\bigcup_{\phi_i\in\mathbb{P}_i}\left\{\sum_{i=1}^n w_i\left(\frac{\max_i\phi_i}{1-\max_i\phi_i}\right)^\lambda\right\}^{\frac{1}{\lambda}} \geq \bigcup_{\phi_i\in\mathbb{P}_i}\left\{\sum_{i=1}^n w_i\left(\frac{\phi_i}{1-\phi_i}\right)^\lambda\right\}^{\frac{1}{\lambda}}$$

$$\geq \underset{\phi_i\in\mathbb{P}_i}{\cup}\left\{\sum_{i=1}^{n} w_i\left(\frac{\min_i\phi_i}{1-\min_i\phi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}$$

$$\underset{\phi_i\in\mathbb{P}_i}{\cup} 1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\max_i\phi_i}{1-\max_i\phi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}} \geq \underset{\phi_i\in\mathbb{P}_i}{\cup} 1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\phi_i}{1-\phi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}$$

$$\geq \underset{\phi_i\in\mathbb{P}_i}{\cup} 1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\min_i\phi_i}{1-\min_i\phi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}$$

$$\underset{\phi_i\in\mathbb{P}_i}{\cup}\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\min_i\phi_i}{1-\min_i\phi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}} \leq \underset{\phi_i\in\mathbb{P}_i}{\cup}\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\phi_i}{1-\phi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}}$$

$$\leq \underset{\phi_i\in\mathbb{P}_i}{\cup}\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\max_i\phi_i}{1-\max_i\phi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}}$$

$$\underset{\phi_i\in\mathbb{P}_i}{\cup}\left(1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\min_i\phi_i}{1-\min_i\phi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}}\right) \leq \underset{\phi_i\in\mathbb{P}_i}{\cup}\left(1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\phi_i}{1-\phi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}}\right) \leq$$

$$\underset{\phi_i\in\mathbb{P}_i}{\cup}\left(1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\max_i\phi_i}{1-\max_i\phi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}}\right)$$

Now,

$$\underset{\varphi_i\in\mathbb{I}_i}{\cup}\min_i(\varphi_i) \leq \underset{\varphi_i\in\mathbb{I}_i}{\cup}(\varphi_i) \leq \underset{\varphi_i\in\mathbb{I}_i}{\cup}\max_i(\varphi_i)$$

$$\underset{\varphi_i\in\mathbb{I}_i}{\cup}(1-\max_i\varphi_i) \leq \underset{\varphi_i\in\mathbb{I}_i}{\cup}(1-\varphi_i) \leq \underset{\varphi_i\in\mathbb{I}_i}{\cup}(1-\min_i\varphi_i)$$

$$\underset{\varphi_i\in\mathbb{I}_i}{\cup}\left(\frac{1-\max_i\varphi_i}{\max_i\varphi_i}\right) \geq \underset{\varphi_i\in\mathbb{I}_i}{\cup}\left(\frac{1-\varphi_i}{\varphi_i}\right) \geq \underset{\varphi_i\in\mathbb{I}_i}{\cup}\left(\frac{1-\min_i\varphi_i}{\min_i\varphi_i}\right)$$

$$\underset{\varphi_i\in\mathbb{I}_i}{\cup}\left(\frac{1-\max_i\varphi_i}{\max_i\varphi_i}\right)^{\lambda} \geq \underset{\varphi_i\in\mathbb{I}_i}{\cup}\left(\frac{1-\varphi_i}{\varphi_i}\right)^{\lambda} \geq \underset{\varphi_i\in\mathbb{I}_i}{\cup}\left(\frac{1-\min_i\varphi_i}{\min_i\varphi_i}\right)^{\lambda}$$

$$\underset{\varphi_i\in\mathbb{I}_i}{\cup}\sum_{i=1}^{n} w_i\left(\frac{1-\max_i\varphi_i}{\max_i\varphi_i}\right)^{\lambda} \geq \underset{\varphi_i\in\mathbb{I}_i}{\cup}\sum_{i=1}^{n} w_i\left(\frac{1-\varphi_i}{\varphi_i}\right)^{\lambda} \geq \underset{\varphi_i\in\mathbb{I}_i}{\cup}\sum_{i=1}^{n} w_i\left(\frac{1-\min_i\varphi_i}{\min_i\varphi_i}\right)^{\lambda}$$

$$\underset{\varphi_i\in\mathbb{I}_i}{\cup}\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\max_i\varphi_i}{\max_i\varphi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}} \geq \underset{\varphi_i\in\mathbb{I}_i}{\cup}\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\varphi_i}{\varphi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}} \geq \underset{\varphi_i\in\mathbb{I}_i}{\cup}\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\min_i\varphi_i}{\min_i\varphi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}$$

$$\underset{\varphi_i\in\mathbb{I}_i}{\cup} 1+\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\max_i\varphi_i}{\max_i\varphi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}} \geq \underset{\varphi_i\in\mathbb{I}_i}{\cup} 1+\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\varphi_i}{\varphi_i}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}$$

$$\geq \bigcup_{\varphi_i \in \mathbb{I}_i} 1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1 - \min_i \varphi_i}{\min_i \varphi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}$$

$$\bigcup_{\varphi_i \in \mathbb{I}_i} \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1-\min_i \varphi_i}{\min_i \varphi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}} \leq \bigcup_{\varphi_i \in \mathbb{I}_i} \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1-\varphi_i}{\varphi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}}$$

$$\leq \bigcup_{\varphi_i \in \mathbb{I}_i} \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1-\max_i \varphi_i}{\max_i \varphi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}}$$

and

$$\bigcup_{\psi_i \in \mathbb{N}_i} \min_i(\psi_i) \leq \bigcup_{\psi_i \in \mathbb{N}_i} (\psi_i) \leq \bigcup_{\psi_i \in \mathbb{N}_i} \max_i(\psi_i)$$

$$\bigcup_{\psi_i \in \mathbb{N}_i} (1 - \max_i \psi_i) \leq \bigcup_{\psi_i \in \mathbb{N}_i} (1 - \psi_i) \leq \bigcup_{\psi_i \in \mathbb{N}_i} (1 - \min_i \psi_i)$$

$$\bigcup_{\psi_i \in \mathbb{N}_i} \left(\frac{1 - \max_i \psi_i}{\max_i \psi_i} \right) \geq \bigcup_{\psi_i \in \mathbb{N}_i} \left(\frac{1 - \psi_i}{\psi_i} \right) \geq \bigcup_{\psi_i \in \mathbb{N}_i} \left(\frac{1 - \min_i \psi_i}{\min_i \psi_i} \right)$$

$$\bigcup_{\psi_i \in \mathbb{N}_i} \left(\frac{1 - \max_i \psi_i}{\max_i \psi_i} \right)^{\lambda} \geq \bigcup_{\psi_i \in \mathbb{N}_i} \left(\frac{1 - \psi_i}{\psi_i} \right)^{\lambda} \geq \bigcup_{\psi_i \in \mathbb{N}_i} \left(\frac{1 - \min_i \psi_i}{\min_i \psi_i} \right)^{\lambda}$$

$$\bigcup_{\psi_i \in \mathbb{N}_i} \sum_{i=1}^{n} w_i \left(\frac{1 - \max_i \psi_i}{\max_i \psi_i} \right)^{\lambda} \geq \bigcup_{\psi_i \in \mathbb{N}_i} \sum_{i=1}^{n} w_i \left(\frac{1 - \psi_i}{\psi_i} \right)^{\lambda} \leq \bigcup_{\psi_i \in \mathbb{N}_i} \sum_{i=1}^{n} w_i \left(\frac{1 - \min_i \psi_i}{\min_i \psi_i} \right)^{\lambda}$$

$$\bigcup_{\psi_i \in \mathbb{N}_i} \left\{ \sum_{i=1}^{n} w_i \left(\frac{1 - \max_i \psi_i}{\max_i \psi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}} \leq \bigcup_{\psi_i \in \mathbb{N}_i} \left\{ \sum_{i=1}^{n} w_i \left(\frac{1 - \psi_i}{\psi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}} \geq \bigcup_{\psi_i \in \mathbb{N}_i} \left\{ \sum_{i=1}^{n} w_i \left(\frac{1 - \min_i \psi_i}{\min_i \psi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}$$

$$\bigcup_{\psi_i \in \mathbb{N}_i} 1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1 - \max_i \psi_i}{\max_i \psi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}} \geq \bigcup_{\psi_i \in \mathbb{N}_i} 1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1 - \psi_i}{\psi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}$$

$$\geq \bigcup_{\psi_i \in \mathbb{N}_i} 1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1 - \min_i \psi_i}{\min_i \psi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}$$

$$\bigcup_{\psi_i \in \mathbb{N}_i} \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1-\min_i \psi_i}{\min_i \psi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}} \leq \bigcup_{\psi_i \in \mathbb{N}_i} \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1-\psi_i}{\psi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}}$$

$$\leq \bigcup_{\psi_i \in \mathbb{N}_i} \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1-\max_i \psi_i}{\max_i \psi_i} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}}$$

So,

$$\mathfrak{x}^{-} \leq SVNHFDWA(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) \leq \mathfrak{x}^{+}$$

□

4.2. SVN Hesitant Fuzzy Dombi Weighted Geometric Operator (SVNHFDWG)

Definition 11. *Consider a collection of SVNHFNs $\mathfrak{x}_i$; then, the SVNHFDWG operator is defined as follows:*

$$SVNHFDWG(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \otimes_{i=1}^{n} (\mathfrak{x}_i^{w_i}) = \mathfrak{x}_1^{w_1} \otimes \mathfrak{x}_2^{w_2} \otimes \cdots \otimes \mathfrak{x}_n^{w_n} \tag{11}$$

The $w = (w_1, w_2, \cdots, w_n)^T$ *in Equation* (11) *is the WV of the SVNHFNs* $\mathfrak{x}_i$, $w_i \in [0,1]$ *and* $\sum_{i=1}^{n} w_i = 1$.

Theorem 7. *Consider a collection of SVNHFNs* $\mathfrak{x}_i$ *with WVs* $w = (w_1, w_2, \cdots, w_n)^T$*; then, the aggregated SVNHFN is*

$$SVNHFDWG(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \otimes_{i=1}^{n}(\mathfrak{x}_i^{w_i}) = \bigcup_{\phi_1 \in \mathbb{P}_1, \phi_2 \in \mathbb{P}_2, \cdots, \phi_n \in \mathbb{P}_n, \varphi_1 \in \mathbb{I}_1, \varphi_2 \in \mathbb{I}_2, \cdots, \varphi_n \in \mathbb{I}_n, \psi_1 \in \mathbb{N}_1, \psi_2 \in \mathbb{N}_2, \cdots, \psi_n \in \mathbb{N}_n}$$

$$\left\{ \frac{1}{1+\left\{\sum_{i=1}^{n} w_i \left(\frac{1-\phi_i}{\phi_i}\right)^{\lambda}\right\}^{1/\lambda}}, 1 - \frac{1}{1+\left\{\sum_{i=1}^{n} w_i \left(\frac{\varphi_i}{1-\varphi_i}\right)^{\lambda}\right\}^{1/\lambda}}, 1 - \frac{1}{1+\left\{\sum_{i=1}^{n} w_i \left(\frac{\psi_i}{1-\psi_i}\right)^{\lambda}\right\}^{1/\lambda}} \right\} \tag{12}$$

Proof. For $n = 2$, since

$$\mathfrak{x}_1^{w_1} = \bigcup_{\phi_1 \in \mathbb{P}_1, \varphi_1 \in \mathbb{I}_1, \psi_1 \in \mathbb{N}_1} \left\{ \frac{1}{1+\left\{w_1\left(\frac{1-\phi_1}{\phi_1}\right)^{\lambda}\right\}^{1/\lambda}}, 1 - \frac{1}{1+\left\{w_1\left(\frac{\varphi_1}{1-\varphi_1}\right)^{\lambda}\right\}^{1/\lambda}}, 1 - \frac{1}{1+\left\{w_1\left(\frac{\psi_1}{1-\psi_1}\right)^{\lambda}\right\}^{1/\lambda}} \right\}$$

$$\mathfrak{x}_2^{w_2} = \bigcup_{\phi_2 \in \mathbb{P}_2, \varphi_2 \in \mathbb{I}_2, \psi_2 \in \mathbb{N}_2} \left\{ \frac{1}{1+\left\{w_2\left(\frac{1-\phi_2}{\phi_2}\right)^{\lambda}\right\}^{1/\lambda}}, 1 - \frac{1}{1+\left\{w_2\left(\frac{\varphi_2}{1-\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}}, 1 - \frac{1}{1+\left\{w_2\left(\frac{\psi_2}{1-\psi_2}\right)^{\lambda}\right\}^{1/\lambda}} \right\}$$

then,

$$\mathfrak{x}_1^{w_1} \otimes \mathfrak{x}_2^{w_2} = \bigcup_{\phi_1 \in \mathbb{P}_1, \phi_2 \in \mathbb{P}_2, \varphi_1 \in \mathbb{I}_1, \varphi_2 \in \mathbb{I}_2, \psi_1 \in \mathbb{N}_1, \psi_2 \in \mathbb{N}_2}$$

$$\left\{ \frac{1}{1+\left(\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{1-\phi_1}{\phi_1}\right)^{\lambda}\right\}^{1/\lambda}}}{\frac{1}{1+\left\{w_1\left(\frac{1-\phi_1}{\phi_1}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda} + \left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{1-\phi_2}{\phi_2}\right)^{\lambda}\right\}^{1/\lambda}}}{\frac{1}{1+\left\{w_2\left(\frac{1-\phi_2}{\phi_2}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda}\right)^{1/\lambda}}, \right.$$

$$1 - \frac{1}{1+\left(\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{\varphi_1}{1-\varphi_1}\right)^{\lambda}\right\}^{1/\lambda}}}{1-1+\frac{1}{1+\left\{w_1\left(\frac{\varphi_1}{1-\varphi_1}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda} + \left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{\varphi_2}{1-\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}}}{1-1+\frac{1}{1+\left\{w_2\left(\frac{\varphi_2}{1-\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda}\right)^{1/\lambda}},$$

$$\left. 1 - \frac{1}{1+\left(\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{\psi_1}{1-\psi_1}\right)^{\lambda}\right\}^{1/\lambda}}}{1-1+\frac{1}{1+\left\{w_1\left(\frac{\psi_1}{1-\psi_1}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda} + \left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{\psi_2}{1-\psi_2}\right)^{\lambda}\right\}^{1/\lambda}}}{1-1+\frac{1}{1+\left\{w_2\left(\frac{\psi_2}{1-\psi_2}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda}\right)^{1/\lambda}} \right\}$$

$$= \bigcup_{\phi_1 \in \mathbb{P}_1, \phi_2 \in \mathbb{P}_2, \varphi_1 \in \mathbb{I}_1, \varphi_2 \in \mathbb{I}_2, \psi_1 \in \mathbb{N}_1, \psi_2 \in \mathbb{N}_2} \left\{ \frac{1}{1+\left(\left(\frac{\frac{1+\left\{w_1\left(\frac{1-\phi_1}{\phi_1}\right)^{\lambda}\right\}^{1/\lambda}-1}{1+\left\{w_1\left(\frac{1-\phi_1}{\phi_1}\right)^{\lambda}\right\}^{1/\lambda}}}{\frac{1}{1+\left\{w_1\left(\frac{1-\phi_1}{\phi_1}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda} + \left(\frac{\frac{1+\left\{w_2\left(\frac{1-\phi_2}{\phi_2}\right)^{\lambda}\right\}^{1/\lambda}-1}{1+\left\{w_2\left(\frac{1-\phi_2}{\phi_2}\right)^{\lambda}\right\}^{1/\lambda}}}{\frac{1}{1+\left\{w_2\left(\frac{1-\phi_2}{\phi_2}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda}\right)^{1/\lambda}}, \right.$$

$$1 - \frac{1}{1+\left(\left(\frac{\frac{1+\left\{w_1\left(\frac{\varphi_1}{1-\varphi_1}\right)^{\lambda}\right\}^{1/\lambda}-1}{1+\left\{w_1\left(\frac{\varphi_1}{1-\varphi_1}\right)^{\lambda}\right\}^{1/\lambda}}}{\frac{1}{1+\left\{w_1\left(\frac{\varphi_1}{1-\varphi_1}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda} + \left(\frac{\frac{1+\left\{w_2\left(\frac{\varphi_2}{1-\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}-1}{1+\left\{w_2\left(\frac{\varphi_2}{1-\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}}}{\frac{1}{1+\left\{w_2\left(\frac{\varphi_2}{1-\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda}\right)^{1/\lambda}},$$

$$1-\frac{1}{1+\left(\left(\frac{\frac{1+\left\{w_1\left(\frac{\psi_1}{1-\psi_1}\right)^{\lambda}\right\}^{1/\lambda}-1}{1+\left\{w_1\left(\frac{\psi_1}{1-\psi_1}\right)^{\lambda}\right\}^{1/\lambda}}}{\frac{1}{1+\left\{w_1\left(\frac{\psi_1}{1-\psi_1}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda}+\left(\frac{\frac{1+\left\{w_2\left(\frac{\psi_2}{1-\psi_2}\right)^{\lambda}\right\}^{1/\lambda}-1}{1+\left\{w_2\left(\frac{\psi_2}{1-\psi_2}\right)^{\lambda}\right\}^{1/\lambda}}}{\frac{1}{1+\left\{w_2\left(\frac{\psi_2}{1-\psi_2}\right)^{\lambda}\right\}^{1/\lambda}}}\right)^{\lambda}\right)^{1/\lambda}}\Bigg\}$$

$$=\bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2}\left\{\frac{1}{1+\left\{w_1\left(\frac{1-\phi_1}{\phi_1}\right)^{\lambda}+w_2\left(\frac{1-\phi_2}{\phi_2}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{w_1\left(\frac{\varphi_1}{1-\varphi_1}\right)^{\lambda}+w_2\left(\frac{\varphi_2}{1-\varphi_2}\right)^{\lambda}\right\}^{1/\lambda}},\right.$$

$$\left.1-\frac{1}{1+\left\{w_1\left(\frac{\psi_1}{1-\psi_1}\right)^{\lambda}+w_2\left(\frac{\psi_2}{1-\psi_2}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

If Equation (12) holds for $n=k$, that is

$$\otimes_{i=1}^{k}(\mathfrak{x}_i^{w_i})=\bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\cdots,\phi_k\in\mathbb{P}_k,\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\cdots,\varphi_k\in\mathbb{I}_k,\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2,\cdots,\psi_k\in\mathbb{N}_k}$$

$$\left\{\frac{1}{1+\left\{\sum_{i=1}^{k}w_i\left(\frac{1-\phi_i}{\phi_i}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{\sum_{i=1}^{k}w_i\left(\frac{\varphi_i}{1-\varphi_i}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{\sum_{i=1}^{k}w_i\left(\frac{\psi_i}{1-\psi_i}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

then, when $n=k+1$, by the Dombi multiplication operation, we have

$$\otimes_{i=1}^{k+1}(\mathfrak{x}_i^{w_i})=\otimes_{i=1}^{k}(\mathfrak{x}_i^{w_i})\otimes(\mathfrak{x}_{k+1}^{w_{k+1}})=\bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\cdots,\phi_k\in\mathbb{P}_k,\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\cdots,\varphi_k\in\mathbb{I}_k,\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2,\cdots,\psi_k\in\mathbb{N}_k}$$

$$\left\{\frac{1}{1+\left\{\sum_{i=1}^{k}w_i\left(\frac{1-\phi_i}{\phi_i}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{\sum_{i=1}^{k}w_i\left(\frac{\varphi_i}{1-\varphi_i}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{\sum_{i=1}^{k}w_i\left(\frac{\psi_i}{1-\psi_i}\right)^{\lambda}\right\}^{1/\lambda}}\right\}\otimes$$

$$\bigcup_{\phi_{k+1}\in\mathbb{P}_{k+1},\varphi_{k+1}\in\mathbb{I}_{k+1},\psi_{k+1}\in\mathbb{N}_{k+1}}\left\{\frac{1}{1+\left\{w_{k+1}\left(\frac{1-\phi_{k+1}}{\phi_{k+1}}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{w_{k+1}\left(\frac{\varphi_{k+1}}{1-\varphi_{k+1}}\right)^{\lambda}\right\}^{1/\lambda}},\right.$$

$$\left.1-\frac{1}{1+\left\{w_{k+1}\left(\frac{\psi_{k+1}}{1-\psi_{k+1}}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

$$=\bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\cdots,\phi_{k+1}\in\mathbb{P}_{k+1},\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\cdots,\varphi_{k+1}\in\mathbb{I}_{k+1},\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2,\cdots,\psi_{k+1}\in\mathbb{N}_{k+1}}\left\{\frac{1}{1+\left\{\sum_{i=1}^{k+1}w_i\left(\frac{1-\phi_i}{\phi_i}\right)^{\lambda}\right\}^{1/\lambda}},\right.$$

$$\left.1-\frac{1}{1+\left\{\sum_{i=1}^{k+1}w_i\left(\frac{\varphi_i}{1-\varphi_i}\right)^{\lambda}\right\}^{1/\lambda}}.1-\frac{1}{1+\left\{\sum_{i=1}^{k+1}w_i\left(\frac{\psi_i}{1-\psi_i}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

i.e., Equation (12) holds for $n=k+1$. Thus, Equation (12) holds for all n. Then,

$$\text{SVNHFDWG}(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n)=\otimes_{i=1}^{n}(\mathfrak{x}_i^{w_i})=\bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\cdots,\phi_n\in\mathbb{P}_n,\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\cdots,\varphi_n\in\mathbb{I}_n,\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2,\cdots,\psi_n\in\mathbb{N}_n}$$

$$\left\{\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{1-\phi_i}{\phi_i}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{\varphi_i}{1-\varphi_i}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{\psi_i}{1-\psi_i}\right)^{\lambda}\right\}^{1/\lambda}}\right\}\quad\square$$

Example 5. *Let us consider three SVNHFNs given in Table 2 and let* $w=\{0.25,0.35,0.40\}^T$ *be the WV of these three SVNHFNs. Then, the aggregated SVNHFN with* $\lambda=2$ *is*

$$SVNHFDWG(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3)=\otimes_{i=1}^{3}(\mathfrak{x}_i^{w_i})=\bigcup_{\phi_1\in\mathbb{P}_1,\phi_2\in\mathbb{P}_2,\phi_3\in\mathbb{P}_3,\varphi_1\in\mathbb{I}_1,\varphi_2\in\mathbb{I}_2,\varphi_3\in\mathbb{I}_3,\psi_1\in\mathbb{N}_1,\psi_2\in\mathbb{N}_2,\psi_3\in\mathbb{N}_3}$$
$$\left\{\frac{1}{1+\left\{\sum_{i=1}^{3}w_i\left(\frac{1-\phi_i}{\phi_i}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{\sum_{i=1}^{3}w_i\left(\frac{\varphi_i}{1-\varphi_i}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{\sum_{i=1}^{3}w_i\left(\frac{\psi_i}{1-\psi_i}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$
$$=\langle\{0.531,0.588,0.573,0.658,0.586,0.684\},\{0.233,0.296,0.374,0.293,0.336,0.398\},\{0.408,0.416\}\rangle$$

Now, we examine the nature of λ, which is an operational parameter, and Table 4 shows some of the SVNHFN's aggregated values.

Table 4. Aggregated SVNHFNs based on SVNHFDWG operator

Parameter	Aggregated SVNHFNs
$\lambda=1$	$\langle\{0.560,0.607,0.614,0.671,0.645,0.708\},\{0.220,0.267,0.325,0.277,0.318,0.368\},\{0.371,0.393\}\rangle$
$\lambda=2$	$\langle\{0.531,0.588,0.573,0.658,0.586,0.684\},\{0.233,0.296,0.374,0.293,0.336,0.398\},\{0.408,0.416\}\rangle$
$\lambda=4$	$\langle\{0.499,0.566,0.516,0.636,0.518,0.645\},\{0.251,0.330,0.420,0.310,0.352,0.426\},\{0.445,0.446\}\rangle$
$\lambda=5$	$\langle\{0.490,0.559,0.500,0.627,0.501,0.632\},\{0.257,0.339,0.431,0.315,0.356,0.434\},\{0.455,0.456\}\rangle$
$\lambda=10$	$\langle\{0.465,0.543,0.466,0.600,0.466,0.601\},\{0.272,0.360,0.453,0.328,0.364,0.453\},\{0.477,0.477\}\rangle$
$\lambda=20$	$\langle\{0.449,0.532,0.449,0.584,0.449,0.584\},\{0.281,0.370,0.465,0.338,0.371,0.465\},\{0.489,0.489\}\rangle$
$\lambda=50$	$\langle\{0.439,0.524,0.439,0.574,0.439,0.574\},\{0.286,0.377,0.471,0.345,0.377,0.471\},\{0.495,0.495\}\rangle$
$\lambda=100$	$\langle\{0.435,0.522,0.435,0.570,0.435,0.570\},\{0.288,0.379,0.474,0.348,0.379,0.474\},\{0.498,0.498\}\rangle$

Theorem 8 (Idempotency property). *Let $\mathfrak{x}_i=\langle\mathbb{P}_i,\mathbb{I}_i,\mathbb{N}_i\rangle$ be a collection of SVNHFNs. If all $\mathfrak{x}_i$ are equal, i.e., $\mathfrak{x}_i=\mathfrak{x}\ \forall i$, then*

$$SVNHFDWG(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n)=\mathfrak{x} \tag{13}$$

Theorem 9 (Monotonicity property). *Let $\mathfrak{x}_i=\langle\mathbb{P}_i,\mathbb{I}_i,\mathbb{N}_i\rangle$ and $\check{\mathfrak{x}}_i=\langle\check{\mathbb{P}}_i,\check{\mathbb{I}}_i,\check{\mathbb{N}}_i\rangle$ be two collections of SVNHFNs. If $\mathfrak{x}_i\leq\check{\mathfrak{x}}_i\ \forall i$, then*

$$SVNHFDWG(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n)\leq SVNHFDWG(\check{\mathfrak{x}}_1,\check{\mathfrak{x}}_2,\cdots,\check{\mathfrak{x}}_n)$$

Theorem 10 (Boundary). *Let $\mathfrak{x}_i=\langle\mathbb{P}_i,\mathbb{I}_i,\mathbb{N}_i\rangle$ be a collection of SVNHFNs, then*

$$\mathfrak{x}^-\leq SVNHFDWG(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n)\leq\mathfrak{x}^+ \tag{14}$$

where $\mathfrak{x}^-=\langle\mathbb{P}^-,\mathbb{I}^-,\mathbb{N}^-\rangle$, $\mathfrak{x}^+=\langle\mathbb{P}^+,\mathbb{I}^+,\mathbb{N}^+\rangle$

$$\mathbb{P}^-=\bigcup_{\phi_i\in\mathbb{P}_i}\min_i(\phi_i),\mathbb{P}^+=\bigcup_{\phi_i\in\mathbb{P}_i}\max_i(\phi_i)$$
$$\mathbb{I}^-=\bigcup_{\varphi_i\in\mathbb{I}_i}\max_i(\varphi_i),\mathbb{I}^+=\bigcup_{\varphi_i\in\mathbb{I}_i}\min_i(\varphi_i)$$
$$\mathbb{N}^-=\bigcup_{\psi_i\in\mathbb{N}_i}\max_i(\psi_i),\mathbb{N}^+=\bigcup_{\psi_i\in\mathbb{N}_i}\min_i(\psi_i)$$

4.3. SVNHFDOWA Operator

Definition 12. *Consider a collection of SVNHFNs $\mathfrak{x}_i$; then, the SVNHFDOWA operator is defined as follows:*

$$SVNHFDOWA(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n)=\oplus_{i=1}^{n}(w_i\mathfrak{x}_{\sigma_i})=w_1\mathfrak{x}_{\sigma_1}\oplus w_2\mathfrak{x}_{\sigma_2}\oplus\cdots\oplus w_n\mathfrak{x}_{\sigma_n} \tag{15}$$

The $w=(w_1,w_2,\cdots,w_n)^T$ in Equation (15) is the WV of the SVNHFNs $\mathfrak{x}_i$, $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$ and σ is a permutation such that $\mathfrak{x}_{\sigma_i}$ ia the largest number in $(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n)$.

Theorem 11. *Consider a collection of SVNHFNs $\mathfrak{x}_i$ with WVs $w=(w_1,w_2,\cdots,w_n)^T$; then, the aggregated SVNHFN is*

$$SVNHFDOWA(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n)=\oplus_{i=1}^{n}(w_i\mathfrak{x}_{\sigma_i})=\bigcup_{\substack{\phi_{\sigma_1}\in\mathbb{P}_{\sigma_1},\phi_{\sigma_2}\in\mathbb{P}_{\sigma_2},\cdots,\phi_{\sigma_n}\in\mathbb{P}_{\sigma_n},\\ \varphi_{\sigma_1}\in\mathbb{I}_{\sigma_1},\varphi_{\sigma_2}\in\mathbb{I}_{\sigma_2},\cdots,\varphi_{\sigma_n}\in\mathbb{I}_{\sigma_n},\\ \psi_{\sigma_1}\in\mathbb{N}_{\sigma_1},\psi_{\sigma_2}\in\mathbb{N}_{\sigma_2},\cdots,\psi_{\sigma_n}\in\mathbb{N}_{\sigma_n}}}$$

$$\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{\phi_{\sigma_i}}{1-\phi_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}},\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{1-\varphi_{\sigma_i}}{\varphi_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}},\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{1-\psi_{\sigma_i}}{\psi_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}}\right\} \tag{16}$$

Example 6. *Let us consider three SVNHFNs given in Table 2 and* $w=\{0.25,0.35,0.40\}^T$ *let be the WV of these three SVNHFNs. Then, the aggregated SVNHFN with* $\lambda=2$ *is*

According to Equation (1), the score function of $\mathfrak{x}_1,\mathfrak{x}_2$ *and* $\mathfrak{x}_3$ *is calculated as follows:*

$$S(\mathfrak{x}_1)=0.685,S(\mathfrak{x}_2)=0.741,S(\mathfrak{x}_3)=0.623$$

Since

$$S(\mathfrak{x}_2)>S(\mathfrak{x}_1)>S(\mathfrak{x}_3)$$

$$\mathfrak{x}_{\sigma_1}=\mathfrak{x}_2=\langle\{0.519,0.678,0.800\},\{0.200,0.350\},\{0.156,0.259\}\rangle$$

$$\mathfrak{x}_{\sigma_2}=\mathfrak{x}_1=\langle\{0.432,0.567\},\{0.111\},\{0.334\}\rangle$$

$$\mathfrak{x}_{\sigma_3}=\mathfrak{x}_3=\langle\{0.750\},\{0.290,0.381,0.476\},\{0.500\}\rangle$$

By using the concept of the SVNHFDOWA operator, we have

$$SVNHFDOWA(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n)=\oplus_{i=1}^{n}(w_i\mathfrak{x}_{\sigma_i})=\bigcup_{\substack{\phi_{\sigma_1}\in\mathbb{P}_{\sigma_1},\phi_{\sigma_2}\in\mathbb{P}_{\sigma_2},\cdots,\phi_{\sigma_n}\in\mathbb{P}_{\sigma_n},\\ \varphi_{\sigma_1}\in\mathbb{I}_{\sigma_1},\varphi_{\sigma_2}\in\mathbb{I}_{\sigma_2},\cdots,\varphi_{\sigma_n}\in\mathbb{I}_{\sigma_n},\\ \psi_{\sigma_1}\in\mathbb{N}_{\sigma_1},\psi_{\sigma_2}\in\mathbb{N}_{\sigma_2},\cdots,\psi_{\sigma_n}\in\mathbb{N}_{\sigma_n}}}$$

$$\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{\phi_{\sigma_i}}{1-\phi_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}},\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{1-\varphi_{\sigma_i}}{\varphi_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}},\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{1-\psi_{\sigma_i}}{\psi_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

$$=\langle\{0.669,0.689,0.736,0.679,0.697,0.741\},\{0.157,0.165,0.160,0.168,0.162,0.170\},\{0.249,0.338\}\rangle$$

When the parameter λ is provided in multiple quantities, several aggregated SVNHFNs can be determined, which are all displayed in Table 5 below.

Table 5. Aggregated SVNHFNs based on SVNHFDOWA operator.

Parameter	Aggregated SVNHFNs
$\lambda=1$	$\langle\{0.634,0.666,0.711,0.658,0.686,0.727\},\{0.173,0.191,0.183,0.203,0.191,0.212\},\{0.290,0.355\}\rangle$
$\lambda=2$	$\langle\{0.669,0.689,0.736,0.679,0.697,0.741\},\{0.157,0.165,0.160,0.168,0.162,0.170\},\{0.249,0.338\}\rangle$
$\lambda=4$	$\langle\{0.705,0.7120.758,0.707,0.713,0.759\},\{0.138,0.139,0.138,0.140,0.138,0.140\},\{0.206,0.314\}\rangle$
$\lambda=5$	$\langle\{0.714,0.718,0.764,0.715,0.719,0.764\},\{0.133,0.133,0.133,0.133,0.133,0.133\},\{0.196,0.306\}\rangle$
$\lambda=10$	$\langle\{0.732,0.733,0.778,0.732,0.733,0.778\},\{0.122,0.122,0.122,0.122,0.122,0.122\},\{0.175,0.286\}\rangle$
$\lambda=20$	$\langle\{0.741,0.741,0.789,0.741,0.741,0.789\},\{0.116,0.116,0.116,0.116,0.116,0.116\},\{0.165,0.273\}\rangle$
$\lambda=50$	$\langle\{0.747,0.747,0.796,0.747,0.747,0.796\},\{0.113,0.113,0.113,0.113,0.113,0.113\},\{0.160,0.264\}\rangle$
$\lambda=100$	$\langle\{0.748,0.748,0.798,0.748,0.748,0.798\},\{0.112,0.112,0.112,0.112,0.112,0.112\},\{0.158,0.262\}\rangle$

Theorem 12 (Idempotency property)**.** *Let* $\mathfrak{x}_i=\langle\mathbb{P}_i,\mathbb{I}_i,\mathbb{N}_i\rangle$ *be a collection of SVNHFNs. If all* $\mathfrak{x}_i$ *are equal, i.e.,* $\mathfrak{x}_i=\mathfrak{x}\ \forall i$*, then*

$$SVNHFDOWA(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n)=\mathfrak{x} \tag{17}$$

Theorem 13 (Monotonicity property). *Let* $\mathfrak{x}_i = \langle \mathbb{P}_i, \mathbb{I}_i, \mathbb{N}_i \rangle$ *and* $\check{\mathfrak{x}}_i = \langle \check{\mathbb{P}}_i, \check{\mathbb{I}}_i, \check{\mathbb{N}}_i \rangle$ *be two collections of SVNHFNs. If* $\mathfrak{x}_i \leq \check{\mathfrak{x}}_i \ \forall i$, *then*

$$SVNHFDOWA(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) \leq SVNHFDOWA(\check{\mathfrak{x}}_1, \check{\mathfrak{x}}_2, \cdots, \check{\mathfrak{x}}_n)$$

Theorem 14 (Boundary). *Let* $\mathfrak{x}_i = \langle \mathbb{P}_i, \mathbb{I}_i, \mathbb{N}_i \rangle$ *be a collection of SVNHFNs, then*

$$\mathfrak{x}^- \leq SVNHFDOWA(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) \leq \mathfrak{x}^+ \tag{18}$$

4.4. SVNHFDOWG Operator

Definition 13. *Consider a collection of SVNHFNs* $\mathfrak{x}_i$; *then, the SVNHFDOWG operator is described as follows:*

$$SVNHFDOWG(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \otimes_{i=1}^{n} (\mathfrak{x}_{\sigma_i}^{w_i}) = \mathfrak{x}_{\sigma_1}^{w_1} \otimes \mathfrak{x}_{\sigma_2}^{w_2} \otimes \cdots \otimes \mathfrak{x}_{\sigma_n}^{w_n} \tag{19}$$

The $w = (w_1, w_2, \cdots, w_n)^T$ *in Equation (19) is the WV of the SVNHFNs* $\mathfrak{x}_i$, $w_i \in [0,1]$ *and* $\sum_{i=1}^{n} w_i = 1$.

Theorem 15. *Consider a collection of SVNHFNs* $\mathfrak{x}_i$ *with WVs* $w = (w_1, w_2, \cdots, w_n)^T$; *then, the aggregated SVNHFN is*

$$SVNHFDOWG(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \otimes_{i=1}^{n} (\mathfrak{x}_{\sigma_i}^{w_i}) = \bigcup_{\substack{\phi_{\sigma_1} \in \mathbb{P}_{\sigma_1}, \phi_{\sigma_2} \in \mathbb{P}_{\sigma_2}, \cdots, \phi_{\sigma_n} \in \mathbb{P}_{\sigma_n}, \\ \varphi_{\sigma_1} \in \mathbb{I}_{\sigma_1}, \varphi_{\sigma_2} \in \mathbb{I}_{\sigma_2}, \cdots, \varphi_{\sigma_n} \in \mathbb{I}_{\sigma_n}, \\ \psi_{\sigma_1} \in \mathbb{N}_{\sigma_1}, \psi_{\sigma_2} \in \mathbb{N}_{\sigma_2}, \cdots, \psi_{\sigma_n} \in \mathbb{N}_{\sigma_n}}$$

$$\left\{ \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1-\phi_{\sigma_i}}{\phi_{\sigma_i}} \right)^{\lambda} \right\}^{1/\lambda}}, 1 - \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{\varphi_{\sigma_i}}{1-\varphi_{\sigma_i}} \right)^{\lambda} \right\}^{1/\lambda}}, 1 - \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{\psi_{\sigma_i}}{1-\psi_{\sigma_i}} \right)^{\lambda} \right\}^{1/\lambda}} \right\} \tag{20}$$

Example 7. *Let us consider three SVNHFNs given in Table 2 and let* $w = \{0.25, 0.35, 0.40\}^T$ *be the WV of these three SVNHFNs. Then, for the aggregated SVNHFN,* $\lambda = 2$.

According to Equation (1), the score function of $\mathfrak{x}_1, \mathfrak{x}_2$ *and* $\mathfrak{x}_3$ *is calculated as follows:*

$$S(\mathfrak{x}_1) = 0.685, S(\mathfrak{x}_2) = 0.741, S(\mathfrak{x}_3) = 0.623$$

Since

$$S(\mathfrak{x}_2) > S(\mathfrak{x}_1) > S(\mathfrak{x}_3)$$

$$\mathfrak{x}_{\sigma_1} = \mathfrak{x}_2 = \langle \{0.519, 0.678, 0.800\}, \{0.200, 0.350\}, \{0.156, 0.259\} \rangle$$

$$\mathfrak{x}_{\sigma_2} = \mathfrak{x}_1 = \langle \{0.432, 0.567\}, \{0.111\}, \{0.334\} \rangle$$

$$\mathfrak{x}_{\sigma_3} = \mathfrak{x}_3 = \langle \{0.750\}, \{0.290, 0.381, 0.476\}, \{0.500\} \rangle$$

Using the SVNHFDOWG operator, we obtain

$$SVNHFDOWG(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \otimes_{i=1}^{n} (\mathfrak{x}_{\sigma_i}^{w_i}) = \bigcup_{\substack{\phi_{\sigma_1} \in \mathbb{P}_{\sigma_1}, \phi_{\sigma_2} \in \mathbb{P}_{\sigma_2}, \cdots, \phi_{\sigma_n} \in \mathbb{P}_{\sigma_n}, \\ \varphi_{\sigma_1} \in \mathbb{I}_{\sigma_1}, \varphi_{\sigma_2} \in \mathbb{I}_{\sigma_2}, \cdots, \varphi_{\sigma_n} \in \mathbb{I}_{\sigma_n}, \\ \psi_{\sigma_1} \in \mathbb{N}_{\sigma_1}, \psi_{\sigma_2} \in \mathbb{N}_{\sigma_2}, \cdots, \psi_{\sigma_n} \in \mathbb{N}_{\sigma_n}}$$

$$\left\{ \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{1-\phi_{\sigma_i}}{\phi_{\sigma_i}} \right)^{\lambda} \right\}^{1/\lambda}}, 1 - \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{\varphi_{\sigma_i}}{1-\varphi_{\sigma_i}} \right)^{\lambda} \right\}^{1/\lambda}}, 1 - \frac{1}{1 + \left\{ \sum_{i=1}^{n} w_i \left(\frac{\psi_{\sigma_i}}{1-\psi_{\sigma_i}} \right)^{\lambda} \right\}^{1/\lambda}} \right\}$$

$$= \langle \{0.518, 0.543, 0.551, 0.595, 0.644, 0.661\}, \{0.229, 0.324, 0.372, 0.276, 0.294, 0.390\}, \{0.413, 0.419\} \rangle$$

When the parameter λ is provided with multiple quantities, several aggregated SVNHFNs can be determined, which are all displayed in Table 6 below.

Table 6. Aggregated SVNHFNs based on SVNHFDOWG operator.

Parameter	Aggregated SVNHFNs
$\lambda = 1$	$\langle\{0.548, 0.584, 0.604, 0.613, 0.658, 0.683\}, \{0.212, 0.298, 0.320, 0.255, 0.261, 0.351\}, \{0.383, 0.399\}\rangle$
$\lambda = 2$	$\langle\{0.518, 0.543, 0.551, 0.595, 0.644, 0.661\}, \{0.229, 0.324, 0.372, 0.276, 0.294, 0.390\}, \{0.413, 0.419\}\rangle$
$\lambda = 4$	$\langle\{0.487, 0.4960.497, 0.573, 0.622, 0.627\}, \{0.249, 0.346, 0.420, 0.298, 0.330, 0.424\}, \{0.446, 0.447\}\rangle$
$\lambda = 5$	$\langle\{0.478, 0.484, 0.484, 0.566, 0.614, 0.617\}, \{0.256, 0.351, 0.431, 0.304, 0.339, 0.433\}, \{0.456, 0.456\}\rangle$
$\lambda = 10$	$\langle\{0.457, 0.458, 0.458, 0.549, 0.592, 0.593\}, \{0.272, 0.363, 0.453, 0.321, 0.360, 0.453\}, \{0.477, 0.477\}\rangle$
$\lambda = 20$	$\langle\{0.445, 0.445, 0.445, 0.536, 0.580, 0.580\}, \{0.281, 0.371, 0.465, 0.334, 0.370, 0.465\}, \{0.486, 0.486\}\rangle$
$\lambda = 50$	$\langle\{0.437, 0.437, 0.437, 0.526, 0.572, 0.572\}, \{0.286, 0.377, 0.471, 0.344, 0.377, 0.471\}, \{0.495, 0.495\}\rangle$
$\lambda = 100$	$\langle\{0.435, 0.435, 0.435, 0.522, 0.570, 0.570\}, \{0.288, 0.379, 0.474, 0.347, 0.379, 0.474\}, \{0.498, 0.498\}\rangle$

Theorem 16 (Idempotency property). *Let $\mathfrak{x}_i = \langle \mathbb{P}_i, \mathbb{I}_i, \mathbb{N}_i \rangle$ be a collection of SVNHFNs. If all $\mathfrak{x}_i$ are equal, i.e., $\mathfrak{x}_i = \mathfrak{x}$ $\forall i$, then*

$$SVNHFDOWG(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \mathfrak{x} \tag{21}$$

Theorem 17 (Monotonicity property). *Let $\mathfrak{x}_i = \langle \mathbb{P}_i, \mathbb{I}_i, \mathbb{N}_i \rangle$ and $\check{\mathfrak{x}}_i = \langle \check{\mathbb{P}}_i, \check{\mathbb{I}}_i, \check{\mathbb{N}}_i \rangle$ be two groups of SVNHFNs. If $\mathfrak{x}_i \leq \check{\mathfrak{x}}_i$ $\forall i$, then*

$$SVNHFDOWG(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) \leq SVNHFDOWG(\check{\mathfrak{x}}_1, \check{\mathfrak{x}}_2, \cdots, \check{\mathfrak{x}}_n)$$

Theorem 18 (Boundary). *Let $\mathfrak{x}_i = \langle \mathbb{P}_i, \mathbb{I}_i, \mathbb{N}_i \rangle$ be a collection of SVNHFNs, then*

$$\mathfrak{x}^- \leq SVNHFDOWG(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) \leq \mathfrak{x}^+ \tag{22}$$

4.5. SVNHFDHA Operator

Definition 14. *Consider a collection of SVNHFNs $\mathfrak{x}_i$, where $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \cdots, \mathfrak{w}_n)^T$ is the WV of the SVNHFNs $\mathfrak{x}_i$, $\mathfrak{w}_i \in [0, 1]$ and $w = \{w_1, w_2, \cdots, w_n\}^T$ are the associated WV of the aggregated documents, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, the SVNHFDHA operator is defined as follows:*

$$SVNHFDHA(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \oplus_{i=1}^{n}(w_i \dot{\mathfrak{x}}_{\sigma_i}) = w_1 \dot{\mathfrak{x}}_{\sigma_1} \oplus w_2 \dot{\mathfrak{x}}_{\sigma_2} \oplus \cdots \oplus w_n \dot{\mathfrak{x}}_{\sigma_n} \tag{23}$$

where $\dot{\mathfrak{x}}_{\sigma_i}$ is the largest number in $\dot{\mathfrak{x}}_i = n\mathfrak{w}_i\mathfrak{x}_i$.

Theorem 19. *Consider a collection of SVNHFNs $\mathfrak{x}_i$ with WVs $w = (w_1, w_2, \cdots, w_n)^T$; then, the aggregated SVNHFN is*

$$SVNHFDHA(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \oplus_{i=1}^{n}(w_i \dot{\mathfrak{x}}_{\sigma_i}) = \bigcup_{\substack{\dot{\phi}_{\sigma_1} \in \dot{\mathbb{P}}_{\sigma_1}, \dot{\phi}_{\sigma_2} \in \dot{\mathbb{P}}_{\sigma_2}, \cdots, \dot{\phi}_{\sigma_n} \in \dot{\mathbb{P}}_{\sigma_n}, \\ \dot{\varphi}_{\sigma_1} \in \dot{\mathbb{I}}_{\sigma_1}, \dot{\varphi}_{\sigma_2} \in \dot{\mathbb{I}}_{\sigma_2}, \cdots, \dot{\varphi}_{\sigma_n} \in \dot{\mathbb{I}}_{\sigma_n}, \\ \dot{\psi}_{\sigma_1} \in \dot{\mathbb{N}}_{\sigma_1}, \dot{\psi}_{\sigma_2} \in \dot{\mathbb{N}}_{\sigma_2}, \cdots, \dot{\psi}_{\sigma_n} \in \dot{\mathbb{N}}_{\sigma_n}}}$$

$$\left\{1 - \frac{1}{1 + \left\{\sum_{i=1}^{n} w_i \left(\frac{\dot{\phi}_{\sigma_i}}{1 - \dot{\phi}_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1 + \left\{\sum_{i=1}^{n} w_i \left(\frac{1 - \dot{\varphi}_{\sigma_i}}{\dot{\varphi}_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1 + \left\{\sum_{i=1}^{n} w_i \left(\frac{1 - \dot{\psi}_{\sigma_i}}{\dot{\psi}_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}}\right\} \tag{24}$$

where $\dot{\mathfrak{x}}_{\sigma_i}$ is the largest number in $\dot{\mathfrak{x}}_i = n\mathfrak{w}_i\mathfrak{x}_i$.

Example 8. *Let us consider three SVNHFNs given in Table 2 and let $\mathfrak{w} = \{0.25, 0.35, 0.40\}^T$ be the WV of these three SVNHFNs and $w = \{0.15, 0.65, 0.20\}^T$ be the associated WV. Then, the aggregated SVNHFN with $\lambda = 2$ is*

$$\dot{\mathfrak{x}}_1 = \langle\{0.397, 0.531\}, \{0.126\}, \{0.367\}\rangle$$

$$\dot{\mathfrak{x}}_2 = \langle\{0.525, 0.683, 0.804\}, \{0.196, 0.344\}, \{0.153, 0.254\}\rangle$$

$$\dot{\mathfrak{x}}_3 = \langle\{0.767\},\{0.272,0.360,0.453\},\{0.477\}\rangle$$

According to Equation (1), the score function of $\dot{\mathfrak{x}}_1, \dot{\mathfrak{x}}_2$ and $\dot{\mathfrak{x}}_3$ is calculated as follows:

$$S(\dot{\mathfrak{x}}_1) = 0.657, S(\dot{\mathfrak{x}}_2) = 0.746, S(\dot{\mathfrak{x}}_3) = 0.643$$

Since

$$S(\dot{\mathfrak{x}}_2) > S(\dot{\mathfrak{x}}_1) > S(\dot{\mathfrak{x}}_3)$$

Then

$$\dot{\mathfrak{x}}_{\sigma_1} = \dot{\mathfrak{x}}_2 = \langle\{0.525,0.683,0.804\},\{0.196,0.344\},\{0.153,0.254\}\rangle$$

$$\dot{\mathfrak{x}}_{\sigma_2} = \dot{\mathfrak{x}}_1 = \langle\{0.397,0.531\},\{0.126\},\{0.367\}\rangle$$

$$\dot{\mathfrak{x}}_{\sigma_3} = \dot{\mathfrak{x}}_3 = \langle\{0.767\},\{0.272,0.360,0.453\},\{0.477\}\rangle$$

Using the concept of the SVNHFDHA operator, we have

$$SVNHFDHA(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n) = \oplus_{i=1}^{n}(w_i\dot{\mathfrak{x}_{\sigma_i}}) = \bigcup_{\substack{\dot{\phi}_{\sigma_1}\in\dot{\mathbb{P}}_{\sigma_1},\dot{\phi}_{\sigma_2}\in\dot{\mathbb{P}}_{\sigma_2},\cdots,\dot{\phi}_{\sigma_n}\in\dot{\mathbb{P}}_{\sigma_n},\\ \dot{\varphi}_{\sigma_1}\in\dot{\mathbb{I}}_{\sigma_1},\dot{\varphi}_{\sigma_2}\in\dot{\mathbb{I}}_{\sigma_2},\cdots,\dot{\varphi}_{\sigma_n}\in\dot{\mathbb{I}}_{\sigma_n},\\ \dot{\psi}_{\sigma_1}\in\dot{\mathbb{N}}_{\sigma_1},\dot{\psi}_{\sigma_2}\in\dot{\mathbb{N}}_{\sigma_2},\cdots,\dot{\psi}_{\sigma_n}\in\dot{\mathbb{N}}_{\sigma_n}}}$$

$$\left\{1-\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{\phi_{\sigma_i}}{1-\phi_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}},\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{1-\varphi_{\sigma_i}}{\varphi_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}},\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{1-\psi_{\sigma_i}}{\psi_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

where $\dot{\mathfrak{x}}_{\sigma_i}$ is the largest number in $\dot{\mathfrak{x}}_i = n\mathfrak{w}_i\mathfrak{x}_i$.

$$= \langle\{0.619,0.639,0.690,0.641,0.658,0.702\},\{0.144,0.148,0.146,0.149,0.146,0.150\},\{0.278,0.149\}\rangle$$

4.6. SVNHFDHG Operator

Definition 15. *Consider a collection of SVNHFNs $\mathfrak{x}_i$, where $\mathfrak{w} = (\mathfrak{w}_1,\mathfrak{w}_2,\cdots,\mathfrak{w}_n)^T$ is the WV of the SVNHFNs $\mathfrak{x}_i$, $\mathfrak{w}_i \in [0,1]$ and $w = \{w_1,w_2,\cdots,w_n\}^T$ is the associated WV of the aggregated documents, $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$; then, the SVNHFDHG operator is defined as follows:*

$$SVNHFDHG(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n) = \otimes_{i=1}^{n}(\tilde{\mathfrak{x}}_{\sigma_i}^{w_i}) = \tilde{\mathfrak{x}}_{\sigma_1}^{w_1}\otimes\tilde{\mathfrak{x}}_{\sigma_2}^{w_2}\otimes\cdots\otimes\tilde{\mathfrak{x}}_{\sigma_n}^{w_n} \tag{25}$$

where $\tilde{\mathfrak{x}}_{\sigma_i}$ is the largest number in $\tilde{\mathfrak{x}}_i = \mathfrak{x}_i^{n\mathfrak{w}_i}(i = 1,2,\cdots,n)$.

Theorem 20. *Consider a collection of SVNHFNs $\mathfrak{x}_i$ with WVs $w = (w_1,w_2,\cdots,w_n)^T$; then, the aggregated SVNHFN is*

$$SVNHFDHG(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n) = \otimes_{i=1}^{n}(\tilde{\mathfrak{x}}_{\sigma_i}^{w_i}) = \bigcup_{\substack{\dot{\phi}_{\sigma_1}\in\dot{\mathbb{P}}_{\sigma_1},\dot{\phi}_{\sigma_2}\in\dot{\mathbb{P}}_{\sigma_2},\cdots,\dot{\phi}_{\sigma_n}\in\dot{\mathbb{P}}_{\sigma_n},\\ \dot{\varphi}_{\sigma_1}\in\dot{\mathbb{I}}_{\sigma_1},\dot{\varphi}_{\sigma_2}\in\dot{\mathbb{I}}_{\sigma_2},\cdots,\dot{\varphi}_{\sigma_n}\in\dot{\mathbb{I}}_{\sigma_n},\\ \dot{\psi}_{\sigma_1}\in\dot{\mathbb{N}}_{\sigma_1},\dot{\psi}_{\sigma_2}\in\dot{\mathbb{N}}_{\sigma_2},\cdots,\dot{\psi}_{\sigma_n}\in\dot{\mathbb{N}}_{\sigma_n}}}$$

$$\left\{\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{1-\tilde{\phi}_{\sigma_i}}{\tilde{\phi}_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{\tilde{\varphi}_{\sigma_i}}{1-\tilde{\varphi}_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}},1-\frac{1}{1+\left\{\sum_{i=1}^{n}w_i\left(\frac{\tilde{\psi}_{\sigma_i}}{1-\tilde{\psi}_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}}\right\} \tag{26}$$

where $\tilde{\mathfrak{x}}_{\sigma_i}$ is the largest number in $\tilde{\mathfrak{x}}_i = \mathfrak{x}_i^{n\mathfrak{w}_i}(i = 1,2,\cdots,n)$.

Example 9. *Let us consider three SVNHFNs given in Table 2 and let $\mathfrak{w} = \{0.25,0.35,0.40\}^T$ be the WV of these three SVNHFNs, and let $w = \{0.15,0.65,0.20\}^T$ be the associated WV. Then, the aggregated SVNHFN with $\lambda = 2$ is*

$$\dot{\mathfrak{x}}_1 = \langle\{0.468,0.602\},\{0.098\},\{0.303\}\rangle$$

$$\dot{\mathfrak{x}}_2 = \langle\{0.513,0.673,0.796\},\{0.204,0.356\},\{0.159,0.264\}\rangle$$

$$\dot{\mathfrak{x}}_3 = \langle\{0.733\},\{0.309,0.403,0.499\},\{0.523\}\rangle$$

According to Equation (1), the score function of $\dot{\mathfrak{x}}_1, \dot{\mathfrak{x}}_2$ and $\dot{\mathfrak{x}}_3$ is calculated as follows:

$$\mathrm{S}(\dot{\mathfrak{x}}_1) = 0.711, \mathrm{S}(\dot{\mathfrak{x}}_2) = 0.737, \mathrm{S}(\dot{\mathfrak{x}}_3) = 0.602$$

Since

$$\mathrm{S}(\dot{\mathfrak{x}}_2) > \mathrm{S}(\dot{\mathfrak{x}}_1) > \mathrm{S}(\dot{\mathfrak{x}}_3)$$

Then

$$\dot{\mathfrak{x}}_{\sigma_1} = \dot{\mathfrak{x}}_2 = \langle\{0.513,0.673,0.796\},\{0.204,0.356\},\{0.159,0.264\}\rangle$$

$$\dot{\mathfrak{x}}_{\sigma_2} = \dot{\mathfrak{x}}_1 = \langle\{0.468,0.602\},\{0.098\},\{0.303\}\rangle$$

$$\dot{\mathfrak{x}}_{\sigma_3} = \dot{\mathfrak{x}}_3 = \langle\{0.733\},\{0.309,0.403,0.499\},\{0.523\}\rangle$$

Using the SVNHFDHG operator, we calculate

$$SVNHFDHG(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_n) = \otimes_{i=1}^{n}(\tilde{\mathfrak{x}}_{\sigma_i}^{w_i}) = \bigcup_{\substack{\dot{\phi}_{\sigma_1}\in\dot{\mathbb{P}}_{\sigma_1},\dot{\phi}_{\sigma_2}\in\dot{\mathbb{P}}_{\sigma_2},\cdots,\dot{\phi}_{\sigma_n}\in\dot{\mathbb{P}}_{\sigma_n},\\ \dot{\varphi}_{\sigma_1}\in\dot{\mathbb{I}}_{\sigma_1},\dot{\varphi}_{\sigma_2}\in\dot{\mathbb{I}}_{\sigma_2},\cdots,\dot{\varphi}_{\sigma_n}\in\dot{\mathbb{I}}_{\sigma_n},\\ \dot{\psi}_{\sigma_1}\in\dot{\mathbb{N}}_{\sigma_1},\dot{\psi}_{\sigma_2}\in\dot{\mathbb{N}}_{\sigma_2},\cdots,\dot{\psi}_{\sigma_n}\in\dot{\mathbb{N}}_{\sigma_n}}}$$

$$\left\{\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{1-\tilde{\phi}_{\sigma_i}}{\tilde{\phi}_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}}, 1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\tilde{\varphi}_{\sigma_i}}{1-\tilde{\varphi}_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}}, 1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\tilde{\psi}_{\sigma_i}}{1-\tilde{\psi}_{\sigma_i}}\right)^{\lambda}\right\}^{1/\lambda}}\right\}$$

where $\tilde{\mathfrak{x}}_{\sigma_i}$ is the largest number in $\tilde{\mathfrak{x}}_i = \mathfrak{x}_i^{n\mathfrak{w}_i} (i = 1,2,\cdots,n)$.

$$= \langle\{0.500,0.619,0.675,0.640,0.656,0.697\},\{0.193,0.234,0.248,0.276,0.317,0.334\},\{0.378,0.382\}\rangle$$

5. An Optimization Method with Proposed Operators

5.1. Numerical Example

Wireless Charging System (WCS) with mobile robot for Electrical Vehicles: In a scenario in which enabling charging stations for an electric vehicle is a critical aspect of expediting the transformation to a higher sustainable energy system, other solutions may be deployed in addition to the charging stations. Inductive charging is one way to overcome the challenge. Everything is extensible from a technical standpoint; nevertheless, as the power transfer rates increase, the complexity and size of the electronics that regulate power must also increase. Most significantly, when the power increases, a number of new issues, such as power dissipation and thermal management, need to be taken into consideration. The higher the power utilization and the level of inefficiency, the greater the amount of heat that is lost and the greater the amount of work that must be completed to control it (https://www.powerelectronicsnews.com/wireless-charging-technology-for-evs/ (accessed on 1 October 2022). WCS is an approach that permits the transmission of electrical power without the requirements of the use of infrastructure that is referred to as wireless charging. This is made possible by WCS. This strategy is designed to be used for the aim of offering sites that are difficult to access. An electromagnetic field, which acts as the basis of the system, is used to enable the transfer of energy. The transmission of energy from a sender to a receiver is made possible by the utilization of an electromagnetic field, which serves as the fundamental component of the system. This field also serves as a foundation of the framework.

In the present application, we utilize an intelligent decision support system (IDSS) to find a best wireless charging system and optimal charging station.

Prakash et al. [47] came up with the innovative idea of an LDF-graph and examined the elements of extending the lifespan of a wireless charging sensor network by employing a mobile robot. In addition to that, they were composed of three major aspects, as demonstrated in Figure 1.

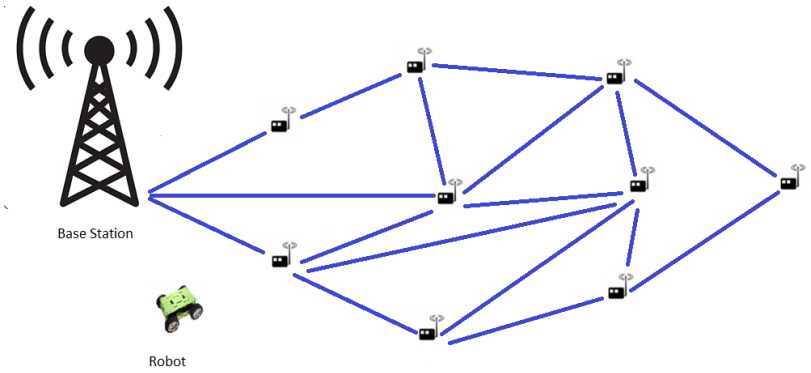

Figure 1. Sensor network for mobile robot charger.

Let us assume that $\mathcal{A}^{\gamma} = \{\mathcal{A}_1^{\gamma}\mathcal{A}_2^{\gamma}\mathcal{A}_3^{\gamma}\mathcal{A}_4^{\gamma}\}$ is the set of four feasible mobile robot chargers, and we determine the ranking of these alternatives as well as the optimal mobile charger. The decision makers select three different criterion $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ that are demonstrated in Table 7.

Table 7. Criterion for mobile robot charger.

Criterion	Significance of Criterion
$\mathcal{C}_1$	Type of mobile charger (MC).
$\mathcal{C}_2$	Each individual node that makes up a wireless sensor system is provided with its own wireless power receiver.
$\mathcal{C}_3$	The energy generation station is located in the base station. It is in charge of determining the amount of energy that is required for each node and supplying that information to the robot along with the nodes' respective positions.

With the WV $w = (0.40, 0.25, 0.35)^T$ given by the decision maker.

Step 1. The decision matrix is expressed in Table 8 with the SVN hesitant fuzzy information.

Step 2. Compute the collective SVNHFN $\mathfrak{x}_i (i = 1, 2, 3, 4)$ for the alternatives $\mathcal{A}_i^{\gamma} (i = 1, 2, 3, 4)$ by utilizing SVNHFDWA operator: $\mathfrak{x}_i = \text{SVNHFDWA}(\mathfrak{x}_{i1}, \mathfrak{x}_{i2}, \cdots, \mathfrak{x}_{im}) = \oplus_{j=1}^{m} (w_j \mathfrak{x}_{ij}) = \bigcup_{\substack{\phi_{i1} \in \mathbb{P}_{i1}, \phi_{i2} \in \mathbb{P}_{i2}, \cdots, \phi_{im} \in \mathbb{P}_{im}, \\ \varphi_{i1} \in \mathbb{I}_{i1}, \varphi_{i2} \in \mathbb{I}_{i2}, \cdots, \varphi_{im} \in \mathbb{I}_{im}, \\ \psi_{i1} \in \mathbb{N}_{i1}, \psi_{i2} \in \mathbb{N}_{i2}, \cdots, \psi_{im} \in \mathbb{N}_{im}}}$

$$\left\{1 - \frac{1}{1 + \left\{\sum_{j=1}^{m} w_j \left(\frac{\phi_{ij}}{1-\phi_{ij}}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1 + \left\{\sum_{j=1}^{m} w_j \left(\frac{1-\varphi_{ij}}{\varphi_{ij}}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1 + \left\{\sum_{j=1}^{m} w_j \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\lambda}\right\}^{1/\lambda}}\right\} \tag{27}$$

$$\mathfrak{x}_1 = \langle\{0.727, 0.775, 0.827, 0.840, 0.755, 0.834, 0.790, 0.845\}, \{0.252, 0.285, 0.299, 0.367\}, \{0.184, 0.405, 0.185, 0.408\}\rangle$$

$$\mathfrak{x}_2 = \langle\{0.657, 0.679, 0.731, 0.670, 0.690, 0.737\}, \{0.151, 0.153, 0.154, 0.157, 0.160, 0.162\}, \{0.247, 0.334\}\rangle$$

$$\mathfrak{x}_3 = \langle\{0.780, 0.796\}, \{0.381\}, \{0.201, 0.238, 0.293, 0.220, 0.276, 0.389\}\rangle$$

$$\mathfrak{x}_4 = \langle\{0.969, 0.969\}, \{0.153, 0.144, 0.133, 0.144, 0.207, 0.208, 0.265, 0.268, 0.224, 0.309, 0.226, 0.313\}, \{0.268\}\rangle$$

Step 3. Calculate the score function of $\mathfrak{x}_i (i = 1, 2, 3, 4)$ by using Equation (1); then, we obtain

$$S(\mathfrak{x}_1) = 0.73$$

$$S(\mathfrak{x}_2) = 0.75$$
$$S(\mathfrak{x}_3) = 0.71$$
$$S(\mathfrak{x}_4) = 0.89$$

Step 4. Rank the alternatives according to score function,

$$\mathfrak{x}_4 \succ \mathfrak{x}_2 \succ \mathfrak{x}_1 \succ \mathfrak{x}_3$$

Step 5. Ranking of alternatives shows that $\mathfrak{x}_4$ is the best alternative among four alternatives.

Table 8. Tabular representation of SVNHFNs.

Alternatives	$\mathcal{C}_1$
$\mathcal{A}_1^{\gamma}$	$\langle\{0.678, 0.801\}, \{0.222, 0.333\}, \{0.444, 0.550\}\rangle$
$\mathcal{A}_2^{\gamma}$	$\langle\{0.432, 0.567\}, \{0.111\}, \{0.334\}\rangle$
$\mathcal{A}_3^{\gamma}$	$\langle\{0.676, 0.768\}, \{0.382\}, \{0.449\}\rangle$
$\mathcal{A}_4^{\gamma}$	$\langle\{0.980\}, \{0.100, 0.230, 0.320\}, \{0.200\}\rangle$
Alternatives	$\mathcal{C}_2$
$\mathcal{A}_1^{\gamma}$	$\langle\{0.810, 0.900\}, \{0.215, 0.356\}, \{0.400\}\rangle$
$\mathcal{A}_2^{\gamma}$	$\langle\{0.519, 0.678, 0.800\}, \{0.200, 0.350\}, \{0.156, 0.259\}\rangle$
$\mathcal{A}_3^{\gamma}$	$\langle\{0.856\}, \{0.358\}, \{0.200, 0.380\}\rangle$
$\mathcal{A}_4^{\gamma}$	$\langle\{0.900\}, \{0.450, 0.550\}, \{0.660\}\rangle$
Alternatives	$\mathcal{C}_3$
$\mathcal{A}_1^{\gamma}$	$\langle\{0.600, 0.750\}, \{0.444\}, \{0.121, 0.344\}\rangle$
$\mathcal{A}_2^{\gamma}$	$\langle\{0.750\}, \{0.290, 0.381, 0.476\}, \{0.500\}\rangle$
$\mathcal{A}_3^{\gamma}$	$\langle\{0.706\}, \{0.400\}, \{0.150, 0.200, 0.350\}\rangle$
$\mathcal{A}_4^{\gamma}$	$\langle\{0.789, 0.890\}, \{0.159, 0.260\}, \{0.370\}\rangle$

If we solve the problem of making decisions based on numerous attributes by employing the SVNHFDWG operator, the process of creating decisions may be outlined in the following way:

Step 2. Compute the collective SVNHFN $\mathfrak{x}_i (i = 1, 2, 3, 4)$ for the alternatives $\mathcal{A}_i^{\gamma} (i = 1, 2, 3, 4)$ by utilizing the SVNHFDWG operator:

$$\mathfrak{x}_i = \text{SVNHFDWG}(\mathfrak{x}_{i1}, \mathfrak{x}_{i2}, \cdots, \mathfrak{x}_{im}) = \otimes_{j=1}^{m} \mathfrak{x}_{ij}^{w_j} = \bigcup_{\substack{\phi_{i1} \in \mathbb{P}_{i1}, \phi_{i2} \in \mathbb{P}_{i2}, \cdots, \phi_{im} \in \mathbb{P}_{im}, \\ \varphi_{i1} \in \mathbb{I}_{i1}, \varphi_{i2} \in \mathbb{I}_{i2}, \cdots, \varphi_{im} \in \mathbb{I}_{im}, \\ \psi_{i1} \in \mathbb{N}_{i1}, \psi_{i2} \in \mathbb{N}_{i2}, \cdots, \psi_{im} \in \mathbb{N}_{im}}}$$

$$\left\{1 - \frac{1}{1 + \left\{\sum_{j=1}^{m} w_j \left(\frac{1-\phi_{ij}}{\phi_{ij}}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1 + \left\{\sum_{j=1}^{m} w_j \left(\frac{\varphi_{ij}}{1-\varphi_{ij}}\right)^{\lambda}\right\}^{1/\lambda}}, \frac{1}{1 + \left\{\sum_{j=1}^{m} w_j \left(\frac{\psi_{ij}}{1-\psi_{ij}}\right)^{\lambda}\right\}^{1/\lambda}}\right\} \quad (28)$$

$$\mathfrak{x}_1 = \langle\{0.662, 0.694, 0.667, 0.700, 0.726, 0.733, 0.782, 0.795\}, \{0.344, 0.366, 0.369, 0.387\}, \{0.397, 0.405, 0.458, 0.473\}\rangle$$

$$\mathfrak{x}_2 = \langle\{0.507, 0.530, 0.537, 0.589, 0.635, 0.651\}, \{0.221, 0.282, 0.358, 0.271, 0.315, 0.377\}, \{0.404, 0.410\}\rangle$$

$$\mathfrak{x}_3 = \langle\{0.715, 0.756\}, \{0.383\}, \{0.351, 0.355, 0.382, 0.378, 0.382, 0.404\}\rangle$$

$$\mathfrak{x}_4 = \langle\{0.856, 0.915\}, \{0.301, 0.317, 0.385, 0.394, 0.317, 0.394, 0.332, 0.402, 0.341, 0.354, 0.408, 0.415\}, \{0.511\}\rangle$$

Step 3. Calculate the score function of $\mathfrak{x}_i (i = 1, 2, 3, 4)$ by using Equation (1); then, we obtain

$$S(\mathfrak{x}_1) = 0.64$$

$$S(\mathfrak{x}_2) = 0.62$$
$$S(\mathfrak{x}_3) = 0.66$$
$$S(\mathfrak{x}_4) = 0.67$$

Step 4. The score function provides real numbers to the alternatives, and these alternatives gained the ranking as follows. The score function assigns real numbers to every alternative, and the order in which these alternatives are arranged follow a usual order from higher values to lower values as follows:

$$\mathfrak{x}_4 \succ \mathfrak{x}_3 \succ \mathfrak{x}_1 \succ \mathfrak{x}_2$$

Step 5. The ranking of alternatives clearly describes that $\mathfrak{x}_4$ is the top alternative among four alternatives.

5.2. Comparative Analysis

The symmetry of top alternative can be seen by using Algorithm 1 for SVNHFWA, SVNHFWG, SVNHFDWA and SVNHFDWG operators. On the other hand, the order in which the options are ranked is not entirely consistent. This demonstrates that the ranking of alternatives is determined to seek the top alternative and positions of remaining alternatives. The comparative analysis of this ranking of alternatives is expressed in Table 9.

Algorithm 1: Algorithm for SVNHF information using Dombi aggregation operators

Consider a set of alternatives $\mathcal{A}^\gamma = \{\mathcal{A}_1^\gamma \mathcal{A}_2^\gamma, \cdots, \mathcal{A}_n^\gamma\}$ and a set of criterion $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$. The decision maker $\mathcal{D}$ gives his/her own decision matrix $\mathfrak{D} = (\mathfrak{x}_{ij})_{n\times m}$ in the form of SVNHFNs, $\mathfrak{x}_{ij}$ is given for alternatives $\mathcal{A}_i^\gamma \in \mathcal{A}^\gamma$ with respect to criterion $\mathcal{C}_j \in \mathcal{C}$.

Step 1. Consider a decision matrix $\mathfrak{D} = (\mathfrak{x}_{ij})_{n\times m}$ in the form of SVNHFNs.

Step 2. Compute the collective SVNHFN $\mathfrak{x}_i (i = 1, 2, \cdots, n)$ for the alternatives $\mathcal{A}_i^\gamma (i = 1, 2, \cdots, n)$ by utilizing the SVNHFDWA operator:

$$\mathfrak{x}_i = \text{SVNHFDWA}(\mathfrak{x}_{i1}, \mathfrak{x}_{i2}, \cdots, \mathfrak{x}_{im}) = \oplus_{j=1}^m (w_j \mathfrak{x}_{ij}) = \bigcup_{\substack{\phi_{i1} \in \mathbb{P}_{i1}, \phi_{i2} \in \mathbb{P}_{i2}, \cdots, \phi_{im} \in \mathbb{P}_{im}, \\ \varphi_{i1} \in \mathbb{I}_{i1}, \varphi_{i2} \in \mathbb{I}_{i2}, \cdots, \varphi_{im} \in \mathbb{I}_{im}, \\ \psi_{i1} \in \mathbb{N}_{i1}, \psi_{i2} \in \mathbb{N}_{i2}, \cdots, \psi_{im} \in \mathbb{N}_{im}}}$$

$$\left\{ 1 - \frac{1}{1 + \left\{ \sum_{j=1}^m w_j \left(\frac{\phi_{ij}}{1-\phi_{ij}} \right)^\lambda \right\}^{1/\lambda}}, \frac{1}{1 + \left\{ \sum_{j=1}^m w_j \left(\frac{1-\varphi_{ij}}{\varphi_{ij}} \right)^\lambda \right\}^{1/\lambda}}, \frac{1}{1 + \left\{ \sum_{j=1}^m w_j \left(\frac{1-\psi_{ij}}{\psi_{ij}} \right)^\lambda \right\}^{1/\lambda}} \right\} \tag{29}$$

or by utilizing the SVNHFDWG operator:

$$\mathfrak{x}_i = \text{SVNHFDWG}(\mathfrak{x}_{i1}, \mathfrak{x}_{i2}, \cdots, \mathfrak{x}_{im}) = \otimes_{j=1}^m \mathfrak{x}_{ij}^{w_j} = \bigcup_{\substack{\phi_{i1} \in \mathbb{P}_{i1}, \phi_{i2} \in \mathbb{P}_{i2}, \cdots, \phi_{im} \in \mathbb{P}_{im}, \\ \varphi_{i1} \in \mathbb{I}_{i1}, \varphi_{i2} \in \mathbb{I}_{i2}, \cdots, \varphi_{im} \in \mathbb{I}_{im}, \\ \psi_{i1} \in \mathbb{N}_{i1}, \psi_{i2} \in \mathbb{N}_{i2}, \cdots, \psi_{im} \in \mathbb{N}_{im}}}$$

$$\left\{ 1 - \frac{1}{1 + \left\{ \sum_{j=1}^m w_j \left(\frac{1-\phi_{ij}}{\phi_{ij}} \right)^\lambda \right\}^{1/\lambda}}, \frac{1}{1 + \left\{ \sum_{j=1}^m w_j \left(\frac{\varphi_{ij}}{1-\varphi_{ij}} \right)^\lambda \right\}^{1/\lambda}}, \frac{1}{1 + \left\{ \sum_{j=1}^m w_j \left(\frac{\psi_{ij}}{1-\psi_{ij}} \right)^\lambda \right\}^{1/\lambda}} \right\} \tag{30}$$

Step 3. Compute the score function of the collective SVNHFNs $\mathfrak{x}_i (i = 1, 2, \cdots, n)$ by utilizing Equation (1).

Step 4. Using a score function, rank the alternatives.

Step 5. Choose the top alternative.

Table 9. Comparison table.

Operator	Ranking	Optimal Alternative
SVNHFWA	$\mathfrak{x}_4 \succ \mathfrak{x}_2 \succ \mathfrak{x}_1 \succ \mathfrak{x}_3$	$\mathfrak{x}_4$
SVNHFWG	$\mathfrak{x}_4 \succ \mathfrak{x}_3 \succ \mathfrak{x}_1 \succ \mathfrak{x}_2$	$\mathfrak{x}_4$
SVNHFDWA	$\mathfrak{x}_4 \succ \mathfrak{x}_2 \succ \mathfrak{x}_1 \succ \mathfrak{x}_3$	$\mathfrak{x}_4$
SVNHFDWG	$\mathfrak{x}_4 \succ \mathfrak{x}_3 \succ \mathfrak{x}_1 \succ \mathfrak{x}_2$	$\mathfrak{x}_4$
SVNHFCOA [43]	$\mathfrak{x}_4 \succ \mathfrak{x}_3 \succ \mathfrak{x}_2 \succ \mathfrak{x}_1$	$\mathfrak{x}_4$
GSVNHFPWA [44]	$\mathfrak{x}_4 \succ \mathfrak{x}_3 \succ \mathfrak{x}_1 \succ \mathfrak{x}_2$	$\mathfrak{x}_4$

The comparison bar chart of ranking with SVNHFWA and SVNHFWG is shown in Figure 2.

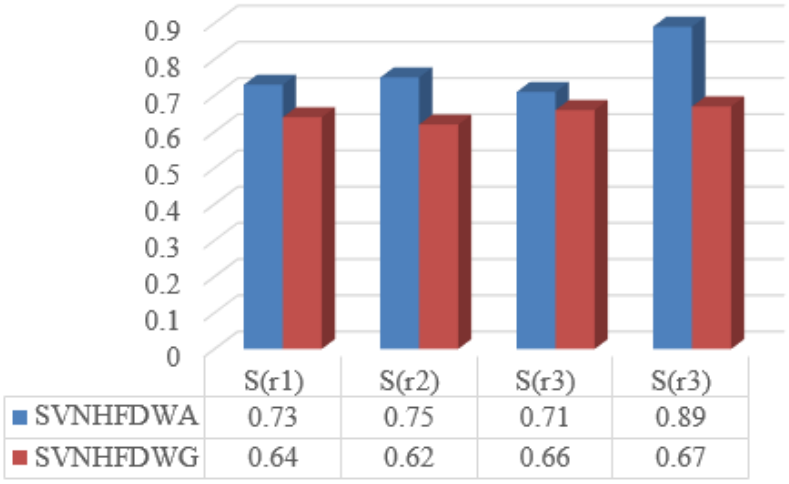

Figure 2. Comparison bar chart of ranking with SVNHFWA and SVNHFWG.

6. Conclusions

SVNHFS is a robust fusion of an SVNS and HFS created for scenarios where each object has some possible values determined by truth-MD, indeterminacy, and falsity-MD. In this article, we propose an SVNHFDWA operator, SVNHFDWG operator, SVNHFDOWA operator, SVNHFDOWG operator, SVNHFDHA operator and SVNHFDHG operator. In addition, we proposed two innovative MADM techniques based on the SVNHFDWA and SVNHFDWG operators. The benefits of these methods are outlined in more detail below.

1. First, the SVNHFDWA and SVNHFDWG operators have significant properties such as idempotency, commutativity as well as boundedness and monotonity.
2. Second, the SVNHFDWA and SVNHFDWG operators can be converted to the previous AOs for SVNHFSs, which identify the versatility of proposed AOs.
3. Third, when compared to other existing approaches for MADM problems in an SVNHF environment, the results achieved by the SVNHFDWA and SVNHFDWG operators are reliable and accurate, which demonstrates their applicability in practical settings.
4. The techniques that are proposed for MADM in this paper are able to further acknowledge more association between attributes and alternatives, which demonstrates that they have a greater accuracy and a larger reference value than the techniques that are currently in use and that are unable to take into account the inter-relationships of attributes in practical applications. This means that the techniques that are proposed for MADM in this paper can further recognize more association between attributes.
5. A practical application of the proposed aggregation operators is also presented to examine symmetrical analysis in the selection of a feasible mobile robot (mobile charger) for vehicles.

6. It would be interesting to use the proposed AOs in future studies to deal with personalized individual semantics-based consistency control consensus problems in IDSS, consensus reaching with non-cooperative behavior management decision-making problems, and two-sided matching decision making with multi-granular and incomplete criteria weight information. In the context of this discussion on the constraints imposed by proposed AOs, there is no interaction between the degrees of membership, abstention, and non-membership. New hybrid structure of interactive and prioritized AOs may be seen being put into place on this side of the planned AOs.

References

1. Zadeh, L.A. Fuzzy sets. *Inform. Control* **1965**, *8*, 338–356. [CrossRef]
2. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Set Syst.* **1986**, *20*, 87–96. [CrossRef]
3. Xu, Z. Intuitionistic fuzzy aggregation operators. *IEEE Trans. Fuzzy Syst.* **2007**, *15*, 1179–1187.
4. Atanassov, K.; Gargov, G. Interval-valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349 [CrossRef]
5. Smarandache, F. *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic*; American Research Press: Rehoboth, DE, USA, 1999.
6. Wang, H.; Smarandache, F.; Zhang, Y.; Sunderraman, R. *Single-Valued Neutrosophic Sets*; Technical Sciences and Applied Mathematics; Infinite Study: Hurstville, Australia, 2012.
7. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*; Hexis: Phoenix, AZ, USA, 2005; Volume 7.
8. Torra, V.; Narukawa, Y. On hesitant fuzzy sets and decision. In Proceedings of the IEEE International Conference on Fuzzy Systems, Tianjin, China, 14–16 August 2009; pp. 1378–1382.
9. Torra, V. Hesitant fuzzy sets. *Int. J. Intell. Syst.* **2010**, *25*, 529–539. [CrossRef]
10. Ye, J. Multiple attribute decision making method under a single-valued neutrosophic hesitant fuzzy environment. *J. Intell. Syst.* **2015**, *24*, 23–36. [CrossRef]
11. Liu, C.F.; Luo, Y.S. New aggregation operators of single-valued neutrosophic hesitant fuzzy set and their application in multi-attribute decision making. *Pattern Anal. Appl.* **2019**, *22*, 417–427. [CrossRef]
12. Dombi, J. A general class of fuzzy operators, the demorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. *Fuzzy Sets Syst.* **1982**, *8*, 149–163. [CrossRef]
13. Seikh, M.R.; Mandal, U. Intuitionistic fuzzy Dombi aggregation operators and their application to multiple attribute decision-making. *Granul. Comput.* **2021**, *6*, 473–488. [CrossRef]
14. Liu, P.D.; Liu, J.L.; Chen, S.M. Some intuitionistic fuzzy Dombi Bonferroni mean operators and their application to multi-attribute group decision making. *J. Oper. Res. Soc.* **2017**, *69*, 1–26. [CrossRef]
15. Wang, W.; Liu, X. Intuitionistic Fuzzy Information Aggregation Using Einstein Operations. *IEEE Trans. Fuzzy Syst.* **2012**, *20*, 923–938. [CrossRef]
16. Yua, X.; Xu, Z. Prioritized intuitionistic fuzzy aggregation operators. *Inf. Fusion* **2013**, *14*, 108–116. [CrossRef]
17. Alkenani, A.; Anjum, R.; Islam, S. Intuitionistic Fuzzy Prioritized Aggregation Operators Based on Priority Degrees with Application to Multicriteria Decision-Making. *J. Funct. Spaces* **2022**, *2022*, 4751835.
18. Tan, C.; Chen, X. Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making. *Expert Syst. Appl.* **2010**, *37*, 149–157. [CrossRef]
19. Xu, Z. Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators. *Knowl.-Based Syst.* **2011**, *24*, 749–760. [CrossRef]
20. Yang, W.; Chen, Z. The quasi-arithmetic intuitionistic fuzzy OWA operators. *Knowl.-Based Syst.* **2012**, *27*, 219–233. [CrossRef]
21. Wu, L.; Wei, G.; Wu, J.; Wei, C. Some Interval-Valued Intuitionistic Fuzzy Dombi Heronian Mean Operators and their Application for Evaluating the Ecological Value of Forest Ecological Tourism Demonstration Areas. *Int. J. Environ. Res. Public Health* **2020**, *17*, 829. [CrossRef]

22. Wu, L.; Wei, G.; Gao, H.; Wei, Y. Some Interval-Valued Intuitionistic Fuzzy Dombi Hamy Mean Operators and Their Application for Evaluating the Elderly Tourism Service Quality in Tourism Destination. *Mathematics* **2018**, *6*, 294. [CrossRef]
23. Wang, Q.; Sun, H. Interval-Valued Intuitionistic Fuzzy Einstein Geometric Choquet Integral Operator and Its Application to Multiattribute Group Decision-Making. *Math. Probl. Eng.* **2018**, *2018*, 9364987. [CrossRef]
24. Wang, W.; Liu, X. Interval-valued intuitionistic fuzzy hybrid weighted averaging operator based on Einstein operation and its application to decision making. *J. Intell. Fuzzy Syst.* **2013**, *25*, 279–290. [CrossRef]
25. Liu, P. Some Hamacher Aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. *IEEE Trans. Fuzzy Syst.* **2013**, *22*, 83–97. [CrossRef]
26. Akram, M.; Dudek, W.A.; Dar, J.M. Pythagorean Dombi fuzzy aggregation operators with application in multi criteria decision-making. *Int. J. Intell. Syst.* **2019**, *34*, 3000–3019. [CrossRef]
27. Jana, C.; Senapati, T.; Pal, M. Pythagorean fuzzy Dombi aggregation operators and its applications in multiple attribute decision-making. *Int. J. Intell. Syst.* **2019**, *34*, 1–21. [CrossRef]
28. Rahman, K.; Abdullah, S.; Ali, A.; Amin, F. Pythagorean Fuzzy Einstein Hybrid Averaging Aggregation Operator and its Application to Multiple-Attribute Group Decision Making. *J. Intell. Syst.* **2018**, *40*, 180–187. [CrossRef]
29. Khan, M.S.A.; Abdullah, S.; Ali, A.; Amin, F. Pythagorean fuzzy prioritized aggregation operators and their application to multi-attribute group decision making. *Granul. Comput.* **2019**, *4*, 249–263. [CrossRef]
30. Farid, H.M.A.; Riaz, M. Pythagorean fuzzy prioritized aggregation operators with priority degrees for multi-criteria decision-making. *Int. J. Intell. Comput. Cybern.* **2022**, *15*, 510–539. [CrossRef]
31. Wang, L.; Li, N. Pythagorean fuzzy interaction power Bonferroni mean aggregation operators in multiple attribute decision making. *Int. J. Intell. Syst.* **2020**, *35*, 150–183. [CrossRef]
32. Liang, D.; Zhang, Y.; Xu, Z.; Darko, A.P. Pythagorean fuzzy Bonferroni mean aggregation operator and its accelerative calculating algorithm with the multithreading. *Int. J. Intell. Syst.* **2018**, *33*, 615–633. [CrossRef]
33. Chen, J.; Ye, J. Some Single-Valued Neutrosophic Dombi Weighted Aggregation Operators for Multiple Attribute Decision-Making. *Symmetry* **2017**, *9*, 82. [CrossRef]
34. Jana, C.; Pal, M. Multi-criteria decision making process based on some single-valued neutrosophic Dombi power aggregation operators. *Soft Comput.* **2021**, *25*, 5055–5072. [CrossRef]
35. Wei, G.; Wei, Y. Some single-valued neutrosophic dombi prioritized weighted aggregation operators in multiple attribute decision making. *J. Intell. Fuzzy* **2018**, *35*, 2001–2013. [CrossRef]
36. Liu, P.; Khan, Q.; Mahmood, T. Multiple-attribute decision making based on single-valued neutrosophic Schweizer-Sklar prioritized aggregation operator. *Cogn. Syst. Res.* **2019**, *57*, 175–196. [CrossRef]
37. Farid, H.M.A.; Garg, H.; Riaz, M.; Garcia, G.S. Multi-criteria group decision-making algorithm based on single-valued neutrosophic Einstein prioritized aggregation operators and its applications. *Manag. Decis.* **2022**. [CrossRef]
38. He, X. Typhoon disaster assessment based on Dombi hesitant fuzzy information aggregation operators. *Nat. Hazards J. Int. Soc. Prev. Mitig. Nat. Hazards* **2018**, *90*, 1153–1175. [CrossRef]
39. Liu, P.; Saha, A.; Dutta, D.; Kar, S. Multi-Attribute Decision-Making Using Hesitant Fuzzy Dombi–Archimedean Weighted Aggregation Operators. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 386–411. [CrossRef]
40. Saha, A.; Dutta, D.; Kar, S. Some new hybrid hesitant fuzzy weighted aggregation operators based on Archimedean and Dombi operations for multi-attribute decision making. *Neural Comput. Appl.* **2021**, *33*, 8753–8776. [CrossRef]
41. Yu, Q.; Hou, F.; Zhai, Y.; Du, Y. Some Hesitant Fuzzy Einstein Aggregation Operators and Their Application to Multiple Attribute Group Decision Making. *Int. J. Intell. Syst.* **2015**, *31*, 722–746. [CrossRef]
42. Zhou, L.; Zhao, X.; Wei, G. Hesitant fuzzy Hamacher Aggregation operators and their application to multiple attribute decision making. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2689–2699. [CrossRef]
43. Li, X.; Zhang, X. Single-Valued Neutrosophic Hesitant Fuzzy Choquet Aggregation Operators for Multi-Attribute Decision Making. *Symmetry* **2018**, *10*, 50. [CrossRef]
44. Wang, R.; Li, Y. Generalized Single-Valued Neutrosophic Hesitant Fuzzy Prioritized Aggregation Operators and Their Applications to Multiple Criteria Decision-Making. *Information* **2018**, *9*, 10. [CrossRef]
45. Wang, L.; Bao, Y.L. Multiple-Attribute Decision-Making Method Based on Normalized Geometric Aggregation Operators of Single-Valued Neutrosophic Hesitant Fuzzy Information. *Complexity* **2021**, *2021*, 5580761. [CrossRef]
46. Hanif, M.Z.; Yaqoob, N.; Riaz, M.; Aslam, M. Linear Diophantine fuzzy graphs with new decision-making approach. *AIMS Math.* **2022**, *7*, 14532–14556. [CrossRef]
47. Prakash, K.; Parimala, M.; Garg, H.; Riaz, M. Lifetime prolongation of a wireless charging sensor network using a mobile robot via linear Diophantine fuzzy graph environment. *Complex Intell. Syst.* **2022**, *8*, 2419–2434. [CrossRef]
48. Mohamed, N.; Aymen, F.; Mouna, B.H.; Alassaad, S. Review on autonomous charger for EV and HEV. In Proceedings of the 2017 IEEE International Conference on Green Energy Conversion Systems (GECS), Hammamet, Tunisia, 23–25 March 2017; pp. 1–6.
49. Tang, Y.M.; Zhang, L.; Bao, G.Q.; Ren, F.J.; Pedrycz, W. Symmetric implicational algorithm derived from intuitionistic fuzzy entropy. *Iran. J. Fuzzy Syst.* **2022**, *19*, 27–44.
50. Wang, Z.J. A Representable Uninorm-Based Intuitionistic Fuzzy Analytic Hierarchy Process. *IEEE Trans. Fuzzy Syst.* **2020**, *28*, 2555–2569. [CrossRef]
51. Lodwick, W.A.; Kacprzyk, J. (Eds.) *Fuzzy Optimization: Recent Advances and Applications*; Springer: Berlin/Heidelberg, Germany, 2010; Volume 254.

52. Riaz, M.; Farid, H.M.A. Picture fuzzy aggregation approach with application to third-party logistic provider selection process. *Rep. Mech. Eng.* **2022**, *3*, 318–327. [CrossRef]
53. Karaaslan, F.; Çağman, N. Parameter trees based on soft set theory and their similarity measures. *Soft Comput.* **2022**, *26*, 4629–4639. [CrossRef]
54. Karamasa, Ç.; Karabasevic, D.; Stanujkic, D.; Kookhdan, A.; Mishra, A.; Ertürk, M. An extended single-valued neutrosophic AHP and MULTIMOORA method to evaluate the optimal training aircraft for flight training organizations. *Facta Univ. Ser. Mech. Eng.* **2021**, *19*, 555–578. [CrossRef]
55. Ali, Z.; Mahmood, T.; Ullah, K.; Khan, Q. Einstein Geometric Aggregation Operators using a Novel Complex Interval-valued Pythagorean Fuzzy Setting with Application in Green Supplier Chain Management. *Rep. Mech. Eng.* **2021**, *2*, 105–134. [CrossRef]
56. Bozanic, D.; Milic, A.; Tešic, D.; Salabun, W.; Pamucar, D. D numbers—FUCOM—Fuzzy RAFSI model for selecting the group of construction machines for enabling mobility. *Facta Univ. Ser. Mech. Eng.* **2021**, *19*, 447–471.
57. Ashraf, A.; Ullah, K.; Hussain, A.; Bari, M. Interval-Valued Picture Fuzzy Maclaurin Symmetric Mean Operator with application in Multiple Attribute Decision-Making. *Rep. Mech. Eng.* **2022**, *3*, 301–317. [CrossRef]
58. Sahu, R.; Dash, S.R.; Das, S. Career selection of students using hybridized distance measure based on picture fuzzy set and rough set theory. *Decis. Mak. Appl. Manag. Eng.* **2021**, *4*, 104–126. [CrossRef]
59. Shi, L.; Ye, J. Dombi aggregation operators of neutrosophic cubic sets for multiple attribute decision-making. *Algorithms* **2018**, *11*, 29. [CrossRef]
60. Biswas, P.; Pramanik, S.; Giri, B.C. GRA Method of Multiple Attribute Decision Making with Single Valued Neutrosophic Hesitant Fuzzy Set Information. *New Trends Neutrosophic Theory Appl.* **2016**, 55–63.

Antipodal Turiyam Neutrosophic Graphs

Takaaki Fujita

Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan

Florentin Smarandache

Department of Mathematics & Sciences, University of New Mexico, Gallup, NM 87301, USA

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Abstract

Graph theory, a mathematical field, investigates the relationships between entities through vertices and edges [29]. Within this discipline, Uncertain Graph Theory emerges to model uncertainties in realworld networks. This paper presents the concept of the Antipodal Turiyam Neutrosophic Graph. In an Antipodal Graph, two nodes are connected if their shortest path distance equals the graph's diameter, emphasizing connections between the farthest nodes. Turiyam Neutrosophic Graphs extend traditional graphs by introducing four membership values-truth, indeterminacy, falsity, and liberal state-assigned to each vertex and edge, enabling a more nuanced representation of complex relationships.

Keywords: Neutrosophic Graph; Fuzzy Graph; Plithogenic Graph; Turiyam Neutrosophic Graph; Antipodal Graph.

1 | Introduction

1.1 | Uncertain Graph Theory

Graph theory is a foundational branch of mathematics that models networks using vertices (nodes) and edges (connections) to represent relationships between entities [19, 29, 85, 91, 123]. In graph theory, parameters such as shortest path distance [7, 35, 93, 98] and diameter [8, 18, 34, 67] are often studied to analyze the mathematical structure of a given graph.

This paper investigates several uncertain graph models, including Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Graphs, which enhance classical graph theory by introducing different layers of uncertainty. These models offer a comprehensive framework for analyzing complex and imprecise relationships, making them applicable to various real-world contexts. Consequently, a variety of related graph classes and applications have emerged [38, 39, 41-43, 46, 48-52]. Foundational concepts such as Fuzzy Sets and Neutrosophic Sets have also been extensively studied and documented in the literature [10, 14, 26] Similarly, parameters like shortest path distance and diameter are actively studied within uncertain graphs as well [75, 115, 116]. For a more comprehensive overview, readers are encouraged to refer to existing survey papers [44, 46, 48].

1.2 | Contributions

This paper introduces the concept of the Antipodal Turiyam Neutrosophic Graph. Antipodal graph is a graph where two nodes are connected if their shortest path distance equals the graph's diameter, highlighting farthest node connections [9, 36, 53, 59, 63, 92]. Turiyam Neutrosophic Graphs expand the traditional graph framework by assigning four membership values-truth, indeterminacy, falsity, and liberal state-to each vertex and edge, allowing for a more comprehensive representation of complex relationships [44, 46, 54, 103]. Note that Turiyam Neutrosophic Set is actually a particular case of the quadripartitioned Neutrosophic Set, by replacing "Contradiction" with "Liberal" [104]. The corresponding graph concept known as quadripartitioned neutrosophic graphs is well-documented [69, 70].

While Antipodal Graphs have been extensively studied in contexts like Fuzzy [4, 58, 87, 97], Vague[81], and Neutrosophic Graphs [78-80], the concept of the Turiyam Neutrosophic Antipodal Graph has not been thoroughly explored. This paper aims to address this gap by defining and analyzing the properties of Antipodal Turiyam Neutrosophic Graphs.

1.3 | The Structure of the Paper

The format of this paper is described below. Section 2 provides the Preliminaries and Definitions. Section 3 introduces results of the antipodal single valued Turiyam neutrosophic graph while and future directions in Section 4.

2 | Preliminaries and Definitions

This section provides an overview of the fundamental definitions and notations used throughout the paper.

2.1 | Basic Graph Concepts

Below are some of the foundational concepts in graph theory. For more comprehensive information on graph theory and its notations, refer to [27-29, 64, 119].

Definition 1 (Graph). [29] A graph G is a mathematical structure that represents relationships between objects. It consists of a set of vertices $V(G)$ and a set of edges $E(G)$, where each edge connects a pair of vertices. Formally, a graph is represented as $G = (V, E)$, where V is the set of vertices and E is the set of edges.

Definition 2 (Degree). [29] Let $G = (V, E)$ be a graph. The degree of a vertex $v \in V$, denoted $\deg(v)$, is defined as the number of edges connected to v. For undirected graphs, the degree is given by:

$$\deg(v) = |\{e \in E \mid v \in e\}|$$

For directed graphs, the in-degree $\deg^-(v)$ refers to the number of edges directed towards v, while the out-degree $\deg^+(v)$ represents the number of edges directed away from v.

2.2 | Uncertain Graph

This paper addresses Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Turiyam, and Plithogenic concepts. Note that Turiyam Neutrosophic Set is actually a particular case of the Quadruple Neutrosophic Set, by replacing "Contradiction" with "Liberal" [104].

Definition 3 (Unified Uncertain Graphs Framework). (cf.[47]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, falsity, and other nuanced measures of uncertainty.

1. Fuzzy Graph [17, 40, 57, 60, 74, 82, 86, 99, 100, 112, 118]

- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0,1]$.
- Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0,1]$.

2. Intuitionistic Fuzzy Graph (IFG) [41,16,24,72,83,114,117,124]:
   - Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0,1]$ (degree of membership) and $v_A(v) \in [0,1]$ (degree of non-membership), such that $\mu_A(v) + v_A(v) \leq 1$.
   - Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0,1]$ and $v_B(u, v) \in [0,1]$, with $\mu_B(u, v) + v_B(u, v) \leq 1$.
3. Neutrosophic Graph [5,6,23,44,49,52,65,68,73,101,109,110] :
   - Each vertex $v \in V$ is assigned a triplet $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where $\sigma_T(v), \sigma_I(v), \sigma_F(v) \in [0,1]$ and $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
   - Each edge $e = (u, v) \in E$ is assigned a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$.
4. Turiyam Neutrosophic Graph 54 56]:
   - Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where each component is in $[0,1]$ and $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.
   - Each edge $e = (u, v) \in E$ is similarly assigned a quadruple.
5. Vague Graph $[2, 3, 20, 22, 95, 96, 102]$:
   - Each vertex $v \in V$ is assigned a pair $(\tau(v), \phi(v))$, where $\tau(v) \in [0,1]$ is the degree of truthmembership and $\phi(v) \in [0,1]$ is the degree of false-membership, with $\tau(v) + \phi(v) \leq 1$.
   - The grade of membership is characterized by the interval $[\tau(v), 1 - \phi(v)]$.
   - Each edge $e = (u, v) \in E$ is assigned a pair $(\tau(e), \phi(e))$, satisfying:

$$\tau(e) \leq \min\{\tau(u), \tau(v)\},\ \phi(e) \geq \max\{\phi(u), \phi(v)\}$$

6. Hesitant Fuzzy Graph [15, 62, 88, 20]:
   - Each vertex $v \in V$ is assigned a hesitant fuzzy set $\sigma(v)$, represented by a finite subset of $[0,1]$, denoted $\sigma(v) \subseteq [0,1]$.
   - Each edge $e = (u, v) \in E$ is assigned a hesitant fuzzy set $\mu(e) \subseteq [0,1]$.
   - Operations on hesitant fuzzy sets (e.g., intersection, union) are defined to handle the hesitation in membership degrees.
7. Single-Valued Pentapartitioned Neutrosophic Graph [25, 69, 71, 94]:
   - Each vertex $v \in V$ is assigned a quintuple $\sigma(v) = (T(v), C(v), R(v), U(v), F(v))$, where:
     - $T(v) \in [0,1]$ is the truth-membership degree.
     - $C(v) \in [0,1]$ is the contradiction-membership degree.
     - $R(v) \in [0,1]$ is the ignorance-membership degree.
     - $U(v) \in [0,1]$ is the unknown-membership degree.
     - $F(v) \in [0,1]$ is the false-membership degree.
     - $T(v) + C(v) + R(v) + U(v) + F(v) \leq 5$.

- Each edge $e = (u, v) \in E$ is assigned a quintuple $\mu(e) = (T(e), C(e), R(e), U(e), F(e))$, satisfying:

$$\begin{cases} T(e) \leq \min\{T(u), T(v)\} \\ C(e) \leq \min\{C(u), C(v)\} \\ R(e) \geq \max\{R(u), R(v)\} \\ U(e) \geq \max\{U(u), U(v)\} \\ F(e) \geq \max\{F(u), F(v)\} \end{cases}$$

Definition 4. [61, 106, 107 111 113] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A Plithogenic Graph PG is defined as:

$$PG = (PM, PN)$$

where:

1. Plithogenic Vertex Set $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - adf : $M \times Ml \to [0,1]^s$ is the Degree of Appurtenance Function (DAF) for vertices.
   - $aCf: Ml \times Ml \to [0,1]^t$ is the Degree of Contradiction Function (DCF) for vertices.
2. Plithogenic Edge Set $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - bdf : $N \times Nm \to [0,1]^s$ is the Degree of Appurtenance Function (DAF) for edges.
   - bCf: Nm $\times Nm \to [0,1]^t$ is the Degree of Contradiction Function (DCF) for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. Edge Appurtenance Constraint: For all $(x, a), (y, b) \in M \times Ml$:

$$bdf((xy), (a, b)) \leq \min\{adf(x, a), adf(y, b)\}$$

   where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.

2. Contradiction Function Constraint: For all $(a, b), (c, d) \in Nm \times Nm$:

$$bCf((a, b), (c, d)) \leq \min\{aCf(a, c), aCf(b, d)\}$$

3. Reflexivity and Symmetry of Contradiction Functions:

$$\begin{aligned} aCf(a, a) &= 0, && \forall a \in Ml \\ aCf(a, b) &= aCf(b, a), && \forall a, b \in Ml \\ bCf(a, a) &= 0, && \forall a \in Nm \\ bCf(a, b) &= bCf(b, a), && \forall a, b \in Nm \end{aligned}$$

Example 5. (cf. [44, 47]) The following examples are provided.

- When $s = t = 1, PG$ is called a Plithogenic Fuzzy Graph.
- When $s = 2, t = 1, PG$ is called a Plithogenic Intuitionistic Fuzzy Graph.
- When $s = 3, t = 1, PG$ is called a Plithogenic Neutrosophic Graph.
- When $s = 4, t = 1, PG$ is called a Plithogenic Turiyam Neutrosophic Graph.

The General Plithogenic Graph is a relax definition of the Plithogenic Graph (cf.[37,44, 84]).

Definition 6 (General Plithogenic Graph). [44] Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges.

A General Plithogenic Graph $G^{GP} = (PM, PN)$ consists of:

1. General Plithogenic Vertex Set PM:

$$PM = (M, l, Ml, adf, aCf)$$

where:

- $M \subseteq V$: Set of vertices.
- l : Attribute associated with the vertices.
- Ml : Range of possible attribute values.
- $adf: M \times Ml \to [0,1]^s$: Degree of Appurtenance Function (DAF) for vertices.
- $aCf: Ml \times Ml \to [0,1]^t$: Degree of Contradiction Function (DCF) for vertices.

2. General Plithogenic Edge Set PN:

$$PN = (N, m, Nm, bdf, bCf)$$

where:

- $N \subseteq E$: Set of edges.
- m : Attribute associated with the edges.
- Nm: Range of possible attribute values.
- bdf: $N \times Nm \to [0,1]^s$: Degree of Appurtenance Function (DAF) for edges.
- $bCf: Nm \times Nm \to [0,1]^t$: Degree of Contradiction Function (DCF) for edges.

The General Plithogenic Graph G^{GP} only needs to satisfy the following Reflexivity and Symmetry properties of the Contradiction Functions:

- Reflexivity and Symmetry of Contradiction Functions:

$$\begin{aligned} aCf(a,a) &= 0, & \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), & \forall a,b \in Ml \\ bCf(a,a) &= 0, & \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), & \forall a,b \in Nm \end{aligned}$$

2.3 | Antipodal Fuzzy Graph and Neutrosophic Graph

In Uncertain Graph Theory, Antipodal Fuzzy Graphs [58, 87, 97] and Neutrosophic Graphs [78-80] are well-known concepts. Their definitions are presented below.

Definition 7 (Antipodal Fuzzy Graph). [58] Let $G = (\sigma, \mu)$ be a fuzzy graph with the underlying set V, where:

- $\sigma: V \to [0,1]$ is a fuzzy subset of V,
- $\mu: V \times V \to [0,1]$ is a symmetric fuzzy relation on σ, satisfying $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$, where $\wedge$ denotes the minimum operation.

The Antipodal Fuzzy Graph of G, denoted as $A(G) = (\sigma_{A(G)}, \mu_{A(G)})$, is defined as follows:

1. The node set of $A(G)$ is the same as the node set of G, i.e., V.
2. The fuzzy subset $\sigma_{A(G)}$ is defined as:

$$\sigma_{A(G)}(u) = \sigma(u), \text{ for all } u \in V$$

3. The fuzzy relation $\mu_{A(G)}$ is defined based on the μ-distance between nodes:

$$\mu_{A(G)}(u,v) = \begin{cases} \mu(u,v), & \text{if } \delta(u,v) = \operatorname{diam}(G) \text{ and } u,v \text{ are neighbors in } G, \\ \sigma(u) \wedge \sigma(v), & \text{if } \delta(u,v) = \operatorname{diam}(G) \text{ and } u,v \text{ are not neighbors in } G, \\ 0, & \text{otherwise.} \end{cases}$$

Here:

- $\delta(u,v)$ denotes the μ-distance between nodes u and v in the fuzzy graph G, defined as the smallest μ-length of any $u - v$ path.
- $\operatorname{diam}(G)$ is the diameter of G, calculated as the maximum eccentricity of any node $v \in V$, where the eccentricity $e(v)$ is given by:

$$e(v) = \max_{u \in V} \delta(u,v)$$

The Antipodal Fuzzy Graph $A(G)$ is a fuzzy graph because:

1. $\sigma_{A(G)}(u) = \sigma(u)$ for all $u \in V$, ensuring that $\sigma_{A(G)}$ is a valid fuzzy subset on V.
2. By the definition of $\mu_{A(G)}$, it remains a fuzzy relation on $\sigma_{A(G)}$, satisfying:

$$\mu_{A(G)}(u,v) \leq \sigma_{A(G)}(u) \wedge \sigma_{A(G)}(v), \text{ for all } u,v \in V$$

Thus, the graph $A(G)$ is called the Antipodal Fuzzy Graph of G, as it connects pairs of nodes whose μ-distance equals the diameter of G.

Definition 8 (Antipodal Single Valued Neutrosophic Graph (ASVNG)). 79] Let $G = (A,B)$ be a single-valued neutrosophic graph (SVNG), where:

- A is the neutrosophic membership function on the vertex set V, with:

$$A(v) = (T_A(v), I_A(v), F_A(v)), \ \forall v \in V,$$

where $T_A(v), I_A(v)$, and $F_A(v)$ represent the truth, indeterminacy, and falsity memberships of vertex v, respectively.

- B is the neutrosophic relation on the edge set E, with:

$$B(e) = (T_B(e), I_B(e), F_B(e)), \ \forall e \in E,$$

where $T_B(e), I_B(e)$, and $F_B(e)$ represent the truth, indeterminacy, and falsity memberships of edge e, respectively.

The Antipodal Single Valued Neutrosophic Graph $(ASVNG)$, denoted as $A(G) = (Q,R)$, is defined as follows:

1. Node Set Q:

$$Q = A, \text{ on } V$$

2. Edge Set R : Let $\delta(p,q)$ be the neutrosophic distance between nodes p and q, and $d(G)$ be the diameter of G. For nodes $p, q \in V$:

$$\text{If } \delta(p,q) = d(G), \text{ then:}$$

a) If p and q are adjacent in G :

$$R = B, \text{ on } E.$$

b) If p and q are not adjacent in G :

$$T_R(p,q) = \min(T_A(p), T_A(q))$$
$$I_R(p,q) = \max(I_A(p), I_A(q))$$
$$F_R(p,q) = \max(F_A(p), F_A(q))$$

3 | Result: Antipodal Single Valued Turiyam Neutrosophic Graph

We consider the Antipodal Single Valued Turiyam Neutrosophic Graph. The following sections present its definition and theorems.

Definition 9 (Antipodal Single Valued Turiyam Neutrosophic Graph (ASVTG)). Let $G = (A, B)$ be a Single Valued Turiyam Neutrosophic Graph (SVTG), where:

- V is a non-empty finite set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- A is the Turiyam Neutrosophic membership function on V, defined as:

$$A(v) = (T_A(v), I_A(v), F_A(v), L_A(v)), \ \forall v \in V$$

where:

- $T_A(v)$ is the truth-membership degree of vertex v.
- $I_A(v)$ is the indeterminacy-membership degree of vertex v.
- $F_A(v)$ is the falsity-membership degree of vertex v.
- $L_A(v)$ is the latent-membership degree of vertex v.
- Each $T_A(v), I_A(v), F_A(v), L_A(v) \in [0,1]$.
- The sum satisfies $0 \leq T_A(v) + I_A(v) + F_A(v) + L_A(v) \leq 1$.

- B is the Turiyam Neutrosophic relation on E, defined as:

$$B(e) = (T_B(e), I_B(e), F_B(e), L_B(e)), \ \forall e \in E.$$

The Antipodal Single Valued Turiyam Neutrosophic Graph $A(G) = (Q, R)$ is defined as follows:

1. Node Set Q:

$$Q = A, \text{ on } V$$

2. Edge Set R:

- Let $\delta(p,q)$ be the Turiyam Neutrosophic distance between nodes p and q in G.
- Let $\text{diam}(G)$ denote the diameter of G.

For nodes $p, q \in V$:

If $\delta(p,q) = \text{diam}(G)$, then:

a) If p and q are adjacent in G :

$$R(p,q) = B(p,q)$$

b) If p and q are not adjacent in G :

$$T_R(p,q) = \min(T_A(p), T_A(q))$$
$$I_R(p,q) = \max(I_A(p), I_A(q))$$
$$F_R(p,q) = \max(F_A(p), F_A(q))$$
$$L_R(p,q) = \max(L_A(p), L_A(q))$$

Theorem 10. The Antipodal Single Valued Turiyam Neutrosophic Graph $A(G) = (Q,R)$ can be transformed into a standard Turiyam Neutrosophic Graph by removing the antipodal conditions.

Proof.

1. Observation: The ASVTG $A(G)$ includes all vertices V with Turiyam Neutrosophic memberships A and edges defined based on antipodal conditions.
2. Transformation Process:
   - Remove the antipodal condition by considering all possible edges.
   - Define the edge set R' where:

$$R'(p,q) = B(p,q), \text{ for all } (p,q) \in E$$

   - For $(p,q) \notin E$, define $R'(p,q)$ using Turiyam Neutrosophic edge definitions.
3. Resulting Graph: The graph $G' = (A,R')$ is a Turiyam Neutrosophic Graph without antipodal constraints. Thus, the ASVTG transforms into a Turiyam Neutrosophic Graph.

Theorem 11. By mapping the Turiyam Neutrosophic memberships to neutrosophic memberships, the ASVTG $A(G)$ can be transformed into an Antipodal Single Valued Neutrosophic Graph (ASVNG).

Proof.

1. Mapping of Memberships:
   - For each vertex $v \in V$:

$$T'_A(v) = T_A(v)$$
$$I'_A(v) = I_A(v) + L_A(v)$$
$$F'_A(v) = F_A(v)$$

   - Normalize $I'_A(v)$ to ensure it lies within [0, 1]:

$$I''_A(v) = \frac{I'_A(v)}{\max(1, I'_A(v))}$$

   - Similar mappings for edges.
2. Adjusted $ASVNG$: The new memberships satisfy neutrosophic conditions.
3. Edge Definitions: Redefine the edge set R using the neutrosophic memberships.

   Therefore, the ASVTG transforms into an ASVNG.

Theorem 12. By reducing the Turiyam Neutrosophic memberships to single membership degrees, the ASVTG $A(G)$ can be transformed into an Antipodal Fuzzy Graph.

Proof.

1. Mapping of Memberships:
   - For each vertex $v \in V$:
   $$\sigma(v) = T_A(v)$$
   - For each edge $e \in E$:
   $$\mu(e) = T_B(e)$$
2. Edge Definitions: Redefine the edge set using fuzzy memberships.
3. Antipodal Conditions: Use the fuzzy distance and diameter analogous to the Turiyam Neutrosophic case. Hence, the ASVTG reduces to an Antipodal Fuzzy Graph.

4 | Future Research

Future research will focus on extending this graph class to Hypergraphs, Directed Hypergraphs, and Superhypergraphs [45,66,108,109]. This extension will involve investigating the mathematical structures and exploring potential applications of these generalized forms.

References

[1] Muhammad Akram and Noura Omair Alshehri. Intuitionistic fuzzy cycles and intuitionistic fuzzy trees. The Scientific World Journal, 2014(1):305836, 2014.

[2] Muhammad Akram, Feng Feng, Shahzad Sarwar, and Youne Bae Jun. Certain types of vague graphs. University Politehnica of Bucharest Scientific Bulletin Series A, 76(1):141-154, 2014.

[3] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. Journal of Intelligent & Fuzzy Systems, 26(2):647-653, 2014.
[4] Muhammad Akram, Sheng-Gang Li, and KP Shum. Antipodal bipolar fuzzy graphs. Italian Journal of Pure and Applied Mathematics, 31(56):425-438, 2013.
[5] Muhammad Akram, Hafsa M Malik, Sundas Shahzadi, and Florentin Smarandache. Neutrosophic soft rough graphs with application. Axioms, 7(1):14, 2018.
[6] Muhammad Akram and Gulfam Shahzadi. Operations on single-valued neutrosophic graphs. Infinite Study, 2017.
[7] Morteza Alamgir and Ulrike von Luxburg. Shortest path distance in random k-nearest neighbor graphs. In International Conference on Machine Learning, 2012.
[8] David F. Anderson and Shashikant Mulay. On the diameter and girth of a zero-divisor graph. Journal of Pure and Applied Algebra, 210:543-550, 2007.
[9] S. Arockiaraj, R. Gurusamy, and KM. Kathiresan. On the steiner antipodal number of graphs. Electron. J. Graph Theory Appl., 7:225-233, 2019.
[10] Krassimir Atanassov. Intuitionistic fuzzy sets. International journal bioautomation, 20:1, 2016.
[11] Krassimir T. Atanassov. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20:87-96, 1986.
[12] Krassimir T Atanassov. On intuitionistic fuzzy sets theory, volume 283. Springer, 2012.
[13] Krassimir T. Atanassov. On interval valued intuitionistic fuzzy sets. Interval-Valued Intuitionistic Fuzzy Sets, 2019.
[14] Krassimir T Atanassov and Krassimir T Atanassov. Intuitionistic fuzzy sets. Springer, 1999.
[15] Wenhui Bai, Juanjuan Ding, and Chao Zhang. Dual hesitant fuzzy graphs with applications to multi-attribute decision making. International Journal of Cognitive Computing in Engineering, 1:18-26, 2020.
[16] T Bharathi, S Felixia, and S Leo. Intuitionistic felicitous fuzzy graphs.
[17] Anushree Bhattacharya and Madhumangal Pal. A fuzzy graph theory approach to the facility location problem: A case study in the indian banking system. Mathematics, 11(13):2992, 2023.
[18] Béla Bollobás and Oliver Riordan. The diameter of a scale-free random graph. Combinatorica, 24:5-34, 2004.
[19] John Adrian Bondy, Uppaluri Siva Ramachandra Murty, et al. Graph theory with applications, volume 290. Macmillan London, 1976.
[20] RA Borzooei and HOSSEIN RASHMANLOU. Degree of vertices in vague graphs. Journal of applied mathematics & informatics, 33(5_6):545-557, 2015.
[21] RA Borzooei and Hossein Rashmanlou. More results on vague graphs. UPB Sci. Bull. Ser. A, 78(1):109-122, 2016.
[22] Rajab Ali Borzooei, Hossein Rashmanlou, Sovan Samanta, and Madhumangal Pal. Regularity of vague graphs. Journal of Intelligent & Fuzzy Systems, 30(6):3681-3689, 2016.
[23] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. Critical Review, XII, 2016:5-33, 2016.
[24] Sujit Das and Samarjit Kar. Intuitionistic multi fuzzy soft set and its application in decision making. In Pattern Recognition and Machine Intelligence: 5th International Conference, PReMI 2013, Kolkata, India, December 10-14, 2013. Proceedings 5, pages 587-592. Springer, 2013.
[25] Suman Das, Rakhal Das, and Surapati Pramanik. Single valued pentapartitioned neutrosophic graphs. Neutrosophic Sets and Systems, 50(1):225-238, 2022.
[26] Supriya Kumar De, Ranjit Biswas, and Akhil Ranjan Roy. Some operations on intuitionistic fuzzy sets. Fuzzy sets and Systems, 114(3):477-484, 2000.
[27] Reinhard Diestel. Graduate texts in mathematics: Graph theory.
[28] Reinhard Diestel. Graph theory 3rd ed. Graduate texts in mathematics, 173(33):12, 2005.
[29] Reinhard Diestel. Graph theory. Springer (print edition); Reinhard Diestel (eBooks), 2024.
[30] Didier Dubois and Henri Prade. Fuzzy sets and systems: theory and applications. In Mathematics in Science and Engineering, 2011.
[31] Didier Dubois and Henri Prade. Fundamentals of fuzzy sets, volume 7. Springer Science & Business Media, 2012.
[32] Didier Dubois, Henri Prade, and Lotfi A. Zadeh. Fundamentals of fuzzy sets. 2000.
[33] PA Ejegwa, SO Akowe, PM Otene, and JM Ikyule. An overview on intuitionistic fuzzy sets. Int. J. Sci. Technol. Res, 3(3):142-145, 2014.
[34] David Eppstein. Diameter and treewidth in minor-closed graph families. Algorithmica, 27:275-291, 1999.
[35] Chenglin Fan, P. Li, and Xiaoyun Li. Private graph all-pairwise-shortest-path distance release with improved error rate. In Neural Information Processing Systems, 2022.
[36] Miguel Angel Fiol. An eigenvalue characterization of antipodal distance regular graphs. Electron. J. Comb., 4, 1997.
[37] Takaaki Fujita. General, general weak, anti, balanced, and semi-neutrosophic graph.
[38] Takaaki Fujita. Contact graphs in fuzzy and neutrosophic graphs. Preprint, May 2024, File available, 2024.
[39] Takaaki Fujita. Extension of bipolar, interval-valued, and complex neutrosophic soft expert graphs to general plithogenic soft rough expert mixed graphs. July 2024.
[40] Takaaki Fujita. Fuzzy directed tree-width and fuzzy hypertree-width. ResearchGate(Preprint), 2024.
[41] Takaaki Fujita. General plithogenic soft rough graphs and some related graph classes, June 2024. License: CC BY 4.0.
[42] Takaaki Fujita. Interval graphs and proper interval graphs in fuzzy and neutrosophic graphs. Preprint, July 2024, File available, 2024.

[43] Takaaki Fujita. Permutation graphs in fuzzy and neutrosophic graphs. Preprint, July 2024. File available.
[44] Takaaki Fujita. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. ResearchGate(Preprint), 2024.
[45] Takaaki Fujita. Short note of supertree-width and n-superhypertree-width. Neutrosophic Sets and Systems, 77:54-78, 2024.
[46] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. ResearchGate, July 2024.
[47] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs, 2024.
[48] Takaaki Fujita. Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs. ResearchGate, July 2024.
[49] Takaaki Fujita. Survey of trees, forests, and paths in fuzzy and neutrosophic graphs. July 2024.
[50] Takaaki Fujita. Uncertain automata and uncertain graph grammar, June 2024. License CC BY 4.0.
[51] Takaaki Fujita. Vague soft expert graph and complex fuzzy soft expert graph. ResearchGate, June 2024. License CC BY 4.0.
[52] Takaaki Fujita. Various properties of various ultrafilters, various graph width parameters, and various connectivity systems. arXiv preprint arXiv:2408.02299, 2024.
[53] Komei Fukuda and Keiichi Handa. Antipodal graphs and oriented matroids. Discret. Math., 111:245-256, 1993.
[54] GA Ganati, VNS Rao Repalle, MA Ashebo, and M Amini. Turiyam graphs and its applications. Information Sciences Letters, 12(6):2423-2434, 2023.
[55] Gamachu Adugna Ganati, VN Srinivasa Rao Repalle, and Mamo Abebe Ashebo. Social network analysis by turiyam graphs. BMC Research Notes, 16(1):170, 2023.
[56] Gamachu Adugna Ganati, VN Srinivasa Rao Repalle, and Mamo Abebe Ashebo. Relations in the context of turiyam sets. BMC Research Notes, 16(1):49, 2023.
[57] A Nagoor Gani and K Radha. On regular fuzzy graphs. 2008.
[58] Nagoor Gani and J. Malarvizhi. On antipodal fuzzy graph. 2010.
[59] Anthony D. Gardiner. Antipodal covering graphs. Journal of Combinatorial Theory, Series B, 16:255-273, 1974.
[60] Puspendu Giri, Somnath Paul, and Bijoy Krishna Debnath. A fuzzy graph theory and matrix approach (fuzzy gtma) to select the best renewable energy alternative in india. Applied Energy, 358:122582, 2024.
[61] S Gomathy, D Nagarajan, S Broumi, and M Lathamaheswari. Plithogenic sets and their application in decision making. Infinite Study, 2020.
[62] Zengtai Gong and Junhu Wang. Hesitant fuzzy graphs, hesitant fuzzy hypergraphs and fuzzy graph decisions. Journal of Intelligent & Fuzzy Systems, 40(1):865-875, 2021.
[63] Oliver Goodman and Vincent Moulton. On the tight span of an antipodal graph. Discret. Math., 218:73-96, 2000.
[64] Jonathan L Gross, Jay Yellen, and Mark Anderson. Graph theory and its applications. Chapman and Hall/CRC, 2018.
[65] Muhammad Gulistan, Naveed Yaqoob, Zunaira Rashid, Florentin Smarandache, and Hafiz Abdul Wahab. A study on neutrosophic cubic graphs with real life applications in industries. Symmetry, 10(6):203, 2018.
[66] Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. Decision Making Based on Valued Fuzzy Superhypergraphs. Infinite Study, 2023.
[67] Carmen Hernando, Mercè Mora, Ignacio M. Pelayo, Carlos Seara, and David R. Wood. Extremal graph theory for metric dimension and diameter. Electron. J. Comb., 17, 2007.
[68] Liangsong Huang, Yu Hu, Yuxia Li, PK Kishore Kumar, Dipak Koley, and Arindam Dey. A study of regular and irregular neutrosophic graphs with real life applications. Mathematics, 7(6):551, 2019.
[69] S Satham Hussain, N Durga, Muhammad Aslam, G Muhiuddin, and Ganesh Ghorai. New concepts on quadripartitioned neutrosophic competition graph with application. International Journal of Applied and Computational Mathematics, 10(2):57, 2024.
[70] S Satham Hussain, N Durga, Rahmonlou Hossein, and Ghorai Ganesh. New concepts on quadripartitioned singlevalued neutrosophic graph with real-life application. International Journal of Fuzzy Systems, 24(3):1515-1529, 2022.
[71] S Satham Hussain, Hossein Rashmonlou, R Jahir Hussain, Sankar Sahoo, Said Broumi, et al. Quadripartitioned neutrosophic graph structures. Neutrosophic Sets and Systems, 51(1):17, 2022.
[72] Chiranjibe Jana, Tapan Senapati, Monoranjan Bhowmik, and Madhumangal Pal. On intuitionistic fuzzy g-subalgebras of g-algebras. Fuzzy Information and Engineering, 7(2):195-209, 2015.
[73] Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. Neutrosophic graphs: a new dimension to graph theory. Infinite Study, 2015.
[74] M. G. Karunambigai, R. Parvathi, and R. Buvaneswari. Arc in intuitionistic fuzzy graphs. Notes on Intuitionistic Fuzzy Sets, 17:37-47, 2011.
[75] Armita Khorsandi, Xiao-Chu Liu, and Bing yuan Cao. A new algorithm to shortest path problem with fuzzy arc lengths. 2016.
[76] Hongxing Li and Vincent C Yen. Fuzzy sets and fuzzy decision-making. CRC press, 1995.
[77] Robert Lin. Note on fuzzy sets. Yugoslav Journal of Operations Research, 24:299-303, 2014.
[78] J. Malarvizhi and G. Divya. Annals of on antipodal single valued neutrosophic graph. 2017.
[79] J. Malarvizhi and G.Divya. On antipodal single valued neutrosophic graph. viXra, pages 235-242, 2017.
[80] M. A. Malik. m-polar neutrosophic graphs. 2021.
[81] S. Monolisa, Lancy A. Arokia, and U. Mary. Self-centered and self-corner vertex on cartesian product of star related vague graphs. 2ND INTERNATIONAL CONFERENCE ON MATHEMATICAL TECHNIQUES AND APPLICATIONS: ICMTA2021, 2022.

[82] John N Mordeson and Sunil Mathew. Advanced topics in fuzzy graph theory, volume 375. Springer, 2019.
[83] Sunil MP and J Suresh Kumar. On intuitionistic hesitancy fuzzy graphs. 2024.
[84] TM Nishad, Talal Ali Al-Hawary, and B Mohamed Harif. General fuzzy graphs. Ratio Mathematica, 47, 2023.
[85] János Pach. Geometric graph theory. Handbook of Discrete and Computational Geometry, 2nd Ed., pages 219-238, 2004.
[86] Madhumangal Pal, Sovan Samanta, and Ganesh Ghorai. Modern trends in fuzzy graph theory. Springer, 2020.
[87] Shelly Pamora, Lucia Ratnasari, and Y. D. Sumanto. Graf fuzzy antipodal. 2010.
[88] Sakshi Dev Pandey, AS Ranadive, and Sovan Samanta. Bipolar-valued hesitant fuzzy graph and its application. Social Network Analysis and Mining, 12(1):14, 2022.
[89] T Pathinathan, J Jon Arockiaraj, and J Jesintha Rosline. Hesitancy fuzzy graphs. Indian Journal of Science and Technology, 8(35):1-5, 2015.
[90] Witold Pedrycz. Fuzzy sets engineering. 1995.
[91] Sriram Pemmaraju and Steven Skiena. Computational discrete mathematics: Combinatorics and graph theory with mathematica®. Cambridge university press, 2003.
[92] Norbert Polat. On antipodal and diametrical partial cubes. Discussiones Mathematicae Graph Theory, 41:1127 - 1145, 2021.
[93] Miao Qiao, Hong Cheng, and Jeffrey Xu Yu. Querying shortest path distance with bounded errors in large graphs. In International Conference on Statistical and Scientific Database Management, 2011.
[94] Shio Gai Quek, Ganeshsree Selvachandran, D Ajay, P Chellamani, David Taniar, Hamido Fujita, Phet Duong, Le Hoang Son, and Nguyen Long Giang. New concepts of pentapartitioned neutrosophic graphs and applications for determining safest paths and towns in response to covid-19. Computational and Applied Mathematics, 41(4):151, 2022.
[95] Yongsheng Rao, Saeed Kosari, and Zehui Shao. Certain properties of vague graphs with a novel application. Mathematics, 8(10):1647, 2020.
[96] Hossein Rashmanlou and Rajab Ali Borzooei. Vague graphs with application. Journal of Intelligent & Fuzzy Systems, 30(6):3291-3299, 2016.
[97] Hossein Rashmanlou and Madhumangal Pal. Antipodal interval-valued fuzzy graphs. ArXiv, abs/1401.0823, 2014.
[98] Fatemeh Salehi Rizi, Jörg Schlötterer, and Michael Granitzer. Shortest path distance approximation using deep learning techniques. 2018 IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining (ASONAM), pages 1007-1014, 2018.
[99] David W Roberts. Analysis of forest succession with fuzzy graph theory. Ecological Modelling, 45(4):261-274, 1989.
[100] Azriel Rosenfeld. Fuzzy graphs. In Fuzzy sets and their applications to cognitive and decision processes, pages 77-95. Elsevier, 1975.
[101] Rıdvan Şahin. An approach to neutrosophic graph theory with applications. Soft Computing, 23(2):569-581, 2019.
[102] Sovan Samanta, Madhumangal Pal, Hossein Rashmanlou, and Rajab Ali Borzooei. Vague graphs and strengths. Journal of Intelligent & Fuzzy Systems, 30(6):3675-3680, 2016.
[103] Prem Kumar Singh, Naveen Surathu, Ghattamaneni Surya Prakash, et al. Turiyam based four way unknown profile characterization on social networks. Full Length Article, 10(2):27-7, 2024.
[104] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). nidus idearum, page 16.
[105] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In Philosophy, pages 1-141. American Research Press, 1999.
[106] Florentin Smarandache. Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic setsrevisited. Infinite study, 2018.
[107] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. arXiv preprint arXiv:1808.03948, 2018.
[108] Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. Nidus Idearum, 7:107-113, 2019.
[109] Florentin Smarandache. Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra. Infinite Study, 2020.
[110] Florentin Smarandache and Said Broumi. Neutrosophic graph theory and algorithms. IGI Global, 2019.
[111] Florentin Smarandache and Nivetha Martin. Plithogenic n-super hypergraph in novel multi-attribute decision making. Infinite Study, 2020.
[112] A Sudha and P Sundararajan. Robust fuzzy graph. Ratio Mathematica, 46, 2023.
[113] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. Journal of ambient intelligence and humanized computing, 14(10):13139-13159, 2023.
[114] AL-Hawary Talal and Bayan Hourani. On intuitionistic product fuzzy graphs. Italian Journal of Pure and Applied Mathematics, page 113.
[115] Mini Tom and M.S.Sunitha. Sum distance in fuzzy graphs. 2014.
[116] Mini Tom and M. S. Sunitha. Strong sum distance in fuzzy graphs. SpringerPlus, 4, 2015.
[117] Vakkas Uluçay and Memet Şahin. Intuitionistic fuzzy soft expert graphs with application. Uncertainty discourse and applications, 1(1):1-10, 2024.
[118] Tong Wei, Junlin Hou, and Rui Feng. Fuzzy graph neural network for few-shot learning. In 2020 International joint conference on neural networks (IJCNN), pages 1-8. IEEE, 2020.
[119] Douglas Brent West et al. Introduction to graph theory, volume 2. Prentice hall Upper Saddle River, 2001.

[120] Zeshui Xu. Hesitant fuzzy sets theory, volume 314. Springer, 2014.
[121] Lotfi A Zadeh. Fuzzy sets. Information and control, 8(3):338-353, 1965.
[122] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh, pages 775-782. World Scientific, 1996.
[123] Ping Zhang and Gary Chartrand. Introduction to graph theory. Tata McGraw-Hill, 2:2-1, 2006.
[124] Hua Zhao, Zeshui Xu, Shousheng Liu, and Zhong Wang. Intuitionistic fuzzy mst clustering algorithms. Computers & Industrial Engineering, 62(4):1130-1140, 2012.

Empowering Artificial Intelligence Techniques with Soft Computing of Neutrosophic Theory in Mystery Circumstances for Plant Diseases

Ahmed El-Massry,[1] **Florentin Smarandache,**[2] **Mona Mohamed**[3]

[1]Department of Computer Science, Faculty of Computers and Informatics, Zagazig University, Zagazig 44519, Egypt
[2]University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA
[3]Higher Technological Institute, 10th of Ramadan City 44629, Egypt

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Abstract

Plant diseases are one of the factors that lead to yield and economic losses, which have a direct effect on national and international food production systems. One of the most essential ways to avoid agricultural product loss or reduction in amount is to diagnose plant diseases promptly and accurately. Hence, the diagnosis process for plants is crucial and should be conducted accurately. Moreover, this study focuses on this process by constructing an Artificiality Diagnostics Framework (ADF) to serve the study's objectives which entailed conducting diagnosis for plants in a professional and precise manner over uncertain environments. Thus, neutrosophic theory is considered the principal ingredient in our ADF. Due to the ability of neutrosophic to divide images into Truth (T), Falsity(F), and Indeterminacy (I). Also, deep learning (DL) is considered another principal ingredient in treating vast samples of datasets. Our comparative analysis of the leaves of potatoes is conducted whether leveraging neutrosophic and without utilizing Neutrosophic. ResNet50, ResNet152, and Mobile Net are the principal ingredients for the training dataset. The findings of implementing these networks indicated that ResNet50 achieved the highest accuracy of 0.915 in the T domain, ResNet152 achieved the highest accuracy of 0.905 in the True(T) domain, and Mobile Net achieved the highest accuracy of 0.915 in Truth(T) domain. Accuracy of 0.863 in Indeterminate(I).

Keywords: Neutrosophic theory, Deep Learning, plant disease

1. Introduction

Economic losses arise from yield losses caused by plant diseases, which have a direct effect on national and international food production systems. Oerke et al. [1] estimated that 13% of the loss in agricultural productivity worldwide is attributed to plant diseases. From the perspective of [2] comprehension of the causes of plant diseases is mandatory. A conducive habitat, the pathogen, and the host are the three elements that help diseases develop in plants. Also, drought and plant disease[3] are factors that impact agricultural productivity. As a result of the spread of many plant diseases throughout the crop after infection, [4] regular crop monitoring is necessary since prompt disease control will stop the disease's progress.

In this evolving context [5], the need to accurately and promptly identify illnesses, including early impediments, has never been higher. For decades[6], computer vision technology has been used in

agriculture to identify weeds, assess crop geometric sizes, diagnose nutritional deficiencies, and predict crop yields.

Overall, plant phenotyping and precision agriculture need the diagnosis of plant diseases. While diagnosing and monitoring plant diseases is important [7], conventional approaches that require a human visual inspection are costly, time-consuming, dependent on specialists, and unsuitable for precision agriculture. Furthermore, human prejudice is likely to have an impact on these methods, reducing their accuracy. Hence [2], solved these issues by examining the utilization of image processing methods with images of plants.in the same vein [8] that employed image processing techniques to measure corn stripe disease, and it was shown that computer-based approaches outperformed traditional visual analysis in terms of accuracy.

Nevertheless [9] decided that the degree of uncertainty in the image's data increases when two-dimensional images are converted from three-dimensional ones through image processing. Furthermore, many of the concepts in the image are also vague and ambiguous. Whilst a crucial part of image processing and pattern recognition [10], image segmentation is one of the trickiest processes which has many uses and uncertainties [9] that contribute to the difficulty of segmenting images.

Wherefore fuzzy theory is used to the field of image segmentation[11] , to effectively express fuzzy concepts and information. In the discipline of image segmentation [12] , fuzzy image segmentation has grown in importance and popularity as a study area. Subsequently [13] established neutrosophic fuzzy clustering algorithm within the conventional fuzzy C-means clustering techniques. In the same vein, the study of [14] confusing data in the diagnosis of skin cancer are grouped using neutrophilic c-means clustering (NCMC). On the other hand, the notion of Neutrosophic theory is highlighted by [15] which proposed by Smarandache, where the basic distinction between fuzzy and intuitionistic fuzzy logic and set is represented by the neutral concept that is known as Neutrosophy [16].

In the realms of artificial intelligence (AI) and deep learning (DL), Neutrosophic theory (NTh)[17] offers the necessary ability to serve as a universal framework for uncertainty analysis in data sets, particularly with images.

Overall, all of these studies are considering catalyst for implementing neutrosophic theory in field of diagnosis the plant and classify it [18] into three sets as Truth (T), Falsity (F),and Indeterminacy (I).Although, we are leveraging ML especially DL [19] for the identification of diseases as a result of enhanced computing power, larger storage capabilities, and the accessibility of enormous datasets.

Herein, Plant diseases can be diagnosed by merging NTh with DL toward establishing Artificiality Diagnostics Framework (ADF). We applied the established ADF to real problem through implementing the framework on dataset consisting of around 4000 images of potato. The findings of ADF are recorded and analyzed into the results and discussion section.

2. Realm of Neutrosophic Theory

In the context of vague and ambiguity about data and circumstances, the notion of neutrosophic was founded by Smarandache[15] which extended to fuzzy logic.

2.1 Preliminary

This theory described in field of image processing [9, 16] as :

- Assum $\Re$ be a universe in this study. A group of image pixels represent as $\vartheta = \omega * \omega$.whilst $\omega \subseteq \Re$, and ω is an argument.
- Herein, neutrosophic image can be described as ∂ , F, ℓ are symbols of truth, false and indeterminacy.
- Each pixel $p(\chi, \nu)$ in the neutrosophic image can be described as $p_{Neu}(\chi, \nu) = \{\partial(\chi, \nu), \ell(\chi, \nu), F(\chi, \nu)\}$.

$$\partial(\chi, \nu) = \frac{\bar{\kappa}(\chi,\nu) - \bar{\kappa}\min}{\bar{\kappa}\max - \bar{\kappa}\min} \quad (1)$$

Where: $\bar{\kappa}(\chi, \nu)$is the region mean value of $\kappa(\chi, \nu)$

$$\bar{\kappa}(\chi, \nu) = \frac{1}{\omega * \omega} \sum_{m=i-\omega/2}^{i+\omega/2} \sum_{n=i-\omega/2}^{i+\omega/2} \kappa(m, n) \quad (2)$$

$$\ell(\chi, \nu) = \frac{\wp(\chi,\nu) - \wp_{\min}}{\wp_{\max} - \wp_{\min}} \quad (3)$$

where: $\wp(\chi, \nu)$is the absolute value of the difference between intensity $\kappa(\chi, \nu)$ and its local mean value at $\kappa(\chi, \nu)$.whilst ℓ is indeterminacy degree of p_{Neu}

$$\wp(\chi, \nu) = \mathrm{abs}\big(\kappa(\chi, \nu) - \bar{\kappa}(\chi, \nu)\big) \quad (4)$$

where: $\kappa(\chi, \nu)$ is the gray value of $p(\chi, \nu)$

$$F(\chi, \nu) = 1 - \partial(\chi, \nu) \quad (5)$$

- whereas, in interval neutrosophic, p_{INeu} described according to interval number set as:

$$p_{INeu} = \{[\partial_1(\chi, \nu), \partial_2(\chi, \nu)], [\ell_1(\chi, \nu), \ell_2(\chi, \nu)], [F_1(\chi, \nu), F_2(\chi, \nu)\} \quad (6)$$

2.2 Applications of Neutrosophic

According to bibliometrics analysis [20], we are conducted analysis based on web of science (WoS) database for prior studies and terms which related to our scope. VoS viewer software is utilized for showcases the findings of queries conducted on WoS for applying neutrosophic in various domains.

2.2.1 Neutrosophic in Agriculture

We conducted bibliometric analysis based on certain keywords such as ("Agriculture" AND "Neutrosophic") The findings of this process showcase as following Figure 1 where query conducted for Co-citation based on Co-Sources for mentioned key words. The findings indicated that 14 items which classified into two clusters. Cluster 1 has seven items with red color and cluster 2 has seven items with green color.

2.2.2 Neutrosophic in Healthcare

Figure 2 illustrated bibliometric analysis which conducted certain keywords such as ("Healthcare" AND "Neutrosophic") for Co-citation based on Co-Sources for mentioned key words. The findings indicated that 12 items which classified into two clusters. Cluster 1 has six items with red color and cluster 2 has six items with green color.

2.2.3 Neutrosophic in Climate

The conducted bibliometric analysis for Co-citation based on Co-Sources for utilizing neutrosophic in climate has been represented in Figure 3. According to this Figure, there are two clusters. Each cluster

has 6 items. Overall, there are 12 items resulted of query (“Climate” AND “Neutrosophic”) for Co-citation based on Co-Sources.

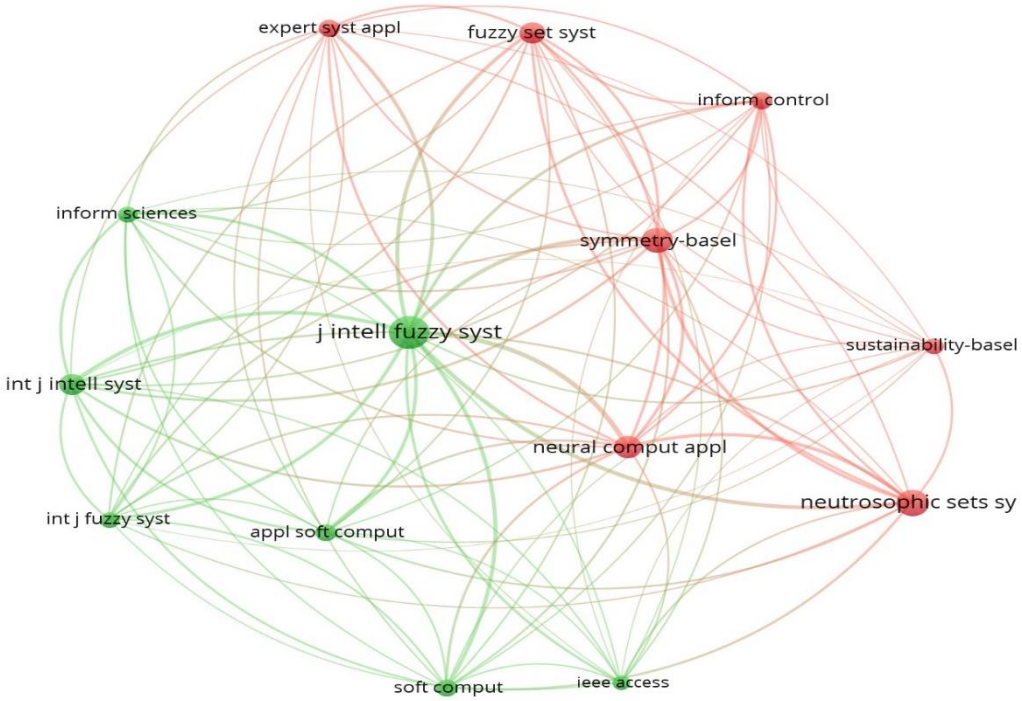


Figure 1. Visualization network for Co-citation based on Co-Sources in agriculture.

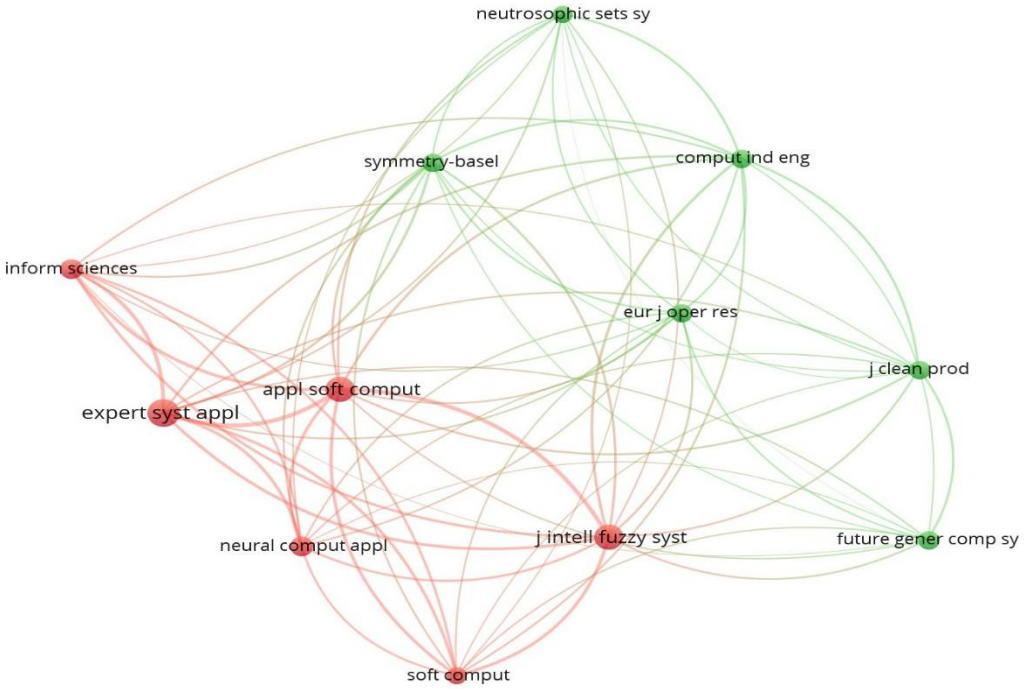


Figure 2. Visualization network for Co-citation based on Co-Sources in healthcare.

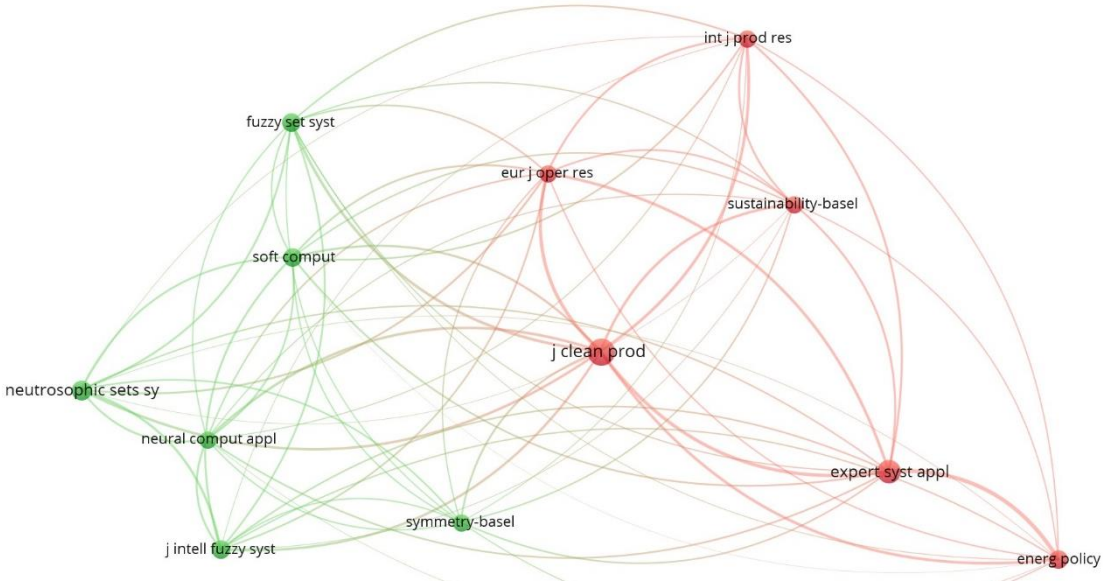


Figure 3. Visualization network for Co-citation based on Co-Sources in climate.

3. Related Work

Herein, we attempted to ensemble the prior studies which related to our scope; through surveys have been conducted. The surveys' findings indicated that not many literary works treat uncertainty and incomplete data. Thereby, this study covered this issue through merging Neutrosophic with DL and established ADF. Table 1 illustrates a set of related studies.

Table 1. previous literary studies

Ref #	Methodology	Objectives	Dataset	Findings
Paul et al. [21]	In this study, the author built a custom lightweight CNN model and compared it with some CNN models based on the transfer learning principle.	Tomato disease detection and classification	The dataset, called multi-source tomato disease, was collected from a single source consisting of 11 varieties and containing 32,535 images.	The proposed model achieved the best with data augmentation accuracy 95% and without 89%.
Memon et al.[22]	Build Meta DL Model and Compare Accuracy it With Custom CNN Model, Vgg16 Transfer learning and ResNet50 Model	Identify and detect cotton leaf diseases	The cotton data set with 2384 samples. The utilized data divided into seven classes: nutrient deficiency, healthy, leafspot, powdery mildew, target spot, verticillium, and leaf curl)	The Proposed Model achieve 98.53%
Rangarajan et al.[23]	Train two Deep learning Model Vgg16 and Alex Net based of Transfer Learning and Compare accuracy of them	Detect Tomato Leaves Disease	The dataset consists of seven categories, including the health category, which consists of 13,262 images extracted from the plant Village Dataset.	Alex Net achieved best Accuracy 97.49 With better performance
Bi et al.[24]	Scholars build mobile phone-based models based on Mobile Net and compare it with InceptionV3 and ResNet152 in terms of speed in image prediction.	Detecting diseases of apple leaves by low-cost model	The data set consists of two categories (Alternaria leaf spots and rust leaves) and contains 334 images. The data set was collected by a group of experts.	The best model in terms of image processing speed is Mobile Net with the fastest speed of .22 seconds
Dahiya et al.[25]	In this research, the author trained a group of the most famous types of deep learning models on plant diseases Dataset such as Google Net and ResNet18 and compared them in terms of accuracy.	Analysis of some deep learning models in terms of accuracy in detecting plant leaf diseases	Plant Village data set consists of 18 classes of 2064 images, divided into 70% for training, 20% for validation, and 10% for testing	The best models achieved Accuracy is ResNet50 and ResNet101
Wei et al.[26]	In this work, the author made a comparison between a group of models such as Alex Net, VGG16, Res Net50, and DenseNet121) on more than one endpoint device (CPU, GPU, VPU).	plant leaf disease identification	The Plant Village dataset, which consists of 55,446 images, is divided into 38 classes	The Best Model archive accuracy is DenseNet121 96.4

Rao et al.[27]	authors implemented pre trained Model Alex Net	Detect Diseases Grape and Mango Leaves	The dataset consists of 7,222 grape leaves from plant Village and 1,216 self-acquired mango leaves	Model archive accuracy in Grape Dataset 99.03% and 89% for Mango Dataset
Belay et al.[28]	proposed model has been built by combining CNN with LSTM and Compare it with server models such a InceptionV3 s VGGNet16 and	Chickpea disease detection	Chickpea Data Set which it consists of 3 Class and contain 1399 image and after augmenting 8391	The Proposed Model achieved highest with 92.55 accuracy

4. Data and Methods

In this study, Potato Dataset has been utilized to compare the performance of a group of models before and after applying the neutrospheric technique to the data. Potato Dataset consists of around 4000 images divided into 3 we used Sample of It in training Model. Table 2 show Statistics of Sample.

The methodology which is employed in this study is to correctly classify potato plant diseases, and this is done in two steps. The first step is to apply the Neutrosophic Domains as an image processing step. The second step is to train the data on deep learning models using the Transfer learning principle. Establised ADF is highlighted in Figure 4.

We used three models for training data ResNet50, ResNet152, and Mobile Net. We trained models with Adam optimizer and lr = .0001.Model training with 50 epochs and evaluate models after each epoch with metrics.

Table 2. Statistics of utilized samples

	Classes	Images	Percentage	Total
Diseased	Early Blight	325	0.349%	638
	Late_Blight	313	0.336%	
Healthy	Healthy leaf (C2)	292	0.313 %	930

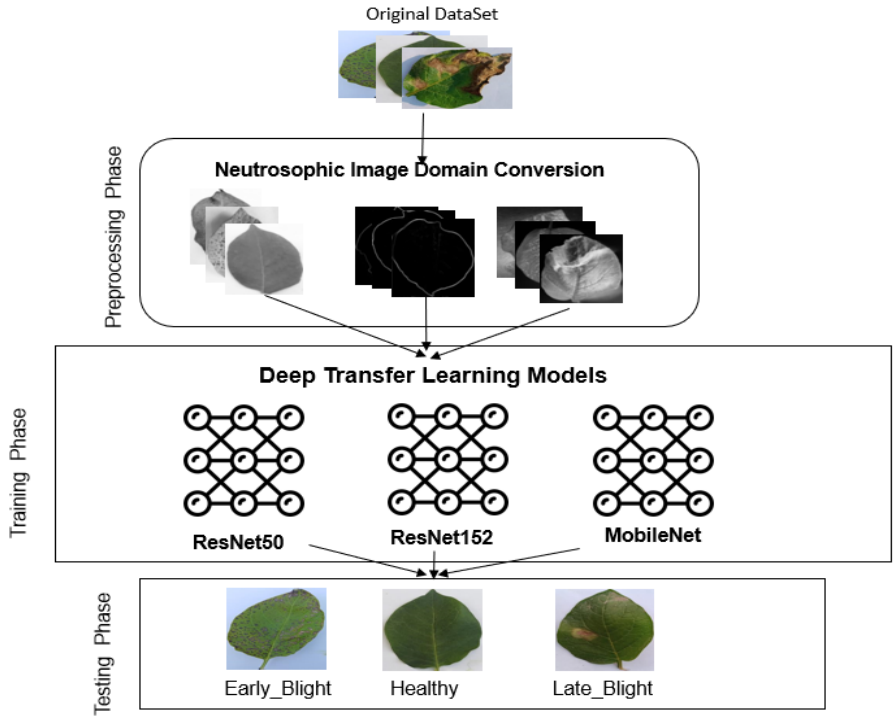


Figure 4. Proposed ADF

5. Result and Discussion

Herein, we illustrate the ADF'S findings and discuss it.

5.1 Performance parameter

Comparison Analysis between models with some matrices has been conducted. We evaluated the model's performance through recall, F1Score, Precision, and Accuracy

5.1.1 Accuracy

This metric is calculated from the number of correct predictions for all categories to the total number of predictions according to Eq.(7).

$$\text{Accuracy} = \frac{(TP+TN)}{(TP+FP+TN+FN)} \quad (7)$$

5.1.2 Precision

This measure is calculated from the number of correct predictions for a category to the total number of predictions in the same category through implementing Eq.(8).

$$\text{Precision} = \frac{TP}{(TP+FP)} \quad (8)$$

5.1.3 Recall

This metric is calculated by harmonic mean based on Eq.(9).

$$\text{F1 Score} = 2 \times \frac{\text{recall} \times \text{precision}}{\text{recall} + \text{precision}} \quad (9)$$

We ustilized Kaggle to train all Models that has GPU Nvidia Tesla P100 With Ram 16 GB.We use Python Version 3.7.6 and Keras Version 2.3.1.

5.2 Hyperparameters For Models

All Models train with Adam optimizer through learning rate .0001, batch size 32, number of epoch 50.

5.2.1 Original Dataset Experiment

Summary performance of models for original dataset are listed in Table 3 without Neutrosophic Domain. We observed that ResNet 50 achieved highest accuracy with .873.

Table 3. Performance o various models based on original dataset

Model	Accuracy	Precision	Recall	F1 Score
MobileNet	0.842	0.842	0.841	0.840
ResNet50	0.873	0.871	0.870	0.870
ResNet152	0.831	0.841	0.841	0.826

5.2.2 Neutrosophic Domains Experimental Results

- In table 4, summary performance of models for True(T) NS Domain has been showcased. whilst ResNet152 achieved highest accuracy with .0.905.
- Whereas performance of models for Falsity(F)has been showcased and listed in Table 5. Therefore, ResNet50 achieved highest accuracy with 0.905.
- Table 6 summary performance of models for Indeterminate (I) has been summarized. Moreover, NS Domain Mobile Net achieved highest accuracy with .863%.
-

Table 4. Dataset True(T)

Model	Accuracy	Precision	Recall	F1 Score
MobileNet	0.852	0.854	0.852	0.853
ResNet50	0.852	0.850	0.848	0.846
ResNet152	0.905	0.907	0.901	0.901

Table 5. Dataset Falsity (F)

Model	Accuracy	Precision	Recall	F1 Score
MobileNet	0.694	0.701	0.693	0.694
ResNet50	0.915	0.914	0.913	0.9134
ResNet152	0.905	0.912	0.900	0.899

Table 6. Dataset (I)

Model	Accuracy	Precision	Recall	F1 Score
MobileNet	0.863	0.862	0.861	0.861
ResNet50	0.778	0.777	0.776	0.776
ResNet152	0.831	0.841	0.827	0.826

5.3 Comparative Result of Neutrosophic Domain with the Original Dominant

Table 7 shows best accuracy of models achieved Mobile Net achieved best accuracy in Indeterminate (I) domain with accuracy .863. ResNet50 and ResNet152 achieved best accuracy in True(T) domain with accuracy 91.5 and 0.905. Figure 5 summarized comparison of models' accuracy.

Table 7. Accuracy of various Models

Model	Accuracy	Precision	Recall	F1 Score
MobileNet	0.863	0.862	0.861	0.861
ResNet50	0.915	0.914	0.913	0.9134
ResNet152	0.905	0.912	0.900	0.899

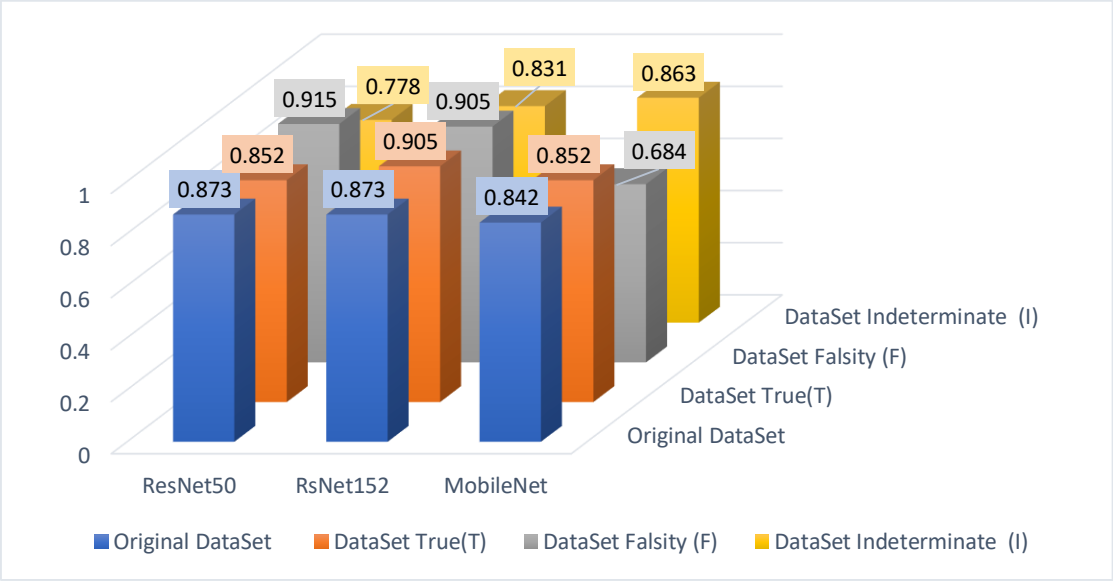
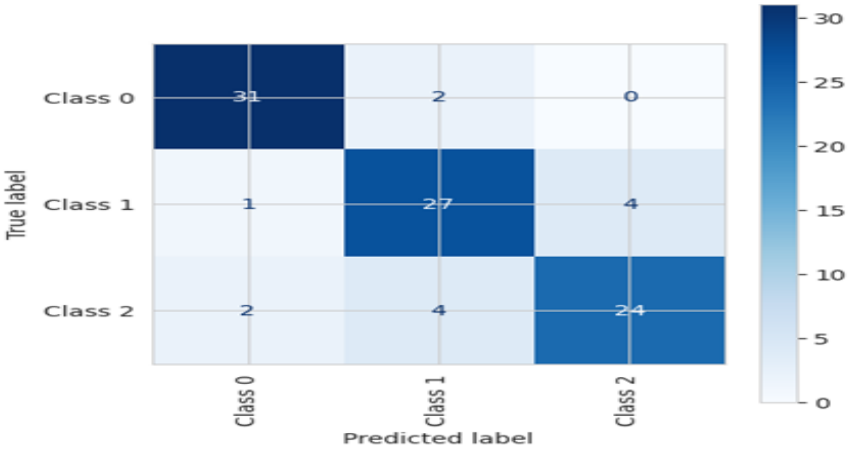


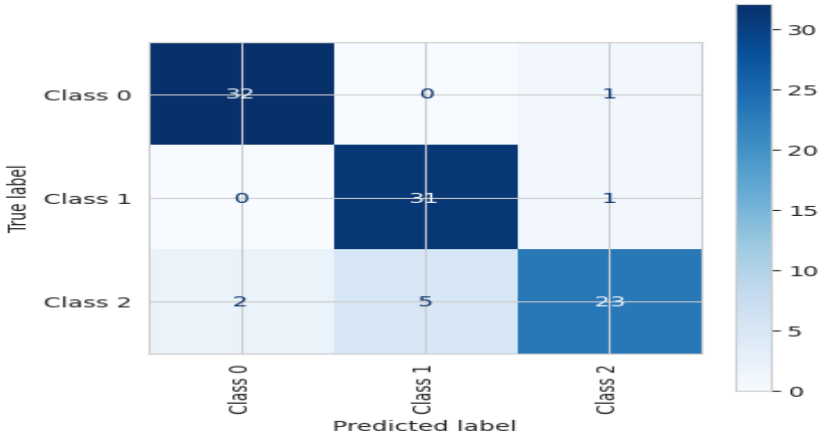
Figure 5. Accuracy of comparative models

5.4 Confusion Matrix of MobileNet

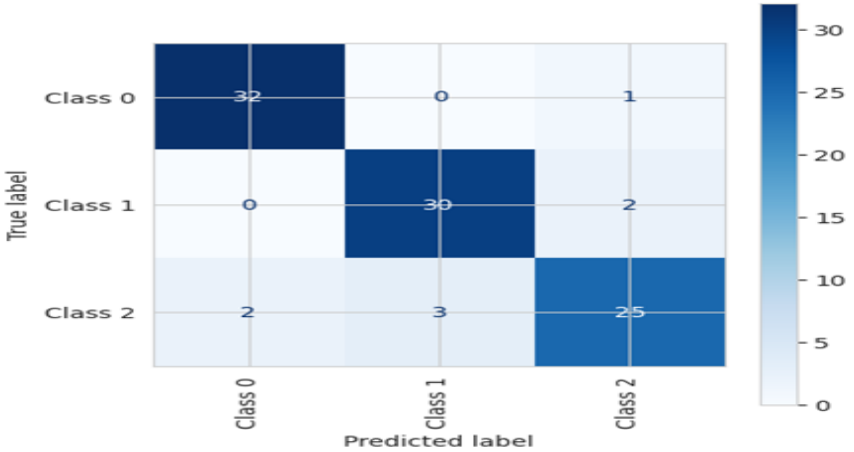
Figure 6. aggregated the confusion matrix for each measurement for truth (T), Indeterminate (I), and Falsity (F).



Confusion Matrix of MobileNet in domain Indeterminate (I)



Confusion Matrix of ResNet152 in domain True(T)



Confusion Matrix of ResNet50 in domain False (F)

Figure 6. Confusion Matrix of Mobile Net for True, False, and Indeterminate

6. Conclusion

This study attempted to cover some of the issues mentioned and determined in the prior studies. One of these issues was incorrect diagnosis for plant diseases. A misdiagnosis might shorten the crop's life or even eradicate it. Moreover, agriculture is diminishing.

Another issue entailed in not many literary studies discuss the uncertainty and incomplete data.

Thereby, these issues are catalysts for developing our ADF which depends on theory which characterized with uncertainty and has ability to treat with ambiguity circumstances. This theory is Neutrosophic which treats with image through three possibilities are T, I, and F. Also, we combined this theory with ML especially DL. Due to the ability of DL to treat with large volume of data. Hence, commenced acquiring an interest in diagnosing diseases especially plants in agriculture domain.

Overall, we applied our constructed ADF potato dataset. Whereas the utilized dataset trained by ResNet50, ResNet152, and Mobile Net models Also, we conducted comparison between the performance of the models before and after applying Neutrosophic theory. The findings of this experiment indicated that Neutrosophic theory proved very effective in improving the accuracy of the models.

References

[1] Oerke, E.-C., & Dehne, H.-W. (2004). Safeguarding production—losses in major crops and the role of crop protection. *Crop protection*, *23*(4), 275–285.

[2] Ahmad, A., Saraswat, D., & El Gamal, A. (2023). A survey on using deep learning techniques for plant disease diagnosis and recommendations for development of appropriate tools. *Smart agricultural technology*, *3*, 100083.

[3] Pantazi, X. E., Moshou, D., & Tamouridou, A. A. (2019). Automated leaf disease detection in different crop species through image features analysis and One Class Classifiers. *Computers and electronics in agriculture*, *156*, 96–104.

[4] DeChant, C., Wiesner-Hanks, T., Chen, S., Stewart, E. L., Yosinski, J., Gore, M. A., … Lipson, H. (2017). Automated identification of northern leaf blight-infected maize plants from field imagery using deep learning. *Phytopathology*, *107*(11), 1426–1432.

[5] Jackulin, C., & Murugavalli, S. (2022). A comprehensive review on detection of plant disease using machine learning and deep learning approaches. *Measurement: sensors*, *24*, 100441.

[6] Senthilkumar, T. P., & Prabhusundhar, P. (n.d.). ENSEMBLE MACHINE LEARNINGAND HYBRID NEUTROSOPHIC COGNITIVE MAPS BASED FEATURE SELECTION FOR RICE DISEASE PREDICTION.

[7] Barbedo, J. G. A. (2019). A review on the use of unmanned aerial vehicles and imaging sensors for monitoring and assessing plant stresses. *Drones*, *3*(2), 40.

[8] Martin, D. P., & Rybicki, E. P. (1998). Microcomputer-based quantification of maize streak virus symptoms in Zea mays. *Phytopathology*, *88*(5), 422–427.

[9] Yuan, Y., Ren, Y., Liu, X., & Wang, J. (2020). *Approach to image segmentation based on interval neutrosophic set*. Infinite Study.

[10] Smarandache, F., Quiroz-Martínez, M. A., Ricardo, J. E., Hernández, N. B., & Vázquez, M. Y. L. (2020). *Application of neutrosophic offsets for digital image processing*. Infinite Study.

[11] Bloch, I. (2015). Fuzzy sets for image processing and understanding. *Fuzzy sets and systems*, *281*, 280–291.

[12] Zhao, J., Wang, X., & Li, M. (2020). A novel Neutrosophic image segmentation based on improved fuzzy C-means algorithm (NIS-IFCM). *International journal of pattern recognition and artificial intelligence*, *34*(05), 2055011.

[13] Alsmadi, M. K. (2018). A hybrid Fuzzy C-Means and Neutrosophic for jaw lesions segmentation. *Ain shams engineering journal*, *9*(4), 697–706.

[14] Abdelhafeez, A., & Mohamed, H. K. (2023). Skin cancer detection using neutrosophic c-means and fuzzy c-means clustering algorithms. *Journal of intelligent systems and internet of things*, *8*(1), 33–42. DOI:10.54216/JISIoT.080103

[15] Abouhawwash, M., Bacanin, N., & Venkatachalam , K. (2024). A Global Analysis of the Policies and Regulations in Place to Address the Impacts of Climate Change. Multicriteria Algorithms With Applications, 2, 1-28. https://doi.org/10.61356/j.mawa.2024.26461

[16] Smarandache, F. (2019). Neutrosophic set is a generalization of intuitionistic fuzzy set, inconsistent intuitionistic fuzzy set (picture fuzzy set, ternary fuzzy set), pythagorean fuzzy set, spherical fuzzy set, and q-rung orthopair fuzzy set, while neutrosophication is a genera. *Journal of new theory*, (29), 1–31.

[17] Ali, A. M., & Broumi, S. (2024). Machine Learning with Multi-Criteria Decision Making Model for Thyroid Disease Prediction and Analysis. Multicriteria Algorithms With Applications, 2, 80-88. https://doi.org/10.61356/j.mawa.2024.26961

[18] Deli, I., Ali, M., & Smarandache, F. (2015). Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. *2015 international conference on advanced mechatronic systems (icamechs)* (pp. 249–254). Ieee.

[19] Krizhevsky, A., Sutskever, I., & Hinton, G. E. (2017). ImageNet classification with deep convolutional neural networks. *Communications of the acm*, *60*(6), 84–90.

[20] Okoli, C., & Schabram, K. (2015). A guide to conducting a systematic literature review of information systems research.

[21] Paul, S. G., Biswas, A. A., Saha, A., Zulfiker, M. S., Ritu, N. A., Zahan, I., … Islam, M. A. (2023). A real-time application-based convolutional neural network approach for tomato leaf disease classification. *Array*, *19*, 100313.

[22] Memon, M. S., Kumar, P., & Iqbal, R. (2022). Meta deep learn leaf disease identification model for cotton crop. *Computers*, *11*(7), 102.

[23] Rangarajan, A. K., Purushothaman, R., & Ramesh, A. (2018). Tomato crop disease classification using pre-trained deep learning algorithm. *Procedia computer science*, *133*, 1040–1047.

[24] Bi, C., Wang, J., Duan, Y., Fu, B., Kang, J.-R., & Shi, Y. (2022). MobileNet based apple leaf diseases identification. *Mobile networks and applications*, 1–9.

[25] Dahiya, S., Gulati, T., & Gupta, D. (2022). Performance analysis of deep learning architectures for plant leaves disease detection. *Measurement: sensors*, *24*, 100581.

[26] Wei, S. J., Al Riza, D. F., & Nugroho, H. (2022). Comparative study on the performance of deep learning implementation in the edge computing: Case study on the plant leaf disease identification. *Journal of agriculture and food research*, *10*, 100389.

[27] Rao, U. S., Swathi, R., Sanjana, V., Arpitha, L., Chandrasekhar, K., & Naik, P. K. (2021). Deep learning precision farming: grapes and mango leaf disease detection by transfer learning. *Global transitions proceedings*, *2*(2), 535–544.

[28] Belay, A. J., Salau, A. O., Ashagrie, M., & Haile, M. B. (2022). Development of a chickpea disease detection and classification model using deep learning. *Informatics in medicine unlocked*, *31*, 100970.

Interpretation of Neutrosophic Soft cubic T-ideal in the Environment of PS-Algebra

Neha Andaleeb Khalid,[1] **Muhammad Saeed,**[1] **Florentin Smarandache**[2]

[1]Department of Mathematics, University of Management and Technology, C-II, Johar Town, Lahore, Pakistan
[2]Department of Mathematics, University of New Mexico, USA

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Abstract. This study provides an innovative approach to neutrosophic algebraic structures by introducing a new structure called Neutrosophic Soft Cubic T-ideal (NSCTID), which combines T-ideal (TID) and Neutrosophic Soft Cubic Sets (NSCSs) within the framework of PS-Algebra. Within the already-existing neutrosophic cubic structures, the addition of soft sets with the characteristics of TID makes this structure more desirable. The theoretical development of the proposed structure includes the application of fundamental ideas as union, intersection, the Cartesian product, and homomorphism. We also introduce the notions of NSCTID-translation and NSCTID-multiplication to further enhance the structure of NSCTID. By applying the idea of translation and multiplication, we offer improved algorithm for neutrosophic cubic sets to deal with different parameters that are satisying the TID's distinctive characteristics. Through this thorough research, we offer an elementary understanding of NSCTID and its capabilities, providing the way to new algebraic structures.

Keywords: Neutrosophic soft cubic set; T-ideal; PS-algebra; Cartesian product; Homomorphism; Translation and Multiplication.

1. Introduction

Zadeh was the first who put up the notion of Fuzzy Sets (FSs) in 1965, which contained a membership degree for each element say "t" [1,2]. The Intuitionistic FSs (IFSs) was established by Atanassov [3], which is a general form of FS on a universe U in which non-membership degree was taken into consideration and presented Interval-Valued IFSs which are undoubtedly both

IVFS and IFS extensions. Rahman et al. [4] employed a fresh approach of a refined intuitionistic fuzzy set to conceptualize its fundamental characteristics through set theoretic procedures such as extended union as well as intersection and same for restricted. The Neutrosophic Sets (NSs) was developed by Smarandache [5] by proposing the concept of the indeterminate degree of an element as an independent element in his 1995 manuscript, which was later published in 1998. The Interval -Valued NSs was first given by Wang et al. [6]. A new methodology for simulating fuzziness and uncertainty was established by Molodtsov in 1999 [7], which is called "soft set theory". Saeed et al. [8] conducted an exhaustive investigation of the idea of soft elements as well as soft members in the respect of soft sets (SSs). Maji et al. [9, 10] expanded SSs to IFSSs and NSSs. Smarandache [11] generalized the soft set to the hypersoft set by transforming the function F into a multi-attribute function and then introduced the hybrids of Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Hypersoft Set. In [12], Distance and similarity measures using Max-Min operators were proposed in the environment of neutrosophic hypersoft sets with application in MCDM. The concept of cubic sets considers only the intervals of membership and ignores the segments of non-membership information. However, expressing the degree of membership in fuzzy sets with exact values can be difficult in real-world situations. In such cases, it may be simple to represent ambiguity both interval and exact values instead of just one. Therefore, Jun et al. [13] proposed the cubic ISs which combines two sets to represent membership degrees: an interval-valued IFS and an IFS. This hybrid approach can be useful in decision-making when dealing with uncertain judgments. A theoretical development of cubic pythagorean fuzzy soft set was established with its application in MADM. Based on the established TOPSIS method and aggregation operators, the decision-making algorithm is proposed under an intuitionistic fuzzy hypersoft environment to resolve uncertain and confusing information [14, 15]. By combining the NS and IVNS, Ali et al. [16] proposed the definition of NCS. In [17], Mumtaz et al. made an adjustable approach to NCS based decision making by similarity measure and is employed in pattern recognition to show that they can be successfully applied to problems that contain uncertainties. The NCSS was first introduced and some of its characteristics were proved by Chinnadurai et al. [18] in 2016. Gulistan et al. [19] developed a more general approach of neutrosophic cubic soft matrices and used it in MCDM-problems. In 1978, Iseki et al. were the first who established the idea of BCK-algebra [20]. In 1980, BCI-algebras were proposed [21]. The algebra named as BCK is a subclass of BCI-algebras. PS-algebras is a generalization of algebras like BCI, BCK, Q and KU and was first explored in [22] by Priya et al.. Shah et al. investigated images as well as pre images of anti-homomorphism for semiprime, strongly prime, irreducible, and highly irreducible fuzzy ideals in rings [23, 24]. They also established the ideas of strongly

primary fuzzy ideals and strongly irreducible fuzzy ideals in ring with unity as well as commutativity and examined their importance in a Leskerian ring. The research conducted by Senapati et al. [25] examined the cubic intuitionistic q-ideals within BCI-algebras. Kandasamy et al. [26] were the first who introduced T- ideals, IFT-ideals, and IF closed T-ideals. Priya et al. [27] investigated the concept of fuzzy translation and fuzzy multiplication in the context of PS-algebra, In [28, 29], Khalid et al. established the MBJ-neutrosophic T-ideal in B-algebra and described several of its properties. The concept of NC translation and multiplication of BF-subalgebra and BF-ideal was also introduced. In [30], they presented the ideas of IF Alpha-Translation as well as multiplication and IF Magnified Beta-Alpha-Translation in the context of PS-algebra. This paper introduces the Neutrosophic Soft Cubic Set (NSCS) and its application in PS-algebra through the concept of T-ideal. Section two provides relevant definitions, while section three presents the concept of NSCTID in PS algebra. Section four examines NSCTID's fundamental properties, followed by an exploration of its translation and multiplication in section five. Finally, section six summarizes the results and suggests future directions for research.

2. **Preliminaries**

To provide a clear understanding of the proposed work, some essential definitions that will be utilized throughout the explanation are given.

Definition 2.1. [22]:Let P be a set with the constant $'0'$ and the binary operation $'*'$ which is nonempty. If P satisfies the axioms below, it is regarded to be PS-algebra.

i. $\vartheta_1 * \vartheta_1 = 0$

ii. $\vartheta_1 * 0 = 0$

iii. $\vartheta_1 * \vartheta_2 = 0$, and $\vartheta_2 * \vartheta_1 = 0$ implies $\vartheta_1 = \vartheta_2$, for all $\vartheta_1, \vartheta_2 \in$ P.

In any PS-Algebra $(\mathrm{P}, *, 0)$, the following characteristics also satisfy for every $\vartheta_1, \vartheta_2 \in$ P.

iv. $\vartheta_1 * (\vartheta_2 * \vartheta_1) = \vartheta_2 * (\vartheta_1 * \vartheta_1)$

v. $\vartheta_2 * (\vartheta_1 * (\vartheta_2 * \vartheta_1)) = 0$

vi. $\vartheta_1 * (\vartheta_1 * (\vartheta_1 * \vartheta_2)) = \vartheta_1 * \vartheta_2$

vii. $\vartheta_2 * (\vartheta_1 * (\vartheta_1 * \vartheta_2)) = 0$

Definition 2.2. [26]:Let I be a nonempty subset of P. I be an ideal of P if:

i. $0 \in$ I,

ii. $\vartheta_1 * \vartheta_2 \in$ I and $\vartheta_2 \in$ I implies $\vartheta_1 \in$ I,

A nonempty subset I be the T-ideal if it satisfies (i) with

iii. $(\vartheta_1 * \vartheta_2) * \vartheta_3 \in$ I and $\vartheta_2 \in$ I implies $\vartheta_1 * \vartheta_3 \in$ I for all $\vartheta_1, \vartheta_2, \vartheta_3 \in$ P.

Definition 2.3. [26]:An IFS is said to be an IFT-ideal of P if it these three conditions holds:

i. $\alpha_{\mathrm{A}}(0) \geq \alpha_{\mathrm{A}}(\vartheta_1), \beta_{\mathrm{A}}(0) \leq \beta_{\mathrm{A}}(\vartheta_1)$,

ii. $\alpha_{\mathrm{A}}(\vartheta_1 * \vartheta_3) \geq \min\{\alpha_{\mathrm{A}}((\vartheta_1 * \vartheta_2) * \vartheta_3), \alpha_{\mathrm{A}}(\vartheta_2)\}$,
iii. $\beta_{\mathrm{A}}(\vartheta_1 * \vartheta_3) \leq \max\{\beta_{\mathrm{A}}((\vartheta_1 * \vartheta_2) * \vartheta_3), \beta_A(\vartheta_2)\}$, for all $\vartheta_1, \vartheta_2 \in \mathrm{P}$.

Definition 2.4. [5]:A set in P represented by

$$\mathrm{A} = \{\langle \vartheta_1, \alpha_{\mathrm{A}}(\vartheta_1), \gamma_{\mathrm{A}}(\vartheta_1), \beta_{\mathrm{A}}(\vartheta_1)\rangle \mid \vartheta_1 \in \mathrm{P}\}$$

is called NS in which the mapping $\alpha_{\mathrm{A}}(\vartheta_1) : \mathrm{P} \to]0^-, 1^+[\gamma_{\mathrm{A}}(\vartheta_1) : \mathrm{P} \to]0^-, 1^+[$ and $\beta_{\mathrm{A}}(\vartheta_1) : \mathrm{P} \to]0^-, 1^+[$ denotes the functions of truth, indeterminate, and falsity membership respectively, and satisfy

$$0^- \leq \alpha_{\mathrm{A}}(\vartheta_1) + \gamma_{\mathrm{A}}(\vartheta_1) + \beta_{\mathrm{A}}(\vartheta_1) \leq 3^+.$$

For an interval-valued NS, $\widehat{\mathrm{A}}$ in P, the mappings $\widehat{\alpha}_{\mathrm{A}}(\vartheta_1), \widehat{\gamma}_{\mathrm{A}}(\vartheta_1), \widehat{\beta}_{\mathrm{A}}(\vartheta_1) \subset [0, 1]$ with $\widehat{\alpha}_{\mathrm{A}}(\vartheta_1) = [\widehat{\alpha}_{\mathrm{A_L}}(\vartheta_1), \widehat{\alpha}_{\mathrm{A_U}}(\vartheta_1)], \hat{\beta}_{\mathrm{A}}(\vartheta_1) = [\hat{\beta}_{\mathrm{A_L}}(\vartheta_1), \hat{\beta}_{\mathrm{A_U}}(\vartheta_1)]$ and $\hat{\gamma}_{\mathrm{A}}(\vartheta_1) = [\hat{\gamma}_{\mathrm{A_L}}(\vartheta_1), \hat{\gamma}_{\mathrm{A_U}}(\vartheta_1)]$.

Definition 2.5. [16]:A NCS in P is defined as $\mathrm{C} = \langle \widehat{\mathrm{A}}, \mathrm{A}\rangle$ in which $\widehat{\mathrm{A}}$ is an interval-valued NS and A is the NS. The collection of all NCSs in P is represented by C(P).

Definition 2.6. [7]:Let E be the collection of parameters and V be the universal set. The set $\mathrm{S_K}$ which is defined as $\hat{\mathrm{S}}_{\mathrm{K}} = \{\langle \mathrm{v}, \mathrm{S(e)}\rangle, \mathrm{v} \in \mathrm{V}, \mathrm{e} \in \mathrm{K}, \mathrm{S(e)} \in \mathrm{P(V)}\}$ is said to be a soft set over V. Where $\mathrm{S} : \mathrm{E} \to \mathrm{P(V)}$ in which the P(V) represents the power set of V and $\mathrm{K} \subset \mathrm{E}$.

Definition 2.7. [18]:A SS denoted by $\widehat{\mathrm{S}}_{\mathrm{K}}$ is said to be a NCSS in P if $\widehat{\mathrm{S}}_{\mathrm{K}}$ is the mapping from E to the set C(P). i.e. $\widehat{\mathrm{S}}_{\mathrm{K}} : \mathrm{E} \to \mathrm{C(P)}$ where $\mathrm{K} \subset \mathrm{E}$. The NCSS is denoted by $\widehat{\mathrm{S}}_{\mathrm{K}} = \langle \widehat{\mathrm{B}}, \mathrm{B}\rangle$. Where $\widehat{\mathrm{B}} = \{\langle \vartheta_1, \alpha_{\widehat{\mathrm{B}}_{\mathrm{e_i}}}(\vartheta_1), \gamma_{\widehat{\mathrm{B}}_{\mathrm{e_i}}}(\vartheta_1), \beta_{\widehat{\mathrm{B}}_{\mathrm{e_i}}}(\vartheta_1)\rangle \mid \vartheta_1 \in \mathrm{P}, \mathrm{e_i} \in \mathrm{E}\}$ is the interval-valued NSS and $\mathrm{B} = \{\langle \vartheta_1, \alpha_{\mathrm{B}_{\mathrm{e_i}}}(\vartheta_1), \gamma_{\mathrm{B}_{\mathrm{e_i}}}(\vartheta_1), \beta_{\mathrm{B_i}}(\vartheta_1)\rangle \mid \vartheta_1 \in \mathrm{P}, \mathrm{e_i} \in \mathrm{E}\}$ is the NSS.

Definition 2.8. [29]:Let $\mathrm{C} = \langle \widehat{\mathrm{A}}, \mathrm{A}\rangle$ be an NCS of P and for the set $\widehat{\mathrm{A}}$, the $\mu, v \in [[0, 0], \Theta]$ and $\lambda \in [[0, 0], \mathrm{I}]$, where for the set $\mathrm{A}, \mu, v \in [0, \Gamma]$ and $\lambda \in [0, \epsilon]$. An object of the form $\mathrm{C}^{\mathrm{T}}_{\mu,v,\lambda} = \langle\ (\widehat{\mathrm{A}})^{\mathrm{T}}_{\mu,v,\lambda}, (\mathrm{A})^{\mathrm{T}}_{\mu,v,\lambda}\rangle$ is called an NC-Translation of C, when

$$(\widehat{\mathrm{A}}_{\mathrm{T}})^{\mathrm{Tr}}_{\mu}(\mathrm{t}_1) = \widehat{\mathrm{A}}_{\mathrm{T}}(\mathrm{t}_1) + \mu, (\widehat{\mathrm{A}}_{\mathrm{I}})^{\mathrm{Tr}}_{v}(\mathrm{t}_1) = \widehat{\mathrm{A}}_{\mathrm{I}}(\mathrm{t}_1) + v, (\widehat{\mathrm{A}}_{\mathrm{F}})^{\mathrm{Tr}}_{\lambda}(\mathrm{t}_1) = \widehat{\mathrm{A}}_{\mathrm{F}}(\mathrm{t}_1) - \lambda$$
$$(\mathrm{A}_{\mathrm{T}})^{\mathrm{Tr}}_{\mu}(\mathrm{t}_1) = \mathrm{A}_{\mathrm{T}}(\mathrm{t}_1) + \mu, (\mathrm{A}_{\mathrm{I}})^{\mathrm{Tr}}_{v}(\mathrm{t}_1) = \mathrm{A}_{\mathrm{I}}(\mathrm{t}_1) + v, (\mathrm{A}_{\mathrm{F}})^{\mathrm{Tr}}_{\lambda}(\mathrm{t}_1) = \mathrm{A}_{\mathrm{F}}(\mathrm{t}_1) - \lambda$$

for all $\mathrm{t}_1 \in \mathrm{P}$.

Definition 2.9. [29]:Let $\mathrm{C} = \langle \widehat{\mathrm{A}}, \mathrm{A}\rangle$ be an NCS of P and $\sigma \in [0, 1]$. A set having the representation as $\mathrm{C}^{\mathrm{Mp}}_{\sigma} = \langle((\kappa_{\mathrm{T}})^{\mathrm{Mp}}_{\sigma}, (\kappa_{\mathrm{I}})^{\mathrm{Mp}}_{\sigma}, (\kappa_{\mathrm{F}})^{\mathrm{Mp}}_{\sigma}), ((v_{\mathrm{T}})^{\mathrm{Mp}}_{\sigma}, (v_{\mathrm{I}})^{\mathrm{Mp}}_{\sigma}, (v_{\mathrm{F}})^{\mathrm{Mp}}_{\sigma})\rangle$ is called an NC-Multiplication of C, when

$$(\kappa_{\mathrm{T}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot \kappa_{\mathrm{T}}(\mathrm{t}_1), (\kappa_{\mathrm{I}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot \kappa_{\mathrm{I}}(\mathrm{t}_1), (\kappa_{\mathrm{F}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot \kappa_{\mathrm{F}}(\mathrm{t}_1)$$
$$(v_{\mathrm{T}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot v_{\mathrm{T}}(\mathrm{t}_1),\ (v_{\mathrm{I}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot v_{\mathrm{I}}(\mathrm{t}_1), (v_{\mathrm{F}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot v_{\mathrm{F}}(\mathrm{t}_1)$$

for all $\mathrm{t}_1 \in \mathrm{P}$.

3. Neutrosophic soft cubic T-ideal

This section aims to present the notion of a NSCTID, accompanied by an illustrative example. Additionally, we explore several properties that are pertinent to this concept.

Definition 3.1. Let an NSC-set which is denoted as $\mathrm{K} = \langle(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}}, (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,F}}\rangle$, where $\widehat{\mathrm{N}}_{\mathrm{K_{e_i}}}$ and $\mathrm{A}_{\mathrm{K_{e_i}}}$ represents an interval-valued NSS and NSS respectively in PS-algebra P. The set K is termed as Neutrosophic soft cubic T-ideal (NSCTID) in P if it achieves the following assertions:

i. $(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(0) \geq (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t}_1), (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(0) \geq (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t}_1), (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(0) \geq (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t}_1),$
$(\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(0) \leq (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t}_1), (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(0) \leq (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t}_1), (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(0) \leq (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t}_1).$

ii. $(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t}_1 * \mathrm{t}_3) \geq \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t}_2)\},$
$(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t}_1 * \mathrm{t}_3) \geq \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t}_2)\},$
$(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t}_1 * \mathrm{t}_3) \geq \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t}_2)\},$

iii. $(\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t}_1 * \mathrm{t}_3) \leq \max\{(\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t}_2)\}$
$(\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t}_1 * \mathrm{t}_3) \leq \max\{(\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t}_2)\},$
$(\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t}_1 * \mathrm{t}_3) \leq \max\{(\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t}_2)\},$

For all $\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in \mathrm{P}$. Where $(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}}$ and $(\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}} \in [0, 1]$.

Example 3.2. Let $\mathrm{P} = \{0, \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3\}$ be a PS-algebra with the following Cayley table. We

TABLE 1. Cayley table of $(\mathrm{P}, *, 0)$.

$*$	0	t_1	t_2	t_3
0	0	t_2	t_1	t_3
t_1	0	0	0	t_2
t_2	0	0	0	t_2
t_3	0	t_2	t_2	0

define a NSCS represented as $\mathrm{K} = \langle\widehat{\mathrm{N}}_{\mathrm{e_i}}, \mathrm{A}_{\mathrm{e_i}}\rangle$ in P as in Table **2** and Table **3**. The set K with the aforementioned values satisfies all of the requirements of the definition **3.1** above.

The calculations below show a few outcomes.

$$[0.5, 0.7] = (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t}_1 * \mathrm{t}_3) = (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t}_2) \geq \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t}_3), (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t}_2)\} = \mathrm{rmin}\{[0.4, 0.6], [0.5, 0.7]\},$$
$$[0.4, 0.6] = (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t}_1 * \mathrm{t}_3) = (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t}_2) \geq \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t}_3), (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t}_2)\} = \mathrm{rmin}\{[0.3, 0.5], [0.4, 0.6]\},$$
$$[0.4, 0.5] = (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t}_1 * \mathrm{t}_3) = (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t}_2) \geq \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t}_3), (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t}_2)\} = \mathrm{rmin}\{[0.2, 0.4], [0.4, 0.5]\},$$

TABLE 2. Interval-valued NSS $(\widehat{\mathrm{N}}_{\mathrm{e_i}})$.

$*$	0	$\mathrm{t_1}$	$\mathrm{t_2}$	$\mathrm{t_3}$
$(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}$	$[0.6, 0.8]$	$[0.3, 0.6]$	$[0.5, 0.7]$	$[0.4, 0.6]$
$(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}$	$[0.5, 0.7]$	$[0.3, 0.4]$	$[0.4, 0.6]$	$[0.3, 0.5]$
$(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}$	$[0.4, 0.9]$	$[0.3, 0.6]$	$[0.4, 0.5]$	$[0.2, 0.4]$

TABLE 3. NSS $(\mathrm{A}_{\mathrm{e_i}})$.

$*$	0	$\mathrm{t_1}$	$\mathrm{t_2}$	$\mathrm{t_3}$
$(\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}$	0.2	0.3	0.5	0.6
$(\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}}$	0.5	0.6	0.8	1
$(\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}}$	0.3	0.7	0.4	0.5

$$0.5 = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_1} * \mathrm{t_3}) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_2}) \leq \max\{(\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_3}), (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_2})\} = 0.6,$$
$$0.8 = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_1} * \mathrm{t_3}) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_2}) \leq \max\{(\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_3}), (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_2})\} = 1,$$
$$0.4 = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_1} * \mathrm{t_3}) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_2}) \leq \max\{(\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_3}), (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_2})\} = 0.5$$

Hence $\mathrm{K} = \langle(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{e_i}, \mathrm{A_K})^{\mathrm{T,I,F}}_{e_i}\rangle$ is NSCTID in P.

Theorem 3.3. *Every NSCTID* $K = \langle(\widehat{N}_K)^{\mathrm{T,I,F}}_{e_i}, (\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}\rangle$ *of* P *fulfills the following inequalities for all* $\mathrm{x}, \mathrm{t_1}, \mathrm{t_2}, \mathrm{t_3} \in \mathrm{P}$.

i. *If* $\mathrm{t_1} * \mathrm{t_2} \leq \mathrm{t_3}$, *then*

$$(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_1} * \mathrm{x}) \geq \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,}}_{\mathrm{e_i}}(\mathrm{t_3}), (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_2})\}$$
$$(\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_1} * \mathrm{x}) \leq \max\{(\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_3}), (\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_2})\}.$$

Proof. Let $\mathrm{x}, \mathrm{t_1}, \mathrm{t_2}, \mathrm{t_3} \in \mathrm{P}$ such that $\mathrm{t_1} * \mathrm{t_2} = \mathrm{t_3}$. Now, $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_1} * \mathrm{x}) \geq \{\mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{x}), (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_2})\} \geq \mathrm{rmin}\{\mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(((\mathrm{t_1} * \mathrm{t_3}) * \mathrm{t_2}) * \mathrm{x}), (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_3})\}, (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,F}}_{\mathrm{e_i}}(\mathrm{t_2})\}$ $= \mathrm{rmin}\{\mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(0), (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_3})\}, (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_2})\} = \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I}}_{\mathrm{e_i}}(\mathrm{t_3}), (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_2})\}$
$(\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_1} * \mathrm{x}) \leq \{\max\{(\mathrm{A_K})^{\mathrm{T,I}}_{\mathrm{e_i}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{x}), (\mathrm{A_K})^{\mathrm{T,IF}}_{\mathrm{e_i}}(\mathrm{t_2})\} \leq \max\{\max\{(\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(((\mathrm{t_1} * \mathrm{t_3}) * \mathrm{t_2}) * \mathrm{x})), (\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_3})\}, (\mathrm{A_K})^{\mathrm{T,F}}_{\mathrm{e_i}}(\mathrm{t_2})\} = \max\{\max\{(\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(0), (\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_3})\}, (\mathrm{A_K})^{\mathrm{T,F}}_{\mathrm{e_i}}(\mathrm{t_2})\} =$ $\max\{(\mathrm{A_K})^{\mathrm{T,,F}}_{\mathrm{e_i}}(\mathrm{t_3}), (\mathrm{A_K})^{\mathrm{T,F}}_{\mathrm{e_i}}(\mathrm{t_2})\}$.

Hence

$$(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_1} * \mathrm{x}) \geq \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,IF}}_{\mathrm{e_i}}(\mathrm{t_3}), (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_2})\}$$
$$(\mathrm{A_K})^{\mathrm{T,IF}}_{\mathrm{e_i}}(\mathrm{t_1} * \mathrm{x}) \leq \max\{(\mathrm{A_K})^{\mathrm{T,IF}}_{\mathrm{e_i}}(\mathrm{t_3}), (\mathrm{A_K})^{\mathrm{T,I,F}}_{e_i}(\mathrm{t_2})\}$$

forall $\mathrm{t_1}, \mathrm{t_2}, \mathrm{t_3} \in \mathrm{P}$. □

ii. *If* $t_1 \leq t_2$, *then*

$$(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * x) \geq (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)$$
$$(A_K)^{T,I,F}_{e_i}(t_1 * x) \leq (A_K)^{T,I,F}_{e_i}(t_2)$$

for all $x, t_1, t_2, t_3 \in P$.

Proof. Let $x, t_1, t_2, t_3 \in P$ such that $t_1 \leq t_2 \to t_1 * t_2 = 0$, Now, $(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * x) \geq \{\text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * t_2) * x), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\} = \{\text{rmin}\{(\widehat{N}_K)^{T,F}_{e_i}(0), (\widehat{N}_K)^{T,F}_{e_i}(t_2)\} = (\widehat{N}_K)^{T,F}_{e_i}(t_2)$.
And
$(A_K)^{T,I,F}_{e_i}(t_1 * x) \leq \{\max\{(A_K)^{T,I,F}_{e_i}((t_1 * t_2) * x), (A_K)^{T,I,F}_{e_i}(t_2)\}. = \{\max\{(A_K)^{T,I,F}_{e_i}(0), (A_K)^{T,I,F}_{e_i}(t_2)\}\} = (A_K)^{T,IF}_{e_i}(t_2)$.
Hence $(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * x) \geq (\widehat{N}_K)^{T_i,IF}_{e_i}(t_2)$ and $(A_K)^{T,I,F}_{e_i}(t_1 * x) \leq (A_K)^{T,I,F}_{e_i}(t_2)$ for all $t_1, t_2, t_3 \in P$.
□

iii.

$$((\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * (t_2 * t_1)) * x) \geq (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)$$
$$((A_K)^{T,I,F}_{e_i}(t_1 * (t_2 * t_1)) * x) \leq (A_K)^{T,I,F}_{e_i}(t_2).$$

Proof. Let K is an NSCTID. Then, $(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * (t_2 * t_1)) * x) \geq \text{rmin}\{(\widehat{N}_K)^{T,I}_{e_i}(((t_1 * (t_2 * t_1)) * t_2) * x), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\} = \text{rmin}\{(\widehat{N}_K)^{T,I}_{e_i}(0), (\widehat{N}_K)^{T,IF}_{e_i}(t_2)\} = (\widehat{N}_K)^{T,I}_{e_i}(t_2)$, $(A_K)^{T,I,F}_{e_i}((t_1 * (t_2 * t_1)) * x) \leq \max\{(A_K)^{T,I,F}_{e_i}(((t_1 * (t_2 * t_1)) * t_2) * x), (A_K)^{T,I,F}_{e_i}(t_2)\} = \max\{(A_K)^{T,F}_{e_i}(0), (A_K)^{T,F}_{e_i}(t_2)\} = (A_K)^{T,I,F}_{e_i}(t_2)$.
Hence

$$(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * (t_2 * t_1)) * x) \geq (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)$$
$$(A_K)^{T,I,F}_{e_i}((t_1 * (t_2 * t_1)) * x) \leq (A_K)^{T,F}_{e_i}(t_2),$$

for all $x, t_1, t_2, t_3 \in P$. □

Theorem 3.4. *Suppose* $K = \langle (\widehat{N}_K)^{T,I,F}_{e_i}, (A_K)^{T,,F}_{e_i} \rangle$ *and* $L = \langle (\widehat{N}_L)^{T,I,F}_{e_i}, (A_L)^{T,I,F}_{e_i} \rangle$ *are NSCTIDs of* P. *Then their union* $K \cup L$ *is also an NSCTID of* P.

Proof. If $t_1, t_2 \in K \cup L$, then $t_1, t_2 \in K$ and $t_1, t_2 \in L$. Then, $(\widehat{N}_{e_i})^{T,I,F}_{KUL}(0) = (\widehat{N}_{e_i})^{T,I,F}_{KUL}(t_1 * t_1) = \text{rmax}\{(\widehat{N}_K)^{T,IF}_{e_i}(t_1 * t_1), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1 * t_1)\} \geq \text{rmax}\{\text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1), (\widehat{N}_K)^{T,I,F}_{e_i}(t_1)\}, \text{rmin}\{(\widehat{N}_L)^{T,I,F}_{e_i}(t_1), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1)\}\} = \text{rmax}\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1)\} = (\widehat{N}_{e_i})^{T,I,F}_{KUL}(t_1)$, And $(A_{e_i})^{T,I,F}_{K\cup L}(0) = (A_{e_i})^{T,I,F}_{K\cup L}(t_1 * t_1) = \min\{(A_K)^{T,I,F}_{e_i}(t_1 * t_1), (A_L)^{T,I,F}_{e_i}(t_1 * t_1)\} \leq \min\{\max\{(A_K)^{T,I,F}_{e_i}(t_1), (A_K)^{T,I,F}_{e_i}(t_1)\}, \max\{(A_L)^{T,I,F}_{e_i}(t_1), (A_L)^{T,I,F}_{e_i}(t_1)\}\} = \min\{(A_K)^{T,I,F}_{e_i}(t_1), (A_L)^{T,I,F}_{e_i}(t_1)\} = (A_{e_i})^{T,I,F}_{KUL}(t_1)$. Thus $(\widehat{N}_{e_i})^{T,I,F}_{K\cup L}(0) \geq (\widehat{N}_{e_i})^{T,I,F}_{K\cup L}(t_1)$ and $(A_{e_i})^{T,I,F}_{KUL}(0) \leq (A_{e_i})^{TI,F}_{K\cup L}(t_1)$.
Now, $(\widehat{N}_{e_i})^{T,I,F}_{KUL}(t_1 * t_3) = \text{rmax}\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * t_3), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1 * t_3)\} \geq \text{rmax}\{\text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\}, \text{rmin}\{(\widehat{N}_L)_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_L)^{T,I,F}_{e_i}(t_2)\}\} = \text{rmin}\{\text{rmax}\{(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_L)^{T,I,F}_{e_i}((t_1 * t_2) * t_3)\}\ \text{rmax}\{(\widehat{N}_K)^{T,IF}_{e_i}(t_2), (\widehat{N}_L)^{T,I,F}_{e_i}(t_2)\}\} = \text{rmin}\{(\widehat{N}_{e_i})^{T,I,F}_{KUL}((t_1 * t_2) * t_3), (\widehat{N}_{e_i})^{T,I,F}_{KUL}(t_2)\}$, And $(A_{e_i})^{T,I,F}_{KUL}(t_1 * t_3) = \min\{(A_K)^{T,I,F}_{e_i}(t_1 * t_3), (A_L)^{T,I,F}_{e_i}(t_1 * t_3)\}$

$\leq \min\{\max\{(A_K)^{T,F}_{e_i}((t_1 * t_2) * t_3), (A_K)^{T,I,F}_{e_i}(t_2)\}, \max\{(A_L)^{T,F}_{e_i}((t_1 * t_2) * t_3), (A_L)^{T,I,}_{e_i}(t_2)\}\}$
$= \max\{\min\{(A_K)^{T,I,F}_{e_i}((t_1 * t_2) * t_3), (A_L)^{T,I,F}_{e_i}((t_1 * t_2) * t_3)\}, \min\{(A_K)^{T,I,F}_{e_i}(t_2), (A_L)^{T,I,F}_{e_i}(t_2)\}\}$
$= \max\{(A_{e_i})^{T,I,F}_{KUL}((t_1 * t_2) * t_3), (A_{e_i})^{T,I,}_{KUL}(t_2)\}$
Hence the union " $K \cup L$ " is also an NSCTID of P. □

Theorem 3.5. *Suppose* $K = \langle(\widehat{N}_K)^{T,I,F}_{e_i}, (A_K)^{T,I,F}_{e_i}\rangle$ *and* $L = \langle(\widehat{N}_L)^{T,I,F}_{e_i}, (A_L)^{T,I,F}_{e_i}\rangle$ *are two NSCTIDs of a PS-algebra* P. *Then the intersection* $K \cap L$ *is also NSCTID of* P.

Proof. Let $t_1, t_2 \in K \cap L$, then $t_1, t_2 \in K$ and $t_1, t_2 \in L$. As $(\widehat{N}_{e_i})^{T,I,F}_{K\cap L}(0) = (\widehat{N}_{e_i})^{T,I,F}_{K\cap L}(t_1 * t_1)$ $= \text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * t_1), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1 * t_1)\} \geq \text{rmin}\{\text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1), (\widehat{N}_K)^{T,I,F}_{e_i}(t_1)\}, \text{rmin}\{(\widehat{N}_L)^{T,I,F}_{e_i}(t_1), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1)\}\} = \text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1)\} = (\widehat{N}_{e_i})^{T,I,F}_{K\cap L}(t_1)$. And $(A_{e_i})^{T,I,F}_{K\cap L}(0) = (A_{e_i})^{T,I,F}_{K\cap L}(t_1 * t_1) = \max\{(A_K)^{T,I,F}_{e_i}(t_1 * t_1), (A_L)^{T,I,F}_{e_i}(t_1 * t_1)\} \leq \max\{\max\{(A_K)^{T,I,F}_{e_i}(t_1), (A_K)^{T,I,F}_{e_i}(t_1)\}, \max\{(A_L)^{T,I,F}_{e_i}(t_1), (A_L)^{T,I,F}_{e_i}(t_1)\}\} = \max\{(A_K)^{T,I,F}_{e_i}(t_1), (A_L)^{T,I,F}_{e_i}(t_1)\} = (A_{e_i})^{T,I,F}_{K\cap L}(t_1)$. Thus$(\widehat{N}_{e_i})^{T,I,F}_{K\cap L}(0) \geq (\widehat{N}_{e_i})^{T,I,F}_{K\cap L}(t_1)$ and $(A_{e_i})^{T,I,F}_{K\cap L}(0) \leq (A_{e_i})^{T,IF}_{K\cap L}(t_1)$.
Now, $(\widehat{N}_{e_i})^{T,I,F}_{K\cap L}(t_1 * t_3) = \text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * t_3), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1 * t_3)\} \geq \text{rmin}\{\text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\}, \text{rmin}\{(\widehat{N}_L)^{T,F}_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_L)^{T,I,F}_{e_i}(t_2)\}\} = \text{rmin}\{\text{rmin}\{(\widehat{N}_K)^{T,IF}_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_L)^{T,I,F}_{e_i}((t_1 * t_2) * t_3)\}$ $\text{rmin}\{(\widehat{N}_K)^{T,IF}_{e_i}(t_2), (\widehat{N}_L)^{T,I,F}_{e_i}(t_2)\}\} = \text{rmin}\{(\widehat{N}_{e_i})^{T,I,F}_{K\cap L}((t_1 * t_2) * t_3), (\widehat{N}_{e_i})^{T,I,F}_{K\cap L}(t_2)\}$, And $A_{e_{i_{K\cap L}}}(t_1 * t_3) = \max\{(A_K)^{T,I,F}_{e_i}(t_1 * t_3), (A_L)^{T,I,F}_{e_i}(t_1 * t_3)\} \leq \max\{\max\{(A_K)^{T,I,F}_{e_i}((t_1 * t_2) * t_3), (A_K)^{T,I,F}_{e_i}(t_2)\}, \max\{(A_L)^{T,I,F}_{e_i}((t_1 * t_2) * t_3), (A_L)^{T,I,F}_{e_i}(t_2)\}\}$
$= \max\{\max\{(A_K)^{T,I,F}_{e_i}((t_1 * t_2) * t_3), (A_L)^{T,I,F}_{e_i}((t_1 * t_2) * t_3)\}, \max\{(A_K)^{T,I,F}_{e_i}(t_2), (A_L)^{T,I,F}_{e_i}(t_2)\}\}$
$= \max\{(A_{e_i})^{T,I,F}_{K\cap L}((t_1 * t_2) * t_3), (A_{e_i})^{T,I,F}_{K\cap L}(t_2)\}$
Hence, $K \cap L$ is an NSCTID of P. □

4. Cartesian product and Homomorphism of NSCTID

In this section, the interpretation of the cartesian product and homomorphism of NSCTID is given by proving some theorems.

Definition 4.1. Let $K = ((\widehat{N}_K)^{T,I,F}_{e_i}, (A_K)^{T,I,F}_{e_i})$ and $L = ((\widehat{N}_L)^{T,I,F}_{e_i}, (A_L)^{T,I,F}_{e_i})$ are two NSCTIDs of R and P respectively. The cartesian product $K \times L = (R \times P, (\widehat{N}_K)^{T,I,F}_{e_i} \times \widehat{N}_L)^{T,I,F}_{e_i}, (A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i})$ is defined as:

$$(\widehat{N}_K)^{T,I,F}_{e_i} \times (\widehat{N}_L)^{T,I,F}_{e_i}(t_1, t_2) \geq \text{rmin}\{(\widehat{N}_K)^{T,IF}_{e_i}(t_1), (\widehat{N}_L)^{T,I,F}_{e_i}(t_2)\}$$
$$(A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i}(t_1, t_2) \leq \max\{(A_K)^{T,I,F}_{e_i}(t_1), (A_L)^{T,I,F}_{e_i}(t_2)\}.$$

Where $(\widehat{N}_K)^{T,I,F}_{e_i} \times (\widehat{N}_L)^{T,I,F}_{e_i} : R \times P \to [0, 1]$ and $(A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i} : R \times P \to [0, 1]$ for all $t_1 \in R$ and $t_2 \in P$.

Theorem 4.2. *Let* K *and* L *be two NSCTIDs of* P, *then* $K \times L$ *is an NSCTID of* $R \times P$.

Proof. Let R and P be two PS-algebras.For any $t_1, t_2 \in R \times P$, We have $((\widehat{N}_K)_{e_i}^{T,I,F} \times (\widehat{N}_L)_{e_i}^{T,I,F})(0,0) = \text{rmin}\{(\widehat{N}_K)_{e_i}^{T,I,F}(0), (\widehat{N}_L)_{e_i}^{T,I,F}(0)\} \geq \text{rmin}\{(\widehat{N}_K)_{e_i}^{T,F}(t_1), (\widehat{N}_L)_{e_i}^{T,F}(t_2)\} = ((\widehat{N}_K)_{e_i}^{T,F} \times (\widehat{N}_L)_{e_i}^{T,I,F})(t_1,t_2)((A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F})(0,0) = \max\{(A_K)_{e_i}^{T,F}(0), (A_L)_{e_i}^{T,I,F}(0)\} \leq \max\{(A_K)_{e_i}^{T,I,F}(t_1), (A_L)_{e_i}^{T,I,F}(t_2)\} = ((A_K)_{e_i}^{T,F} \times (A_L)_{e_i}^{T,I,F})(t_1,t_2)$ That is $((\widehat{N}_K)_{e_i}^{T,I,F} \times (\widehat{N}_L)_{e_i}^{T,F})(0,0) \geq ((\widehat{N}_K)_{e_i}^{T,I,F} \times (\widehat{N}_L)_{e_i}^{T,I})(t_1,t_2)\ ((A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,F})(0,0) \leq ((A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,F})(t_1,t_2)$.

Now, Let $(x_1,x_2),(t_1,t_2)$ and $(y_1,y_2) \in R \times P$.Then, $((\widehat{N}_K)_{e_i}^{T,I} \times (\widehat{N}_L)_{e_i}^{T,I})(t_1 * x_1, t_2 * x_2) = \text{rmin}\{(\widehat{N}_K)_{e_i}^{T,I,F}(t_1 * x_1), (\widehat{N}_L)_{e_i}^{T,IF}(t_2 * x_2)\} \geq \text{rmin}\{\text{rmin}\{(\widehat{N}_K)_{e_i}^{T,I}((t_1 * y_1) * x_1), (\widehat{N}_K)_{e_i}^{T,IF}(y_1)\}, \text{rmin}\{(\widehat{N}_L)_{e_i}^{T,I,F}((t_2 * y_2) * x_2), (\widehat{N}_L)_{e_i}^{T,I,F}(y_2)\}\} = \text{rmin}\{\text{rmin}\{(\widehat{N}_K)_{e_i}^{T,I,F}((t_1 * y_1) * x_1), (\widehat{N}_L)_{e_i}^{T,I,F}((t_2 * y_2) * x_2)\}\ \text{rmin}\{(\widehat{N}_K)_{e_i}^{T,I,F}(y_1), (\widehat{N}_L)_{e_i}^{T,I,F}(y_2)\}\} = \text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F} \times (\widehat{N}_L)_{e_i}^{T,I,F})(((t_1 * y_1)x_1)\ ((t_2 * y_2) * x_2)), ((\widehat{N}_K)_{e_i}^{T,IF} \times (\widehat{N}_L)_{e_i}^{T,IF})(y_1,y_2)\} = \text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F} \times (\widehat{N}_L)_{e_i}^{T,I,F})((t_1 * x_1, t_2 * x) * (y_1,y_2)),\ ((\widehat{N}_K)_{e_i}^{T,I,F} \times (\widehat{N}_L)_{e_i}^{T,I,F})(y_1,y_2)$.

And $(A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F})(t_1 * x_1, t_2 * x_2) = \max\{(A_K)_{e_i}^{T,I,F}(t_1 * x_1), (A_L)_{e_i}^{T,I,F}(t_2 * x_2)\} \leq \max\{\max\{(A_K)_{e_i}^{T,I,F}((t_1 * y_1) * x_1), (A_K)_{e_i}^{T,F}(y_1)\}, \max\{(A_L)_{e_i}^{T,I}((t_2 * y_2) * x), (A_L)_{e_i}^{T,I,F}(y_2)\}\} = \max\{\max\{(A_K)_{e_i}^{T,I,F}((t_1 * y_1) * x_1), (A_L)_{e_i}^{T,I,F}((t_2 * y_2) * x_2)\}, \max\{(A_K)_{e_i}^{T,I,F}(y_1), (A_L)_{e_i}^{T,I,F}(y_2)\} = \max\{((A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F})(((t_1 * y_1) * x_1), ((t_2 * y_2) * x_2)),\ (A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F})(y_1,y_2)\} = \max\{(A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F})((t_1 * x_1, t_2 * x_2) * (y_1,y_2)),\ (A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F})(y_1,y_2)$.

Thus $((\widehat{N}_K)_{e_i}^{T,I,F} \times (\widehat{N}_L)_{e_i}^{T,I,F})(t_1 * x_1, t_2 * x_2) \geq \text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F} \times (\widehat{N}_L)_{e_i}^{T,I,F})((t_1 * x_1, t_2 * x_2) * (y_1,y_2)), ((\widehat{N}_K)_{e_i}^{T,I,F} \times (\widehat{N}_L)_{e_i}^{T,I,F})(y_1,y_2)$.$(A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F})(t_1 * x_1, t_2 * x_2) \leq \max\{(A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F})((t_1 * x_1, t_2 * x_2) * (y_1,y_2)), (A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F})(y_1,y_2)$. □

Theorem 4.3. *Let* K *and* L *are two NSCSs of* R *and* P *such that* K × L *is an NSCTID of* R × P*, Then*

i. *For all* $t_1 \in R$ *and* $t_2 \in P$,

$$(\widehat{N}_K)_{e_i}^{T,I,F}(0) \geq (\widehat{N}_K)_{e_i}^{T,I,F}(t_1), (A_K)_{e_i}^{T,I,F}(0) \leq (A_K)_{e_i}^{T,I,F}(t_2),$$
$$(\widehat{N}_L)_{e_i}^{T,I}, F(0) \geq (\widehat{N}_L)_{e_i}^{T,I,F}(t_1), (A_L)_{e_i}^{T,I,F}(0) \leq (A_L)_{e_i}^{T,I,F}(t_2).$$

ii. *For all* $t_1 \in P$, *If* $(\widehat{N}_K)_{e_i}^{T,I,F}(0) \geq (\widehat{N}_K)_{e_i}^{T,I,F}(t_1)$ *then,*

$$(\widehat{N}_L)_{e_i}^{T,I,F}(0) \geq (\widehat{N}_K)_{e_i}^{T,I,F}(t_1) \text{ and } (\widehat{N}_L)_{e_i}^{T,I,F}(0) \geq (\widehat{N}_L)_{e_i}^{T,I,F}(t_1),$$

Also if $(A_K)e_{e_i}^{T,F}(0) \leq (A_K)_{e_i}^{T,I,F}(t_1)$ *then*

$$(A_L)_{e_i}^{T,I,F}(0) \leq (A_K)_{e_i}^{T,I,F}(t_1) \text{ and } (A_L)_{e_i}^{T,I,F}(0) \leq (A_L)_{e_i}^{T,I,F}(t_1).$$

iii. *For all* $t_1 \in P$, *If* $(\widehat{N}_L)_{e_i}^{T,I,F}(0) \geq (\widehat{N}_L)_{e_i}^{T,I,F}(t_1)$ *then,*

$$(\widehat{N}_K)_{e_i}^{T,I,F}(0) \geq (\widehat{N}_K)_{e_i}^{T,I,F}(t_1) \text{and } (\widehat{N}_K)_{e_i}^{T,I,F}(0) \geq (\widehat{N}_L)_{e_i}^{T,I,F}(t_1),$$

Also if $(\mathrm{A_L})_{e_i}^{T,I,F}(0) \leq (\mathrm{A_L})_{e_i}^{T,I,F}(t_1)$ *then*

$$(\mathrm{A_K})_{e_i}^{T,I,F}(0) \leq (\mathrm{A_K})_{e_i}^{T,I,F}(t_1) and (\mathrm{A_K})_{e_i}^{T,I,F}(0) \leq (\mathrm{A_L})_{e_i}^{T,I,F}(t_1).$$

Proof. ***i.*** Suppose $(\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(t_1) > (\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(0)$ or $(\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F}(t_2) > (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F}(0)$ for all $t_1 \in R$ and $t_2 \in P$. Then, $(\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F})(t_1,t_2) = \mathrm{rmin}\{(\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(t_1), (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F}(t_2)\} > \mathrm{rmin}\{(\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(0), (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F}(0)\} = ((\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F})(0,0)$. Thus $((\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F})(t_1,t_2) > ((\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F})(0,0)$ for all $t_1 \in \mathrm{R}$ and $t_2 \in \mathrm{P}$, which is the contradiction to $((\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F})$ is a NSCTID of $\mathrm{R} \times \mathrm{P}$.
Also, if $(\mathrm{A_K})_{e_i}^{T,I,F}(t_1) < (\mathrm{A_K})_{e_i}^{T,I,F}(0)$ or $(\mathrm{A_L})_{e_i}^{T,I,F}(t_2) < (\mathrm{A_L})_{e_i}^{T,I,F}(0)$ for all $t_1 \in \mathrm{R}$ and $t_2 \in \mathrm{P}$. Then, $(\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})(t_1,t_2) = \max\{(\mathrm{A_K}))_{e_i}^{T,I,F}(t_1), (\mathrm{A_L})_{e_i}^{T,I,F}(t_2)\} > \max\{(\mathrm{A_K})_{e_i}^{T,I,F}(0), (\mathrm{A_L})_{e_i}^{T,I,F}(0)\} = ((\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})(0,0)$.
Thus $(\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})(t_1,t_2) < (\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})(0,0)$ for all $t_1 \in \mathrm{R}$ and $t_2 \in \mathrm{P}$, which is the contradiction to $(\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})$ is an NSCTID of $\mathrm{R} \times \mathrm{P}$. □

Proof. ***ii.*** Suppose $(\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T_iI,F}(0) < (\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(t_1)$ or $(\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(0) < (\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(t_1)$ for all $t_1 \in \mathrm{R,P}$. Then, $((\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F})(0,0) = \mathrm{rmin}\{(\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I}(0), (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,IF}(0)\} = (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F}(0)$, And $((\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I})(t_1,t_1) = \mathrm{rmin}\{(\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(t_1), (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F}(t_1)\} > (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,F}(0) = ((\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,F} \times (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F})(0,0)$. This implies $((\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,IF})(t_1,t_1) = ((\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,IF} \times (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F})(0,0)$. Which is the contradiction to $((\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{TI,F} \times (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,F,F})$ is an NSCTID of $\mathrm{R} \times \mathrm{P}$. Hence if $(\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(0) \geq (\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(t_1)$ then $(\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F}(0) \geq (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}^{T,I,F}(t_1)$ and $(\widehat{\mathrm{N}}_\mathrm{L})_{e_i}(0) \geq (\widehat{\mathrm{N}}_\mathrm{L})_{e_i}(t_1)$ for all $t_1 \in \mathrm{R,P}$. Now suppose $(\mathrm{A_L})_{e_i}^{T,I,F}(0) > (\mathrm{A_K})_{e_i}^{T,I,F}(t_1)$ or $(\mathrm{A_K})_{e_i}^{T,I,F}(0) > (\mathrm{A_K})_{e_i}^{T,I,F}(t_1)$ for all $t_1 \in \mathrm{R,P}$. Then, $(\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})(0,0) = \max\{(\mathrm{A_K})_{e_i}^{T,I,F}(0), (\mathrm{A_L})_{e_i}^{T,I}(0)\} = (\mathrm{A_L})_{e_i}(0)$.
And $(\mathrm{A_K})_{e_i}^{T,IF} \times (\mathrm{A_L})_{e_i}^{T,I,F})(t_1,t_1) = \max\{(\mathrm{A_K})_{e_i}^{T,I,F}(t_1), (\mathrm{A_L})_{e_i}^{T,I,F}(t_1)\} > (\mathrm{A_L})_{e_i}^{T,I,F}(0) = (\mathrm{A_K})_{e_i}^{T,IF} \times (\mathrm{A_L})_{e_i}^{T,I,F})(0,0)$.
This implies $(\mathrm{A_K})_{e_i}^{T,IF} \times (\mathrm{A_L})_{e_i}^{T,I,F})(t_1,t_1) = (\mathrm{A_K})_{e_i}^{T,IF} \times (\mathrm{A_L})_{e_i}^{T,I,F})(0,0)$. Which is a contradiction to $(\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i^{T,I,F}})$ is an NSCTID of $\mathrm{R} \times \mathrm{P}$. □

Proof. ***iii.*** The proof is quite the same as ***ii***. □

Theorem 4.4. *Let* $\Sigma : \mathrm{P} \to \mathrm{R}$ *is a homomorphism of PS -algebra. If* $\mathrm{K} = \langle (\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}, (\mathrm{A_K})_{e_i}^{T,I,F} \rangle$ *be an NSCTID of* R *then the pre-image* $\Sigma^{-1}(\mathrm{K}) = (\Sigma^{-1}(\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}), \Sigma^{-1}((\mathrm{A_K})_{e_i}^{T,I,F}))$ *of* P *under* Σ *is an NSCTID in* P.

Proof. For any $t_1 \in \mathrm{P}$, we have
$\Sigma^{-1}((\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F})(t_1) = (\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(\Sigma(t_1)) \leq (\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(0) = (\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(\Sigma(0)) = \Sigma^{-1}(\widehat{\mathrm{N}}_\mathrm{K})_{e_i}^{T,I,F}(0)$,
$\Sigma^{-1}((\mathrm{A_K})_{e_i}^{T,I,F})(t_1) = (\mathrm{A_K})_{e_i}^{T,I,F}(\Sigma(t_1)) \geq (\mathrm{A_K})_{e_i}^{T,I,F}(0) = (\mathrm{A_K})_{e_i}^{T,I,F}(\Sigma(0)) = \Sigma^{-1}(\mathrm{A_K})_{e_i}^{T,I,F}(0)$.
Also,

$\Sigma^{-1}(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})(\mathrm{t}_1 * \mathrm{x}) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\Sigma(\mathrm{t}_1 * \mathrm{x})) \geq \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,F}}_{\mathrm{e_i}}((\Sigma(\mathrm{t}_1) * \Sigma(\mathrm{t}_2)) * \Sigma(\mathrm{x})), (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}$ $(\Sigma(\mathrm{t}_2))\}$ $= \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\Sigma((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{x}), (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,F}}_{\mathrm{e_i}}(\Sigma(\mathrm{t}_2))\} = \mathrm{rmin}\{\Sigma^{-1}((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})((\mathrm{t}_1 * \mathrm{t}_2)$ $\mathrm{x}), \Sigma^{-1}((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})(\mathrm{t}_2)\}$ $,\Sigma^{-1}((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})(\mathrm{t}_1 * \mathrm{x}) = (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\Sigma(\mathrm{t}_1 * \mathrm{x})) \leq \max\{(\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\Sigma(((\mathrm{t}_1)$ $\Sigma(\mathrm{t}_2)) * \Sigma(\mathrm{x})), (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,F}}_{\mathrm{e_i}}(\Sigma(\mathrm{t}_2))\} = \max\{(\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\Sigma((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{x}), (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,F}}_{\mathrm{e_i}}(\Sigma(\mathrm{t}_2))\} = \max\{\Sigma^{-1}$ $((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,F}}_{\mathrm{e_i}})((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{x}), \Sigma^{-1}((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})(\mathrm{t}_2)\}$. □

Theorem 4.5. *Let* $\Sigma : \mathrm{P} \to \mathrm{R}$ *be an epimorphism of PS-algebra. Then* $\mathrm{K} = \langle(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}\rangle$ *is an NSCTID of* P, *if* $\Sigma^{-1}(\ \mathrm{K}) = ((\Sigma^{-1}((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}), \Sigma^{-1}((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}$ $))$ *of* P *under* Σ *is an NSCTID in* P.

Proof. For any $\mathrm{t}_1 \in \mathrm{R}$, there exist $\mathrm{a} \in \mathrm{P}$ such that $\Sigma(\mathrm{a}) = \mathrm{t}_1$. Then, $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t}_1) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\Sigma(\mathrm{a})) = \Sigma^{-1}((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})(\mathrm{a}) \leq \Sigma^{-1}((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})(0) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\Sigma(0)) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(0)$, And $(\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t}_1) = (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\Sigma(\mathrm{a})) = \Sigma^{-1}((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})(\mathrm{a}) \geq \Sigma^{-1}((\ \mathrm{A}_{\mathrm{K}})^{\mathrm{T,F}}_{\mathrm{e_i}})(0) = (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,IF}}_{\mathrm{e_i}}(\Sigma(0)) = (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(0)$.
Let $\mathrm{y}, \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{R}$ then $\Sigma(\mathrm{x}) = \mathrm{y}, \Sigma(\mathrm{a}) = \mathrm{t}_1$ and $\Sigma(\mathrm{b}) = \mathrm{t}_2$ for some $\mathrm{a}, \mathrm{b}, \mathrm{x} \in \mathrm{P}$.
Now, $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t}_1 * \mathrm{y}) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,}}_{\mathrm{e_i}}(\Sigma(\mathrm{a} * \mathrm{x})) = \Sigma^{-1}((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})(\mathrm{a} * \mathrm{x}) \geq \mathrm{rmin}\{\Sigma^{-1}((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,F}}_{\mathrm{e_i}}((\mathrm{a} *$ $\mathrm{b}) * \mathrm{x})), \Sigma^{-1}((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{b}))\} = \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,F}}_{\mathrm{e_i}}(\Sigma((\mathrm{a} * \mathrm{b}) * \mathrm{x})), (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I}}_{\mathrm{e_i}}(\Sigma(\mathrm{b}))\} = \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}$ $(\Sigma(\mathrm{a}) * \Sigma(\mathrm{b})) * \Sigma(\mathrm{x})), (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}(\Sigma(\mathrm{b}))\} = \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{y}), (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t}_2)\}$.
And $(\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t}_1 * \mathrm{y}) = (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\Sigma(\mathrm{a} * \mathrm{x})) = \Sigma^{-1}((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,F}}_{\mathrm{e_i}}(\mathrm{a} * \mathrm{x}) \leq \max\{\Sigma^{-1}((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}((\mathrm{a} *$ $\mathrm{b}) * \mathrm{x})), \Sigma^{-1}((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I}}_{\mathrm{e_i}}(\mathrm{b}))\} = \max\{(\mathrm{A}_{\mathrm{K}})^{\mathrm{TI,F}}_{\mathrm{e_i}}(\Sigma((\mathrm{a} * \mathrm{b}) * \mathrm{x})), (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\Sigma(\mathrm{b}))\} = \max\{(\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}$ $(\Sigma(\mathrm{a}) * \Sigma(\mathrm{b})) * \Sigma(\mathrm{x}))\}, (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\Sigma(\mathrm{b}))\} = \max\{(\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{y}), (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t}_2)\}$ □

5. Translation and Multiplication of Neutrosophic Soft Cubic T-Ideal

In this section, the interpretation of NSCTID-translation and NSCTID-multiplication is given. For the simplicity, the notation $\mathrm{K} =< \mathrm{t}_1, (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t}_1), (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I}}_{\mathrm{e_i}}(\mathrm{t}_1) \mid \mathrm{t}_1 \in \mathrm{P} >$ for the NSCS is used.
In this paper, we use $\xi = [1,1] - \mathrm{rsup}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t}_1) \mid \mathrm{t}_1 \in \mathrm{P}\}$, $\Psi = \mathrm{r\,inf}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t}_1) \mid \mathrm{t}_1 \in \mathrm{P}\}$, $\zeta = 1- \sup\{(\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I}}_{\mathrm{e_i}}(\mathrm{t}_1) \mid \mathrm{t}_1 \in \mathrm{P}\}, \Phi = \inf\{(\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t}_1)\} \mid \mathrm{t}_1 \in \mathrm{P}$ for any NSCS $\mathrm{K} = \langle(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}\rangle$ of P.

Definition 5.1. Let $\mathrm{K} = \langle(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,,F}}_{\mathrm{e_i}}, (\mathrm{A}_{\mathrm{K}})^{\mathrm{T,IF}}_{\mathrm{e_i}}\rangle$ be an NSCS of P. For $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, \alpha, \beta \in [[0,0], \xi]$ and $\gamma \in [[0,0], \Psi]$, where for $(\mathrm{A}_K)^{\mathrm{T,I,F}}_{\mathrm{e_i}}, \alpha, \beta \in [0, \zeta]$ and $\gamma \in [0, \Phi]$. A set of the form $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma} = \langle((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}\rangle$ is called an Neutrosophic Soft Cubic Translation **NSCTr** of K, As for all $\mathrm{t}_1 \in \mathrm{P}$.
$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t}_1) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t}_1)+\alpha, ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t}_1) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t}_1)+\beta, ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t}_1) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t}_1)-\gamma,$
$((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t}_1) = (\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t}_1)+\alpha, ((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{Tr}_{\beta}(\mathrm{t}_1) = (\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t}_1)+\beta, ((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t}_1) = (\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t}_1)-\gamma.$

Definition 5.2. Let K be an NSCS of P and $\eta \in [0,1]$. An object having the form $\widetilde{\mathrm{K}}_{\eta}^{\mathrm{Mp}} = \langle(((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}})_{\eta}^{\mathrm{Mp}}, ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}})_{\eta}^{\mathrm{Mp}}, ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}})_{\eta}^{\mathrm{Mp}}), (((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}})_{\eta}^{\mathrm{Mp}}, ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}})_{\eta}^{\mathrm{Mp}}, ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}})_{\eta}^{\mathrm{Mp}})\rangle$ is called a Neutrosophic Soft Cubic Translation **NSCMp** of K, Where

$((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1}) = \eta \cdot (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1}) = \eta \cdot (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1}) = \eta \cdot (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t_1}),$

$((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1}) = \eta \cdot \mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t_1}), (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1}) = \eta \cdot (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t_1}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1}) = \eta \cdot (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t_1})$

for all $\mathrm{t_1} \epsilon P$.

5.1. *Neutrosophic Soft Cubic T-Ideal Translation*

This section defines neutrosophic soft cubic T-Ideal translation with a theorem and example.

Theorem 5.3. *If* K *is an NSCTID of* P*, then NSCTr* $\widetilde{\mathrm{K}}_{\alpha,\beta,\gamma}^{\mathrm{Tr}}$ *of K is an NSCTID of* P*.*

Proof. Let $\mathrm{K} = \langle(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}}, (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}}\rangle$ of P be an NSCTID of P. Then we have $((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}})_{\alpha}^{\mathrm{Tr}}(0) = (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(0) + \alpha \geq (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t_1}) + \alpha = ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\alpha}^{\mathrm{T}})^{\mathrm{Tr}}(\mathrm{t_1})$, $((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}})_{\beta}^{\mathrm{Tr}}(0) = (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(0) + \beta \geq (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t_1}) + \beta = ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}})_{\beta}^{\mathrm{Tr}}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}})_{\gamma}^{\mathrm{Tr}}(0) = (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}}(0) + \gamma \geq (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t_1}) + \gamma = ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\gamma}^{\mathrm{F}})^{\mathrm{Tr}}(\mathrm{t_1})$,
And $((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}})_{\alpha}^{\mathrm{Tr}}(0) = (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}}(0) + \alpha \leq ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t_1}) + \alpha = ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}})_{\alpha}^{\mathrm{Tr}}(\mathrm{t_1})((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}})_{\beta}^{\mathrm{Tr}}(0) = (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}}(0) + \beta \leq (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t_1}) + \beta = ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}})_{\beta}^{\mathrm{Tr}}(\mathrm{t_1})$ $((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}})_{\gamma}^{\mathrm{Tr}}(0) = (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}}(0) + \gamma \leq (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t_1}) + \gamma = ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}})_{\gamma}^{\mathrm{Tr}}(\mathrm{t_1})$ Now $((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}})_{\alpha}^{\mathrm{Tr}}(\mathrm{t_1} * \mathrm{t_3}) = \widehat{\mathrm{N}}_{\mathrm{e_i}}(\mathrm{t_1} * \mathrm{t_3}) + \alpha \geq \mathrm{rmin}\{\widehat{\mathrm{N}}_{\mathrm{e_i}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), \widehat{\mathrm{N}}_{\mathrm{e_i}\mathrm{T}}(\mathrm{t_2})\} + \alpha = \mathrm{rmin}\{\widehat{\mathrm{N}}_{\mathrm{e_i}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}) + \alpha, \widehat{\mathrm{N}}_{\mathrm{e_i}}(\mathrm{t_2}) + \alpha\} = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}})_{\alpha}^{\mathrm{Tr}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}})_{\alpha}^{\mathrm{Tr}}(\mathrm{t_2})\}$, $((\widehat{N}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}})_{\beta}^{\mathrm{Tr}}(\mathrm{t_1} * \mathrm{t_3}) = (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t_1} * \mathrm{t_3}) + \beta \geq \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t_2})\} + \beta = \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}) + \beta, (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t_2}) + \beta\} = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}})_{\beta}^{\mathrm{Tr}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}})_{\beta}^{\mathrm{Tr}}(\mathrm{t_2})\}((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}})_{\gamma}^{\mathrm{Tr}}(\mathrm{t_1} * \mathrm{t_3}) = (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t_1} * \mathrm{t_3}) + \gamma \geq \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t_2})\} + \gamma = \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}) + \gamma, (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t_2}) + \gamma\} = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}})_{\gamma}^{\mathrm{Tr}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}})_{\gamma}^{\mathrm{Tr}}(\mathrm{t_2})\}$ And $((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}})_{\alpha}^{\mathrm{Tr}}(\mathrm{t_1} * \mathrm{t_3}) = (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t_1} * \mathrm{t_3}) + \alpha \leq \max\{(\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t_2})\} + \alpha = \max\{(\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}) + \alpha, (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t_2}) + \alpha\} = \max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}})_{\alpha}^{\mathrm{Tr}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T}})_{\alpha}^{\mathrm{Tr}}(\mathrm{t_2})\}$, $((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}})_{\beta}^{Tr}(\mathrm{t_1} * \mathrm{t_3}) = (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t_1} * \mathrm{t_3}) + \beta \leq \max\{(\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t_2})\} + \beta = \max\{(\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}) + \beta, (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t_2}) + \beta\} = \max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}})_{\beta}^{\mathrm{Tr}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{I}})_{\beta}^{\mathrm{Tr}}(\mathrm{t_2})\}$, $((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}})_{\gamma}^{\mathrm{Tr}}(\mathrm{t_1} * \mathrm{t_3}) = (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t_1} * \mathrm{t_3}) + \gamma \leq \max\{(\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t_2})\} + \gamma = \max\{(\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}) + \gamma, (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t_2}) + \gamma\} = \max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}})_{\gamma}^{\mathrm{Tr}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{F}})_{\gamma}^{\mathrm{Tr}}(\mathrm{t_2})\}$, Hence $\widetilde{K}_{\alpha,\beta,\gamma}^{\mathrm{Tr}}$ of K is an NSCTID of P. □

Example 5.4. Let $P = \{0, \mathrm{t_1}, \mathrm{t_2}, \mathrm{t_3}\}$ be a PS-algebra with the cayley's table as shown in **Table 1**. The NSCS $\mathrm{K} = \langle(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e}_i}^{\mathrm{T,I,F}}, (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}}\rangle$ of P is defined as

$$(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T}}(\mathrm{t_i}) = \{[0.7, 0.9] \text{ if } \mathrm{t_i} = 0 \text{ and } [0.3, 0.6] \text{ if otherwise}$$

$$(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{I}}(\mathrm{t_i}) = \{[0.6, 0.8] \text{ if } \mathrm{t_i} = 0 \text{ and } [0.4, 0.5] \text{ if otherwise}$$

$$(\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{F}}(\mathrm{t_i}) = \{[0.5, 1] \text{ if } \mathrm{t_i} = 0 \text{ and } [0.2, 0.7] \text{ if otherwise}$$

And

$$(\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t}_i) = \{0.2 \text{ if } \mathrm{t}_i = 0 \text{ and } 0.7 \text{ if otherwise}$$

$$(\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t}_i) = \{0.5 \text{ if } \mathrm{t}_i = 0 \text{ and } 0.9 \text{ if otherwise}$$

$$(\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t}_i) = \{0.4 \text{ if } \mathrm{t}_i = 0 \text{ and } 1 \text{ if otherwise.}$$

The set K is an NSCTID of P as it satisfies (i) but fulfills (ii) and (iii) with the condition: If $\mathrm{i} = \mathrm{j}$ then $\mathrm{t_i} * \mathrm{t_j} = 0$ with

$$(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i} * \mathrm{t_j}) = [0.7, 0.9], (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t_i} * \mathrm{t_j}) = [0.6, 0.8], (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_i} * \mathrm{t_j}) = [0.5, 1],$$

Otherwise

$$(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i} * \mathrm{t_j}) = [0.3, 0.6], (\mathrm{t_i} * \mathrm{t_j}) = [0.4, 0.5], (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_i} * \mathrm{t_j}) = [0.2, 0.7]$$

And

$$(\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) = 0.2, (\ \mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t_i}) = 0.5, (\ \mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_i}) = 0.4,$$

Otherwise $(\mathrm{A_K})^{T}_{\mathrm{e_i}}(\mathrm{t}_i) = 0.7, (\mathrm{A_K})^{I}_{\mathrm{e_i}}(\mathrm{t}_i) = 0.9, (\mathrm{A_K})^{F}_{\mathrm{e_i}}(\mathrm{t}_i) = 1$ given in definition 3.1. Now, for $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}$ we choose $\alpha = [0.04, 0.08], \beta = [0.05, 0.09], \gamma = [0.03, 0.07]$ and for $(\mathrm{A})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, \alpha = 0.03, \beta = 0.04, \gamma = 0.05$. Then the mapping $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \mid \mathrm{P} \to [0, 1]$ is given by

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.04,0.08]}(0) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(0) + [0.04, 0.08] = [0.74, 0.98]$$

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.05,0.09]}(0) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(0) + [0.05, 0.09] = [0.65, 0.89]$$

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.03,0.07]}(0) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(0) - [0.03, 0.07] = [0.47, 0.93]$$

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.04,0.08]}(\mathrm{t_i}) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) + [0.04, 0.08] = [0.34, 0.68]$$

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.05,0.09]}(\mathrm{t_i}) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) + [0.05, 0.09] = [0.45, 0.59]$$

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.03,0.07]}(\mathrm{t_i}) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_i}) - [0.03, 0.07] = [0.17, 0.63]$$

$$((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.03}(0) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(0) + 0.03 = [0.23]$$

$$((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.04}(0) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(0) + 0.04 = [0.54]$$

$$((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.05}(0) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(0) - 0.05 = [0.35]$$

$$((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.03}(\mathrm{t_i}) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) + 0.03 = [0.73]$$

$$((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.04}(\mathrm{t_i}) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) + 0.04 = [0.94]$$

$$((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.05}(\mathrm{t_i}) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) - 0.05 = [0.95]$$

Hence $\widetilde{K}^{\mathrm{Tr}}_{\alpha,\beta,\gamma}$ is an NSCTID of P.

Theorem 5.5. *The union of any two NSC-translations of an NSCTID is an NSCTID of P.*

Proof. Suppose $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma}$ and $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$ are two NSC-translations of NSCTID of P respectively. In $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma}$ for $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I}}_{\mathrm{e_i}}, \alpha, \beta \in [[0, 0], \xi]$ and $\gamma \in [[0, 0], \Psi]$, where for $(\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, \alpha, \beta \in [0, \zeta]$ and $\gamma \in [0, \Phi]$, and in $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$, for $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, \alpha', \beta' \in [[0, 0], \xi]$ and $\gamma' \in [[0, 0], \Psi]$, where for

$(\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, \alpha', \beta' \in [0, \zeta]$ and $\gamma' \in [0, \Phi]$. and $\alpha \geq \alpha', \beta \geq \beta', \gamma \geq \gamma'$ as we know that, $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma}$ and $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$ are NSCTID of P. Then

$(((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(0) = (((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(\mathrm{t_1} * \mathrm{t_1}) = \mathrm{rmax}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t_1} * \mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t_1} * \mathrm{t_1})\} \geq \mathrm{rmax}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t_1})\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t_1})\}\} = \mathrm{rmax}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t_1})\} = \mathrm{rmax}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_1}) + \alpha, (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_1}) + \alpha'\} = ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cup (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(\mathrm{t_1}),\ (((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'})(0) = (((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'})(\mathrm{t_1} * \mathrm{t_1}) = \mathrm{rmax}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t_1} * \mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t_1} * \mathrm{t_1})\} \geq \mathrm{rmax}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t_1})\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t_1})\}\} = \mathrm{rmax}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t_1})\} = \mathrm{rmax}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t_1}) + \beta, (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t_1}) + \beta'\} = ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta} \cup (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'})(\mathrm{t_1}), (((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(0) = (((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(\mathrm{t_1} * \mathrm{t_1}) = \mathrm{rmax}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t_1} * \mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t_1} * \mathrm{t_1})\} \geq \mathrm{rmax}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t_1})\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t_1})\}\} = \mathrm{rmax}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t_1}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t_1})\} = \mathrm{rmax}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_1}) + \gamma, (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_1}) + \gamma'\} = ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cup (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(\mathrm{t_1}), (((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cup ((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(0) = (((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cup ((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(\mathrm{t_1} * \mathrm{t_1}) = \min\{((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t_1} * \mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t_1} * \mathrm{t_1})\} \leq \min\{\max\{((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t_1})\} \max\{((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t_1})\}\} = \min\{((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t_1})\} = \min\{(\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_1}) + \alpha, (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_1}) + \alpha'\} = ((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cup (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(\mathrm{t_1}).(((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta} \cup ((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'})(0) = (((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta} \cup ((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'})(\mathrm{t_1} * \mathrm{t_1}) = \min\{((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t_1} * \mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t_1} * \mathrm{t_1})\} \leq \min\{\max\{((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t_1})\}, \max\{((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t_1})\}\} = \min\{((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t_1})\} = \min\{(\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t_1}) + \beta, (\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t_1}) + \beta'\} = ((\mathrm{A_K})^{I}_{\mathrm{e_i}})^{Tr}_{\beta} \cup (\mathrm{A_K})^{I}_{\mathrm{e_i}})^{Tr}_{\beta'})(\mathrm{t_1})$ $(((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cup ((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(0) = (((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cup ((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(\mathrm{t_1} * \mathrm{t_1}) = \min\{((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t_1} * \mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t_1} * \mathrm{t_1})\} \leq \min\{\max\{((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t_1})\} \max\{((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t_1})\}\} = \min\{((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t_1})\} = \min\{(\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_1}) + \gamma, (\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_1}) + \gamma'\} = ((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cup (\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(\mathrm{t_1})$.

Now

$(((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,IF}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'})(\mathrm{t_1} * \mathrm{t_3}) = \mathrm{rmax}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{TI,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}(\mathrm{t_1} * \mathrm{t_3}),\ ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}(\mathrm{t_1} * \mathrm{t_3})\} \geq \mathrm{rmax}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}(\mathrm{t_2}),\ \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{TI,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,IF}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}(\mathrm{t_2})\},\ = \mathrm{rmin}\{\mathrm{rmax}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3})\},\ \mathrm{rmax}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}(\mathrm{t_2}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}(\mathrm{t_2})\}\} = \mathrm{rmin}\{(((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'})((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}),\ (((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,IF}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'})(\mathrm{t_2})\}$.

And

$((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \cup ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'})(\mathrm{t_1} * \mathrm{t_3}) = \min\{((\mathrm{A_K})^{\mathrm{T,I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}(\mathrm{t_1} * \mathrm{t_3}),\ ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}(\mathrm{t_1} * \mathrm{t_3})\} \leq \min\{\max\{((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}(\mathrm{t_2})\}, \max\{((\mathrm{A_K})^{\mathrm{T,IF}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}(\mathrm{t_2})\}\} = \max\{\min\{((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})^{\mathrm{T,IF}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3})\}, \min\{((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}(\mathrm{t_2}), ((\mathrm{A_K})^{\mathrm{TI,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}(\mathrm{t_2})\}\} = \max\{(((\mathrm{A_K})^{\mathrm{T,IF}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \cup ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'})((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}),\ (((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \cup ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'})(\mathrm{t_2})\}$.

Hence $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \cup \widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$ is an NSCTID of P. □

Theorem 5.6. *The intersection of any two NSC-translations of an NSCTID is an NSCTID of P.*

Proof. Suppose $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma}$ and $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$ are two NSC-translations of NSCTID of P respectively. Where in $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma}$ for $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, \alpha, \beta \in [[0,0],\xi]$ and $\gamma \in [[0,0],\Psi]$, where for $(\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F,F}}_{\mathrm{e_i}}, \alpha, \beta \in [0,\zeta]$ and $\gamma \in [0,\Phi]$, and in $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$, for $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,F}}_{\mathrm{e_i}}, \alpha', \beta' \in [[0,0],\xi]$ and $\gamma' \in [[0,0],\Psi]$, where for $(\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, \alpha', \beta' \in [0,\zeta]$ and $\gamma' \in [0,\Phi]$. and $\alpha \leq \alpha', \beta \leq \beta', \gamma \leq \gamma'$ as we know that, $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma}$ and $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$ are NSCTID of P.Then

$(((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(0) = (((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(\mathrm{t}_1 * \mathrm{t}_1) = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t}_1 * \mathrm{t}_1), (($
$\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t}_1 * \mathrm{t}_1)\} \geq \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t}_1)\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}$
$(\mathrm{t}_1)\}\} = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t}_1)\} = \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t}_1) + \alpha, (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t}_1) + \alpha'\} = ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cap (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(\mathrm{t}_1)$, $(((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'})(0) = (((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'})(\mathrm{t}_1 * \mathrm{t}_1)$
$= \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t}_1 * \mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t}_1 * \mathrm{t}_1)\} \geq \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t}_1)\},$
$\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t}_1)\}\} = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t}_1)\} = \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t}_1$
$) + \beta, (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t}_1) + \beta'\} = ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta} \cap (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'})(\mathrm{t}_1)$, $(((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(0) =$
$(((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(\mathrm{t}_1 * \mathrm{t}_1) = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t}_1 * \mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t}_1 * \mathrm{t}_1)\} \geq$
$\mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t}_1)\}, \mathrm{rmin}\{((\widehat{N}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t}_1)\}\} = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}}$
$)^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t}_1) = \mathrm{rmin}\{(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t}_1) + \gamma, (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t}_1) + \gamma'\} = ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cap (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(\mathrm{t}_1)$,
$(((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cap ((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(0) = (((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cap ((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(\mathrm{t}_1 * \mathrm{t}_1) = \max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t}_1 * \mathrm{t}_1), ($
$\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t}_1 * \mathrm{t}_1)\} \leq \max\{\max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t}_1)\}, \max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}$
$(\mathrm{t}_1)\}\} = \max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha}(\mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'}(\mathrm{t}_1)\} = \max\{(\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t}_1) + \alpha, (\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t}_1) + \alpha'\} =$
$((\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha} \cap (\mathrm{A}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha'})(\mathrm{t}_1)$. $(((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta} \cap ((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'})(0) = (((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta} \cap ((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'})(\mathrm{t}_1 * \mathrm{t}_1)$
$= \max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t}_1 * \mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t}_1 * \mathrm{t}_1)\} \leq \max\{\max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t}_1)\},$
$\max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t}_1)\}\} = \max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta}(\mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'}(\mathrm{t}_1)\} = \max\{(\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t}_1)$
$+ \beta, (\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t}_1) + \beta'\} = ((\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta} \cap (\mathrm{A}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\beta'})(\mathrm{t}_1)$ $(((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cap ((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(0) = (((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cap$
$((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(\mathrm{t}_1 * \mathrm{t}_1) = \max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t}_1 * \mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t}_1 * \mathrm{t}_1)\} \leq \max\{\max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t}_1),$
$((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t}_1)\}, \max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t}_1)\}\} = \max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma}(\mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'}(\mathrm{t}_1)\} =$
$\max\{(\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t}_1) + \gamma, (\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t}_1) + \gamma'\} = ((\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma} \cap (\mathrm{A}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\gamma'})(\mathrm{t}_1)$.

Now

$(((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'})(\mathrm{t}_1 * \mathrm{t}_3) = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}(\mathrm{t}_1 * \mathrm{t}_3) \geq ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$
$(\mathrm{t}_1 * \mathrm{t}_3)\} \geq \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}(\mathrm{t}_2)\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})$
$^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}(\mathrm{t}_2)\}\} = \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), ((\widehat{\mathrm{N}}_{\mathrm{K}})$
$^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}((\mathrm{t}_1 * \mathrm{t}_2) *\mathrm{t}_3)\}$, $\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}(\mathrm{t}_2), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}(\mathrm{t}_2)\}\} = \{rmin\{(((($
$\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'})((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), (((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{Tr,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'})(\mathrm{t}_2)\}$.

And

$(((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \cap ((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'})(\mathrm{t}_1 * \mathrm{t}_3) = \max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}(\mathrm{t}_1 * \mathrm{t}_3), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$
$(\mathrm{t}_1 * \mathrm{t}_3)\} \leq \max\{\max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), ((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha,\beta,\gamma}(\mathrm{t}_2)\} \max\{((\mathrm{A}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$

$((t_1 * t_2) * t_3), ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_2)\}\} = \max\{\max\{((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}((t_1 * t_2) * t_3), ((A_K)^{T,I,F}_{e_i})$
$^{Tr}_{\alpha',\beta',\gamma'}((t_1 * t_2)\ t_3)\}, \max\{((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_2), ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_2)\ \{\max\{(((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cap$
$((A_K)^{T,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})((t_1 * t_2) * t_3),\ (((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cap ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})(t_2)\}$.
Hence $\widetilde{K}^{Tr}_{\alpha,\beta,\gamma} \cap \widetilde{K}^{Tr}_{\alpha',\beta',\gamma'}$ is an NSCTID of P. □

5.2. *Neutrosophic Soft Cubic T-Ideal Multiplication*

This section defines neutrosophic soft cubic T-Ideal multiplication with a theorem and example.

Theorem 5.7. *If* K *is an NSCTID of* P*, then NSCMp* $\widetilde{K}^{Mp}_{\eta}$ *of* K *is an NSCTID of* P*, for all* $\eta\epsilon\ [0,1]$.

Proof. Let K be an NSCTID of P and $\eta \in [0,1]$. Then we have $((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (\widehat{N}_K)^T_{e_i}(0) \geq \eta \cdot (\widehat{N}_K)^T_{e_i}(t_1) \rightarrow ((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(0) \geq ((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_1)$, $((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (\widehat{N}_K)^I_{e_i}(0) \geq \eta \cdot (\widehat{N}_K)^I_{e_i}(t_1) \rightarrow ((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(0) \geq ((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_1)$, $((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (\widehat{N}_K)^F_{e_i}(0) \geq \eta \cdot (\widehat{N}_K)^F_{e_i}(t_1) \rightarrow ((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(0) \geq ((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_1)$,
And $((A_K)^T_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (A_K)^T_{e_i}(0) \leq \eta \cdot (A_K)^T_{e_i}(t_1) \rightarrow ((A_K)^T_{e_i})^{Mp}_{\eta}(0) \leq ((A_K)^T_{e_i})^{Mp}_{\eta}(t_1)$, $((A_K)^I_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (A_K)^I_{e_i}(0) \leq \eta \cdot (A_K)^I_{e_i}(t_1) \rightarrow ((A_K)^I_{e_i})^{Mp}_{\eta}(0) \leq ((A_K)^I_{e_i})^{Mp}_{\eta}(t_1)$, $((A_K)^F_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (A_K)^F_{e_i}(0) \leq \eta \cdot (A_K)^F_{e_i}(t_1) \rightarrow ((A_K)^F_{e_i})^{Mp}_{\eta}(0) \leq ((A_K)^F_{e_i})^{Mp}_{\eta}(t_1)$, Now $((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (\widehat{N}_K)^T_{e_i}(t_1 * t_3) \geq \eta \cdot \text{rmin}\{(\widehat{N}_K)^T_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_K)^T_{e_i}(t_2)\} = \text{rmin}\{\eta \cdot (\widehat{N}_K)^T_{e_i}((t_1 * t_2) * t_3), \eta \cdot (\widehat{N}_K)^T_{e_i}(t_2)\}$ $((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \text{rmin}\{((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_2)\}$ $((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) \geq \text{rmin}\{((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_2)\}$,
$((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (\widehat{N}_K)^I_{e_i}(t_1 * t_3) \geq \eta \cdot \text{rmin}\{(\widehat{N}_K)^I_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_K)^I_{e_i}(t_2)\} = \text{rmin}\{\eta \cdot (\widehat{N}_K)^I_{e_i}((t_1 * t_2) * t_3), \eta \cdot (\widehat{N}_K)^I_{e_i}(t_2)\}$ $((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \text{rmin}\{((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_2)\}$ $((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_1 * t_3) \geq \text{rmin}\{((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_2)\}$,
$((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (\widehat{N}_K)^F_{e_i}(t_1 * t_3) \geq \eta \cdot \text{rmin}\{(\widehat{N}_K)^F_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_K)^F_{e_i}(t_2)\} = \text{rmin}\{\eta \cdot (\widehat{N}_K)^F_{e_i}((t_1 * t_2) * t_3), \eta \cdot (\widehat{N}_K)^F_{e_i}(t_2)\}$ $((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \text{rmin}\{((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_2)\}$ $((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_1 * t_3) \geq \text{rmin}\{((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_2)\}$
And $((A_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (A_K)^T_{e_i}(t_1 * t_3) \leq \eta \cdot \max\{(A_K)^T_{e_i}((t_1 * t_2) * t_3), (A_K)^T_{e_i}(t_2)\} = \max\{\eta \cdot (A_K)^T_{e_i}((t_1 * t_2) * t_3), \eta \cdot (A_K)^T_{e_i}(t_2)\}$ $((A_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \max\{((A_K)^T_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((A_K)^T_{e_i})^{Mp}_{\eta}(t_2)\}$ $((A_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) \leq \max\{((A_K)^T_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((A_K)^T_{e_i})^{Mp}_{\eta}(t_2)\}$,
$((A_K)^I_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (A_K)^I_{e_i}(t_1 * t_3) \leq \eta \cdot \max\{(A_K)^I_{e_i}((t_1 * t_2) * t_3), (A_K)^I_{e_i}(t_2)\} = \max\{\eta \cdot (A_K)^I_{e_i}((t_1 * t_2) * t_3), \eta \cdot (A_K)^I_{e_i}(t_2)\}$ $((A_K)^I_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \max\{((A_K)^I_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((A_K)^I_{e_i})^{Mp}_{\eta}(t_2)\}$ $((A_K)^I_{e_i})^{Mp}_{\eta}(t_1 * t_3) \leq \max\{((A_K)^I_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((A_K)^I_{e_i})^{Mp}_{\eta}(t_2)\}$c,
$((A_K)^F_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (A_K)^F_{e_i}(t_1 * t_3) \leq \eta \cdot \max\{(A_K)^F_{e_i}((t_1 * t_2) * t_3), (A_K)^F_{e_i}(t_2)\} = \max\{\eta \cdot (A_K)^F_{e_i}((t_1 * t_2) * t_3), \eta \cdot (A_K)^F_{e_i}(t_2)\}$ $((A_K)^F_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \max\{((A_K)^F_{e_i})^{Mp}_{\eta}((t_1 * t_2) *$

$t_3), ((A_K)^F_{e_i})^{Mp}_{\eta}(t_2)\}$ $((A_K)^F_{e_i})^{Mp}_{\eta}(t_1 * t_3) \leq \max\{((A_K)^F_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((A_K)^F_{e_i})^{Mp}_{\eta}(t_2)\}$.
Hence $\widetilde{K}^{Mp}_{\eta}$ of K is an NSCTID of P, for all $\eta \in [0, 1]$. □

Example 5.8. Let $P = \{0, t_1, t_2, t_3\}$ be a PS-algebra with the cayley's table as shown in Table 1 . The NSC-set $K = \langle(\widehat{N}_K)^{T,I,F}_{e_i}, (A_K)^{T,I,F}_{e_i}\rangle$ of P is defined as

$$(\widehat{N}_K)^T_{e_i}(t_i) = \{[0.7, 0.9] \text{ if } t_i = 0 \text{ and } [0.3, 0.6]. \text{ if otherwise}$$

$$(\widehat{N}_K)^I_{e_i}(t_i) = \{[0.6, 0.8] \text{ if } t_i = 0 \text{ and } [0.4, 0.5]. \text{ if otherwise}$$

$$(\widehat{N}_K)^F_{e_i}(t_i) = \{[0.5, 1] \text{ if } t_i = 0 \text{ and } [0.2, 0.7]. \text{ if otherwise.}$$

And

$$(A_K)^T_{e_i}(t_i) = \{0.2 \text{ if } t_i = 0 \text{ and } 0.7. \text{ if otherwise}$$

$$(A_K)^I_{e_i}(t_i) = \{0.5 \text{ if } t_i = 0 \text{ and } 0.9. \text{ if otherwise}$$

$$(A_K)^F_{e_i}(t_i) = \{0.4 \text{ if } t_i = 0 \text{ and } 1. \text{ if otherwise.}$$

The set K is an NSCTID of P as it satisfies (i) but fulfills (ii) and (iii) with the condition: If $i = j$ then $t_i * t_j = 0$ with $(\widehat{N}_K)^T_{e_i}(t_i * t_j) = [0.7, 0.9], (\widehat{N}_K)^I_{e_i}(t_i * t_j) = [0.6, 0.8], (\widehat{N}_K)^F_{e_i}(t_i * t_j) = [0.5, 1]$
Otherwise

$$(\widehat{N}_K)^T_{e_i}(t_i * t_j) = [0.3, 0.6], (\widehat{N}_K)^I_{e_i}(t_i * t_j) = [0.4, 0.5], (\widehat{N}_K)^F_{e_i}(t_i * t_j) = [0.2, 0.7],$$

And

$$(A_K)^T_{e_i}(t_i) = 0.2, (\ A_K)^I_{e_i}(t_i) = 0.5, (\ A_K)^F_{e_i}(t_i) = 0.4$$

Otherwise $(A_K)^T_{e_i}(t_i) = 0.7, (A_K)^I_{e_i}(t_i) = 0.9, (A_K)^F_{e_i}(t_i) = 1$ given in definition 3.1. Now, for $(\widehat{N}_K)^{T,I,F}_{e_i}\eta \in [0.2, 0.5]$ and for $(A)^{T,I,F}_{e_i}, \eta = 0.2$. Then the mapping $\widetilde{K}^{Mp}_{\eta} \mid Parrow[0, 1]$ is given by

$$((\widehat{N}_K)^T_{e_i})^{Mp}_{[0.2,0.5]}(0) = [0.2, 0.5] \cdot (\widehat{N}_K)^T_{e_i}(0) = [0.14, 0.45]$$

$$((\widehat{N}_K)^I_{e_i})^{Mp}_{[0.2,0.5]}(0) = [0.2, 0.5] \cdot (\widehat{N}_K)^I_{e_i}(0) = [0.12, 0.4]$$

$$((\widehat{N}_K)^F_{e_i})^{Mp}_{[0.2,0.5]}(0) = [0.2, 0.5] \cdot (\widehat{N}_K)^F_{e_i}(0) = [0.1, 0.5]$$

$$((\widehat{N}_K)^T_{e_i})^{Mp}_{[0.2,0.5]}(t_i) = [0.2, 0.5] \cdot (\widehat{N}_K)^T_{e_i}(t_i) = [0.06, 0.3]$$

$$((\widehat{N}_K)^I_{e_i})^{Mp}_{[0.2,0.5]}(t_i) = [0.2, 0.5] \cdot (\widehat{N}_K)^I_{e_i}(t_i) = [0.08, 0.25]$$

$$((\widehat{N}_K)^F_{e_i})^{Mp}_{[0.2,0.5]}(t_i) = [0.2, 0.5].(\widehat{N}_K)^F_{e_i}(t_i) = [0.04, 0.35],$$

And

$$((A_K)^T_{e_i})^{Mp}_{0.03}(0) = 0.2.(A_K)^T_{e_i}(0) = [0.04]$$

$$((A_K)^F_{e_i})^{Mp}_{0.2}(0) = 0.2.(A_K)^T_{e_i}(0) = [0.08]$$

$$((A_K)^T_{e_i})^{Mp}_{0.2}(t_i) = 0.2.(A_K)^T_{e_i}(t_i) = [0.14]$$

$$((A_K)^I_{e_i})^{Mp}_{0.2}(t_i) = 0.2.(A_K)^T_{e_i}(t_i) = [0.18]$$

$$((A_K)^F_{e_i})^{Mp}_{0.2}(t_i) = 0.2.(A_K)^T_{e_i}(t_i) = [0.2]$$

Hence $\widetilde{\mathrm{K}}_{\eta}^{\mathrm{Mp}}$ is an NSCTID of P.

Theorem 5.9. *The union of any two NSC-multiplications of an NSCTID is an NSCTID of* P.

Proof. Suppose $\widetilde{\mathrm{K}}_{\eta}^{\mathrm{Mp}}$ and $\widetilde{\mathrm{K}}_{\eta'}^{\mathrm{Mp}}$ are two NSC-multiplications of an NSCTID is an NSCTID of P, where $\eta, \eta' \in (0,1]$ and $\eta \leq \eta'$. As we know that $\widetilde{K}_{\eta}^{\mathrm{Mp}}$ and $\widetilde{K}_{\eta'}^{\mathrm{Mp}}$ are NSCTIDs of P. Then $(((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(0) = (((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t}_1 * \mathrm{t}_1) = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1 * \mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1 * \mathrm{t}_1)\} \geq \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1)\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1)\}\} = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1)\} = \mathrm{rmin}\{\eta \cdot (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}}(\mathrm{t}_1), \eta' \cdot (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}}(\mathrm{t}_1)\} = ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{TI,F}})_{\eta}^{\mathrm{Mp}} \cup (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t}_1)$.

And

$(((\mathrm{A}_{\mathrm{K}})_{\mathrm{e}_i}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cup ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(0) = (((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cup ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{Mp})(\mathrm{t}_1 * \mathrm{t}_1) = \max\{((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}{}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1 * \mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1 * \mathrm{t}_1)\} \leq \max\{\max\{((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{TI,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1)\}, \max\{((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1)\}\} = \max\{((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,IF}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1), ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1)\} = \max\{\eta \cdot (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}}(\mathrm{t}_1), \eta' \cdot (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}}(\mathrm{t}_1)\} = ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cup (\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t}_1)$. $(((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,IF}})_{\eta}^{\mathrm{Mp}} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}{}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t}_1 * \mathrm{t}_3) = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1 * \mathrm{t}_3), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1 * \mathrm{t}_3)\} \geq \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_2)\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_2)\}\} = \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3)\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_2), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_2)\}\} = \{\mathrm{rmin}\{(((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), (((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cup ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t}_2)\}$.

And

$(((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{Mp} \cup ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{Mp})(\mathrm{t}_1 * \mathrm{t}_3) = \max\{((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1 * \mathrm{t}_3), ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{Mp}(\mathrm{t}_1 * \mathrm{t}_3)\} \leq \max\{\max\{((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_2)\}. \max\{((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_2)\}\} = \max\{\max\{((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3), ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}((\mathrm{t}_1 * \mathrm{t}_2) * \mathrm{t}_3)\}, \max\{((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_2), ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_2)\}\} = \{\max\{(((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \mathrm{U} ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})((\mathrm{t}_1 \mathrm{t}_2) * \mathrm{t}_3), \ (((\mathrm{A}_{\mathrm{K}})_{\mathrm{e}_i}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cup ((\mathrm{A}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t}_2)\}$

Hence $\widetilde{\mathrm{K}}_{\eta}^{\mathrm{Mp}} \cup \widetilde{\mathrm{K}}_{\eta'}^{\mathrm{Mp}}$ is an NSCTID of P. □

Theorem 5.10. *The intersection of any two NSC-multiplications of an NSCTID is an NSC-TID of* P.

Proof. Suppose $\widetilde{\mathrm{K}}_{\eta}^{\mathrm{Mp}}$ and $\widetilde{\mathrm{K}}_{\eta'}^{\mathrm{Mp}}$ are two NSC-multiplications of an NSCTID is an NSCTID of P, where $\eta, \eta' \in (0,1]$ and $\eta \leq \eta'$. As we know that $\widetilde{\mathrm{K}}_{\eta}^{Mp}$ and $\widetilde{\mathrm{K}}_{\eta'}^{Mp}$ are NSCTIDs of P. Then $(((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(0) = (((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t}_1 * \mathrm{t}_1) = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}{}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1 * \mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1 * \mathrm{t}_1)\} \geq \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1)\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1)\}\} = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t}_1), ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t}_1)\} = \mathrm{rmin}\{\eta \cdot (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}}(\mathrm{t}_1), \eta' \cdot (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}}(\mathrm{t}_1)\} = ((\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap (\widehat{\mathrm{N}}_{\mathrm{K}})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t}_1)$.

And
$(((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta} \cap ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'})(0) = (((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta} \cap ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'})(\mathrm{t_1} * \mathrm{t_1}) = \max\{((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}(\mathrm{t_1} * \mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}(\mathrm{t_1} * \mathrm{t_1})\} \leq \max\{\max\{((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}(\mathrm{t_1})\}, \max\{((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}(\mathrm{t_1})\}\} = \max\{((\mathrm{A_K})^{\mathrm{T,I}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}(\mathrm{t_1}), ((\mathrm{A_K})^{\mathrm{T,IF}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}(\mathrm{t_1})\} = \max\{\eta \cdot (\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_1}), \eta' \cdot (\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}(\mathrm{t_1})\} = ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta} \cap (\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'})(\mathrm{t_1}). (((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'})(\mathrm{t_1} * \mathrm{t_3}) = \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}(\mathrm{t_1} * \mathrm{t_3}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}(\mathrm{t_1} * \mathrm{t_3})\} \geq \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}(\mathrm{t_2})\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}(\mathrm{t_2})\}\} = \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3})\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}(\mathrm{t_2}), ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}(\mathrm{t_2})\}\} = \{\mathrm{rmin}\{(((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'})((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}) (((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta} \cap ((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'})(\mathrm{t_2})\}$.
And
$(((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta} \cap ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'})(\mathrm{t_1} * \mathrm{t_3}) = \max\{((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}(\mathrm{t_1} * \mathrm{t_3}), ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}(\mathrm{t_1} * \mathrm{t_3})\} \leq \max\{\max\{((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}(\mathrm{t_2})\}, \max\{((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}(\mathrm{t_2})\}\} = \max\{\max\{((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3})\}, \max\{((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta}(\mathrm{t_2}), ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'}(\mathrm{t_2})\}\} = \{\max\{(((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta} \cap ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'})((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), (((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta} \cap ((\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}})^{\mathrm{Mp}}_{\eta'})(\mathrm{t_2})\}$.
Hence $\widetilde{\mathrm{K}}^{\mathrm{Mp}}_{\eta} \cap \widetilde{\mathrm{K}}^{\mathrm{Mp}}_{\eta'}$ is an NSCTID of P. □

6. **Conclusion**

This paper extensively explores the application of the neutrosophic soft cubic set to investigate specific properties of the t-ideal in a PS-algebra. The derived definitions and fundamental outcomes hold potential for broader use in other algebraic structures like Lie algebras and lattices in the future. Moreover, there are several areas where this proposed structure can be advantageous, such as DNA identification, formalizing procedures in genetic algorithms, genomics, fuzzy logic-based networks, and particularly complex networks. Within the already-existing neutrosophic cubic structures, the addition of soft sets with T-ideal properties makes this structure a better choice for future implementations. This can also be extended to the hypersoft sets and can be advantegeous for MCDM algorithms in various real life applications such as transportation, healthcare etc.

One potential limitation of our proposed structure NSCTID is the ground algebra because the results and properties in this study may or may not be satisfy for other algebras in future. Also the practical implementation and computational efficiency of the NSCTID framework in real-world applications may pose challenges. The translation of theoretical concepts into efficient algorithms and the handling of computational complexities could require further investigation and optimization to fully realize the benefits of NSCTID in practical scenarios.

References

[1] L.A Zadeh, Fuzzy sets, Information and control **8** (1965) 338-353.

[2] L.A Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Information science **8** (1975) 199-249.

[3] K.T. Atanassov,Intuitionistic fuzzy sets: Theory and Applications,Studies in Fuzziness and Soft Computing Physica-verlag, Heidelberg/Newyork **35** (1999) 1-137.

[4] A.U. Rahman, M.R. Ahmad, M. Saeed, M. Ahsan, M. Arshad, M. Ihsan, A study on fundamentals of refined intuitionistic fuzzy set with some properties, Journal of fuzzy extension and applications **1** (2020) 279-292.

[5] F. Smarandache, Neutrosophic set a generalization of the intuitionistic fuzzy set, International Journal of Pure and Applied Mathematics **24** (2005), 287-297.

[6] F. Smarandache, A unifying field in logics: neutrosophic logic. neutrosophy, neutrosophic set, neutrosophic probability, American Research Press, Rehoboth (1998).

[7] D. Molodtsov, Soft set theory - first results, Computers and mathematics with applications **37** (1999) 19-31.

[8] M. Saeed, M. Hussain, A.A. Mughal, A study of soft sets with soft members and soft elements: A new approach, Punjab University Journal of Mathematics **52** (2020) 1-15.

[9] P.K. Maji, Neutrosophic soft set, Computers and mathematics with applications **45** (2013) 555-562.

[10] P.K. Maji, R. Biswas, A. Roy, Fuzzy soft sets, Journal of Fuzzy Mathematics **9** (2001) 589-602.

[11] F. Smarandache, Extension of soft set to hypersoft set, and then to plithogenic hypersoft set, Neutrosophic sets and systems, **22** (2018) 168-170.

[12] M. N. Jafar, M. Saeed, K. M. Khan, F. S. Alamri, and H. A. E. W. Khalifa, Distance and similarity measures using max-min operators of neutrosophic hypersoft sets with application in site selection for solid waste management systems, IEEE Access, **10** (2022) 11220-11235.

[13] Y.B. Jun, C.S. Kim, K.O. Yang, Cubic sets, Annals of Fuzzy Mathematics and Informatics **4** (2012) 83-98.

[14] M. Saeed, M. H. Saeed, R. Shafaqat, S. Sessa, U. Ishtiaq, and F. di Martino, A Theoretical Development of Cubic Pythagorean Fuzzy Soft Set with Its Application in Multi-Attribute Decision Making, Symmetry **14** (2022) 2639.

[15] R. M. Zulqarnain, X. L. Xin, and M. Saeed, Extension of TOPSIS method under intuitionistic fuzzy hypersoft environment based on correlation coefficient and aggregation operators to solve decision making problem. AIMS mathematics, **6** (2020) 2732-2755.

[16] A. Ali, Y.B. Jun, M. Khan, F.-G. Shi, S. Anis, Genralized cubic soft sets and their applications to algebraic structres, Italian Journal of Pure and Applied Mathematics **35** (2015) 393-414.

[17] M.Ali, I. Deli, and F. Smarandache, The theory of neutrosophic cubic sets and their applications in pattern recognition, Journal of intelligent and fuzzy systems, **30** (2016) 1957-1963.

[18] V. Chinnadurai, A. Swaminathan, B. Anu, Some properties of neutrosophic cubic soft set, International Journal of Computational Research and Development **1** (2016) 113-119.

[19] M. Gulistan, I. Beg, and N. Yaqoob, A new approach in decision making problems under the environment of neutrosophic cubic soft matrices, Journal of Intelligent and Fuzzy Systems, **36** (2019) 295-307.

[20] K. Iséki, An introduction to the theory of BCK-algebras, Math. Japonica **28** (1978) 1-20.

[21] K. Iseki, On BCI-algebras, in: Math. Seminar Notes **8** (1980) 125-130.

[22] T. Priya, T. Ramachandran, A note on fuzzy PS-ideals in PS-algebra and its level subsets, International Journal of Advanced Mathematical Sciences **2** (2014) 101-106.

[23] T. Shah, M. Saeed, Fuzzy ideals in Laskerian rings, Matematički Vesnik **65** (2013) 74-81.

[24] T. Shah, M. Saeed, On Fuzzy ideals in rings and Anti-homomorphism **7** (2012) 753-759

[25] T. Senapati, C. Jana, M. Pal, Y.B. Jun, Cubic intuitionistic q-ideals of BCI-algebras, Symmetry **10** (2018) 752.

[26] M. Kandasamy, T. Angamuthu, On Intutionistic Fuzzy T-ideals in TM-Algebra, Communications in Computer and Information Science, Springer **260** (2011) 328-335.

[27] T. Priya, T. Ramachandran, Fuzzy translation and fuzzy multiplication on PS-algebras, International Journal of Innovation in Science and Mathematics **2** (2014) 485-489.

[28] M. Khalid, N.A. Khalid, R. Iqbal, MBJ-neutrosophic T-ideal on B-algebra, International Journal of Neutrosophic Science **1** (2020) 29-39.

[29] M. Khalid, F. Smarandache, N.A. Khalid, S. Broumi, Translative and Multiplicative Interpretation of Neutrosophic Cubic Set, Neutrosophic Sets and Systems **35** (2020) 299-339.

[30] M. Khalid, Z. Iqbal, R. Iqbal, Intuitionistic Fuzzy Magnified Translation of PS-Algebra, Journal of New Theory **35** (2021) 32-47.

Neutrosophic speech recognition Algorithm for speech under stress by Machine learning

D. Nagarajan,[1] S. Broumi[2,3] Florentin Smarandache[4]

[1] Department of Mathematics, Rajalakshmi Institute of Technology, Chennai, Tamil Nadu, India
[2] Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco
[3] Regional Center for the Professions of Education and Training (C.R.M.E.F), Casablanca-Setat, Morocco
[4] The University of New Mexico Mathematics, Physics, and Natural Science Division, 705 Gurley Ave., Gallup, NM 87301, USA
E-mail: *dnrmsu2002@yahoo.com, broumisaid78@gmail.com, smarand@unm.edu

Nagarajan, D., Said Broumi, and Florentin Smarandache. "Neutrosophic speech recognition Algorithm for speech under stress by Machine learning." *Neutrosophic Sets and Systems*, vol. 55, 2023, pp. 46–57.

Abstract

It is well known that the unpredictable speech production brought on by stress from the task at hand has a significant negative impact on the performance of speech processing algorithms. Speech therapy benefits from being able to detect stress in speech. Speech processing performance suffers noticeably when perceptually produced stress causes variations in speech production. Using the acoustic speech signal to objectively characterize speaker stress is one method for assessing production variances brought on by stress. Real-world complexity and ambiguity make it difficult for decision-makers to express their conclusions with clarity in their speech. In particular, the Neutrosophic speech algorithm is used to encode the language variables because they cannot be computed directly. Neutrosophic sets are used to manage indeterminacy in a practical situation. Existing algorithms are used except for stress on Neutrosophic speech recognition. The creation of algorithms that calculate, categorize, or differentiate between different stress circumstances. Understanding stress and developing strategies to combat its effects on speech recognition and human-computer interaction system are the goals of this recognition.

Keywords

speech recognition, categorization of stress in speech, linguistic technology, Neutrosophic, Machine learning, Neutrosophic Logic, Speech Recognition, Deep Learning, Uncertainty Modeling, Feature Extraction, Noisy Environments

1.Introduction

In order to produce speech, a series of intricately synchronised articulator movements, respiratory system airflow, and timing of the vocal system physiology are all required. While the posture of the articulator changes to create speech, not all utterances made by a speaker will be identical in every way. This is due to the fact that the subject is frequently experiencing some sort of emotional stress, which will affect the utterance and cause an error in the articulator motions. Listeners can handle or interpret these subtle variations in human communications much better than the automatic human-machine interface. The features of stress, and its effects on human speech production, perception, and automatic speed systems, are still not fully understood. Speech is therefore a complex signal that contains information about the speaker. The speaker's intent, language history, features of their accent and dialect, and additional paralinguistic information. Stress can cause a change in speech output that can large and will consequently affect how well speech processing apps function [1],[2]. Numerous research has examined how stress affects speech production variability[3],[4], and [5]. Moreover, a stress-based expansion of multi-style training Additionally, token generation has improved anxious speech recognition [6]. Then, five stress-sensitive targeted feature sets are chosen.

stress situations such as the cockpit of an Apache helicopter, anger, clarity, the Lombard effect, loudness, etc. features that are frequently employed Include the cepstral characteristic for speaker identification [7]. When doing cepstral analysis, speaker recognition software often ignores the excitation source data that appears as a high-time component of the cestrum[8]. The Mel-Frequency Cepstral Coefficient, a phonetic characteristic, was retrieved from the voice signals, and the stress was identified using a neural network that was programmed into the system using Python[9]. serve as a resource for decision-makers in many real-world scenarios and application domains, particularly from a technical standpoint, for both academic and business experts[10]. In the research described in this paper, stress during applicant screening interviews is identified via voice analysis. The mean energy, mean intensity, and Mel-Frequency Cepstral Coefficients are employed as classification features in machine learning to identify stress in speech[11]. This study uses an EEG signal to suggest a stress classification system. 35 individuals' EEG signals were analyzed after being collected using a commercially available 4-electrode Muse EEG headgear

with four EEG sensors [12]. In this study, it is expected that risk factors would, both cross-sectionally and longitudinally, predict mental health issues after controlling for sociodemographic traits and intent to become pregnant[13]. This investigation uses brain signals to look at how stress levels are affected by English and Urdu language music tracks[14]. This project looks into methods for sensing stress that is used to identify hardware[15]. The high-level features are combined into one unified representation using a proposed model-level fusion technique, which classifies the stress states into baseline, stress, and amusement[16]. The heart rate was measured and classified into three categories of positive, negative, and neutral emotions using the Geneva affective picture database. The support vector machine is a machine learning technique that has been built to predict the mental stress situation from the measured heart rate[17]. The development of a model for measuring stress levels makes use of several sensors, including those that measure body temperature, blood pressure (BP), heart rate, and CO2 concentration[18]. Studies show that combining IoT and AI with deep learning (DL) technology makes it possible to take preventative measures. Recognise stress well before its effects on human health become apparent[19]. In order to assess teaching effectiveness, enhance education, and limit risks from human errors that could occur as a result of workers' stressful circumstances, stress detection is crucial in both education and industry [20]. has good classification performance in this study and is able to gauge the stress levels of kids with accuracy. The growth of students' mental health has a strong foundation thanks to the precise measurement of stress, which also has important practical ramifications[21]. This research explores the concept of the intervention effect of physical activity on college students using an integrated evaluation-based algorithm. College students are used as an example of stress groups. The findings indicate that regular physical activity can significantly reduce college students' stress levels[22]. This study employs Neutrosophic logicto provide a valid ranking of hospital construction assets based on their changeable criticality and to lessen the subjectivity pertaining to expert-driven judgements[23]. This document compiles all research on machine learning mapping.

methods from the sharp number space to the neutrosophic environment. We also talk about contributions and combining single-valued neutrosophic numbers with machine learning methods

Modeling faulty information using (SVNs)[24]. In this paper, a brand-new paradigm for incorporating neutrosophy into deep learning models is given. To further comprehend the feelings, we quantified them using three membership functions as opposed to simply predicting a single class as the outcome. The two

components of our suggested model are feature extraction and feature categorization[25]. The proposed framework would be an appropriate progression in the future by eliminating ineffective qualities through feature selection [26]. Stress is a psychological condition that results from an alleged threat or work demand and is accompanied by a variety of feelings. Finding linguistic cues of stress could be one of the verbal signs of stress. verbal indicators of stress are perceived by the listener, markers range in visibility from very visible to invisible. Consciously and unconsciously, these signals are watched continuously [27]. Speech recognition is the ability of a system to recognise the words and phrases of the speech and convert them to readable or written format. Speech recognition is typically carried out through processes including call routing, speech-to-text conversion, voice dialling, voice audibility, and language modelling. Although there are numerous techniques and algorithms for voice recognition, none of them is handling all factors including word length, speaker independence, a wide vocabulary, comprehension of speech, time complexity, noisy surroundings, and conversational speech. Neutrosophic can be integrated to analyse the acoustic signal of an unknown speaker and the decision-making process when indeterminacy occurs, respectively, to solve these issues.

2. Preliminaries

A neutrosophic set $\tilde{\mathcal{A}}_N$ in $\mathcal{U}$ (Universe of discourse) is catogorized as functions of a truth membership $T_{\tilde{\mathcal{A}}_N}(\mathcal{g})$, an indeterminacy membership $I_{\tilde{\mathcal{A}}_N}(\mathcal{g})$ and a falsity membership $F_{\tilde{\mathcal{A}}_N}(\mathcal{g})$and is given by

$\tilde{\mathcal{A}} = \{\mathcal{g}, \langle T_{\tilde{\mathcal{A}}_N}(\mathcal{g}),\ I_{\tilde{\mathcal{A}}_N}(\mathcal{g}), F_{\tilde{\mathcal{A}}_N}(\mathcal{g})\rangle \mid \mathcal{g} \in \mathcal{U}\}$.

Here $T_{\tilde{\mathcal{A}}_N}(\mathcal{g}),\ I_{\tilde{\mathcal{A}}_N}(\mathcal{g}), F_{\tilde{\mathcal{A}}_N}(\mathcal{g}) \in [0,1]$ and the relation $0 \leq \sup T_{\tilde{\mathcal{A}}_N}(\mathcal{g}) \leq \sup I_{\tilde{\mathcal{A}}_N}(\mathcal{g}) \leq \sup F_{\tilde{\mathcal{A}}_N}(\mathcal{g}) \leq 3$ holds for all $\mathcal{g} \in \mathcal{U}$.

Definition 2.1[27,28 and 29]

Let X be the universal set, then Neutrosophic set is defined as $S = \{(T_S(x), I_S(x), F_S(x)), x \in X\}$ where $T_S(x), I_S(x), F_S(x) \in [0,1]$ and $0 \leq T_S(x) + I_S(x) + F_S(x) \leq 3$.

3.Database

The assessments carried out in this study are based on information previously gathered for speech analysis in noise and stress analysis and algorithm formulation. Because the task at hand entails mapping audio single value Neutrosophic sets(SVNS) to text SVNS for comparison, a dataset that included audio translation was necessary. LibriSpeech dataset was chosen as a result. The following two folders were utilised for the project demonstration: Dev-clean (337 MB) and Train-clean-100 (6.3 GB).

4.Methodology

4.1 Audio

converting.flac audio files to.wav

The dataset could be downloaded in FLAC format. These files had to be converted into.wav format in order to be processed further and have features extracted.

4.2 Features Extraction and Preprocessing

The python feature extraction script was then run on the audio files, extracting 193 features for each audio file. As a result, the npy files X dev.npy (2703 x 193) and X train.npy were created (28539 x 193). Then, sklearn was used to normalise these files.

4.3 Text

Using VADER, analyse the sentiment of translated text.

For each input sentence, the sentiment analysis programme VADER delivers a score for the truth,indeterminacy and falsity. Each audio file's text translation was examined using VADER, and SVNS were produced.

5. Speech recognition in to text conversion

5.1 Algorithm:1
Step 1: Import library
Step 2:Import speech recognition
Step 3:Initialize recognizer class
Step 4; Reading Microphone source
Step 5: Convert audio to text
Step 6: Adjust for ambient noise.
Step 7: Recognize the error
Step 8:Type the text.

5.2 Programme for Speech to Text

```
r = sr.Recognizer()
   print("Talk")
```

```
r.adjust_for_ambient_noise(source, duration=0.2)
audio_text = r.listen(source)
print("Time over, thanks")
  print("Text: "+r.recognize_google(audio_text))
    print("Sorry, I did not get that")
```

Once the programe is over, then run the programe. The out put is
Talk
Speak through microphone then it will showing. In this experiment speech word is “very good”
Time over, thanks
The output in the screen is
Text: very good

6. Neutrosophic speech stress analysis
6.1 Algorithm:2
Step 1: Import SentimentIntensityAnalyzer class
Step 2: Function to print sentiments
Step 3: Score for sentiment speech

Step 4: Which contains Truth, Falsity, Indeterminacy, and compound scores.

Step 5: Decide sentiment as Truth, Falsity and Indeterminacy se.
Step 6: Print the value of the compound score
Step 7: Print overall the stress statement is truth ,falsity or indeterminacy.

6.2 Programme for text to stress analysis by Neutrosophic speech algorithm

```
def sentiment_scores(sentence):

sid_obj = SentimentIntensityAnalyzer()
C = sid_obj.polarity_scores(sentence)
print
("Overall sentiment dictionary is : ", C)
Print
("sentence was rated as ", C['falsity']*100, "% Negative")
Print
("sentence was rated as ", C['indeterminacy']*100, "% Neutral")
Print
("sentence was rated as ", C['Truth']*100, "% Positive")
Print
("Sentence Overall Rated As", end = " ")
```

The following sentence "Very Good.",”Not bad”,”Bad”, "happy birth day."
"god bless you.","beautiful."

In this algorithm 2 , include the output of the algorithm 1 statements. Once run the programme.

6.3 The output of the programme

1st statement is Very Good the output of the programme is
{'Falsity': 0.0, 'Indeterminacy': 0.238, 'Truth': 0.762}
0.0 % Falsity, 23.799999999999997 % Indeterminacy, 76.2 % Truth and the speech is not under strees
2nd Statement : Not bad
{'Falsity': 0.0, 'Indeterminacy': 0.26, 'Truth': 0.74}
0.0 % Falsity, 26.0 % Indeterminacy, 74.0 % Truth and the speech is not under stress.
3rd Statement :Bad
{'Falsity': 1.0, 'indeterminacy': 0.0, 'Truth': 0.0}
100.0 % Falsity, 0.0 % Indeterminacy, 0.0 Truth and the speech is under stress.

4th statement :Happy Birthday
{'Falsity': 0.0, 'Indeterminacy': 0.351, 'Truth': 0.649}
0.0 % Falsity, 35.099999999999994 %Indeterminacy, 64.9 % Truth and the speech is not under stress.

5th Statement : god bless you
{'Falsity': 0.0, 'Indeterminacy': 0.169, 'Truth': 0.831}
0.0 % Falsity,16.900000000000002 Indeterminacy, 83.1 Truth and the speech is not under stress.
6th Statement : beautiful
{'Falsity': 0.0, 'Indeterminacy': 0.0, 'truth': 1.0}
0.0 Falsity, 0.0 % Indeterminacy, 100.0 % Truth the speech is not under stress.

7th Statement :Please help me
{'Falsity': 0.0, 'Indeterminacy': 0.167, 'Truth': 0.833}
0.0 % Falsity,16.7 % Indeterminacy, 83.3 % Truth and the speech is not under stress.

8th Statement :hate
{'Falsity': 1.0, 'Indeterminacy': 0.0, 'Truth': 0.0}
100.0 % Falsity,0.0 % Indeterminacy, 0.0 % Truth and the speech is under stress.

9th Statement :Great
{'Falsity': 0.0, 'Indeterminacy': 0.0, 'Truth': 1.0}
0.0 % Falsity, 0.0 % Indeterminacy,100.0 % Truth and the speech is not under stress.
Fig:1 Stress Analysis using Neutrosophic speech recognition

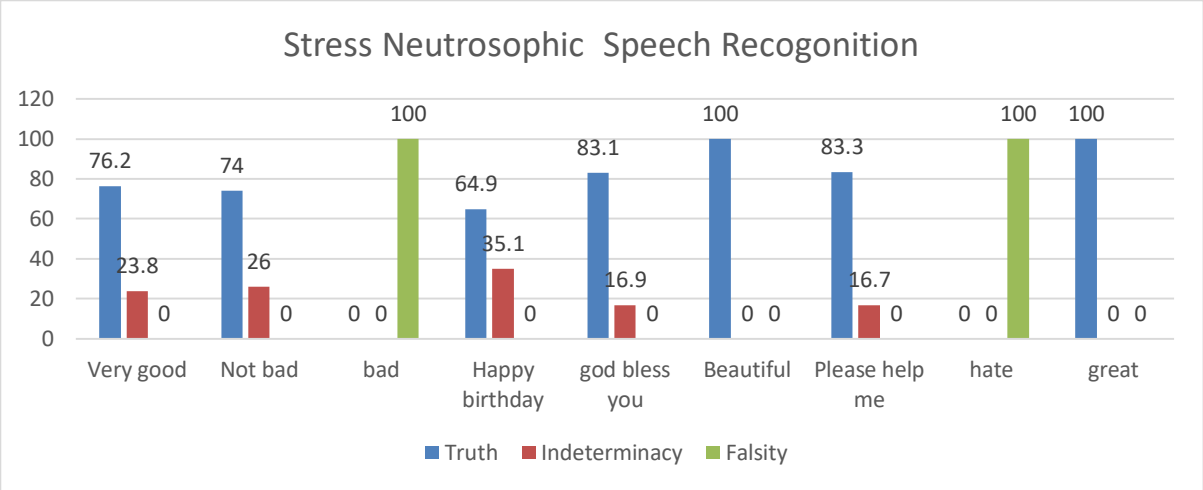

Fig:1 reveal that the percentage of the speech shows the truth, indeterminacy and falsity value .That means the probability of the stress in the speech. The probability value is give the statement is the speech is under stress or not.

Fig:2 Overall rated for Speech

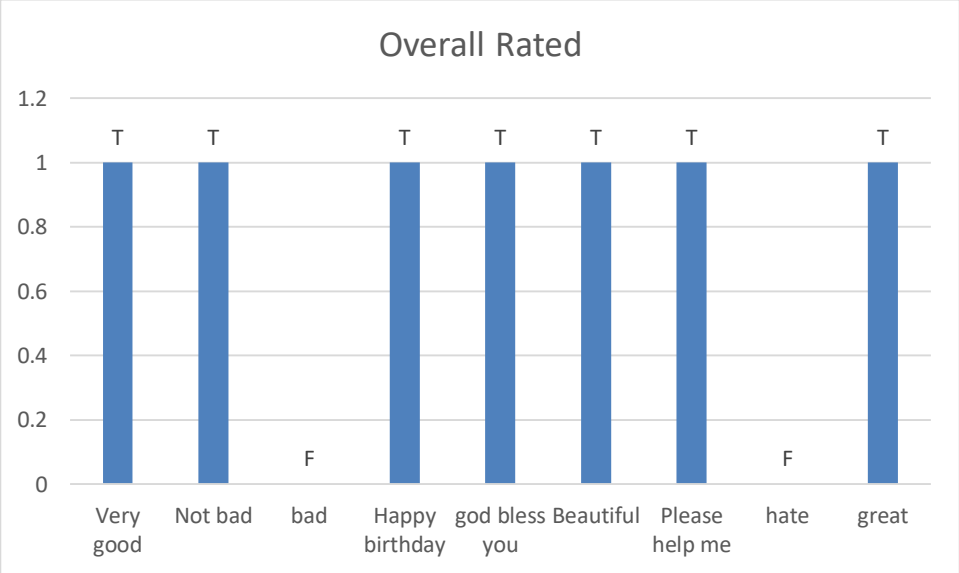

Fig :2 shows that the speech is under stress or not under stress. From the analysis of the speech verygood, notbad, happy birthday, god bless you, beautiful and great is positive speech text. Bad and hate is negative speech .

Conclusion

The requirement to accurately analyse, model, encode, identify, and categorise speech under stress will become increasingly important as speech and language technology develops. The condition of the speaker can be useful information for human-machine and dialogue systems that use voice interaction. This information can be utilised to create speaker and speech recognition technologies, leading to the development of systems that function better in actual multi-tasking environments.The difficulty, though, lies in finding a framework that can effectively analyse and model such speech technologies. The issue of better stress classification utilising targeted speech features has been taken into consideration in this work. categorization of stress The estimation of a probability vector that represents the level of speaker stress is proposed using neurosophic algorithms. Machine learning has demonstrated context-sensitive stress classification. The output stress probability vector can also be used to quantify combinations of speaker stress, such as speech that is both fast and loud. It is claimed that a stress mixture model could be helpful for tasks like sorting emergency phone messages or enhancing the efficiency of traditional speech processing systems. In conclusion, it has been demonstrated that stress classification utilising focused features in Neutrosophic speech recognition algorithm is effective for estimating the level of speaker stress and for providing helpful information for enhancing the performance of a voice recognition algorithm.

Reference

[1] J. H. L. Hansen and S. E. Bou-Ghazale, “Robust speech recognition training via duration and spectral-based stress token generation,” ZEEE Trans. Speech Audio Processing, vol. 3, pp. 415421, Sept. 1995.

[2] B. D. Womack and J. H. L. Hansen, “Stress independent robust HMM speech recognition using neural network stress classification,” in P roc. EuroSpeech, pp. 1999-2002.1995.

[3] D. A. Cairns and J. H. L. Hansen, “Nonlinear analysis and detection of speech under stressed conditions,” J. Acous. Soc. Amer., vol. 96, no. 6, pp. 3392-3400, 1994.

[4] J. C. Junqua, “The Lombard reflex and its role on human listeners and automatic speech recognizers,” J. Acous. Soc. Amer., vol. 93, pp, 510-524, Jan. 1993.

[5] B. J. Stanton, L. H. Jamieson, and G. D. Allen, “Robust recognition of loud and Lombard speech in the fighter cockpit environment,” in P roc. ICASSP, May 1989, pp. 675-678.

[6] J. H. L. Hansen and S. E. Bou-Ghazale, “Robust speech recognition training via duration and spectral-based stress token generation,” ZEEE Trans. Speech Audio Processing, vol. 3, pp. 415421, Sept. 1995.

[7] Shaughnessy, D. O. (1986). Speaker recognition. IEEE ASSP Magazine, 3(4), 4–17.

[8] Murty, K. S. R., & Yegnanarayana, B. (2006). Combining evidence from residual phase and MFCC features for speaker recognition. IEEE Signal Processing Letters, 13(1), 52–55.

[9] M. S. Hafiy Hilmy *et al*., "Stress Classification based on Speech Analysis of MFCC Feature via Machine Learning," *2021 8th International Conference on Computer and Communication Engineering (ICCCE)*, 2021, pp. 339-343, doi: 10.1109/ICCCE50029.2021.9467176.

[10] Sarker, I.H. Machine Learning: Algorithms, Real-World Applications and Research Directions. *SN* COMPUT. SCI. 2, 160 (2021). https://doi.org/10.1007/s42979-021-00592-x.

[11] K. Tomba, J. Dumoulin, E. Mugellini, O. A. Khaled and S. Hawila, "Stress detection through speech analysis", *ICETE 2018 - Proc. 15th Int. Jt. Conf. E-bus. Telecommun*, vol. 1, pp. 394-398, 2018.

[12] Nishtha Phutela, Devanjali Relan, Goldie Gabrani, Ponnurangam Kumaraguru, Mesay Samuel, "Stress Classification Using Brain Signals Based on LSTM Network", *Computational Intelligence and Neuroscience*, vol. 2022, Article ID 7607592, 13 pages, 2022. https://doi.org/10.1155/2022/7607592.

[13] J. Eichler, R. Schmidt, A. Hiemisch, W. Kiess, and A. Hilbert, “Gestational weight gain, physical activity, sleep problems, substance use, and food intake as proximal risk factors of stress and depressive symptoms during pregnancy,” *BMC Pregnancy and Childbirth*, vol. 19, no. 1, p. 175, 2019.

[14] A. Asif, M. Majid, and S. M. Anwar, “Human stress classification using eeg signals in response to music tracks,” *Computers in Biology and Medicine*, vol. 107, pp. 182–196, 2019.

[15] Hatem S. A. Hamatta, Kakoli Banerjee, Harishchander Anandaram, Mohammad Shabbir Alam, C. Anand Deva Durai, B. Parvathi Devi, Hemant Palivela, R. Rajagopal, Alazar Yeshitla, "Genetic Algorithm-Based Human Mental Stress Detection and Alerting in Internet of

Things", *Computational Intelligence and Neuroscience*, vol. 2022, Article ID 4086213, 7 pages, 2022. https://doi.org/10.1155/2022/4086213.

[16] A. Kumar, K. Sharma, and A. Sharma, "Hierarchical deep neural network for mental stress state detection using IoT based biomarkers," *Pattern Recognition Letters*, vol. 145, pp. 81–87, 2021.

[17] N. E. J. Asha and R. Khan, "Low-cost heart rate sensor and mental stress detection using machine learning," in *Proceedings of the 2021 5th International Conference on Trends in Electronics and Informatics (ICOEI)*, pp. 1369–1374, IEEE, Tirunelveli, India, June 2021.

[18] N. Padmaja, A. Anusha, M. Dvs, and B. S. Kumar, "IOT based stress detection and health monitoring system," *Helix-The Scientific Explorer| Peer Reviewed Bimonthly International Journal*, vol. 10, no. 2, pp. 161–167, 2020.

[19] D. Raval, "Stress detection using convolutional neural network and internet of things," *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*, vol. 12, no. 12, pp. 975–978, 2021.

[20] O. Attallah, "An effective mental stress state detection and evaluation system using minimum number of frontal brain electrodes," *Diagnostics*, vol. 10, no. 5, p. 292, 2020.

[21] Li Liu, Yunfeng Ji, Yun Gao, Tao Li, Wei Xu, "A Novel Stress State Assessment Method for College Students Based on EEG", *Computational Intelligence and Neuroscience*, vol. 2022, Article ID 4565968, 11 pages, 2022. https://doi.org/10.1155/2022/4565968.

[22] Bo Yang, Zhizhong Ding, "Algorithm for Evaluating the Intervention Effect of Physical Exercise on Stress Groups", *Mobile Information Systems*, vol. 2022, Article ID 1667814, 9 pages, 2022. https://doi.org/10.1155/2022/1667814.

[23] ReemAhmedaFuzhanNasiriaTarekZayed, A novel Neutrosophic-based machine learning approach for maintenance prioritization in healthcare facilities, Journal of BuildingEngineering,Volume 42, October 2021, 102480, https://doi.org/10.1016/j.jobe.2021.102480

[24] Azeddine Elhassouny, Soufiane Idbrahim, F. Smarandache: Machine learning in Neutrosophic Environment: A Survey, Neutrosophic Sets and Systems, vol. 28, 2019, pp. 58-68. DOI: 10.5281/zenodo.3382515.

[25] MayukhSharmaIlanthenralKandasamyW.B.Vasantha, Comparison of neutrosophic approach to various deep learning models for sentiment analysis, Knowledge-Based Systems, Volume 223, 8 July 2021, 107058, https://doi.org/10.1016/j.knosys.2021.107058.

[26] Belal Amin, A. A. Salama, I. M. El-Henawy, Khaled Mahfouz, Mona G. Gafar, "Intelligent Neutrosophic Diagnostic System for Cardiotocography Data", *Computational Intelligence and Neuroscience*, vol. 2021, Article ID 6656770, 12 pages, 2021. https://doi.org/10.1155/2021/6656770.

[27] F. Smarandache (2005) A unifying feld in logic. Neutrosophy: neutrosophic probability, set, logic, 4th edn. American Research Press, Rehoboth.

[28] H.Wang , F. Smarandache , Y. Zhang , R. Sunderraman (2010) Single valued neutrosophic sets. MultispMultistruct 4:410–413.

[29]H. Wang , F. Smarandache , Zhang YQ, R. Sunderraman (2005) Interval neutrosophic sets and logic: theory and applications in computing. Hexis, Phoenix.

Theory on Duplicity of Finite Neutrosophic Rings

T. Chalapathi,[1*] **K. Kumaraswamy Naidu,**[2] **D. Harish Babu,**[3] **F. Smarandache**[4]

[1,2] Dept. of Mathematics, Mohan Babu University, Tirupati-517 102, A.P, INDIA
[3] Dept. of Mathematics, SASL, VIT Bhopal University, Bhopal-466114, M P, INDIA
[4] Dept. Math and Sciences, University of New Mexico, Gallup, NM, USA
Emails: chalapathi.tekuri@gmail.com; kumarnaidu.kolla@gmail.com; harish225babu@gmail.com; smarand@unm.edu

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Abstract

This article introduces the notion of duplex elements of the finite rings and corresponding neutrosophic rings. The authors establish duplex ring $Dup(R)$ and neutrosophic duplex ring $Dup(R(I))$ by way of various illustrations. The tables of different duplicities are constructed to reveal the comparison between rings $Dup(Z_n)$, $Dup(Dup(Z_n))$ and $Dup(Dup(Dup(Z_n)))$ for the cyclic ring Z_n. The proposed duplicity structures have several algebraic systems with dissimilar consequences. Author's characterize finite rings with $R+R$ is different from the duplex ring $Dup(R)$. However, this characterization supports that $R+R=Dup(R)$ for some well known rings, namely zero rings and finite fields.

Keywords: Multiplicative function, Duplex form; Duplex ring, neutrosophic duplex element, neutrosophic duplex ring

1. Introduction

In the most general sense, elementary number theory deals with and manages the results and properties of different sets of numbers. In this paper, we will examine and discuss some significant sets of numbers in Z_n, called duplexity. We will briefly present the notion of duplexity of Z_n and enumerate how many number of duplex elements are there in Z_n. For the integer x, the element form $x+x$ is called a *duplex form* of x. The most important problem in the elementary theory of integers is to determine the possible forms of duplexes among the integers. For instance, it is clear to see that any duplex form must be of form $2k$, or $2k+2$ in Z, because every even integer is a multiple of 2. This illustration specifies that the ring of integers Z satisfies the conditions: $x+x=2x$, $x+2x=3x$ and so on, but $Z+Z\neq 2Z$, $Z+2Z\neq 3Z$ and so on, where the operation addition '$+$'defined on Z. In general, a duplex form $x+x$ exists in the ring Z of integers. Now, we shall study the enumeration of duplex elements in the finite commutative ring Z_n, and which are finitely many duplex forms $x+x$ in Z_n, where the operation addition '$+$' defined on Z_n.

First, we can generally describe a ring R is an algebraic structure $(R,+,\cdot)$ as an additive abelian group with a multiplicative binary operation such that the structure $(R,+,\cdot)$ is associative and fulfils distributive axioms $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$. A ring R is finite commutative if $|R|<\infty$ and $ab=ba$ for all a, b in R, see [1]. An element u in a commutative ring with unity 1 is called a unit if there exists an element x in R such that $xu=1=ux$, and specifically x is called a multiplicative inverse of u, and vice versa.

All the elements in R which are not multiplicative inverse elements are said to be zero divisors. Note that set of units and zero divisors of R are usually denoted by $R^{\times}$ and $Z(R)$, respectively, and R can be partitioned by the disjoint sets $Z(R)$ and $R^{\times}$. For any subring R of, the set R/S denotes the quotient ring. Now our attention is shifted to focus on the ring Z_n which is isomorphic to the quotient ring Z/nZ, which are the main tools in this paper for various values of $n \geq 1$. For $a, n \in Z$ with $n > 0$, we represent the congruence class of modulo n by the notion $[a]_n$, and the ring Z_n is the set $\{[a]_n : a \in Z\}$, or equivalently $\{[0],[1],[2],...,[n-1]\}$.

But there is a one-to-one correspondence between the complete residue systems $\{[0],[1],[2],...,[n-1]\}$ and $\{0,1,2,...,n-1\}$, and thus Z_n can be written simply as $Z_n = \{0,1,2,...,n-1\}$ for complete residue systems modulo n [2]. It is worth clarifying that the ring $Z_n = \{0,1,2,...,n-1\}$ is a commutative ring with unity 1 under addition and multiplication modulo n. We are happy to say that the ring Z_n has countably many applications in various fields such as algebraic number theory, algebraic coding theory, Cryptography, algebraic circuit theory, Antenna theory and algebraic design theory [3-8]. Further, the problem of enumeration of various types of elements in Z_n up to countably finite has received considerable attention in recent years; see for examples [9-14].

Now starts the basic notions, definitions and results of classical rings.

Let R be a finite commutative ring with nonzero identity and $R^{\times}$ be the set of group units of R. Given a finite commutative ring R, the ring $R + R = \{r + r : r \in R, r \in R\}$ is known as *duplex form* of R. However, the problem of characterizing finite commutative rings up to isomorphism has established considerable attention in recent years [see 15 and 16] initiating from the research works of Eldridge [17]. In this chapter, authors characterize finite commutative rings in terms of their duplexes. First, write the notion $Char(R)$ to denote a positive integer n such that $na = 0$ for every a in R, where $na = a + a + ... + a$ (n copies). Recall that the ring Z_n is a finite commutative ring with nonzero unity 1 under addition and multiplication modulo n. Also, the number of the form $a + ib$, $a, b \in Z_n$, is called Gaussian integer, and the set of Gaussian integers represented by $Z_n[i]$, and defined as $Z_n[i] = \{a + bi : a, b \in Z_n, i^2 = -1\}$. Further, note that $|Z_n| = n$ and $|Z_n[i]| = n^2$.

Neutrosophic Duplex elements are the solutions of some specific neutrosophic equation, and which are main mathematical tools for studying additive elements and their additive reciprocals of an object and their mutual symmetries, which are logically related to neutrosophic systems and their automorphisms. The characterizations of the duplex elements of any finite commutative ring have not been done in general theory of neutrosophic mathematics. But in recent years, the interplay between additive self inverses and group units of a classical ring and its corresponding neutrosophic ring was studied by Chalapathi and co-authors [18-20].

Now reconsider some notations, preliminaries and results of neutrosophic ring theory.

Let 0, 1 and I be three distinct components of any neutrosophic logical system with $0^2 = 0$, $1^2 = 1$ and $I^2 = I$. Then the component I is called the *indeterminate* of a system with some specific algebraic axioms: $0I = 0$, $1I = I$, $I + I = 2I$, and I^{-1} does not exists under usual neutrosophic addition and neutrosophic Multiplication defined on the required system. The component I is a concrete mathematical tool to deal with inconsistent, incomplete and indeterminate information which exist in the real world systems. A nonempty set

N together with I is denoted by $N(I)$ and defined as $N(I)=\{a+bI : a,b\in N, I^2=I\}$, which is called neutrosophic set. Neutrosophic is an innovative research field of philosophy with the composition of indeterminacy founded by Smarandache to develop and deal indeterminacy of a system in nature and science [21]. In addition, the neutrosophic set and their interactions play an important role in classical and modern algebra, and generate a specific theory in modern mathematics called *neutrosophic algebraic theory*, and it contains many algebraic structures, like neutrosophic groups, neutrosophic rings, neutrosophic Boolean rings, neutrosophic zero rings and neutrosophic field [22-25].First, classical rings and their useful results are standard and follow those from [26]. Next, the other neutrosophic concepts and further terminology with corresponding notions will be explained in detail as follows. For any finite commutative ring R, the nonempty neutrosophic set $R(I)=\{a+bI : a,b\in R, I^2=I\}$ is called a Neutrosophic ring generated by R and I under the following neutrosophic binary operations:

$$(a+bI)+(c+dI)=(a+c)+(b+d)I,\ (a+bI)(c+dI)=(ac)+(ad+bc+bd)I.$$

Particularly, $0=0+0I$, $1=1+0I$, $I=0+1I$ are main components of the neutrosophic ring $R(I)$ with $R(I)=R+RI=\langle R\cup I\rangle$. Note that, if R is finite, and then $|R|$ denotes the number of elements in R, consequently that $|R(I)|=|R|^2$.

The contributions of this manuscript are three folds.

First, we propose the use of modular arithmetic to determine the duplex elements for the finite ring Z_n. The number of duplex elements $D(n)$ over Z_n is distributed in $Z_m\times Z_n$. Thus, the enumeration of this procedure is suitable for enumerating the number of duplex elements in $Z_m\times Z_n$. *Second*, we thoroughly characterize finite rings over their duplicities. We provide necessary constructive conditions on various finite rings and weights to achieve their related consequences. *Third* and finally, we establish systematic procedure to construct neutrosophic duplex rings over given classical rings. We prove that neutrosophic duplex rings generated by our basic neutrosophic rule $R(I)=R+RI=\langle R\cup I\rangle$ exhibit a specific structure, and maintain the basic neutrosophic properties of $R(I)$.

2. **Enumeration of Duplex Elements in** Z_n

As the heading suggests, the present section has as its goal is another simple contribution of Z_n, called duplex of Z_n. For those who consider the theory of integers and basic number theory. The intrinsic beauty of the duplex of Z_n has a strange fascination for modern mathematicians. Generally speaking, the duplex of Z_n deals with the characterization of Z_n with $Z_n+Z_n\neq 2Z_n$, or $Z_n+Z_n=2Z_n$.

This section enumerates all duplex elements which are in Z_n, and also demonstrate a number-theoretic connection between the finite number of positive integers and duplex elements in Z_n. Also, this section generates the function $D(n)$ which is a multiplicative function but not complete. Additionally, prove that $|D(Z_n)|=\dfrac{n}{(2,n)}$ and $|D(Z_m\times Z_n)|=\dfrac{mn}{(2,m)(2,n)}$.

Before moving on to the other important concepts and results of the duplex of Z_n, let us define duplex elements of Z_n with different illustrations.

First, we prove that $D(n)$ is a multiplicative function but not complete with an illustration.

Definition 2.1.

An element a in Z_n is called a *duplex element* in Z_n if and only if the equation $x+x=a$ has a solution in Z_n.

The set of all duplex elements in Z_n is denoted by $D(Z_n)$, and $D(n)$ denotes the number of duplex elements in Z_n with $D(n) \neq 0$, since $x+x=0$ is solvable in Z_n. The function $D(n)$ is called the *duplex function* of n.

For any $n>1$, we have $D(Z_{2n}) \neq Z_{2n}$ but $D(Z_{2n-1}) = Z_{2n-1}$. This means that the units of Z_{2n} are not the duplex elements in Z_{2n}. For example $D(8)=4$ since the equations $x+x=0$, $x+x=2$, $x+x=4$ and $x+x=6$ have an individual solution in Z_8, but $x+x=1$, $x+x=3$, $x+x=5$ and $x+x=7$ do not have a solution in Z_8.

The following table illustrates the number of duplex elements in Z_1, Z_2, Z_3,..., Z_{10}, respectively.

n	1	2	3	4	5	6	7	8	9	10
$D(n)$	1	1	3	2	5	3	7	4	9	5

For any positive integers m and n, the notation $gcd(m,n)$, or (m,n) denotes the greatest common divisor of m and n. Particularly, $gcd(m,n)=1$ if and only if m and n are called relatively prime. Suppose $gcd(m,n)=1$. Then the function $f: N \to \mathbb{R}$ is called a Number-Theoretic function, and it is called *multiplicative* if $f(mn)=f(m)f(n)$. Naturally, many number-theoretic functions exist in the theory of numbers [2], and which are completely characterized by its value of n when $n \geq 1$. Now we show that the duplex function $D(n)$ is a *multiplicative* function.

Theorem 2.2. Let $gcd(m,n)=1$. Then the number theoretic relation is $D(mn)=D(m)D(n)$.

Proof: First of all we adopt the notation: $D(mn)$ is the number of duplex elements in Z_{mn} and $D(m)D(n)$ is the number of duplex elements in $Z_m \times Z_n$. Because $gcd(m,n)=1$, the ring Z_{mn} is isomorphic to the ring $Z_m \times Z_n$ by the ring isomorphism ψ: $Z_{mn} \to Z_m \times Z_n$ related by $\psi(t) = (t \ mod \ m, t \ mod \ n)$ for every element t in Z_{mn} (see [1]).

First, we prove that $D(mn) \leq D(m)D(n)$. For this let a be a duplex element in Z_{mn}, then the equation $x+x=a$ is solvable in Z_{mn}. Consequently, there is an element b in Z_{mn} such that $b+b=a$ is solvable in Z_{mn}. Since ψ is an injective map from Z_{mn} onto $Z_m \times Z_n$, so there exists an element (x,y) in $Z_m \times Z_n$ such that $\psi(b)=(x,y)$. Therefore,

$$\psi(a) = \psi(b+b) = \psi(b) + \psi(b) = (x,y) + (x,y) = (x+x, y+y)$$

is solvable in $Z_m \times Z_n$. This implies that $\psi(a)$ is also a duplex element in $Z_m \times Z_n$. Hence, $D(mn) \leq D(m)D(n)$. On the other hand, we can show that $D(mn) \geq D(m)D(n)$. Suppose c is a duplex element in Z_m and d is a duplex element in Z_n. Then there exists u in Z_m and v in Z_n such that

$$(u+u, v+v) = (c,d) \text{ in } Z_m \times Z_n.$$

So, we have

$$\psi^{-1}[(c,d)] = \psi^{-1}[(u+u, v+v)] = \psi^{-1}[(u,v)+(u,v)] = \psi^{-1}[(u,v)] + \psi^{-1}[(u,v)]$$

is solvable in Z_{mn}. This implies that the element $\psi^{-1}[(c,d)]$ is also a duplex element in Z_{mn}. This shows that $D(mn) \geq$

$D(m)D(n)$. Combination of inequalities $D(mn) \leq D(m)D(n)$ and $D(mn) \geq D(m)D(n)$ yields that the equality $D(mn) = D(m)D(n)$, and this shows that $D(n)$ is a number-theoretic multiplicative function. ■

Now we continue our study by verifying other generalizations of duplex function. This requires the following:

Example 2.3. Consider $m = 2, n = 4$, we find that $D(2) = 1, D(4) = 2$ and $D(8) = 4$ with $D(8) \neq D(2)D(4)$.

Corollary 2.4. Prove that $D(1) = 1$.

Proof: Because of $0 + 0 = 0$, the element 0 is a duplex element in Z_n. So there exists an n such that $D(n) \neq 0$. But by the Theorem [2.2],

$$D(n) = D(n1) = D(n)D(1).$$

Being $D(n)$ non-zero, $D(n)$ may be cancelled from both sides of the above equation to give $D(1) = 1$. ■

The following theorem plays an important role in studying the duplexity of the ring Z_n.

Theorem 2.5. For every $n \geq 1$, the units of Z_{2n} are not the duplex elements in Z_{2n}.

Proof: Suppose $u \in Z_{2n}^{\times}$ be a duplex element in Z_{2n}. Then there exists an element x in Z_{2n} such that $x + x = u$ is solvable in Z_{2n} . By the basic celebrations of Z_{2n}, the number $2n$ divides the element $x + x - u$. So, there exists q in Z such that $x + x - u = 2nq$. But $u \epsilon Z_{2n}^{\times}$ implies that $gcd(u, 2n) = 1$, and it implies that $gcd(x + x - 2nq,\ 2n) = 1$, which is not true because $gcd(x + x - 2nq,\ 2n) > 1$ for every x in Z_{2n}. Hence every unit in Z_{2n} is not a duplex element in Z_{2n}. Particularly, $D(Z_{2n}^{\times}) = \emptyset$. ■

Our next goal is to establish a formula for enumerating the number of duplex elements in Z_n. Once this is established, enumerating formulas in a simple form for the different values of n will complete our enumerating procedure. We start with the trivial observation that the duplex element in Z_1 is 0, so that $D(1) = 1$ because $x + x = 0$ is solvable in Z_1. We are now ready to prove that $D(2^n) = 2^{n-1}$, where $n \geq 2$. Because $x + x = 2x(mod\ \ 2^n)$ for all x in Z_{2^n}, it follows that the duplex element in $Z_2 n$ is a multiple of 2 under multiplication modulo 2^n, but the total number of multiples of 2 in Z_{2^n}, is 2^{n-1} since $2^n + 2^n \equiv 0(mod\ \ 2^n)$ and thus $D(2^n) = 2^{n-1}$.

Further, we start with the simple observation that for every x in Z_{p^n}, where $p > 2$ is a prime. This concludes that every element in Z_{p^n} is a duplex element in Z_{p^n}, and thus $D(p^n) = p^n$. Finally, we aim to establish a formula for enumeration number of duplex elements in Z_n whenever $n \geq 1$. For every x in Z_n, we have

$$(2, n)x = (2x, nx) = a(2x) + b(nx) \text{ for some } a \text{ and } b \text{ in } Z_n$$
$$= 2ax \text{ in } Z_n = ax + ax \text{ in } Z_n.$$

This observation shows that x is a duplex element in Z_n if and only if $(2, n)x$ is also a duplex element in Z_n.

As we explored duplex elements in Z_n we were led to specify how many there are. We found the answer in the following way.

Theorem 2.6. The number of duplex elements in Z_n is $D(n) = \frac{n}{(2,n)}$.

Proof: Suppose there is an element x in Z_n such that the duplex form $x + x$ can be written as $x + x = nq + (2, n)r$ in Z. By the Bezout's Theorem (ref.[2]),

$$x + x = nq + (2x + ny)r \text{ for some } x, y \text{ in } Z.$$
$$= nq + 2xr + nyr = n(q + yr) + 2xr.$$

Now $x + x < n$, so $x + x = 2xr$ is a duplex in Z_n. Conversely, suppose that there is an element y in Z_n such that $y + y = mn + (2,n)s$ in Z. Then the number $(2,n)$ divides y. Thus there is an element t such that $y = (2,n)t$, and hence an element $(2,n)t$ is a duplex element in Z_n. Therefore the number of duplex elements in Z_n is

$$D(n) = \frac{|Z_n|}{(2,n)} = \frac{n}{(2,n)} \cdot \blacksquare$$

The following example demonstrates the preceding theorem.

Example 2.7. Because $(2,n) = 1 \ or \ 2$, the number of duplex elements in Z_9 is 9 and the number of duplex elements in Z_{10} is 5.

Our next aim is to enumerate the number of duplex elements in the ring $Z_m \times Z_n$ for every positive integer m and n. We recall that $Z_m \times Z_n \cong Z_{mn}$ if and only if $(m,n) = 1$. This relation explores that $D(mn) = D(m)D(n)$. Further, if $(m,n) \neq 1$, then by the Theorem [2.2], the number of duplex elements in $Z_m \times Z_n$ is

$$D(m)D(n) = \frac{m}{(2,m)} \cdot \frac{n}{(2,n)} = \frac{mn}{(2,m)(2,n)}.$$

However, we observe that

$$D(Z_m \times Z_n) = D(Z_m) \times D(Z_n) \neq D(Z_{mn}) \text{ whenever} (m,n) \neq 1.$$

Subsequently, $D(m)D(n) = \frac{mn}{(2,m)(2,n)}$ is not equal to $D(mn) = \frac{mn}{(2,mn)}$. For instance, $D(2) = \frac{2}{(2,2)} = 1$, $D(4) = \frac{4}{(2,4)} = 2$, $D(8) = \frac{8}{(2,8)} = 4$ but $D(2 \cdot 4) \neq D(2)D(4)$.

Theorem 2.8. Let $m, n \in N$. Then $D(Z_m \times Z_n) = D(Z_m) \times D(Z_n)$. Particularly, we have $|D(Z_m \times Z_n)| = \frac{mn}{(2,m)(2,n)}$.

Proof: Because of Definition [2.1], the duplex of $Z_m \times Z_n$ is defined as

$$\begin{aligned} D(Z_m \times Z_n) &= \{(a,b)\epsilon \ Z_m \times Z_n : (x,\mathrm{y}) + (x,\mathrm{y}) = (\mathrm{a},\mathrm{b}) \text{ is solvable in } Z_m \times Z_n\} \\ &= \{(a,b)\epsilon \ Z_m \times Z_n : (x+x,\mathrm{y}+\mathrm{y}) = (\mathrm{a},\mathrm{b}) \text{ is solvable in } Z_m \times Z_n\} \\ &= \{(a,b)\epsilon \ Z_m : x + x = a \text{ is solvable in } Z_m\} \times \{(a,b)\epsilon \ Z_n : y + y = b \text{ is solvable in } Z_n\} \\ &= D(Z_m) \times D(Z_n). \end{aligned}$$

This result has summarized the cardinality of $D(Z_m) \times D(Z_n)$. So, we have

$$|D(Z_m \times Z_n)| = |D(Z_m) \times D(Z_n)| = |D(Z_m)||D(Z_n)| = \frac{m}{(2,m)} \cdot \frac{n}{(2,n)} = \frac{mn}{(2,m)(2,n)}.$$

3. Duplicity of finite Rings

A ring R is *cyclic* if the structure $(R,+)$ is a cyclic group, where the additive operation $+$ is defined over the ring R. In [27], the author Buck introduced a special ring structure, called cyclic ring. This algebraic structure establishes various results and it explore different algebraic concepts. Generally, every cyclic ring is commutative but it is a ring with unity or without unity. For instance, Z_9 is a cyclic ring with unity but $R^0 = \{0,3,6\}$ is also a cyclic ring without unity under addition and multiplication modulo 9. Further, if R is a cyclic ring then obviously the Cartesian product ring $R \times R$ is not a cyclic ring. For instance, $Z_9 \times Z_9$ is not a

cyclic ring, in view of the fact that the structure $(Z_9 \times Z_9, +)$ is not a cyclic group under addition modulo 9. Throughout the paper, authors consider the ring Z_n as a cyclic ring of order n.

Recall that the element $x + x$ is called duplex form an element in Z_n under addition and multiplication defined over Z_n. Under this duplex form, we explore the following connections over Z_n : $x + x = 2x$, $x + x + x = 3x$ and so on, but $Z_n + Z_n \neq 2Z_n$, $Z_n + Z_n + Z_n \neq 3Z_n$, and so on, where the addition ' + ' defined over the ring Z_n. Now summarize these concepts in the following definitions.

Definition 3.1. An element a in a ring R is called duplex element in R if the equation $x + x = a$ has a solution in R.

For instance, the element 0 is a duplex element in every ring R, since $x + x = 0$ is solvable in R.

Definition 3.2. The duplex ring of a ring R is denoted by $Dup(R)$ and defined as

$$Dup(R) = \{a : x + x = a \text{ is solvable in } R\}.$$

For instance, the following short table illustrates the duplex rings of the rings $Z_1, Z_2, \ldots, Z_{10}$.

R	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
$Dup(R)$	Z_1	$\{0\}$	Z_3	$\{0,2\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0,2,4,6\}$	Z_9	$\{0,2,4,6,8\}$

With this information available, it is an easy task to prove the following result.

Theorem 3.3. The duplicity of R is a subring of R.

Proof. Let a and b be any two elements in $Dup(R)$. Then there exists x and y in R such that $a = x + x$ and $b = y + y$. It is clear that

$a + b = (x + x) + (y + y) = (x + y) + (x + y)$, $ab = (x + x)(y + y) = xy + xy + xy + xy = (xy + xy) + (xy + xy)$, which shows that $a + b$ and ab are both elements in $Dup(R)$, and thus $Dup(R)$ is a subring of R. ■

With the support of the preceding theorem, let us define duplex of duplex.

Definition 3.4. The duplex of duplex of a ring R is denoted by $Dup(Dup(R))$ and defined as

$$Dup(Dup(R)) = \{d \in Dup(R) : x + x = d \text{ is solvable in } Dup(R)\}.$$

Similarly, define $Dup(Dup(Dup(R)))$ as follows.

$$Dup(Dup(Dup(R))) = \{y \in Dup(Dup(R)) : x + x = y \text{ is solvable in } Dup(Dup(R))\}.$$

These notions lead directly to the following tabular information.

R	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
$Dup(R)$	Z_1	$\{0\}$	Z_3	$\{0,2\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0,2,4,6\}$	Z_9	$\{0,2,4,6,8\}$
$Dup(Dup(R))$	Z_1	$\{0\}$	Z_3	$\{0\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0,4\}$	Z_9	$\{0,2,4,6,8\}$
$Dup(Dup(Dup(R)))$	Z_1	$\{0\}$	Z_3	$\{0\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0\}$	Z_9	$\{0,2,4,6,8\}$

In vision of the preceding table, authors conclude the following.

1. $Dup(Dup(Dup(Z_n))) = Z_n \Leftrightarrow n$ is odd.

2. $Dup\left(Dup\left(Dup(Z_n)\right)\right) = Dup\left(Z_n\right) \Leftrightarrow n$ is a perfect square.

3. $Dup\left(Dup\left(Dup(Z_n)\right)\right) = (0) \Leftrightarrow n = 2^k$ for some positive integer k.

Under this information, the following theorem provides the structure of duplex ring of the ring Z_n.

Theorem 3.5. For any positive integer n, there exists duplex ring $Dup(Z_n)$ of the ring Z_n such that $Dup(Z_n) = (2,n)Z_n$, where $(2,n)$ is the greatest common divisor of the numbers 2 and n.

Proof. It is well known that $Z_n + Z_n \neq 2Z_n$ but the duplex equation $x + x = a$ is solvable in the ring Z_n for every positive integer n. So, the calculations

$$Dup(Z_n) = (2,n)\left(\frac{2}{(2,n)}\right)Z_n \subseteq (2,n)Z_n$$

conform that the first set inclusion $Dup(Z_n) \subseteq (2,n)Z_n$ is true. Before proving another way of this result, consider Bezout's Theorem [2], the number $(2,n)$ can be written as $(2,n) = 2x + ny$ for some integers x and y. Applying this Bezout's result,

$$\begin{aligned}(2,n)Z_n &= (2x+ny)Z_n = 2xZ_n, \text{ since } ny \equiv 0 \pmod n \\ &= (x+x)Z_n \subseteq Dup\left(Z_n\right).\end{aligned}$$

Two set inclusions $Dup(Z_n) \subseteq (2,n)Z_n$ and $(2,n)Z_n \subseteq Dup\left(Z_n\right)$ finalize that the duplicity of the ring Z_n as $Dup(Z_n) = (2,n)Z_n$.■

As an immediate application of preceding theorem, authors deduce the following results.

Corollary 3.6. $Dup(Z_n) = Z_n$ if and only if n is odd.

Proof. Noting that $(2,n) = 1$ if and only if n is odd, so may write $Dup(Z_n) \subseteq (2,n)Z_n = 1Z_n = Z_n$.■

In the same way, the relation $Dup(Z_n) \subseteq (2,n)Z_n$ yields the following corollary, and it is another basic fact regarding the order of the duplex ring $Dup(Z_n)$.

Corollary 3.7. Let $n \in N$. Then the cardinality of the duplex ring $Dup(Z_n)$ is $\left|Dup(Z_n)\right| = \frac{n}{(2,n)}$.

Proof. It is clear from the Theorem [section2], and additionally there is a one to one correspondence $(a+a) \mapsto \frac{n}{(2,n)}a$ for every element a in Z_n.■

Theorem 3.8. Let $m,n \in N$. Then $Dup(Z_m \times Z_n) = (2,m)(2,n)(Z_m \times Z_n)$.

Proof. By the Theorem [3.5], we have $Dup(Z_m) = (2,m)Z_m$ and $Dup(Z_n) = (2,n)Z_n$. So, it is clear from the calculations $Dup(Z_m \times Z_n) = Dup(Z_m) \times Dup(Z_n) = (2,m)Z_m \times (2,n)Z_n = (2,m)(2,n)(Z_m \times Z_n)$.■

There is an attractive illustration of the finite fields. First, notice that $Dup(Z_2) \neq Z_2$. For any odd prime, it is well known that $Dup(Z_p) = Z_p$. Particularly, if $p \equiv 3 \pmod 4$ then $Z_p[i]$ is a field of Gaussian integers and $Dup\left(Z_p[i]\right) = Z_p[i]$. Even if R is not a field then there exists R such that $Dup\left(R\right) = R$. For instance, $Z_p \times Z_p$

is not a field but $Dup\left(Z_p \times Z_p\right) = Z_p \times Z_p$. Further, there is another attractive ring R with $Dup(R) = R$, those types of rings are called zero rings and it is denoted by $R = R^0$. Now, we show that $Dup\left(R^0\right) = R^0$.

Theorem 3.9. The duplex ring of any zero rings is itself a zero ring.

Proof. The theorem is certainly true for $\left|R^0\right| = 1$, because $\left|R^0\right| = 1$ if and only if $R^0 = (0)$. Thus we may hereafter restrict our attention to nontrivial zero ring $R^0 \neq (0)$. Let $\left|R^0\right| > 1$. Then we have to prove that $Dup\left(R^0\right) = R^0$. By virtue of the first theorem of this section, $Dup\left(R^0\right) \subseteq R^0$.we makes a start by showing that $R^0 \subseteq Dup\left(R^0\right)$. For a proof by contradiction, assume that R^0 Ú $Dup\left(R^0\right)$. Then the element a is in R^0 and the equation $a = x + x$ is not solvable in R^0. Accordingly, $x + x \neq a$ for some x is in R^0. Squaring on both sides of $x + x \neq a$, it gives $(x+x)^2 \neq a^2 \Rightarrow x^2 + x^2 + x^2 + x^2 \neq a^2 \Rightarrow 0 \neq 0$, it is not true in R^0, and thus our assumption is not true. Hence, $R^0 \subseteq Dup\left(R^0\right)$. So, we finish that $Dup\left(R^0\right) = R^0$.

4.Duplicity of Neutrosophic Rings

In this section, we establish duplex rings and their corresponding neutrosophic duplex rings. On the other hand, first we prove some results of this duplicity and which are useful for subsequent results as well as for the next concepts.

Now, this study is going to define duplicity of R and $R(I)$, and study their properties with different illustrations. We notice that $R + R \neq 2R$, $R + R + R \neq 3R$, and so on.

Definition 4.1. Let R be a finite commutative ring. Then the structure $Dup(R)$ is called duplex ring, and it is defined as $Dup(R) = \{a : x + x = a \text{ is solvable in } R\}$.

For any ring R, there is a neutrosophic duplex ring $Dup(R(I))$ of the neutrosophic ring $R(I)$, and it is defined as $Dup(R(I)) = \{\alpha : \beta + \beta = \alpha \text{ is solvable in } R(I)\}$, where $\alpha = a + Ib$ and $\beta = c + Id$ are neutrosophic elements in $R(I)$.

For example, $Dup(\mathrm{Z}_2(I)) = \{0 + 0I\}$, $Dup(\mathrm{Z}_5(I)) = \mathrm{Z}_5(I)$ but $Dup(\mathrm{Z}_5(i, I)) \neq \mathrm{Z}_5(i, I)$ where $\mathrm{Z}_5(i)$ is the ring of Gaussian integers and $\mathrm{Z}_5(i, I)$ is the neutrosophic ring of Gaussian integers.

The following is a basic result to the preceding analysis of duplicity.

Theorem 4.2. The duplicity of $R(I)$ is a neutrosophic subring of $R(I)$.

Proof. By the Theorem [3.3], $Dup(R)$ is a subring of R. Further, we have $R(I) = R + RI$, and therefore, $Dup(R(\mathrm{I})) = Dup(R) + Dup(R)I$. This relation explore that $Dup(R(I))$ is generated by $Dup(R)$ and I, and hence $Dup(R(I))$ is a neutrosophic subring of $R(I)$.■

Corollary 4.3. The duplicity of $R(I)$ is a neutrosophic ideal of $R(I)$.

Proof. It is clear from the observation that $Dup(R)$ is an ideal R, and thus $Dup(R(I))$ is a neutrosophic ideal of $R(I)$. ■

The following examples are interesting illustrations of the preceding results. Here note that $Dup(Z_2(I)) = \{0+0I\}$.

Example 4.4. For any odd prime p, the neutrosophic duplex ring of $Z_p(I)$ is again $Z_p(I)$, that is $Dup(Z_p(I)) = Z_p(I)$.

Example 4.5. $Dup(Z_4(I)) = \{0, 2, 2I, 2+2I\}$, $Dup(Z_9(I)) = Z_9(I)$.

In this illustration we observed a connection of duplicity of rings and some fundamental concepts of rings. Under this observation, the following theorem provides a necessary and sufficient condition for the characteristic and duplicity of rings.

Theorem 4.6. Let $Char(R(I))$ be the characteristic of $R(I)$ with $R(I) \neq (0)$. Then, $Dup(R(I)) = (0)$ if and only if $Char(R(I)) = 2$.

Proof. It is well known that $|R| > 1$ if and only if $|R(I)| \geq 4$. So, we have $R \neq (0)$ if and only if $R(I) \neq (0)$, and additionally $Char(R) = Char(R(I))$. Thus we finish that

$Dup(R(I)) = (0) \Leftrightarrow Dup(R) + Dup(R)I = (0) \Leftrightarrow Dup(R) = (0) \Leftrightarrow r + r = 0$ is solvable in the ring R
$\Leftrightarrow 2r = 0$ for every r in $R \Leftrightarrow Char(R) = 2$. ■

The following example explores this theorem.

Example 4.7. $Dup(Z_2) = (0)$; $Dup(Z_2(I)) = (0)$, $Dup(Z_2[i]) = (0)$; $Dup(Z_2(i, I)) = (0)$,

$Dup(Z_2[x]) = (0)$; $Dup(Z_2(x, I)) = (0)$, where $Z_2[i]$ and $Z_2[x]$ are both rings of Gaussian integers and polynomials under addition and multiplication modulo 2, respectively.

The following theorem plays a significant role in characterizing finite neutrosophic rings and neutrosophic fields in terms of their corresponding duplicity of systems. Given a finite field F, there exists a neutrosophic field $F(I)$ with $F(I) = F + FI$. For instance, $Z_2(I)$, $Z_3(I)$, $Z_5(I)$ are all finite neutrosophic fields. Make a note of that $F(I) + F(I) \neq 2F(I)$.

Theorem 4.8. For any finite field F, the system $F(I) + F(I)$ is also equal to itself the neutrosophic field $F(I)$, where $F(I) + F(I)$ is defined as $F(I) + F(I) = \{\alpha + \alpha : \alpha \in F(I), \alpha \in F(I)\}$.

Proof. Because the element $x + 0$ in $F + F$, we have $x + 0 = x$, which is in F. This implies that $F + F \subseteq F$. To go the other way, let us suppose that $F \not\subset F + F$. Then, $x \in F$ implies that $x \notin F + F$. So, there is an element a in F such that $x \neq a + a$. It is not true for any finite field F, because the structure $(F, +)$ is an abelian group and the equation $x = a + a$ is solvable in $(F, +)$. Thus our point of view $F \subseteq F + F$ is also true. Hence, $F + F = F$. Suppose that F has the duplex form. We end up with the computations
$F(I) + F(I) = (F + FI) + (F + FI) = (F + F) + (IF + FI) = (F + F) + (F + F)\mathrm{I} = F + FI = F(I)$,where $IF = FI$. ■

Corollary 4.9. $Dup(F) = F$ and $Dup(F(I)) = F(I)$ whenever $Char(F) \neq 2$.

Proof. It is simply proved from Theorem [above] and Theorem [above]. ■

The following is an example to the preceding analysis of the duplex of F and $F(I)$.

Example 4.10. $Z_2 + Z_2 = Z_2$ but $Dup(Z_2) \neq Z_2$. However, $Dup(Z_2) = (0)$.

$Z_2(I) + Z_2(I) = Z_2(I)$ but $Dup(Z_2(I)) \neq Z_2(I)$. However, $Dup(Z_2(I)) = (0+0I)$.

With these results among our tools, we know that the necessary information to now carry out a proof of the fact that duplicity of a zero ring is again itself zero ring. For more information about zero rings, redder refer [24, 27]. A ring $R^0 = (R^0, +, \cdot)$ is called a zero ring if $ab = 0$ for every a and b in R^0. Every finite zero rings is commutative, and also zero ring is a ring without unity. For instance, the ring $R^0 = \{0,5,10,15,20\}$ is a finite commutative ring without unity under addition and multiplication modulo 25. Additionally, the authors Chalapathi and Madhavi introduced and studied the extended structure of zero rings, called, neutrosophic zero rings [24]. For any zero ring R^0, there exists corresponding neutrosophic zero ring $R^0(I)$, which is also commutative and without unity.

Theorem 4.11. Let R^0 be a finite zero ring. Then, $R^0(I) + R^0(I) = R^0(I)$.

Proof. The theorem is certainly true for $R^0 = (0)$, because $R^0 = (0)$ if and only if $R^0(I) = (0)$. Thus we may here after restrict our attention to nontrivial zero ring $R^0 \neq (0)$. Suppose $|R^0| > 1$ be the positive integer such that $|R^0(I)| \geq 4$. Then, first of all we prove that $R^0 + R^0 = R^0$ for any finite zero ring R^0. By virtue of addition of two rings, $R^0 + R^0 = \{a + a : a \in R^0\}$. The crux of our argument is that $R^0 + R^0$ is a subring of R^0, and this fact fallows that $R^0 + R^0 \subseteq R^0$. For a proof by a contradiction, assume that $R^0 \not\subset R^0 + R^0$ .For some a in R^0, there exists $x \in R^0$ such that $x \neq a + a$. Now squaring on both sides of $x \neq a + a$, we calculate

$$x^2 \neq (a+a)^2 \Rightarrow 0 \neq (a+a)^2, \text{ since } x^2 = 0 \Rightarrow a^2 + a^2 + (a+a) \neq 0$$

$$\Rightarrow a + a \neq 0, \text{ since } a^2 = 0$$

$$\Rightarrow a \neq -a.$$

This means that every element in a finite zero ring R^0 has not mutually additive inverse. This violates the basic condition of the zero rings [24], that means that every nonzero element in nontrivial zero ring has mutually additive inverse, giving us our contradiction. Thus, we have $R^0 \subseteq R^0 + R^0$, and hence $R^0 + R^0 = R^0$. Finally, the theorem follows the following calculations.

$$R^0(I) + R^0(I) = (R^0 + R^0 I) + (R^0 + R^0 I) = (R^0 + R^0) + (R^0 I + R^0 I) = (R^0 + R^0) + (R^0 + R^0) I$$

$$= R^0 + R^0 I = R^0(I). \blacksquare$$

Remark 4.12. From Theorem [4.11], $R^0(I)+R^0(I)=R^0(I)$ but $R^0(I)+R^0(I)=Dup\left(R^0(I)\right)$. This explores that the neutrosophic equation $\alpha+\alpha=\beta$ is solvable in $R^0(I)$ for every neutrosophic elements α and β in $R^0(I)$.

Corollary 4.13. For any zero rings R^0, we have $Dup\left(R^0(I)\right)=R^0(I)$.

Proof. It is full fill from the following calculations.

$$Dup\left(R^0(I)\right)=Dup\left(R^0\right)+Dup\left(R^0\right)I=R^0+R^0I=R^0(I).$$

5. Conclusions

In this paper, we have determined and counted all duplex elements in the finite cyclic ring Z_n. We have established that there is a number theoretic connection between the duplex function $D(n)$ and elements in Z_n, and also prove that $D(n)=\frac{n}{(2,n)}$. More importantly, we have shown that $D(Z_m\times Z_n)=\frac{mn}{(2,m)(2,n)}$. We have also discussed duplicity of finite rings and neutrosophic rings. A short discussion about how this duplex ring could be applied to the neutrosophic rings.

References

1. Judson, T. W. Abstract Algebra: Theory and Applications, Orthogonal Publishing L3c, (2016) 1-434.
2. Rosen, K.H. Elementary Number Theory and Its Application (6th edition), Pearson Publishers, (2003) 1-700.
3. Frazer, J. Algebraic Number Theory, Springer Publications, (2014) 1-292.
4. Steven T. Dougherty, Algebraic Coding Theory Over Finite Commutative Rings, Springer Publishers, USA, (2017) 1-103.
5. Jeffrey Hoffstein, Jill Pipher and Joseph H. Silverman, An Introduction to Mathematical Cryptography, Springer Publishers,USA, (2008) 1-524.
6. Antonio, L. R., Encarnación, C. M., Luis, P. R., Antonio, G. R., Algebraic Circuits, Springer Publishers,USA, (2014) 1-394.
7. Hubregt, J. Visser, Antenna Theory and Applications, Wiley Publications,New York city,USA, (2012) 1-280.
8. Charles, J. Colbourn, Algebraic Design Theory and Hadamard Matrices, Springer Publishers, (2014) 1-259.
9. Zun, S and Wang, E.T.H. Mutual Multiples in Z_n, MAA, 72 (2) (1999) 143-145.
10. Musukwa, A. I. A Special Subring associated with Irreducible Elements in the Ring of integers modulo n, MAYFEB Journal of Mathematics, 2 (2017) 1-7.
11. Chalapathi, T., Sajana, S and Bharathi, D. Classical pairs in Z_n, Notes on Number Theory and Discrete Mathematics, Bulgaria, 26 (1) (2020) 59-69.
12. Samuel, S. G., Harrington, J. Special Numbers in the Ring Z_n, Journal of Integer Sequences, Waterloo, Canada,18 (2015) 1-11.
13. Osama, A and Emad A. O. On the regular elements in Z_n, Turk J Math, Turkey, 32 (2008) 31-39.
14. Kassar, A.N. EI and Kojok, B. The number of irreducible elements in Z_n, Mathematical Sciences Research Journal,USA, 10 (2002) 512-519.

15. Mahmood, B., Reza, B., Amir, H and Hossein, K. Classification of finite rings: Theory and Algorithms. Czechoslovak Mathematical Journal, 64(3), (2014) 641-658.
16. Dutta, J., Dhiren, K.B and Rajat, K.N. Characterizing some rings of finite order. arXiv:1510.08207v1, (2015) 1-13.
17. Eldridge, K.E. Orders for Finite Noncommutative Rings with Unity. The American Mathematical Monthly, 75(5), (1968) 512-514.
18. Chalapathi, T and Kiran Kumar, R.V. Self Additive Inverse Elements of Neutrosophic Rings and Fields. Annals of Pure and Applied Mathematics, 13(1), (2017) 63-72.
19. Chalapathi, T and Kiran Kumar, R.V. Neutrosophic Units of Neutrosophic Rings and Fields. Neutrosophic Sets and Systems, 21, (2018) 5-12.
20. Chalapathi, T and Sajana, S. Unitary Invertible Graphs of Finite Rings. General Algebra and Applications, 41, (2021) 195-208.
21. Smarandache, F. A Unifying Filed in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic set, Neutrosophic Probability and Statistics. American Research Press, (2003) 1-157.
22. Vasantha Kandasamy, W.B and Smarandache, F. Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures. Hexis, Church Rock, (2006) 1-219.
23. Abobola, M. On Some Special Substructures of Neutrosophic Rings and Their Properties. International Journal of Neutrosophic Science, 4, (2020) 72-81.
24. Chalapathi, T and Madhavi, L. A study on Neutrosophic Zero Rings, Neutrosophic Sets and Systems, 30, (2019) 191-201.
25. Chalapathi, T and Madhavi, L. Neutrosophic Boolean Rings, Neutrosophic Sets and Systems, 33, (2020) 59-66.
26. Sepanski, M.R. Algebra. American Mathematical Society, 11, (2010) 1-256.
27. Warren, K.B. Cyclic Rings. Eastern Illinois Univesity-Master Theses., (2004) 1-36.

Real Examples of NeutroGeometry & AntiGeometry

Florentin Smarandache

Dept. Math and Sciences, University of New Mexico, Gallup, NM, USA

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Abstract: For the classical *Geometry*, in a geometrical space, all items (concepts, axioms, theorems, etc.) are totally (100%) true. But, in the real world, many items are not totally true. The *NeutroGeometry* is a geometrical space that has some items that are only partially true (and partially indeterminate, and partially false), and no item that is totally false. The *AntiGeometry* is a geometrical space that has some item that are totally (100%) false. While the Non-Euclidean Geometries [hyperbolic and elliptic geometries] resulted from the total negation of only one specific axiom (Euclid's Fifth Postulate), the AntiGeometry results from the total negation of any axiom [and in general: theorem, concept, idea etc.] and even of more axioms [theorem, concept, idea, etc.] and in general from any geometric axiomatic system (Euclid's five postulates, Hilbert's 20 axioms, etc.), and the NeutroAxiom results from the partial negation of any axiom (or concept, theorem, idea, etc.). Clearly, the AntiGeometry is a generalization of Non-Euclidean Geometries. [5]

Keywords: Non-Euclidean Geometries, Euclidean Geometry, Lobachevski-Bolyai-Gauss Geometry, Riemannian Geometry, NeutroManifold, AntiManifold, NeutroAlgebra, AntiAlgebra, NeutroGeometry, AntiGeometry, Hybrid (Smarandache) Geometry, NeutroAxiom, AntiAxiom, NeutroTheorem, AntiTheorem, Partial Function, NeutroFunction, AntiFunction, NeutroOperation, AntiOperation, NeutroAttribute, AntiAttribute, NeutroRelation, AntiRelation, NeutroStructure, AntiStructure.

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1. Introduction

This is a review paper on the newly emerging field of NeutroStructures and AntiStructures, introduced by Smarandache [1] since 2019 and developed [2, 3, 4] in 2020-2021, inspired from our real world since the laws and regulations do not equally apply to all citizens, but in different degrees.

Let T = true, I = indeterminacy, F = false,
where $T, I, F \in [0, 1]$ and $(T, I, F) \notin \{(1, 0, 0), (0, 0, 1)\}$.

The following neutrosophic triplets occur in our real world:

2. <Structure(1, 0, 0), NeutroStructure(T, I, F), AntiStructure(0, 0, 1)>

In any theoretical field of knowledge, the classical **Structures** have all items (concepts, axioms, theorems, properties, ideas, relationships, etc.) totally (100%) true.

But in the real world, most structures have items that are only partially true (and partially indeterminate, or partially false) and no item that is totally false (as in NeutroStructure), we call them **NeutroStructures**.

And structures that have some items that are totally (100%) false, we call them **AntiStructures**.

3. <Algebra(1, 0, 0), NeutroAlgebra(T, I, F), AntiAlgebra(0, 0, 1)>

As particular cases, when the structures are algebras or geometries, one gets the above neutrosophic triplets.

The Classical Algebraic Structures [Algebra] have all operations totally (100%) well-defined, and all axioms [theorems, concepts, ideas, etc.] totally (100%) true.

The NeutroAlgebraic Structures have operations or axioms (and in general: theorems, concepts, ideas, etc.) that are not totally (100%) well-defined or respectively totally (100%) true, but only partially well-defined or partially true [and none of them is 0% well-defined or respectively 0% true as in AntiAlgebraic Structures]. The NeutroAlgebraic Structures are in between Classical Algebraic Structures and AntiAlgebraic Structures.

And the AntiAlgebraic Structures have at least one operation or one axiom that is 0% well-defined or respectively 0% true.

4. <Geometry(1, 0, 0), NeutroGeometry(T, I, F), AntiGeometry(0, 0, 1)>

1) A geometric structure whose all axioms (and theorems, propositions, etc.) are totally true is called a classical Geometric Structure (or **Geometry**).

2) A geometric structure that has at least one NeutroOperation or one NeutroAxiom (and no AntiOperation and no AntiAxiom) is called a NeutroAlgebraic Structure (or **NeutroGeometry**).

3) A geometric structure that has at least one AntiOperation or one Anti Axiom is called an AntiAlgebraic Structure (or **AntiGeometry**).

Therefore, a neutrosophic triplet is formed: <*Geometry, NeutroGeometry, AntiGeometry*>, where "Geometry" can be any classical Euclidean, Projective, Affine, Discrete, Differential, etc. geometric structure.

*

Similarly, for any field of knowledge, the axioms (and theorems, propositions, concepts, ideas etc.) are categorized in three groups [1 – 4]:

5. <Axiom(1, 0, 0), NeutroAxiom(T, I, F), AntiAxiom(0, 0, 1)>

An axiom, defined on a given set, endowed with some operation(s). When we define an axiom on a given set, it does not automatically mean that the axiom is true for all set's elements. We have three possibilities again:

i) The axiom is true for all set's elements (totally true) [degree of truth T = 1] (as in classical algebraic structures; this is a classical **Axiom**). Neutrosophically we write: Axiom(1,0,0).

ii) The axiom if true for some elements [degree of truth T], indeterminate for other elements [degree of indeterminacy I], and false for other elements [degree of falsehood F], where (T,I,F) is different from (1,0,0) and from (0,0,1) (this is **NeutroAxiom**). Neutrosophically we write NeutroAxiom(T,I,F).

iii) The axiom is false for all set's elements [degree of falsehood F = 1](this is **AntiAxiom**). Neutrosophically we write AntiAxiom(0,0,1).

And, of course, the Axiom may be replaced by Theorem, Property, Concept, etc.

6. Examples of AntiGeometry

6.1. The Hyperbolic (Non-Euclidean) Geometry [or Lobachevski-Bolyai-Gauss Geometry] resulted from the total negation of the axiom called Euclid's Fifth Postulate [through a point exterior to a line only one parallel can be drawn to that line] by the AntiAxiom: through a point exterior to a line many parallels can be drawn to that line.

6.2. The Elliptic (Non-Euclidean) Geometry [or Riemannian Geometry] resulted from the total negation of the axiom called Euclid's Fifth Postulate [through a point exterior to a line only one parallel can be drawn to that line] by another AntiAxiom: through a point exterior to a line no parallel can be drawn to that line.

6.3. The second class of the Hybrid (Smarandache) Geometry (or SG)[1] where an axiom is totally denied but in multiple different ways in the same geometric space, which combined the Hyperbolic and Elliptic Geometries into the same geometric space, by totally denying Euclid's Fifth Postulate in two different ways:

a) there are lines and points exteriors to them such that through a point exterior to a line many parallels can be drawn to that line; and

b) there are other lines and points exteriors to them such that through a point exterior to a line no parallel can be drawn to that line.

6.4. New example of AntiGeometry that is not a Non-Euclidean Geometry

Let us have on a plane (π) all circles of radius $r > 0$ and centered into the origin (0, 0).

For example, the below drawn circles (Fig. 1).

By "point" we understand any classical point, and by "line" we understand the circumference of a circle.

Let's take any three distinct points on the circumference of the small circle (similarly it will be for all other circles).

Clearly, the points A, B, C lie on the same line (circumference), and:
the point B lies between the point A and point C;
the point C lies between the point B and point A;
and the point A lies between the point C and point B.

Therefore, Hilbert's Postulate B.3 of the Axioms of Betweenness, stated as follows:

"If A, B and C are three distinct points lying on the same line, then one and only one of the points lies between the other two."

is totally denied, because for any three distinct points lying on a line one has any point lies between the other two.

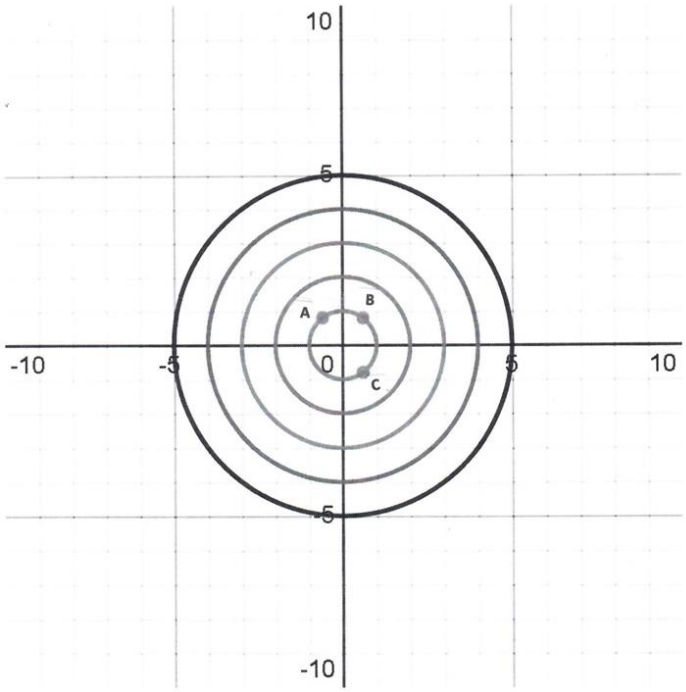

Figure 1. New Example of AntiGeometry that is not a Non-Euclidean Geometry.

[1] Linfan Mao, *Pseudo-Manifold Geometries with Applications*, Cornell University, New York City, USA, 2006. Abstract: https://arxiv.org/abs/math/0610307, Full paper: https://arxiv.org/pdf/math/0610307. "A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom (1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways."

This geometrical model does not represent a Non-Euclidean Geometry because the Euclid's Fifth Postulate is 100% true. Two lines are considered parallel if they do not intersect.

See the proof below:

Giving a line (circumference C_1 centered in the origin) and a point P_1 that does not lie on it, there exists a unique line (circumference C_2 centered in the origin) that passes though the point P_1 and does not intersect C_1.

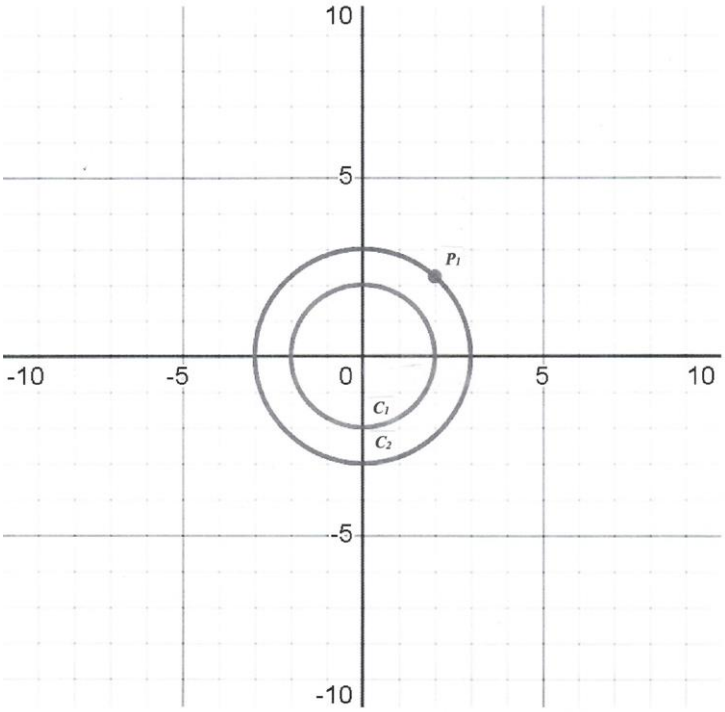

Figure 2. Euclid's Fifth Postulate is totally true.

7. Examples of NeutroGeometry

7.1. The first class of the Hybrid (or Smarandache) Geometry[1] where an axiom is partially true and partially false in the same geometric space.

For example, there are two distinct points that determine a single line, and other two distinct points that determine no line in the same geometric space.

Thus, *Hilbert's Postulate I.1.* of the Axioms of Incidence, announced as follows:

> "For every point P and every point Q not equal to P, there exists a unique line incident with the points P and Q"

becomes partially true and partially false.

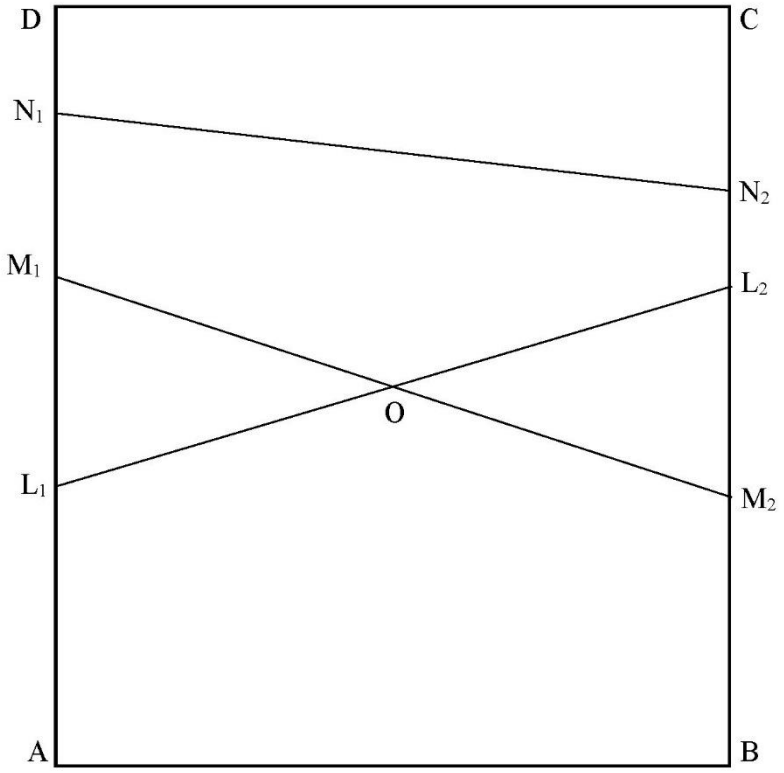

Figure 3: *Example of a NeutroGeometry that is an SG*

Assume the rectangle ABCD is a geometric space, where "point" means any classical point on the sides AB and CD or interior to this rectangle, and "line" is any segment of line connecting a point from AB with a point of CD and passing through the center O of the rectangle.

For example, L_1L_2 is a line since it connects the point L_1 lying on AB, and point L_2 lying on CD, and passes through the center O. Similarly for the line M_1M_2.

But N_1N_2 is not a line, since it does not pass through the center O.

7.2. Example of NeutroGeometry that is not an SG

We consider the previous model of the rectangular geometric space ABCD, but adding some indeterminacy (I), as in our everyday life, i.e. the dark spot below, which represents some marsh area, so M_1M_2 although it is a line since it passes through the origin O, but it has also some degree of indeterminacy when crossing through the indeterminate zone (I).

While L_1L_2 is a totally determinate line, M_1M_2 is partially determinate and partially indeterminate (as in neutrosophy).

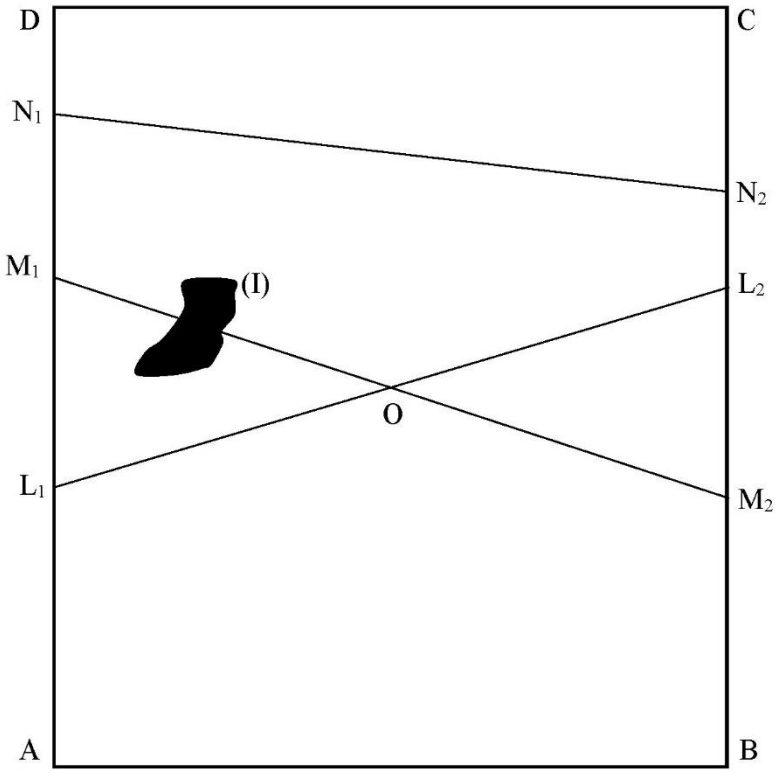

Figure 34: *Example of a NeutroGeometry that is not an SG*

5. Conclusions

In this paper we presented simple examples of NeutroGeometry, AntiGeometry, SG, and Non-Euclidean Geometries..

References

1. F. Smarandache, Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures [http://fs.unm.edu/NA/NeutroAlgebraicStructures-chapter.pdf], in his book Advances of Standard and Nonstandard Neutrosophic Theories, Pons Publishing House Brussels, Belgium, Chapter 6, pages 240-265, 2019; http://fs.unm.edu/AdvancesOfStandardAndNonstandard.pdf
2. Florentin Smarandache: NeutroAlgebra is a Generalization of Partial Algebra. International Journal of Neutrosophic Science (IJNS), Volume 2, 2020, pp. 8-17. DOI: http://doi.org/10.5281/zenodo.3989285 http://fs.unm.edu/NeutroAlgebra.pdf
3. Florentin Smarandache: Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited). Neutrosophic Sets and Systems, vol. 31, pp. 1-16, 2020. DOI: 10.5281/zenodo.3638232 http://fs.unm.edu/NSS/NeutroAlgebraic-AntiAlgebraic-Structures.pdf
4. Florentin Smarandache, Generalizations and Alternatives of Classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraic Structures, Journal of Fuzzy Extension and Applications (JFEA), J. Fuzzy. Ext. Appl. Vol. 1, No. 2 (2020) 85–87, DOI: 10.22105/jfea.2020.248816.1008 http://fs.unm.edu/NeutroAlgebra-general.pdf

5. Florentin Smarandache, NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries, Neutrosophic Sets and Systems, vol. 46, pp. 456-477, 2021. DOI: 10.5281/zenodo.5553552
6. Carlos Granados, A note on AntiGeometry and NeutroGeometry and their application to real life, Neutrosophic Sets and Systems, Vol. 49, 2022, pp. 579-593. DOI: 10.5281/zenodo.6466520
7. A.A.A. Agboola, M.A. Ibrahim, E.O. Adeleke: Elementary Examination of NeutroAlgebras and AntiAlgebras viza-viz the Classical Number Systems. International Journal of Neutrosophic Science (IJNS), Volume 4, 2020, pp. 16-19. DOI: http://doi.org/10.5281/zenodo.3989530 http://fs.unm.edu/ElementaryExaminationOfNeutroAlgebra.pdf
8. 17. F. Smarandache, Structure, NeutroStructure, and AntiStructure in Science, International Journal of Neutrosophic Science (IJNS), Volume 13, Issue 1, PP: 28-33, 2020; http://fs.unm.edu/IJNS/NeutroStructure.pdf
9. F. Smarandache, Universal NeutroAlgebra and Universal AntiAlgebra, Chapter 1, pp. 11-15, in the collective book NeutroAlgebra Theory, Vol. 1, edited by F. Smarandache, M. Sahin, D. Bakbak, V. Ulucay, A. Kargin, Educational Publ., Grandview Heights, OH, United States, 2021, http://fs.unm.edu/NA/UniversalNeutroAlgebra-AntiAlgebra.pdf
10. Madeleine Al-Tahan, NeutroOrderedAlgebra: Theory and Examples, 3rd International Workshop on Advanced Topics in Dynamical Systems, University of Kufa, Iraq, March 1st, 2021, http://fs.unm.edu/NA/NeutroOrderedAlgebra.pdf
11. F. Smarandache, Indeterminacy in Neutrosophic Theories and their Applications, International Journal of Neutrosophic Science (IJNS), Vol. 15, No. 2, PP. 89-97, 2021, http://fs.unm.edu/Indeterminacy.pdf
12. L. Mao, Smarandache Geometries & Map Theories with Applications (I), Academy of Mathematics and Systems, Chinese Academy of Sciences, Beijing, P. R. China, 2006, http://fs.unm.edu/CombinatorialMaps.pdf
13. Linfan Mao, Automorphism Groups of Maps, Surfaces and Smarandache Geometries (first edition - postdoctoral report to Chinese Academy of Mathematics and System Science, Beijing, China; and second editions - graduate textbooks in mathematics), 2005 and 2011, http://fs.unm.edu/Linfan.pdf, http://fs.unm.edu/Linfan2.pdf
14. L. Mao, Combinatorial Geometry with Applications to Field Theory (second edition), graduate textbook in mathematics, Chinese Academy of Mathematics and System Science, Beijing, China, 2011, http://fs.unm.edu/CombinatorialGeometry2.pdf
15. Yuhua Fu, Linfan Mao, and Mihaly Bencze, Scientific Elements - Applications to Mathematics, Physics, and Other Sciences (international book series): Vol. 1, ProQuest Information & Learning, Ann Arbor, MI, USA, 2007, http://fs.unm.edu/SE1.pdf
16. Howard Iseri, Smarandache Manifolds, ProQuest Information & Learning, Ann Arbor, MI, USA, 2002, http://fs.unm.edu/Iseri-book.pdf
17. Linfan Mao, Smarandache Multi-Space Theory (partially post-doctoral research for the Chinese Academy of Sciences), Academy of Mathematics and Systems Chinese Academy of Sciences Beijing, P. R. China, 2006, http://fs.unm.edu/S-Multi-Space.pdf
18. Yanpei Liu, Introductory Map Theory, ProQuest Information & Learning, Michigan, USA, 2010, http://fs.unm.edu/MapTheory.pdf
19. L. Kuciuk & M. Antholy, An Introduction to the Smarandache Geometries, JP Journal of Geometry & Topology, 5(1), 77-81, 2005, http://fs.unm.edu/IntrodSmGeom.pdf
20. S. Bhattacharya, A Model to A Smarandache Geometry, http://fs.unm.edu/ModelToSmarandacheGeometry.pdf
21. Howard Iseri, A Classification of s-Lines in a Closed s-Manifold, http://fs.unm.edu/Closed-s-lines.pdf
22. Howard Iseri, Partially Paradoxist Smarandache Geometries, http://fs.unm.edu/Howard-Iseri-paper.pdf
23. Chimienti, Sandy P., Bencze, Mihaly, "Smarandache Paradoxist Geometry", Bulletin of Pure and Applied Sciences, Delhi, India, Vol. 17E, No. 1, 123-1124, 1998.
24. David E. Zitarelli, Reviews, Historia Mathematica, PA, USA, Vol. 24, No. 1, p. 114, #24.1.119, 1997.
25. Marian Popescu, "A Model for the Smarandache Paradoxist Geometry", Abstracts of Papers Presented to the American Mathematical Society Meetings, Vol. 17, No. 1, Issue 103, 1996, p. 265.
26. Popov, M. R., "The Smarandache Non-Geometry", Abstracts of Papers Presented to the American Mathematical Society Meetings, Vol. 17, No. 3, Issue 105, 1996, p. 595.

27. Brown, Jerry L., "The Smarandache Counter-Projective Geometry", Abstracts of Papers Presented to the American Mathematical Society Meetings, Vol. 17, No. 3, Issue 105, 595, 1996.
28. Florentin Smarandache, "Collected Papers" (Vol. II), University of Kishinev Press, Kishinev, pp. 5-28, 1997.
29. Florentin Smarandache, "Paradoxist Mathematics" (lecture), Bloomsburg University, Mathematics Department, PA, USA, November 1995.
30. F. Smarandache, Degree of Negation of Euclid's Fifth Postulate, http://fs.unm.edu/DegreeOfNegation.pdf
31. M. Antholy, An Introduction to the Smarandache Geometries, New Zealand Mathematics Colloquium, Palmerston North Campus, Massey University, 3-6 December 2001. http://fs.unm.edu/IntrodSmGeom.pdf
32. L. Mao, Let's Flying by Wing — Mathematical Combinatorics & Smarandache Multi-Spaces / 让我们插上翅膀飞翔 -- 数学组合与 Smarandache 重叠空间, English Chinese bilingual, Academy of Mathematics and Systems, Chinese Academy of Sciences, Beijing, P. R. China, http://fs.unm.edu/LetsFlyByWind-ed3.pdf
33. Ovidiu Sandru, Un model simplu de geometrie Smarandache construit exclusiv cu elemente de geometrie euclidiană, http://fs.unm.edu/OvidiuSandru-GeometrieSmarandache.pdf
34. L. Mao, A new view of combinatorial maps by Smarandache's notion, Cornell University, New York City, USA, 2005, https://arxiv.org/pdf/math/0506232
35. Linfan Mao, A generalization of Stokes theorem on combinatorial manifolds, Cornell University, New York City, USA, 2007, https://arxiv.org/abs/math/0703400
36. Linfan Mao, Combinatorial Speculations and the Combinatorial Conjecture for Mathematics, Cornell University, New York City, USA, 2006, https://arxiv.org/pdf/math/0606702
37. Linfan Mao, Geometrical Theory on Combinatorial Manifolds, Cornell University, New York City, USA, 2006, https://arxiv.org/abs/math/0612760
38. Linfan Mao, Parallel bundles in planar map geometries, Cornell University, New York City, USA, 2005, https://arxiv.org/pdf/math/0506386
39. Linfan Mao, Pseudo-Manifold Geometries with Applications, Cornell University, New York City, USA, 2006, Paper's abstract: https://arxiv.org/abs/math/0610307, Full paper: https://arxiv.org/pdf/math/0610307
40. Cornel Gingăraşu, Arta Paradoxistă, Sud-Est Forum, Magazine for photography, culture and visual arts, https://sud-estforum.ro/wp/2020/06/paradoxismul-curent-cultural-romanesc-sau-cand-profu-de-mate-isi-gaseste-libertatea-inpoezie
41. F. Smarandache, Neutrosophy, A New Branch of Philosophy, Multiple Valued Logic / An International Journal, United States, Vol. 8, No. 3, pp. 297-384, 2002.

A Refined Neutrosophic Components into Subcomponents with Plausible Applications to Long Term Energy Planning Predominated by Renewable Energy

Victor Christianto,[1] **Florentin Smarandache**[2]

[1] Malang Institute of Agriculture, East Java, Indonesia; victorchristianto@gmail.com.
[2] Department of Mathematics & Sciences, University of New Mexico, Gallup, NM 87301, USA; smarand@unm.edu.

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Abstract

It is known that in the long-term planning of future energy supplies, in most countries, it is most likely that the energy mix will be predominated by renewable energy generation. While this feature of a predominating renewable energy mix appears too far fetched, such long term planning has been suggested, for instance, by Mark Jacobson from Stanford and also by John Blackburn. Here we consider two things, the first would be how to consider the long-term energy planning based on neutrosophic logic split into subcomponents.

Keywords: Neutrosophic Logic, Long Term Planning, Renewable Energy, Renewable Energy Generation, Decentralized Energy, Decentralization and Clean Energy.

1 | Introduction

The global energy landscape is undergoing a significant transformation, driven by the urgent need to transition towards sustainable and reliable renewable energy sources. This shift presents a complex challenge, requiring robust and adaptable planning strategies that can navigate the inherent uncertainties and imprecisions associated with long-term forecasting. In this quest for effective solutions, the field of neutrosophic logic is likely to emerge as a powerful tool for analysis and prediction, offering a framework capable of handling imprecise, incomplete, and indeterminate information.

This article delves into the exciting potential of refined neutrosophic components and subcomponents, particularly their ability to be decomposed into subcomponents, for long-term energy planning dominated by renewable energy sources. By harnessing the strengths of neutrosophic logic and its refined components, we aim to shed light on plausible applications that can empower stakeholders to make informed decisions in the face of complex and evolving energy scenarios.

The following sections will explore the theoretical underpinnings of refined neutrosophic components, delve into their decomposition process, and showcase their practical applicability in the context of long-term renewable energy planning.

It is known that in the long term planning of future energy supplies, in most countries, it is most likely that the energy mix will be predominated by renewable energy generation. While this feature of predominating renewable energy mix appears too far fetched, such long-term planning has been suggested, for instance, by Mark Jacobson from Stanford and also by John Blackburn. Here we consider two things, the first would be how to consider long term energy planning based on Neutrosophic logic split into subcomponents. And secondly, how to consider 4 stages of energy mix development to be considered by national energy planners (sometimes it is called as energy transition plan).

Embracing Uncertainty: Why Indeterminacy and Intermittency Are Crucial for Long-Term Renewable Energy Planning.

The transition towards a renewable energy future holds immense promise for environmental sustainability and energy security. However, achieving this goal necessitates a paradigm shift in how we approach long-term energy planning. Traditionally, planning has relied on deterministic models, assuming precise data and predictable outcomes. However, the inherent characteristics of renewable energy sources, such as solar and wind, demand a new approach that embraces **indeterminacy and intermittency**.

Indeterminacy refers to the inherent limitations in our ability to perfectly predict future events. Renewable energy sources, like wind and solar, are dependent on variable environmental factors like wind speed and sunlight availability. These factors introduce a degree of uncertainty that deterministic models struggle to capture. For instance, accurately predicting solar power generation for the next decade is inherently challenging due to unpredictable weather patterns.

Intermittency signifies the fluctuating nature of renewable energy production. Unlike traditional fossil fuels, which can be reliably dispatched on demand, solar and wind power generation inherently fluctuates based on environmental conditions. This intermittency poses a challenge for grid stability, as it necessitates additional measures to balance supply and demand in real-time.

Ignoring these crucial aspects can lead to several pitfalls:

- Overly optimistic planning: Underestimating the impact of indeterminacy and intermittency can lead to unrealistic expectations regarding the contribution of renewables to the energy mix. This can result in inadequate infrastructure investments and planning shortfalls when faced with real-world variability.
- Inefficient resource allocation: Failing to account for intermittency can lead to inefficiencies in resource allocation. For example, overbuilding solar capacity in a region with limited sunshine hours might be inefficient if not coupled with adequate storage solutions.
- Grid instability: Ignoring the intermittent nature of renewables can exacerbate grid stability challenges. Without proper planning and integration of technologies like energy storage, fluctuating renewable energy supplies can disrupt grid operations.

Therefore, embracing indeterminacy and intermittency within long-term renewable energy planning is not a limitation, but rather a critical step towards achieving a sustainable and secure energy future. By incorporating these factors, we can:

- Develop more realistic and robust plans: By acknowledging the inherent uncertainties and fluctuations, we can create plans that are more adaptable and resilient to unforeseen circumstances.
- Optimize resource utilization: By understanding the limitations and strengths of different renewable sources, we can allocate resources more efficiently, focusing on technologies best suited for specific regions and grid conditions.
- Enhance grid stability: By integrating technologies like energy storage and demand-side management, we can mitigate the challenges posed by intermittency and ensure a stable and reliable electricity grid.

In conclusion, acknowledging and incorporating indeterminacy and intermittency is not a sign of weakness but a testament to our commitment to building a resilient and sustainable energy future. By embracing the inherent complexities of renewable energy sources, we can pave the way for a future powered by clean, reliable, and affordable energy for generations to come.

2 | Basic Definitions

In 2013, Smarandache Refined / Split the Neutrosophic Components (T, I, F) into Neutrosophic SubComponents (T1, T2, ...; I1, I2, ...; F1, F2, ...). Such a Neutrosophic SubComponents is likely to play significant role in the case of long term energy planning, i.e. we only split the Indeterminacy I into I1 = proper indeterminacy, I2 = intermittency, This is called Quadruple Neutrosophic Set/Logic (4 values), but we can have as many values as needed, i.e. n-valued Neutrosophic Set. More recently, that is called Plithogenics Logic, with more than 3 components (including indeterminacy and intermittency).

2.1 | Enhancing Renewable Energy Planning with Fine-Grained Uncertainty Handling

The growing demand for sustainable energy solutions has placed renewable sources like solar and wind power at the forefront. However, their inherent variability presents a significant challenge for power grid stability and long-term planning. Traditional approaches often struggle to adequately capture this multifaceted uncertainty. This is where Neutrosophic Logic, with its ability to handle indeterminacy, emerges as a powerful tool.

2.2 | Neutrosophic Logic: Beyond True and False

Classical logic operates within a binary framework – a statement is either true or false. Fuzzy logic introduced a degree of truth, allowing for statements to be partially true and partially false. Neutrosophic Logic goes further, introducing a third component: indeterminacy. This represents the unknown or the degree to which the truth or falsity of a statement cannot be determined.

2.3 | Limitations of Single Indeterminacy

While Neutrosophic Logic offers a significant improvement over traditional logic for dealing with uncertainty, it may not be fully equipped to handle the complexities of renewable energy sources. The single indeterminacy component might not adequately capture the distinction between inherent randomness and predictable fluctuations.

2.4 | Introducing Subcomponents: Quadruple Neutrosophic Logic

This article proposes a refinement: Quadruple Neutrosophic Logic. It splits the single indeterminacy component into two subcomponents:

- Proper Indeterminacy (I1): Represents genuine uncertainty, similar to the original neutrosophic indeterminacy.
- Intermittency (I2): Captures the predictable fluctuations inherent in renewable energy sources, such as the daily and seasonal variations in solar and wind power generation.

This distinction allows for a more nuanced understanding of uncertainty. For example, a statement like "solar power will be available tomorrow" can be evaluated based on both its inherent unpredictability (I1) and the predictable daily cycle of sunshine (I2).

Benefits of Quadruple Neutrosophic Logic for Long-Term Energy Planning:

By incorporating intermittency as a distinct component, Quadruple Neutrosophic Logic offers several advantages for long-term energy planning:

- Improved forecasting: By separating intermittency from general indeterminacy, models can better predict the expected fluctuations in renewable energy sources, leading to more accurate forecasts.
- Enhanced decision-making: A clearer understanding of the different types of uncertainty allows policymakers and energy planners to make informed decisions about grid infrastructure, energy storage solutions, and integration with other energy sources.
- Increased efficiency: By accounting for predictable fluctuations, the utilization of renewable energy sources can be optimized, reducing energy waste and improving the overall efficiency of the power grid.

2.5 | Beyond Quadruple Neutrosophic Logic: Exploring the Potential of Plithogenics

The concept of Quadruple Neutrosophic Logic serves as a stepping stone towards even more sophisticated uncertainty management tools. Plithogenic Logic, a generalization of Neutrosophic Logic, allows for the inclusion of more than three components, enabling an even finer-grained analysis of complex phenomena.

While Quadruple Neutrosophic Logic offers a promising approach for enhancing renewable energy planning, further research and development are needed to fully explore its potential and integrate it into existing planning frameworks. As we continue to explore advanced logic systems like Plithogenics, we can unlock new possibilities for managing uncertainty and building a more sustainable energy future.

3 | Outline of Plausible Energy Transition, A Roadmap to Decentralization and Clean Power

The way we generate and distribute energy is undergoing a fundamental shift. Traditionally reliant on centralized fossil fuel power plants, the world is transitioning toward cleaner, more decentralized energy sources. This shift is complex, spanning technological advances, policy changes, and shifts in consumer preferences. Here, we outline four key stages of this energy transition:

Stage I: Centralized Power Producers

The current dominant model relies heavily on large-scale power plants burning fossil fuels like coal, oil, and natural gas. These plants feed into extensive national grids, distributing electricity over long distances. While providing energy security, this model contributes significantly to climate change and pollution.

Stage II: Integrated Power Producers

This stage marks the beginning of the clean energy revolution. Renewable sources like solar, wind, and hydropower begin to gain significant traction, supplementing – but not replacing – traditional power generation. National grids still play a crucial role, but the energy mix starts to diversify. Challenges with intermittency and storage emerge during this phase.

Stage III: Rise of Renewable Energy

Renewable energy sources become increasingly cost-effective and widely adopted. Households, businesses, and utility-scale operations turn to solar panels, wind turbines, and other clean technologies. The reliance on the national grid lessens, while smart grids and local energy storage solutions ensure stability. This stage sees a pivotal shift in the energy landscape.

Stage IV: Individual Power Producers

The pinnacle of the energy transition sees a decentralized model take hold. Communities, cities, and even individual households generate much of their own electricity through renewable sources like:

- **Solar:** Harnessing energy from the sun is a mainstay.

- **Biomass:** Utilizing organic matter for power.
- **Geothermal:** Tapping into Earth's internal heat.
- **Water:** Hydropower and smaller-scale water vortex systems.
- **OTEC:** Ocean Thermal Energy Conversion for coastal areas.

This distributed system is less reliant on large-scale transmission infrastructure, increasing energy independence and resilience.

3.1 | Paving the Way to Individual Power Producers: Key Preparations

The final stage of the energy transition, "Individual Power Producers," envisions a future where communities and even individual households generate most of their own electricity through renewable sources. While this stage holds immense promise for a cleaner, more resilient energy future, achieving it requires extensive preparation across various sectors. Let's delve into the key areas that require significant groundwork:

3.1.1 | Technological Advancements

- **Enhanced storage solutions:** Efficient and affordable battery storage technologies are crucial to manage the inherent variability of renewable energy sources like solar and wind. This will ensure reliable power supply even during periods of low generation.
- **Smart grid infrastructure:** Integrating renewable energy effectively requires intelligent grid management systems that can optimize distribution, integrate different sources, and handle fluctuations. Bi-directional communication and real-time data analysis will be vital.
- **Microgrid development:** Localized power grids, known as microgrids, can play a crucial role in enabling communities and individual users to manage their own energy needs. Advancements in microgrid technology are necessary to ensure seamless integration with the larger grid and ensure reliability.
- **Advancements in renewable technologies:** Ongoing research and development in renewable energy fields are essential to improve efficiency, reduce costs, and explore diverse renewable sources suitable for different geographical contexts. Solar panel efficiency, wind turbine capacity, and advancements in technologies like geothermal and ocean energy conversion are key areas for development.

3.1.2 | Policy and Regulatory Frameworks

- **Financial incentives:** Creating attractive financial incentives, such as feed-in tariffs and tax breaks, can encourage individuals and communities to invest in renewable energy technologies. This can significantly lower the upfront costs and promote wider adoption.
- **Streamlined permitting and regulations:** Simplifying the process for permitting and installing individual and community renewable energy systems can significantly increase adoption rates. Clear and supportive regulations are essential to balance innovation and safety concerns.
- **Grid access and interconnection:** Establishing clear and fair regulations for connecting individual and community-owned renewable energy systems to the national grid or microgrids is crucial. This allows surplus power generated to be fed back into the system, promoting efficiency and grid stability.

3.1.3 | Social and Community Engagement

- **Public awareness and education:** Raising public awareness about the benefits of clean energy and individual power generation is vital. Educational initiatives can empower individuals to make informed energy choices and participate in the transition.

- **Community building and financing models:** Developing innovative financing models, such as community ownership of renewable energy projects, can provide access to clean energy for individuals and communities with limited financial resources. Fostering collaboration and knowledge sharing can accelerate progress.
- **Capacity building and skill development:** Providing training and educational opportunities to establish a skilled workforce that can install, maintain, and manage individual and community renewable energy systems is essential. This ensures the long-term sustainability and success of the transition.

3.1.4 | Benefits of a Decentralized Energy Future

The transition to clean and decentralized energy has numerous advantages:

- **Reduced emissions:** A drastic decline in greenhouse gas emissions is critical for fighting climate change.
- **Improved air quality:** Cleaner energy sources improve public health by reducing pollution.
- **Energy independence:** Communities and individuals gain control over their energy production.
- **Resilience:** Local power generation makes systems less vulnerable to large-scale disruptions.
- **Economic opportunity:** Growth in clean energy industries creates jobs and boosts economies.

3.1.5 | Challenges and Considerations

The energy transition also faces hurdles:

- **Storage solutions:** Managing the intermittency of renewable sources requires advanced storage technologies.
- **Investment:** Significant investments in infrastructure and clean energy technologies are needed.
- **Regulatory frameworks:** Policies and regulations need to adapt to support decentralized power systems.

4 | Concluding Remark

The journey toward a clean and decentralized energy future is an ongoing one. Governments, businesses, and individuals must work together to promote innovation, investment, and supportive policies. The stages outlined here provide a framework for understanding this complex transition and the opportunities and challenges it presents. In this review article, we have discussed plausible stages toward fully clean energy along with individual power producers and the necessity of introducing neutrosophic subcomponents to include both indeterminacy and intermittency in long term energy planning. Achieving the "Individual Power Producers" stage is a long-term endeavor requiring comprehensive and sustained efforts across various sectors. By focusing on technological advancements, establishing supportive policy frameworks, and fostering community involvement, we can pave the way for a future powered by clean, locally-generated renewable energy. This journey towards a decentralized and sustainable energy future holds immense potential for a cleaner environment, greater energy independence, and a more equitable energy landscape for all.

References

[1] John O. Blackburn. The renewable energy alternative. Durham: Duke University Press, 1987

[2] F. Smarandache. Plithogeny, Plithogenic set, Logic, Probability, Statistics. Brussels: Pons, 2017

[3] F. Smarandache. n-Valued Refined Neutrosophic Logic and Its Applications to Physics. Progress in Physics. 2013. url: https://fs.unm.edu/n-ValuedNeutrosophicLogic-PiP.pdf

Inspection Assignment Form for Product Quality Control Using Neutrosophic Logic

Maissam Jdid [1,*], Florentin Smarandache [2], and Said Broumi [3]

[1,*] Faculty member, Damascus University, Faculty of Science, Department of Mathematics, Syria; maissam.jdid66@damascusuniversity.edu.sy.

[2] University of New Mexico, Mathematics, Physics and Natural Sciences Division 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu.

[3] Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Morocco; broumisaid78@gmail.com.

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Abstract: During the production process, production companies need to monitor the finished products and ensure their quality, which imposes on them the appointment of inspectors for auditing, and this appointment costs the company amounts that affect the general profit, so it strives to make this cost as low as possible and that the audit process is carried out with high accuracy because in case that the finished products do not conform to the basic specifications of the product, the company is required to pay a fine. In this research, we will formulate the text of the problem appropriately for such a case and then we will build the appropriate mathematical model through which we can obtain the lowest possible cost of inspection, and we will present the study using classical values and neutrosophic values, and we will demonstrate them through an example that shows us the difference between them.

Keywords: Mathematical models; Neutrosophic science; Inspection issue; Product quality control.

1. Introduction

Since the genesis of the science of operations research, it has been providing solutions to the problems faced by companies through studies presented by scientists and researchers in all fields using the methods of this science, in this research we will use the method of linear programming to build a mathematical model that enables us to ensure the quality of products using inspectors and at the lowest possible cost where we will formulate the appropriate issue for that and build the mathematical model that by finding the optimal solution to it and using the methods of solution provided by the science of operations research we get on the lowest cost of inspection and high-quality products, where the matter will be displayed

First : Using classical values based on references [1,2,3]

Second: Neutrosophic values based on what researchers and those interested in this science have presented and developed through studies and research that have been published in references [4, 16]

Discussion:

1- Studying using classical values:

Formulating the problem and building the mathematical model:

Text of the issue: Through the available information about the functioning of companies, we can develop the following text:

The company has a rank for inspectors and wants to assign the task of quality control to them, and a piece of product should be audited daily during an hour of work per day, in the following table we explain the full information about the inspectors and for all mattresses nKS.

Table 1. Information on inspectors using classical values.

About the Inspector Inspector rank	Number of pieces checked (hour)	Accuracy (percent)	Inspector's remuneration (Monetary Unit per Hour)	Number of inspectors	The fine paid by the company for each fault to the inspector
1	M_1	D_1	G_1	A_1	R
2	M_2	D_2	G_2	A_2	R
----			-		-----
n	M_n	D_n	G_n	A_n	R

Required: Formulate the appropriate mathematical model through which we can assign the optimal assignment to the inspectors so that the cost of inspection is as low as possible.

Building the Mathematical Model:

To build the mathematical model, we impose the number of inspectors of each rank on the order assigned to the inspection task, then the following inequality must be met:

$x_1, x_2, - - - - x_n$

$$x_j \leq A_j \quad ; \quad j = 1,2, --, n$$

Since the company needs to audit K piece daily within S working hour per day, the following set of restrictions must be met:

$$\sum_{j=1}^{n} SM_j x_j \geq K$$

To obtain the target follower we note that the company bears two types of costs during the inspection process, the inspector's wage and the fine corresponding to the error committed by the inspector to write the target follower note the following:

The cost of the inspector from rank j hourly salary is calculated through the following relationship

$$C_j = G_j + M_j R_j \left(\frac{100 - D_j}{100}\right) \quad ; \quad j = 1,2, - - -, n$$

The total costs for all inspectors assigned to the task of quality control per hour shall be given by the following relationship:

$$TC_j = \sum_{j=1}^{n} \left[G_j + M_j R_j \left(\frac{100 - D_j}{100}\right)\right] x_j$$

The target function is then written as follows:

$$Z = S \sum_{j=1}^{n} \left[G_j + M_j R_j \left(\frac{100 - D_j}{100}\right)\right] x_j$$

From the above, we can develop the following mathematical model:
We want to find the smallest possible value for the function:

$$Z = S\sum_{j=1}^{n}\left[G_j + M_j R_j\left(\frac{100 - D_j}{100}\right)\right]x_j \to Min$$

Within Restrictions

$$x_j \le A_j \quad ; \quad j = 1,2,--,n$$

$$\sum_{j=1}^{n} SM_j x_j \ge K$$

$$x_j \ge 0 \ ; \ j = 1,2,----n$$

Example:
The company has three ranks for inspectors and wants to assign the task of quality control to them, and 1500piece should be audited daily during 8working hours per day, in the following table we explain the full information about the inspectors and for all mattresses.

Table 2. Information on inspectors using classical values.

About the Inspector Inspector rank	Number of pieces checked (hour)	Accuracy (percent)	Inspector's remuneration (Monetary Unit per Hour)	Number of inspectors	The fine paid by the company for each fault to the inspector
1	15	95	4	10	2
2	10	90	3	6	2
3	25	98	5	8	2

Required : Formulate the appropriate mathematical model through which we can assign the optimal assignment to the inspectors so that the cost of inspection is as low as possible
To build the mathematical model, we impose the number of inspectors from the three ranks in the order assigned to the inspection task, then the following in equations must be fulfilled. x_1, x_2, x_3

$$x_1 \le 10$$
$$x_2 \le 6$$
$$x_3 \le 8$$

Since the company needs to audit 1500 pieces daily during 8working hours a day, the following set of restrictions must be met:

$$\sum_{j=1}^{n} 8M_j x_j \ge 1500$$

That is

$$8(M_1x_1 + M_2x_2 + M_3x_3) \ge 1500$$

From it we get the following restriction:

$$120x_1 + 80x_2 + 200x_3 \ge 1500$$

To obtain the target function, we note that the company bears two types of costs during the inspection process, the inspector's fee and the fine corresponding to the error committed by the inspector for each piece then the target function will be written as follows:

Then the cost of the inspector's hourly salary from rank j is calculated through the following relationship :

$$C_j = G_j + M_j R_j \left(\frac{100 - D_j}{100}\right) \;\; ; \;\; j = 1,2,---,n$$

From them we get

$$C_1 = 4 + 15 \times 2 \times \left(\frac{100 - 95}{100}\right) = 5.5$$

$$C_2 = 3 + 10 \times 2 \times \left(\frac{100 - 90}{100}\right) = 5$$

$$C_3 = 5 + 25 \times 2 \times \left(\frac{100 - 98}{100}\right) = 6$$

The total costs for all inspectors assigned to the task of quality control per hour shall be given by the following relationship:

$$TC_j = \sum_{j=1}^{n} \left[G_j + M_j R_j \left(\frac{100 - D_j}{100}\right)\right] x_j$$

We substitute the values available to us and we get:

$$TC_j = 5.5x_1 + 5x_2 + 6x_3$$

substituting the following target phrase:

$$Z = S \sum_{j=1}^{n} \left[G_j + M_j R_j \left(\frac{100 - D_j}{100}\right)\right] x_j$$

We get:

$$Z = 44x_1 + 40x_2 + 48x_3$$

From the above, we can develop the following mathematical model:

We want to find :

$$MinZ = 44x_1 + 40x_2 + 48x_3$$

Within Restrictions

$$x_1 \leq 10$$

$$x_2 \leq 6$$

$$x_3 \leq 8$$

$$120x_1 + 80x_2 + 200x_3 \geq 1500$$

$$x_j \geq 0 \;\; ; \;\; j = 1,2,3$$

To obtain the optimal solution, we use the simplex method, which is sufficiently explained in the references [1,2,3]

2- Formulation of the problem and the construction of mathematical model according to neutrosophic values:

The study concluded in the research [12] shows us how to construct neutrosophic linear models, (the linear model is a neutrosophic model if at least one of the likes of variables in the target function or neutrosophic value constraints)

The text of the issue:

The company has n rank for inspectors and wants to assign the task of quality control to them, and K pieces should be audited daily during an S hour of work per day, in the following table we explain the full information about the inspectors and for all ranks:

Table 3. Information on inspectors using neutrosophic values

About the Inspector Inspector rank	Number of pieces checked (hour)	Accuracy (percent)	Inspector's remuneration (Monetary Unit per Hour)	Number of inspectors	The fine paid by the company for each fault to the inspector
1	NM_1	ND_1	G_1	A_1	R
2	NM_2	ND_2	G_2	A_2	R
-----	--		-		-----
n	NM_n	ND_n	G_n	A_n	R

The number of pieces is a neutrosophic value$NM_j = M_j + \varepsilon_j$ where ε_j is the indeterminacy on the number of pieces, it can take one of the shapes $[\lambda_{j1}, \lambda_{j2}]$ or $\{\lambda_{j1}, \lambda_{j2}\}$or any value close to M_j as well as the precision, neutrosophic values $ND_j = D_j + \delta_j$ where δ_j is the indeterminacy on the precision that can take one of the shapes $[\mu_{j1}, \mu_{j2}]$ or $\{\mu_{j1}, \mu_{j2}\}$ or any value close to D_j.

Required: Formulate the appropriate mathematical model through which we can assign the optimal support to the inspectors so that the cost of inspection is as low as possible

Building the neutrosophic mathematical model:

To build the mathematical model, we impose $x_{1,}x_2, - - --, x_n$ the number of inspectors of each rank on the order assigned to the inspection task, then the following inequality must be met:

$$x_j \leq A_j \quad ; \quad j = 1,2, --, n$$

Since the company needs to audit K piece daily within S working hour per day, the following set of restrictions must be met:

$$\sum_{j=1}^{n} S(NM_j)x_j \geq K$$

To obtain the target function, we note that the company bears two types of costs during the inspection process, the inspector's fee and the fine corresponding to the error committed by the inspector for each piece then the target follower writes as follows:

$$Z = S\sum_{j=1}^{n} G_j + NM_j R_j \left[\frac{100 - ND_j}{100}\right] x_j$$

Then the mathematical model is written as follows:

$$Z = S\sum_{j=1}^{n} G_j + NM_j R_j \left[\frac{100 - ND_j}{100}\right] x_j \ \rightarrow Min$$

Within Restrictions

$$x_j \leq A_j \quad ; \quad j = 1,2,--,n$$

$$\sum_{j=1}^{n} S(NM_j)x_j \geq K$$

$$x_j \geq 0 \ ; \ j = 1,2,----n$$

Example 1:

A company has three ranks for inspectors and wants to assign the task of quality control to them, and 1500 pieces should be audited daily during 8 working hours per day, in the following table we explain the full information about the inspectors and for all ranks, in this example we will take the number of pieces checked by the inspectors from each rank as neutrosophic values

Table 4. Information on inspectors using neutrosophic values.

About the Inspector Inspector rank	Number of pieces checked (hour)	Accuracy (percent)	Inspector's remuneration (Monetary Unit per Hour)	Number of inspectors	The fine paid by the company for each fault to the inspector
1	{15,16}	95	4	10	2
2	{10,11}	90	3	6	2
3	{25,26}	98	5	8	2

Required : Formulate the appropriate mathematical model through which we can assign the optimal assignment to the inspectors so that the cost of inspection is as low as possible

To build the mathematical model, we impose $x_{1,}x_2, x_3$ as the number of inspectors from the three ranks in the order assigned to the inspection task, then the following inequality must be met:

$$x_1 \leq 10$$
$$x_2 \leq 6$$
$$x_3 \leq 8$$

Since the company needs to audit K pieces daily within S working hour per day, the following set of restrictions must be met:

$$\sum_{j=1}^{n} 8M_j x_j \geq 1500$$

That is

$$8(M_1x_1 + M_2x_2 + M_3x_3) \geq 1500$$

From it we get the following restriction:

$$8\{15,16\}x_1 + 8\{10,11\}x_2 + 8\{25,26\}x_3 \geq 1500$$

To obtain the target function, we note that the company bears two types of costs during the inspection process, the inspector's fee and the fine corresponding to the error committed by the inspector for each piece then the target follower writes as follows:

Then the cost of the inspector is calculated from j the hourly rank through the following relationship:

$$C_j = G_j + M_j R_j \left(\frac{100 - D_j}{100}\right) \;\; ; \;\; j = 1,2, - - -, n$$

From that we get

$$C_1 = 4 + \{15,16\} \times 2 \times \left(\frac{100 - 95}{100}\right) = \{5.5,5.6\}$$

$$C_2 = 3 + \{10,11\} \times 2 \times \left(\frac{100 - 90}{100}\right) = \{5,5.2\}$$

$$C_3 = 5 + \{25,26\} \times 2 \times \left(\frac{100 - 98}{100}\right) = \{6,6.04\}$$

The total costs for all inspectors assigned to the task of quality control per hour shall be given by the following relationship:

$$TC_j = \sum_{j=1}^{n} \left[G_j + M_j R_j \left(\frac{100 - D_j}{100}\right)\right] x_j$$

$$TC_j = \{5.5,5.6\}x_1 + \{5,5.2\}x_2 + \{6,6.04\}x_3$$

substituting the following target phrase:

$$Z = S \sum_{j=1}^{n} \left[G_j + M_j R_j \left(\frac{100 - D_j}{100}\right)\right] x_j$$

We get:

$$Z = \{44,44.8\}x_1 + \{40,41,6\}x_2 + \{48,48.32\}x_3$$

From the above, we can develop the following mathematical model:

We want to find:

$$MinZ = \{44,44.8\}x_1 + \{40,41,6\}x_2 + \{48,48.32\}x_3$$

Within Restrictions

$$x_1 \leq 10$$

$$x_2 \leq 6$$

$$x_3 \leq 8$$

$$8\{15,16\}x_1 + 8\{10,11\}x_2 + 8\{25,26\}x_3 \geq 1500$$

$$x_j \geq 0 \;\; ; \;\; j = 1,2,3$$

Example 2:

A company has three ranks for inspectors and wants to assign the task of quality control to them, and 1500 pieces should be checked daily during 8working hours per day, in the following table we explain the full information about inspectors and for all ranks, in this example we will take the accuracy of inspection for each inspector as neutrosophic values in the form of areas whose minimum range is less accurate and the highest range is the highest accuracy that the inspector reaches by rank.

Table 5. Information on inspectors using neutrosophic values.

About the Inspector Inspector rank	Number of pieces checked (hour)	Accuracy (percent)	Inspector's remuneration (monetary unit per hour)	Number of inspectors	The fine paid by the company for each fault to the inspector
1	15	[95,97]	4	10	2
2	10	[90,92]	3	6	2
3	25	[98,99.5]	5	8	2

Required : Formulate the appropriate mathematical model through which we can assign the optimal assignment to the inspectors so that the cost of inspection is as low as possible.

To build the mathematical model, we impose $x_{1,}x_2, x_3$ the number of inspectors from the three ranks in the order assigned to the inspection task, then the following inequality must be met:

$$x_1 \leq 10$$
$$x_2 \leq 6$$
$$x_3 \leq 8$$

Since the company needs to audit K pieces daily within S working hour per day, the following set of restrictions must be met:

$$\sum_{j=1}^{n} 8M_j x_j \geq 1500$$

That is

$$8(M_1x_1 + M_2x_2 + M_3x_3) \geq 1500$$

From it, we get the following entry:

$$120x_1 + 80x_2 + 200x_3 \geq 1500$$

To obtain the target function, we note that the company bears two types of costs during the inspection process, the inspector's fee and the fine corresponding to the error committed by the inspector for each piece then the target follower writes as follows:

Then the cost of the inspector is calculated from j the hourly rank through the following relationship:

$$C_j = G_j + M_j R_j \left(\frac{100 - ND_j}{100}\right) \;\; ; \;\; j = 1,2, - - -, n$$

From that we get

$$C_1 = 4 + 15 \times 2 \times \left(\frac{100 - [95,97]}{100}\right) = [4.9,5.5]$$

$$C_2 = 3 + 10 \times 2 \times \left(\frac{100 - [90,92]}{100}\right) = [4.6,5]$$

$$C_3 = 5 + 25 \times 2 \times \left(\frac{100 - [98,99.5]}{100}\right) = [5.25,6]$$

The total costs for all inspectors assigned to the task of quality control per hour shall be given by the following relationship:

$$TC_j = \sum_{j=1}^{n} \left[G_j + M_j R_j \left(\frac{100 - ND_j}{100} \right) \right] x_j$$

$$TC_j = [4.9,5.5]x_1 + [4.6,5]x_2 + [5.25,6]x_3$$

Substituting the following target phrase:

$$Z = S \sum_{j=1}^{n} \left[G_j + M_j R_j \left(\frac{100 - ND_j}{100} \right) \right] x_j$$

We get:

$$Z = [39.2,44]x_1 + [36.8,40]x_2 + [42,48]x_3$$

From the above, we can develop the following mathematical model:

We want to find:

$$MinZ = [39.2,44]x_1 + [36.8,40]x_2 + [42,48]x_3$$

Within Restrictions

$$x_1 \leq 10$$
$$x_2 \leq 6$$
$$x_3 \leq 8$$
$$120x_1 + 80x_2 + 200x_3 \geq 1500$$
$$x_j \geq 0 \ ; \ j = 1,2,3$$

In the two examples, and two for the optimal solution we use the neutrosophic simplex method sufficiently explained in the reference [13].

2. Conclusion and Results

Through the previous study, we note that by using the linear programming method, we can provide the optimal solution to most of the problems that can face the production companies by formulating the situation under treatment with an issue that can be converted into a linear model by solving it using the ideal solution for it, the company achieves the highest profit, and in order to obtain solutions that enjoy a margin of freedom, the concepts of neutrosophic science can be used because the indeterminacy enjoyed by the neutrosophic values can be responsible for managing the company and developing alternative plans that suit all working conditions.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

References

1. Bukajah J.S. - Edward. Mualla, W... and others - Operations Research Book translated into Arabic - The Arab Center for Arabization, Translation, Authoring and Publishing -Damascus -1998. . (Arabic version).
2. Maissam Jdid, Operations Research, Faculty of Informatics Engineering, Al-Sham Private University Publications, 2021
3. The machine. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004. (Arabic version).
4. Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Prospects for Applied Mathematics and Data Analysis (PAMDA), Vol .2, No.1, 2023.
5. Maissam Jdid, A. A. Salam, Using the Inverse Transformation Method to Generate Random Variables that follow the Neutrosophic Uniform Probability Distribution. Journal of Neutrosophic and Fuzzy Systems (JNFS), Vo .6, No. 2, 2023
6. Mohammed Alshikho, Maissam Jdid, Said Broumi ,Artificial Intelligence and Neutrosophic Machine learning in the Diagnosis and Detection of COVID 19, Journal Prospects for Applied Mathematics and Data Analysis ,Vol 01, No,02 USA,2023
7. Mohammed Alshikho, Maissam Jdid, Said Broumi, A Study of a Support Vector Machine Algorithm with an Orthogonal Legendre Kernel According to Neutrosophic logic and Inverse Lagrangian Interpolation, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .5,No .1, 2023
8. Maissam Jdid, Khalifa Alshaqsi, Optimal Value of the Service Rate in the Unlimited Model $M\backslash M\backslash 1$, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .6,No .1, 2023
9. Maissam jdid- Hla Hasan ,The state of Risk and Optimum Decision According to Neutrosophic Rules, International Journal of Neutrosophic Science (IJNS),Vol. 20, No.1,2023
10. Maissam new ,Important Neutrosophic Economic Indicators of the Static Model of Inventory Management without Deficit, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .5,No .1, 2023
11. Maissam Jdid , Huda E. Khalid , Neutrosophic Mathematical formulas of Transportation Problems Neutrosophic sets and Systems, NSS, Vol .5 1, 2022
12. Maissam Jdid, Huda E Khalid ,Mysterious Neutrosophic Linear Models, International Journal of Neutrosophic Science, Vol. 18,No. 2, 2022
13. Maissam Jdid, AA Salama, Huda E Khalid ,Neutrosophic Handling of the Simplex Direct Algorithm to Define the Optimal Solution in Linear Programming ,International Journal of Neutrosophic Science, Vol. 18,No. 1, 2022
14. Maissam Jdid ,Neutrosophic Nonlinear Models, Journal Prospects for Applied Mathematics and Data Analysis, Vo .2,No .1, 2023
15. Maissam Jdid, Neutrosophic Mathematical Model of Product Mixture Problem Using Binary Integer Mutant, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vo .6,No .2, 2023
16. Maissam Jdid, The Use of Neutrosophic linear Programming Method in the Field of Education, Handbook of Research on the Applications of Neutrosophic Sets Theory and Their Extensions in Education, Chapter 15, IGI-Global,2023.

Converting Some Zero-One Neutrosophic Nonlinear Programming Problems into Zero-One Neutrosophic Linear Programming Problems

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract

The science of operations research is the applied aspect of mathematics and one of the most important modern sciences that is concerned with practical issues and meets the desire and request of decision makers to obtain ideal decisions through the methods it presents that are appropriate for all issues, such as linear programming, nonlinear programming, dynamic programming, integer programming, etc. The basic essence of this science is to build mathematical models consisting of an objective function and constraints. In these models, the objective function is a maximization function or a minimization function for a specific quantity. This quantity depends on a number of decision variables that may be independent of each other or related to each other. Through a set of constraints, we obtain values for these variables by solving the mathematical model that we obtain. Given the great importance of operations research methods, we have in previous research presented a neutrosophic vision for some of these methods, such as neutrosophic linear models, neutrosophic nonlinear models, dynamic programming, neutrosophic programming with binary integers, etc. In this research, we present a neutrosophical study of some of the procedures used to convert some zero-one neutrosophic nonlinear programming problems into zero-one neutrosophic linear programming problems.

Keywords: Neutrosophic Operations Research, Neutrosophic Linear Programming, Neutrosophic Nonlinear Programming, Zero-One Neutrosophic Nonlinear Programming Problems, Zero-One Neutrosophic Linear Programming Problems.

1 | Introduction

To keep pace with scientific development and provide new studies of operations research methods that rely on concepts of logic different from classical logic, and after the emergence of neutrosophic logic, many researchers turned to reformulating many scientific concepts using the concepts of this neutrosophic logic, so we have neutrosophic numbers ,neutrosophic groups ,neutrosophic probabilities ,neutrosophic statistics, neutrosophic differentiation, neutrosophic integration, neutrosophic decision theory, neutrosophic linear programming, neutrosophic nonlinear programming, neutrosophic dynamic programming, neutrosophic simulation, neutrosophic binary integer programming... [1-23], in this research we present some procedures

used to converting some zero-one neutrosophic nonlinear programming problems into zero-one neutrosophic linear programming problems. We know that nonlinear programming problems are characterized by the presence of a set of terms that implicitly call for the use of nonlinear functions such as: $lnx, sinx, e^x, x^n$ … , nonlinearity also appears as a result of a mutual influence between two or more variables, such as: x_1x_2, x_1lnx_2, …, If there are such expressions in the objective function, constraints, or both, then the mathematical model is a nonlinear model, and if some or all of the variables are restricted to be one or zero , then we get nonlinear models with binary integers. The mathematical model is a neutrosophic model if it has some indeterminacy. When collecting data that describes a realistic issue and this data is incomplete or contains something ambiguous or may contain some contradictions or may be unclear, we will then be unable to build an accurate traditional model. This will lead us to building an approximate model, in neutrosophic model, the variables in the objective function and constraints, some or all of them, are neutrosophic numbers. The neutrosophic number is a number written in the following standard form, $N = a + bI$, where $a\ and\ b$ are real or complex coefficients. a represents the definite part and bI represents the indefinite part (indeterminacy) of the number N, and it is possible that $[\lambda_1\ ,\ \lambda_2]$ or $\{\lambda_1,\ \lambda_1\}$ or...otherwise it is any set close to the real value a.

2 | Discussion

Since the aim of this research is to converting some zero-one neutrosophic nonlinear programming problems into zero-one neutrosophic linear programming problems, based on the information contained in references [24-26].

1. From reference [24], we find that what is meant by neutrosophic data are values that are not completely defined, written in the following standard format:

   $$N = a + bI$$

   where $a\ and\ b$ are real or complex coefficients, a represents the definite part and bI the indeterminate part of the number N and it can be $[\lambda_1\ ,\ \lambda_2]$ or $\{\lambda_1,\ \lambda_1\}$ or...otherwise it is any set close to the real value a, and the neutrosophic model, the numbers that precede the variables in the objective function and in the constraints are neutrosophic values, as well as the second side of the relationships that represent the constraints, and it is written in the following general formula:

   $$Nf = Nf(x_1\ , x_2, --, x_n) \rightarrow (Max\ )or\ (Min)$$

   Constraints:

   $$Ng_i(x_1\ , x_2, --, x_n)\begin{pmatrix}\leq\\ \geq\\ =\end{pmatrix}Nb_i\ \ ; i = 1,2, - - -, m$$

   $$x_1\ , x_2, --, x_n \geq 0$$

2. In references [25-26] the neutrosophic linear models and the neutrosophic nonlinear models were defined, and we mention that the mathematical model is a nonlinear model if any of the components of the objective function or constraints are nonlinear expressions such as $lnx, sinx, e^x, x^n, x_1x_2, x_1lnx_2, \ldots$,and the nonlinear expressions may be in both.

3. The Neutrosophic nonlinear programming problem is a zero-one problem if the decision variables x_j are restricted to belonging to the set $\{0,1\}$

   Then the neutrosophic mathematical model is written in the following general form:

   $$Nf = Nf(x_1\ , x_2, --, x_n) \rightarrow (Max\ )or\ (Min)$$

   Constraints:

$$Ng_i(x_1, x_2, --, x_n)\begin{pmatrix}\leq\\ \geq\\ =\end{pmatrix}Nb_i \;\; ; i = 1,2, - - -, m$$

$$x_j \in \{0,1\} \;\; ; j = 1,2, \dots, n$$

4. From the study reported in the classical reference [27], we find that if the nonlinearity in any zero–one model results from the presence of nonlinearity expressions of the following two forms:

$$x^n, x_1x_2\, x_3x_4 \dots\, x_{\mathrm{n}}$$

We can convert this model into zero-one linear model by relying on the validity of the following relation:

i. For any positive number k and any binary variable we have:

$$x_j \in \{0,1\} \;\; then\, x_j^k = x_j \;\; ; j = 1,2, \dots, n$$

ii. For binary values of any variables x_r, x_s, the product x_rx_s is always "equal" to 0 or 1, so we replace the product with a new variable, $y_1 = x_rx_s$ achieves the following:

$$y_1 = \begin{cases}1 & x_r = x_s = 1\\ 0 & x_r\ or\ x_s = 0\end{cases}$$

We achieve this by adding the following two constraints:

$$x_r + x_s - y_1 \leq 1$$

$$-x_r - x_s + 2y_1 \leq 0$$

We note that when $x_{\mathrm{r}} = x_{\mathrm{s}} = 1$ reduces the previous two constraint to $y_1 \geq 1$ and $y_1 \leq 1$ Therefore $y_1 = 1$, when $x_{\mathrm{r}} = 0\ or\ x_{\mathrm{s}} = 0\ or\ x_{\mathrm{r}} = x_{\mathrm{s}} = 0$,we find that the second constraint forces y_1 to be equal to zero because:

$$-x_r - x_s + 2y_1 \leq 0 \Longrightarrow y_1 \leq \frac{x_{\mathrm{r}} + x_{\mathrm{s}}}{2} \Longrightarrow y_1 = 0$$

This procedure requires adding an integer variable for every product, and since the time to solve integer programming problems increases proportionally with the number of integer variables, this procedure was replaced by another procedure by (F. Glover and E. Woolsey) that was presented in reference [27] which states the following: Instead of adding an integer variable, a continuous variable is added. The product is replaced by the continuous variable x_{rs} and the following constraints are added:

$$x_r + x_s - x_{rs} \leq 1$$

$$x_{rs} \leq x_r$$

$$x_{rs} \leq x_s$$

$$x_{rs} \geq 0$$

If $x_{\mathrm{r}}\ or\ x_{\mathrm{s}} = 0\ \ or\ x_{\mathrm{r}} = x_{\mathrm{s}} = 0$,the constraints impose that x_{rs} be equal to one. What is taken into account with this procedure is that it adds more constraints than the previous procedure.

The procedure for dealing with the product of two variables can be extended to include the product of any number of variables, if we take the product $x_1x_2\, x_3x_4 \dots\, x_{\mathrm{n}}$.

According to the first procedure, we replace the product with the variable y_1 and add the following two constraints:

$$\sum_{j=1}^{n} x_j - y_1 \leq n - 1$$

$$-\sum_{j=1}^{n} x_j - ny_1 \leq 0$$

$$y_1 \in \{0,1\}$$

$$x_j \in \{0,1\}$$

According to the second procedure, we replace the product with the non-negative variable y_0 and add the following $n+1$ constraints:

$$\sum_{j=1}^{n} x_j - y_0 \leq n-1$$

$$y_0 \leq x_j \quad ; j = 1,2,\dots,n$$

2.1 | Example 1

We have the following zero-one neutrosophic nonlinear programming model:

Find:

$$MaxZ = c_{N1}x_1^2 + c_{N2}x_1x_2 - c_{N3}x_3^3$$

Constraints:

$$-a_{N1}x_1 + a_{N2}x_2 + a_{N3}x_3 \leq b_N$$

$$x_1, x_2, x_3 \in \{0,1\}$$

Where: $c_{N1}, c_{N2}, c_{N3}, a_{N1}, a_{N2}, a_{N3}, b_N$,neutrosophic values are taken in proportion to the nature of the issue under study.

Using the previous study, this zero-one neutrosophic nonlinear model is converting into a zero-one neutrosophic linear model.

1. Using the following transformations:
   a. For $x_1^2 = x_1$and also $x_3^3 = x_3$
   b. For the product x_2x_3, we take instead the variable $y_1 = x_2x_3$ and add the following two constraints:

$$x_2 + x_3 - y_1 \leq 1$$

$$-x_2 - x_3 + 2y_1 \leq 0$$

   We obtain the following zero-one neutrosophic linear model:

   Find:

$$MaxZ = c_{N1}x_1 + c_{N2}y_1 - c_{N3}x_3$$

   Constraints:

$$-a_{N1}x_1 + a_{N2}x_2 + a_{N3}x_3 \leq b_N$$

$$x_2 + x_3 - y_1 \leq 1$$

$$-x_2 - x_3 + 2y_1 \leq 0$$

$$x_1, x_2, x_3 \in \{0,1\}$$

2. Using the transformation (F. Glover and E. Woolsey):

a. For $x_1^2 = x_1$and also $x_3^3 = x_3$

b. We substitute the product x_2x_3 into the continuous variable x_{23} and add the constraints:

$$x_2 + x_3 - x_{23} \leq 1$$

$$x_{23} \leq x_2$$

$$x_{23} \leq x_3$$

$$x_{23} \geq 0$$

We obtain the following zero-one neutrosophic linear model:

Find:

$$MaxZ = c_{N1}x_1 + c_{N2}x_{23} - c_{N3}x_3$$

Constraints:

$$-a_{N1}x_1 + a_{N2}x_2 + a_{N3}x_3 \leq b_N$$

$$x_2 + x_3 - x_{23} \leq 1$$

$$x_{23} \leq x_2$$

$$x_{23} \leq x_3$$

$$x_{23} \geq 0$$

$$x_1, x_2, x_3 \in \{0,1\}$$

3 | Conclusion and Results

As a continuation of what we presented in previous research of neutrosophic studies on operations research topics, we presented in this research a neutrosophic study specifically for zero-one programming problems, which is procedures that we can follow to convert some neutrosophic zero-one nonlinear models into zero-one neutrosophic linear models., with the aim of taking advantage of the easy-to-apply algorithms that have been developed by scholars in the field of operations research to obtain the optimal solution for zero-one linear models. Restricting the decision variables to be either zero or one helps us convert some nonlinear expressions into linear expressions, and these procedures can be used. In any study that contains binary integer variables.

Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Doi :https://doi.org/10.54216/PAMDA.020102

[2] Maissam Jdid, Said Broumi, Neutrosophical Rejection and Acceptance Method for the Generation of Random Variables, Neutrosophic Sets and Systems, Vol. 56, 2023, pp. 153-166. DOI: 10.5281/zenodo.8194749

[3] Victor Christianto, Robert N. Boyd, Florentin Smarandache, Three possible applications of Neutrosophic Logic in Fundamental and Applied Sciences, International Journal of Neutrosophic Science, Volume 1, Issue 2, PP: 90-95, 2020

[4] Maissam Jdid, Rafif Alhabib, Neutrosophical dynamic programming, Doi: https://doi.org/10.54216/IJNS.1803013

[5] Maissam Jdid, Studying Transport Models with the Shortest Time According to the Neutrosophic Logic, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 631-638. DOI: 10.5281/zenodo.8404545

[6] Broumi S. and Smarandache, F., Correlation coefficient of interval neutrosophic set, Appl. Mech. Mater., 436:511–517, 2013.

[7] Madeleine Al- Tahan, Some Results on Single Valued Neutrosophic (Weak) Polygroups, International Journal of Neutrosophic Science, Volume 2, Issue 1, PP: 38-46, 2020.

[8] Aslam, M., Khan, N. and AL-Marshadi, A. H. (2019). Design of Variable Sampling Plan for Pareto Distribution Using Neutrosophic Statistical Interval Method, Symmetry, 11 (1), 80.

[9] Mahmoud Ismail, Ahmed M. Ali, Ahmed Abdelhafeez, Ahmed Abdel-Rahim EI-Douh, Mahmoud Ibrahim and Ayman H. Abdel-aziem, Neutrosophic Multi-Criteria Decision Making for Sustainable Procurement in Food Business, Neutrosophic Sets and Systems, Vol. 57, 2023, pp. 283-293. DOI: 10.5281/zenodo.8271435

[10] Maissam Jdid and Nada A. Nabeeh, Generating Random Variables that follow the Beta Distribution Using the Neutrosophic Acceptance-Rejection Method, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 139-148. DOI: 10.5281/zenodo.8404445

[11] Maissam Jdid, F. Smarandache, Neutrosophic Static Model without Deficit and Variable Prices, Neutrosophic Sets and Systems, Vol. 60, 2023, pp. 124-132. DOI: 10.5281/zenodo.10224148

[12] Maissam Jdid, Florentin Smarandache, Finding a Basic Feasible Solution for Neutrosophic Linear Programming Models: Case Studies, Analysis, and Improvements. DOI: https://doi.org/10.61356/j.nswa.2024.130

[13] Florentin Smarandache, Maissam Jdid, Neutrosophic linear models and algorithms to find their optimal solution, Biblio Publishing, ISBN, 978_1 _59973_784_3, USA,2024

[14] Maissam Jdid, Florentin Smarandache, An Efficient Optimal Solution Method for Neutrosophic Transport Models: Analysis, Improvements, and Examples, DOI: https://doi.org/10.61356/j.nswa.2023.111,

[15] Maissam Jdid , Florentin Smarandache, A Study of Systems of Neutrosophic Linear Equations, Doi: https://doi.org/10.54216/IJNS.230202

[16] https://fs.unm.edu/NeutrosophicOperationsResearch.pdf, Biblio Publishing, Research on the topics of neutrosophic operations research, Volume (1),2023

[17] Maissam Jdid, Some Important Theories about Duality and the Economic Interpretation of Neutrosophic Linear Models and Their Dual Models, Doi :https://doi.org/10.54216/PAMDA.020203

[18] Florentin Smarandache, Maissam Jdid, NEUTROSOPHIC TRANSPORT AND ASSIGNMENT ISSUES, Publisher: Global Knowledge's: 978_1 _59973_769_0

[19] Maissam Jdid, Florentin Smarandache, The Graphical Method for Finding the Optimal Solution for Neutrosophic linear Models and Taking Advantage of Non-Negativity Constraints to Find the Optimal Solution for Some Neutrosophic linear Models in Which the Number of Unknowns is More than Three, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 68-80. DOI: 10.5281/zenodo.8404429

[20] Maissam Jdid, Florentin Smarandache, Graphical Method for Solving Neutrosophical Nonlinear Programming Models, DOI: https://doi.org/10.61356/j.nswa.2023.67

[21] Maissam Jdid, Florentin Smarandache, The Use of Neutrosophic Methods of Operation Research in the Management of Corporate,Work, DOI: https://doi.org/10.61356/j.nswa.2023.11
[22] Maissam Jdid, Neutrosophic Mathematical Model of Product Mixture Problem Using Binary Integer Mutant, Doi :https://doi.org/10.54216/JNFS.060206
[23] Maissam Jdid, Florentin Smarandache, Lagrange Multipliers and Neutrosophic Nonlinear Programming Problems Constrained by Equality Constraints, DOI: https://doi.org/10.61356/j.nswa.2023.35
[24] Maissam Jdid, Huda E. Khalid, Mysterious Neutrosophic Linear Models, Doi: https://doi.org/10.54216/IJNS.180207
[25] Maissam Jdid, Neutrosophic Nonlinear Models, Doi: https://doi.org/10.54216/PAMDA.020104
[26] Smarandache, F., Introduction to Neutrosophic statistics, Sitech & Education Publishing, 2014.
[27] Bugaha J.S, Mualla.W, and others -Operations Research Translator into Arabic, The Arab Center for Arabization, Translation, Authoring and Publishing, Damascus,1998. (Arabic version).

Pura Vida Neutrosophic Algebra

Ranulfo Paiva Barbosa (Sobrinho),[1] **Florentin Smarandache**[2]

[1] Web3 Blockchain Entrepreneur; 37 Dent Flats, Monte de Oca, 11501, San José, Costa Rica; ranulfo17@gmail.com.
[2] Math and Science Department, University of New Mexico, Gallup, NM, 87301, USA; smarand@unm.edu.

* Correspondence: ranulfo17@gmail.com.

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Abstract: We introduce Pura Vida Neutrosophic Algebra, an algebraic structure consisting of neutrosophic numbers equipped with two binary operations namely addition and multiplication. The addition can be calculated sometimes with the function min and other times with the max function. The multiplication operation is the usual sum between numbers. Pura Vida Neutrosophic Algebra is an extension of both Tropical Algebra (also known as Min-Plus, or Min-Algebra) and Max-Plus Algebra (also known as Max-algebra). Tropical and Max-Plus algebras are algebraic structures included in semirings and their operations can be used in matrices and vectors. Pura Vida Neutrosophic Algebra is included in Neutrosophic semirings and can be used in Neutrosophic matrices and vectors.

Keywords: Tropical Algebra; Max-Plus Algebra; Pura Vida Neutrosophic Logic; Neutrosophic Number.

1. Introduction

Uncertain, indeterminacy, imprecise, and vague are common characteristics of data in real-life problems like decision-making, engineering, computer science, finance, etc. Several theories have been proposed to deal with these data characteristics, fuzzy set theory [1], intuitionistic fuzzy sets [2], rough set theory [3], Soft set [4], and Neutrosophy theory [5]. Since Smarandache introduced Neutrosophy to study the basis, nature, and range of neutralities as well as their contact with ideational spectra in the 1990s, we have seen the emergence of neutrosophic algebraic structures [6], neutrosophic probability and statistics [7, 8] neutrosophic numbers [8], single-valued neutrosophic sets (SVNSs) [9, 21], and several algebraic structures such as neutrosophic semirings [10], among others theoretical advances [11] and also applications [12].

Through neutrosophic semirings, we introduce Pura Vida (PV) Neutrosophic Algebra, an algebraic structure consisting of neutrosophic numbers equipped with two binary operations namely addition and multiplication. Pura Vida Neutrosophic Algebra is an extension of both Tropical Algebra (also known as Min-Plus) [13] and Max-Plus Algebra [14]. Both Tropical and Max-Plus algebra are algebraic structures included in semirings and were discovered independently by several researchers [13, 14]. They were defined on the real number domain and for the first time, we extended them to the neutrosophic domain.

2. Preliminaries

2.1. Semiring

A semiring [15] denoted $(V, \oplus, \otimes, 0, 1)$ is a set V equipped with two binary operations, addition:

$$\oplus : V \times V \rightarrow V$$

And multiplication:

$$\otimes: V \times V \rightarrow V$$

Which satisfies the following axioms for any u, v, w $\in$ V:

1. (V, $\oplus$, 0) is a commutative monoid and (V, $\otimes$, 1) is a monoid.

2. u $\otimes$ (v $\oplus$ w) = (u $\otimes$ v) $\oplus$ (u $\otimes$ w) and (v $\oplus$ w) $\otimes$ u = (v $\otimes$ u) $\oplus$ (w $\otimes$ u) (distributivity).

3. 0 annihilates V: v $\otimes$ 0 = 0 $\otimes$ v = 0.

When (V, $\otimes$, 1) is a commutative monoid, the semiring (V, $\oplus$, $\otimes$, 0, 1) is said to be a commutative semiring.

2.2 Tropical algebra

The tropical algebra is also referred to as tropical semiring T, which consists of the set of real numbers, R, extended with infinity, equipped with the operations of taking minimums (as semiring addition) and addition (as semiring multiplication) [14, 16]. Tropical algebra is also known as min-plus algebra. With minimum replaced by maximum, we get the isomorphic max-plus algebra [17]. According to [17], the adjective "tropical" was coined by French mathematicians to honor their Brazilian colleague Imre Simon [16], who pioneered the use of min-plus algebra in optimization theory.

$$T = (R \cup \{ \infty \}, \oplus, \otimes)$$

Addition operation:

$$a \oplus b = \min (a, b)$$

Multiplication operation:

$$a \otimes b = a + b$$

the operations of R, are extended to T in the usual way and the identities of $\oplus$ and $\otimes$ are, respectively, ∞ and 0. The element ∞ represents plus-infinity [13]. Given a real number, x $\in$ T, its addition and multiplication identity are given, respectively:

$$x \oplus \infty = x$$

$$x \otimes 0 = x$$

Michaleck points out the following equations involving the two identity elements:

$$x \otimes \infty = \infty \qquad \text{and} \qquad x \oplus 0 = \begin{cases} 0, & if \ x \geq 0 \\ x, & if \ x < 0 \end{cases}$$

Michaleck said there is no subtraction in tropical arithmetic. Tropical division $\oslash$ is defined to be classical subtraction.

Tropical division, x $\oslash$ y = x, exists if and only if y $\otimes$ z = x [20].

In Tropical algebra the pairs of operations ($\oplus$,$\otimes$) is extended to matrices and vectors similarly as in linear algebra. That is if A = (a_{ij}), B = (b_{ij}) and C = (c_{ij}) are matrices with elements from R of compatible sizes, we write:

C = A $\oplus$ B if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j

C = A $\otimes$ B if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$ for all i, j

$\alpha \otimes A = A \otimes \alpha = (\alpha \otimes a_{ij})$ for all $\alpha \in R$.

2.3 Max-Plus algebra

The Max-Plus algebra is an algebraic structure semiring MP, which consists of the set of real numbers, R, extended with infinity, equipped with the operations of taking maximums (as semiring addition) and addition (as semiring multiplication) [14].

$$MP = (R \cup \{ -\infty \}, \oplus', \otimes)$$

Addition operation:

$$a \oplus' b = \max (a, b)$$

Multiplication operation:

$$a \otimes b = a + b$$

the operations of R, are extended to MP in the usual way and the identities of $\oplus'$ and $\otimes$ are, respectively, $-\infty$ and 0.

In max-plus algebra the pairs of operations $(\oplus', \otimes)$ is extended to matrices and vectors similarly as in linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices with elements from R of compatible sizes, we write:

$C = A \oplus' B$ if $c_{ij} = a_{ij} \oplus' b_{ij}$ for all i, j

$C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus'} a_{ik} \oplus b_{kj} = \max_k (a_{ik} + b_{kj})$ for all i, j

$\alpha \otimes A = A \otimes \alpha = (\alpha \otimes a_{ij})$ for all $\alpha \in R$.

2.4 Neutrosophic Set

Smarandache [5] defined Neutrosophic set as a set of elements composed of tripart structure: a Truth membership (T), an Indeterminacy membership (I) and a False membership (F). These parts are independent each other and can be represented by different functions. Together, <T, I, F>, these parts compose an element of Neutrosophic set.

2.5 Neutrosophic Number

According to [18] the neutrosophic number (NN) is a number which structure is given by "X = a + bI", where I represents the indeterminacy component of X, and 'a' and 'b' are real or complex numbers [19].

2.6 Neutrosophic Semiring

An algebraic structure $(S \cup I, \oplus, \otimes)$ is called neutrosophic semiring [10] if $\oplus$ and $\otimes$ are the closed and associative binary operations and $\otimes$ is distributive over $\oplus$, where S is semiring with respect to $\oplus$ and $\otimes$ and I is the neutrosophic element $(I = I^2)$ and $< S \cup I> = \{ a + bI; a, b \in S\}$.

2.7 Neutrosophic field [6]

Let K be the field of reals. We call the field generated by $K \cup I$ to be the neutrosophic field for it involves the indeterminacy (I) factor in it. We define I2 = I, I + I = 2I, i.e., I + … + I = nI, and if $k \in K$ then kI = Ik, 0I = 0. We denote the neutrosophic field by K(I).

2.8 Neutrosophic matrix [6]

Let $M_{nxm} = \{(a_{ij}) / a_{ij} \in K(I) \}$, where K(I), is a neutrosophic field. We call Mnxm to be the neutrosophic matrix.

3. Pura Vida Neutrosophic Algebra

The Pura Vida Neutrosophic Algebra, PV, is an extension of the Tropical algebra and Max-Plus Algebra.

Pura Vida Neutrosophic Algebra is included in a Neutrosophic semiring, i.e., it has both associative binary operations, addition $\oplus$ and multiplication $\otimes$ where $\otimes$ is distributive over $\oplus$, and S is semiring with respect to $\oplus$ and $\otimes$ and I is the neutrosophic element $(I = I^2)$ and $< S \cup I> = \{ a = bI; a, b \in S\}$. The addition operation can use either the min function, $\oplus$, or the max function, $\oplus'$, depending on the situation.

$$PV = (S \cup I \{ -\infty, +\infty \}, \oplus, \oplus', \otimes)$$

Pura Vida Neutrosophic Algebra operations addition ($\oplus$, or, $\oplus'$) and multiplication ($\otimes$) are given:

3.1 Addition operation$\oplus$, or, $\oplus'$

Depending on the real-life applications, the addition operation can use the min or max function. Given two neutrosophic numbers x = a + bI, and z = c + dI $\in$ S, the addition of x and z:

3.1.1. $x \oplus z = (a \oplus c) + (b \oplus d)I = \min(a, c) + \min(b, d)I$

or,

3.1.2. $x \oplus' z = (a \oplus' c) + (b \oplus' d)I = \max(a, c) + \max(b, d)I$

3.2 Multiplication operation $\otimes$

Given two neutrosophic numbers $x = a + bI$, and $z = c + dI \in S$, the multiplication of x and z:

$x \otimes z = a \otimes c + (b \otimes d)I = (a + c) + (b + d)I$

3.3 Identities

In Pura Vida Neutrosophic Algebra, PV, the identities of the operators $\oplus$, $\oplus'$ and $\otimes$ are, respectively, ∞, $-\infty$ and 0.

3.4 Properties

Next, we show that the PV attends the closure property and distributive and associative laws. We use min for the addition operation, but, one could use the max function to show that PV verifies the mentioned properties.

3.4.1 Closure property:

Let $(a + bI)$ and $(c + dI) \in S \cup I$ then,

$(a + bI) \oplus (c + dI) = (a \oplus c) + (b \oplus d)I = \min(a, b) + \min(c, d)I, \in S \cup I$. The addition operation verifies the closure property.

$(a + bI) \otimes (c + dI) = a \otimes c + (b \otimes d)I = (a + c) + (b + d)I \in S \cup I$. Which shows that the closure property is satisfied for the multiplication operation.

3.4.2 Distributive law:

Let $(a + bI)$, $(c + dI)$ and $(e + fI) \in S \cup I$, then:

$(a + bI) \otimes [(c + dI) \oplus (e + fI)] = (a + bI) \otimes [\min(c, e) + \min(d, f)I] =$

$= [a + \min(c, e)] + [b + \min(d, f)]I.$

And $[(a + bI) \otimes (c + dI)] \oplus [(a + bI) \otimes (e + f)I] =$

$= [(a + c) + (b + d)I] \oplus [(a + e) + (b + f)I] =$

$= \min \{(a + c), (a + e)\} + \min \{(b + d) + (b + f)\}I =$

$= [a + \min(c, e)] + [b + \min(d, f)]I.$

3.4.3 Associative law:

Let $(a + bI)$, $(c + dI)$ and $(e + fI) \in S \cup I$, then:

$[(a + bI) \oplus (c + dI)] \oplus (e + fI) =$

$[\min(a, c) + \min(b, d)I] \oplus (e + fI) = \min[\min(a, c), e] + \min[\min(b, d), f]I =$

$= (a \oplus c \oplus e) + (b \oplus d \oplus f)I.$

$(a + bI) \oplus [(c + dI) \oplus (e + fI)] =$

$(a + bI) \oplus [\min(c, e) + \min(d, f)I] = \min[a, \min(c, e)] + \min[b, \min(d, f)]I =$

$= (a \oplus c \oplus e) + (b \oplus d \oplus f)I.$

Again:

$[(a + bI) \otimes (c + dI)] \otimes (e + fI) = [a \otimes c + (b \otimes d)I] \otimes (e + fI) =$

$[(a + c) + (b + d)I] \otimes (e + fI) = (a + c) \otimes e + [(b + d) \otimes f]I =$

$(a + c + e) + (b + d + f)I.$

$(a + bI) \otimes [(c + dI)] \otimes (e + fI)] = (a + bI) \otimes [(c + e) + (d + f)I] =$

$= a \otimes (c + e) + b \otimes (d + f)I = (a + c + e) + (b + d + f)I.$

3.5 Pura Vida Neutrosophic Algebra on Matrices

In Pura Vida Neutrosophic Algebra the pairs of operations $(\oplus, \oplus', \otimes)$ is extended to matrices and vectors similarly as in linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices with elements from R of compatible sizes, we write:

$C = A \oplus' B$ if $c_{ij} = a_{ij} \oplus' b_{ij}$ for all i, j

$C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus'} a_{ik} \oplus b_{kj} = \max_k (a_{ik} + b_{kj})$ for all i, j

$\alpha \otimes A = A \otimes \alpha = (\alpha \otimes a_{ij})$ for all $\alpha \in R$.

3.5.1 Matrices Addition using $\oplus$ operator

Given P and Q, both square neutrosophic matrices 2x2, their sum is $D = P \oplus Q$.

$$P = \begin{vmatrix} -8+I & 5-I \\ 3+8I & 23-2I \end{vmatrix} \quad \text{and} \quad Q = \begin{vmatrix} 3+2I & 13+3I \\ 7+9I & 3+5I \end{vmatrix}$$

$$D = \begin{vmatrix} \text{Min}(-8,3)+\text{Min}(1,2)I = -8+I & \text{Min}(5,13)+\text{Min}(-1,3)I = 5-I \\ \text{Min}(3,7)+\text{Min}(8,9)I = 3+8I & \text{Min}(23,3)+\text{Min}(-2,5)I = -2+3I \end{vmatrix}$$

3.5.2 Matrices Addition using $\oplus'$ operator

Given X and Z, both square neutrosophic matrices 2x2, their sum is $W = X \oplus' Z$.

$$X = \begin{vmatrix} -8+I & 5-I \\ 3+8I & 23-2I \end{vmatrix} \quad \text{and} \quad Z = \begin{vmatrix} 3+2I & 13+3I \\ 7+9I & 3+5I \end{vmatrix}$$

$$W = \begin{vmatrix} \text{Max}(-8,3)+\text{Max}(1,2)I = 3+2I & \text{Max}(5,13)+\text{Max}(-1,3)I = 13+3I \\ \text{Max}(3,7)+\text{Max}(8,9)I = 7+9I & \text{Max}(23,3)+\text{Max}(-2,5)I = 23+5I \end{vmatrix}$$

3.5.3 Matrices Multiplication using $\otimes$ operator

Given A and B, both rectangular neutrosophic matrices, their multiplication is $C = A \otimes B$.

$$A = \begin{vmatrix} -1 & 2 & -I \\ 3 & I & 0 \end{vmatrix} \quad \text{and} \quad B = \begin{vmatrix} I & 1 & 2 & 4 \\ 1 & I & 0 & 2 \\ 5 & -2 & 3I & -I \end{vmatrix}$$

$$C = A \otimes B = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \end{vmatrix}$$

Where,

$C_{11} = (-1 \quad 2 \quad -I) \otimes (I \quad 1 \quad 5) = (-1 \otimes I + 2 \otimes 1 + -I \otimes 5) = (-1+I + 2+1 + -I+5) = 7$

$C_{21} = (3 \; I \; 0) \otimes (I \; 1 \; 5) = (3 \otimes I + I \otimes 1 + 0 \otimes 5) = (3 + I + 1 + I + 5) = 9 + 2I.$

$C_{12} = (-1 \quad 2 \quad -I) \otimes (1 \; I \quad -2) = (-1 \otimes 1 + 2 \otimes I + -I \otimes -2) = (0 + 2 + I - I - 2) = 0.$

$C_{22} = (3 \; I \; 0) \otimes (1 \; I \quad -2) = 3 + 1 + I + I - 2 = 2 + 2I.$

$C_{13} = (-1 \quad 2 \quad -I) \otimes (2 \quad 0 \quad 3I) = 1 + 2 + 2I = 3 + 2I.$

$C_{23} = (3 \; I \; 0) \otimes (2 \quad 0 \quad 3I) = 5 + I + 3I = 5 + 4I.$

$C_{14} = (-1 \quad 2 \quad -I) \otimes (4 \; 2 \; -I) = 3 + 4 - 2I = 7 - 2I.$

$C_{24} = (3 \; I \; 0) \otimes (4 \; 2 \; -I) = 7 + I + 2 - I = 9.$

$$C = A \otimes B = \begin{vmatrix} 7 & 0 & 3+2I & 7-2I \\ 9+2I & 2+2I & 5+4I & 9 \end{vmatrix}$$

4. Conclusion

We introduced Pura Vida Neutrosophic Algebra through neutrosophic numbers and explored some its properties and applied to neutrosophic matrices.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Zadeh, L. Fuzzy sets. Information and Control 1965, 338-353.
2. Atanassov, K., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1) (1986) 87 − 96.
3. Pawlak, Z.: Rough sets. Int. J. Comput. Inform. Sci. 11, 341–356, 1982.
4. Molodtsov, D.: Soft set theory-first results. Comput. Math. Appl. 37, 19–31, 1999.
5. Smarandache, F. A unifying field in Logics: Neutrosophic Logic. In Philosophy. American Research Press, 1-141, 1992.
6. Kandasamy, W.B.V, and Smarandache, F. Basic Neutrosophic algebraic structures and their applications to fuzzy and Neutrosophic models. Hexis, 2004.
7. Smarandache, F. Neutrosophy neutrosophic probability, set, and logic, American Research Press, Rehoboth, USA, 1998.
8. Smarandache, F. Introduction to neutrosophic statistics. 2014.
9. Wang, H.; Smarandache, F.; Zhang, Y.; and Sunderraman, R. 2010. Single valued SVNS sets, Multisp Multistruct 4: 410–413.
10. Gulistan, M., Rehman, I., Shazad, M., Nawaz, S., Khan, S. Generalized Neutrosophic semirings. Neutrosophic Sets and Systems, 47: 533-540, 2021.
11. https://fs.unm.edu/NSS.
12. El-Hefenaway, N., Metwally, M.A., Ahmed, Z.M., El-Henawy, I.M. A review on the applications of neutrosophic sets. J.Comp.Theor. Nanoscience, 13, 936-944, 2016
13. Michalek, M., Sturmfels, B. Tropical Algebra in Invitation to nonlinear algebra. 2021.
14. Butkovic, P. Max-linear systems: theory and algorithms. Springer Monographs in Mathematics. 2010.
15. Golan, J. S. Semirings and their applications. Springer Science & Business Media, 2013
16. Simon, I. 1988. On semigroups of matrices over the Tropical semiring. Theoretical Informatics and Applications, 28(3-4): 277-294.
17. Maclagan, D., Sturmfels, B. Introduction to tropical geometry. AMS. 2015.
18. Kandasamy WBV, Smarandache F. Fuzzy cognitive maps and neutrosophic cognitive maps. ProQuest Information and Learning, Ann Arbor, 2003.
19. Kandasamy WBV, Smarandache F. Neutrosophic rings. 2006 https://arxiv.org/ftp/math/papers/0607/0607765.pdf.
20. Grigoriev, D., Shpilrain, V. Tropical cryptography. Comm. Algebra, 6: 2624-2632, 2014.
21. Jdid, M., Smarandache, F., & Broumi, S. (2023). Inspection Assignment Form for Product Quality Control Using Neutrosophic Logic. Neutrosophic Systems With Applications, 1, 4–13. https://doi.org/10.61356/j.nswa.2023.2

Introduction to algorithmic ethics under a neutrosophic framework: Re-mapping the debate

Antonios Paraskevas,[1] **Florentin Smarandache**[2]

[1]University of Macedonia, Department of Applied Informatics, 156, Egnatia Str., 54636, Thessaloniki, Greece; aparaskevas@uom.edu.gr;
[2]University of New Mexico, Mathematics Department, Gallup, NM 87301, USA; smarand@unm.edu;

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Abstract: Algorithms have become an integral part of professional life and social interactions, in health, transport, commerce and industry. Moreover, they are bringing about changes in the natural, social and human sciences, enriching our knowledge and testing the limits of technology. In the age of artificial intelligence (AI), the ethical behavior of algorithms is facing increasing scrutiny. Traditional models that attempt to evaluate algorithm ethics using binary or probabilistic methods often fall short in addressing the complexity and uncertainties present in real-world situations. This study introduces a novel approach to assessing algorithmic ethics by utilizing neutrosophic indeterminacy. Neutrosophic logic, which considers three parameters - truth (T), falsity (F), and indeterminacy (I) - provides a robust framework for capturing the ambiguity and inconsistencies that arise in ethical decision-making processes. By utilizing this approach, we attempt to develop a method for measuring ethical ambiguity in algorithmic decision-making. By applying the proposed conceptual framework to an illustrative example we highlight the capacity of neutrosophic logic to capture and measure ethical uncertainties in a more comprehensive manner, thus offering a new tool for evaluating the ethical integrity of algorithms in complex environments.

Keywords: Neutrosophic ethical integrity score; neutrosophic logic; algorithmic ethics; fairness; transparency; accountability.

1. Introduction

Algorithms subtly influence our lives in various ways. Operations, judgements, and choices that were previously left to people are increasingly being delegated to algorithms, which may advise, if not determine, how data should be evaluated and what actions should be performed in response. Examples are abundant. Profiling and categorization algorithms influence how people and groups are formed and managed [1]. Recommendation systems guide users on when and how to exercise, what to buy, which route to take, and who to contact [2]. Data mining techniques are believed to have promise for making sense of growing streams of behavioral data provided by the 'Internet of Things' [3]. Personalization and filtering algorithms continue to be used by online service providers to manage information access [4-5]. Machine learning algorithms automatically detect misleading, biased, or erroneous knowledge at the time of generation.

In this paper we adapt the following definition about an algorithm: "a finite, abstract, effective, compound control structure, imperatively given, accomplishing a given purpose under given provisions" [6].

In the broader context, the rapid expansion of algorithms in crucial industries such as healthcare, banking, and law enforcement has raised significant ethical concerns. Algorithms have the potential to worsen biases, diminish transparency, and weaken accountability. There are many factors that make it difficult to determine the potential and actual ethical influence of an algorithm. Identifying the role of human subjectivity in algorithm design and configuration often requires an examination of long-term, multi-user development processes. Learning algorithms, which are frequently cited as the "future" of algorithms and analytics [7], add uncertainty into how and why judgements are made because of their ability to change operational parameters and decision-making rules "in the wild" [8]. The potential for algorithms to improve individual and social welfare comes with significant ethical risks [9]. In conclusion, algorithms are not ethically neutral.

In recent years, researchers have attempted to identify and categorize the ethical problems that algorithms give rise to and the solutions that have been proposed in recent relevant literature. The conceptual map proposed by [10] (Fig.1) remains a fruitful framework for reviewing the current debate on the ethics of algorithms.

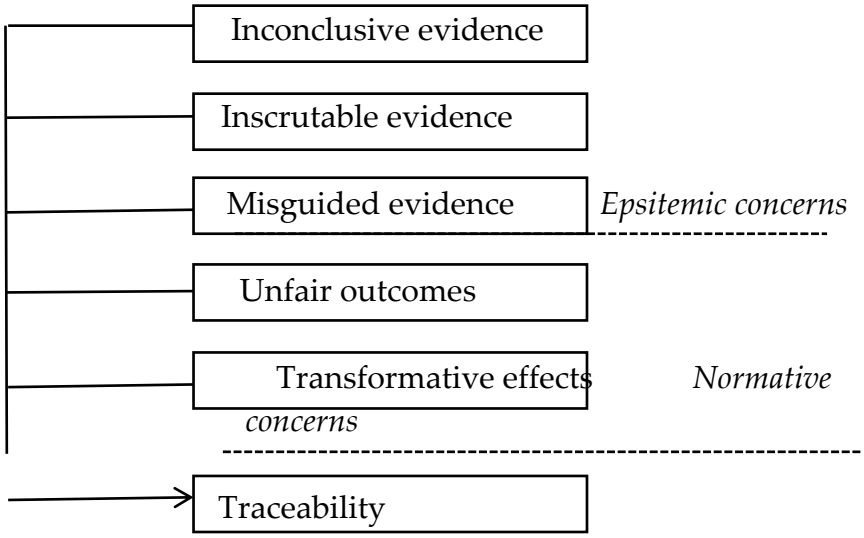

Figure 1. Six types of ethical concerns raised by algorithms [10]

Inconclusive evidence refers to the inevitably uncertain knowledge when algorithms draw conclusions from the data they process using inferential statistics and/or machine learning techniques.

Inscrutable evidence refers to the accessibility between the data and the conclusion in the case data are used as (or processed to produce) evidence for a conclusion.

Misguided evidence means that conclusions can only be as reliable (but also as neutral) as the data they are based on.

Unfair outcomes refers to the observer-dependent 'fairness' of the action and its effects.

Transformative effects denotes the unintended and significant changes that algorithms can impose on individuals or society, such as altering behaviors, decision-making processes, or social structures, often without transparency or user consent.

Traceability implies that harm caused by algorithmic activity is hard to debug (i.e. to detect the harm and find its cause), but also that it is rarely straightforward to identify who should be held responsible for the harm caused.

Driven by the definition of inconclusive evidence and its observed uncertainty, we are re-mapping the debate by introducing the concept of indeterminacy into all aspects of the ethical map (Fig.1). This approach strengthens our ability to address the complexity and ambiguity inherent in ethical decision-making for algorithms. By providing a structured approach to assessing how uncertainty affects ethical decisions, we increase accountability and transparency. This research

introduces a new neutrosophic methodological framework for evaluating ethical correctness (truth), ethical breaches (falsehood), and uncertainty (indeterminacy) in algorithms, offering a more comprehensive approach to ethical assessment. Thus, the conceptual idea of our study is depicted in Figure 2 and is posed as an organizing structure that allows a neutrosophic rigorous diagnosis of ethical challenges related to the use of algorithms.

Furthermore, aligning the ethical evaluation process with the intricacies of real-world scenarios makes the framework more useful and effective for guiding responsible and informed algorithm design and implementation. Introducing an "ethical score" metric that deals with indeterminacy can help tackle problematic situations such as ambiguous outcomes, data quality and representation, complex interactions, conflicting ethical principles, and human interpretation.

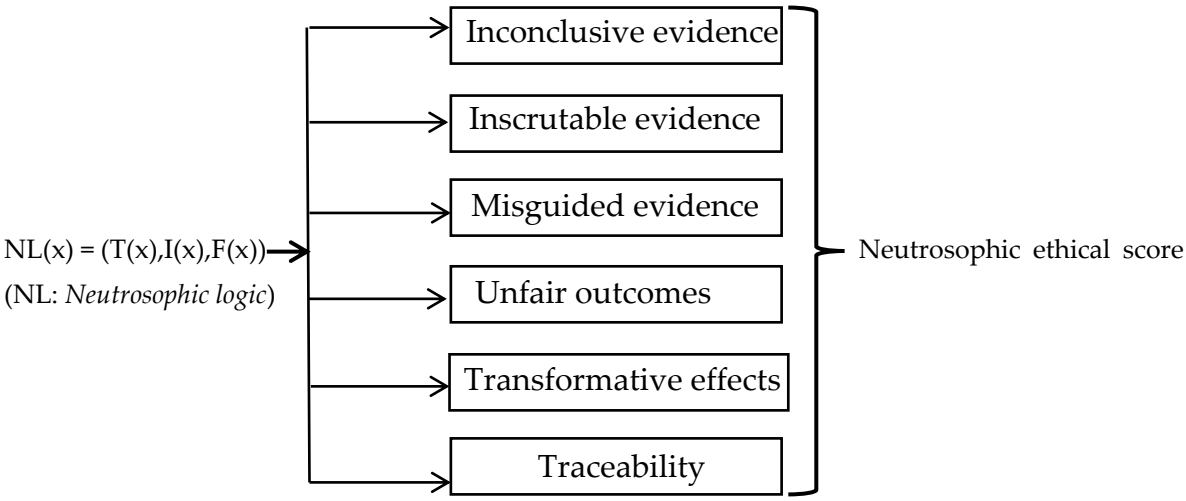

Figure 2. Neutrosophic algorithmic ethical map

Our review of the literature has revealed no prior studies that utilize this approach to systematically evaluate ethical ambiguity in algorithms, particularly in scenarios where principles like fairness, transparency, and accountability are in conflict. For this reason, this paper examines the significance of indeterminacy in ethical evaluations of algorithms and explores how a neutrosophic approach could impact the design of future algorithms. Actually, in our study, we aim to map out the ethical problems that arise from algorithmic decision-making by posing two main questions: 1) how can we systematically evaluate ethical uncertainty in algorithmic decision-making using neutrosophic logic? and 2) how does indeterminacy manifest in the conflicting ethical principles of justice, transparency, and accountability in algorithmic design?

While significant emphasis has been placed on concerns regarding prejudice, fairness, and openness [10-13], existing ethical frameworks sometimes struggle to navigate situations of uncertainty or conflicting ethical principles. In this way, our study also pinpoints the overlooked concept of indeterminacy and offers strategies for identifying ethical dilemmas, addressing a gap in the literature where traditional binary and fuzzy logic methods may fall short.

The structure of this article is as follows: In Section 2, we define and explain the neutrosophic mathematical framework needed to "construct" our proposed algorithmic “ethical index”, namely the Overall Neutrosophic Ethical Integrity Score (OvNEIS). This score will help us quantify the ethical performance of an algorithm in a neutrosophic environment. Next, in Section 3, we highlight the applicability of the suggested OvNEIS in an illustrative example from the field of healthcare. We examine an algorithmic decision-making process under three ethical criteria: fairness, transparency, and accountability and we briefly comment on the obtained results. Lastly, the "Concluding Remarks" section wraps up the key points of our study and proposes potential research work.

2. Materials and Methods

In neutrosophic logic, a concept A is T% true, I% indeterminate, and F% false, with (T, I, F) ⊂]]-0, 1+[[^{3}, where]]-0, 1+[[is an interval of hyperreals.

In this model, truth, falsehood, and indeterminacy may coexist, allowing for a more comprehensive representation of complex and ambiguous information. Sets containing neutrosophic components are employed in neutrosophic logic, with constituents having degrees of truth, falsehood, and indeterminacy. Its capacity to deal with ambiguity and uncertainty makes it useful in circumstances where standard logic systems may fail to offer correct representations.

In this framework, a formula φ is characterized by a triplet of truth-values, called the neutrosophical value defined as [14]:

NL(φ) = (T(φ), I(φ), F(φ)) where (T(φ), I(φ), F(φ)) ⊂]]-0, 1+[[^{3} (1)

For each algorithmic ethical criterion C_i $(i = 1,2 \dots, n)$, we can represent its performance by utilizing the concept of neutrosophic set in a similar way as given in (1):

$$C_i = \{(x, T_i(x), I_i(x), F_i(x)) | x \in U\} \quad (2)$$

where U : universe of discourse (e.g. algorithm outputs)

Next we define a formula, namely Neutrosophic Ethical Integrity Score (NEIS), which will be utilized for evaluating the ethical performance of algorithms based on the principles of neutrosophic logic. It is designed in such a way so as to balance positive and negative aspects (subtraction of falsehood) and capture the complexity of ethical evaluation (indeterminacy as a modifier)

Definition 1. The Neutrosophic Ethical Integrity Score (NEIS) for a given ethical criterion $\boldsymbol{i}$ is defined as:

$NEIS_i : \mathbb{R}^3 \to \mathbb{R}$

where $\mathbb{R}^3$ represents the three-dimensional space of the components of ethical evaluation, specifically truth, falsehood, and indeterminacy.

The $\boldsymbol{NEIS_i}$ function is given as follows:

$$NEIS_i = T_i\text{-}F_i\text{+}I_i \quad (3)$$

where :

$T_i \in [0,1]$ represents the degree to which the ethical criterion is satisfied by the algorithm. It is obvious that a value of T_i = 1 means complete satisfaction and T_i = 0 indicates complete dissatisfaction.

Following the same logic, F_i = 1 indicates complete violation to ethical criterion i and F_i = 0 means no violation.

Extra care should be given when evaluating the value of I_i. In our context, I_i= 1 indicates complete indeterminacy (uncertainty or ambiguity) in the ethical evaluation, while I_i= 0 means no indeterminacy.

Once the individual NEIS values are calculated for each criterion, the overall NEIS can be computed. This may involve taking a weighted average or simple average of the individual scores, depending on the specific context and importance of each criterion.

Definition 2. Let n be the number of criteria for assessing algorithmic ethics. The Overall NEIS can be defined as:

$$\text{Overall NEIS (OvNEIS)} = \sum_{i=1}^{n} w_i * NEIS_i \quad (4)$$

where w_i is the weight assigned to algorithmic ethical criterion i such as $\sum_{i=1}^{n} w_i = 1$.

3. Results

Let us now study an illustrative example that applies the proposed Neutrosophic Ethical Integrity Score (NEIS) to an algorithmic decision-making process in the context of healthcare. Let us assume that we wish to measure the performance of an algorithm designed to recommend treatment plans for patients in the following three ethical criteria: (i) *fairness*, (ii) *transparency*, and (iii) *accountability*.

To assess its ethical integrity, we evaluate its performance using the three aforesaid criteria in the next framework:

(i) Fairness: Does the algorithm treat all patients similarly, regardless of demographics (such as age, gender, or ethnicity)?

(ii) Transparency: Are healthcare practitioners able to comprehend the algorithm's decision-making process?

(iii)Accountability: Can the algorithm's outputs and decisions be tracked back to the responsible stakeholders?

Next, we will calculate the NEIS for each criterion.

Let us assume that based on historical data or surveys and expert evaluations we have the following assignment of values for each criterion (Equation 2):

(i) Fairness: C_f = (0.8, 0.1, 0.1)

(ii) Transparency: C_t = (0.7, 0.1, 0.2)

(iii) Accountability: C_a = (0.6, 0.1, 0.3)

The NEIS for each criterion is calculated using Equation (3):

$NEIS_{C_f} = T_{C_f} - F_{C_f} + I_{C_f}$ = 0.8

In the same way, we get :

$NEIS_{C_t}$ = 0.6 and

$NEIS_{C_a}$ = 0.4

Let us know make the fair assumption that the weights of each ethical criterion is equal, i.e. $w_{C_f} = w_{C_t} = w_{C_a}$ =0.33.

Remark: depending on the context of the problem we are studying, different weights could be assigned to the criteria. This could be useful in cases where the fairness criteria is considered more important than the accountability and transparency criteria (e.g. hiring algorithms, loan approval algorithms, etc.). Therefore, a higher weight value for the fairness criterion will be assigned.

In our scenario, the overall NEIS is calculated by applying Equation (4):

OvNEIS = $w_{C_f} * NEIS_{C_f} + w_{C_t} * NEIS_{C_t} + w_{C_a} * NEIS_{C_a}$ = 0.6

The Overall NEIS of 0.6 reflects an average ethical performance of the algorithm across all three criteria (Fairness, Transparency, and Accountability), given that all criteria are equally important.

A NEIS of 0.6 indicates moderate ethical integrity, with some space for improvement, notably in criteria such as accountability, which scored lower (0.4) compared to ther ethical criteria.

Furthermore, we can make the following observations based on our proposed method:

- The high fairness score indicates that the algorithm is unlikely to perpetuate considerable prejudice or discrimination, a critical worry in many algorithmic systems.
- The algorithm's decision-making process is only partly transparent, weakening confidence and making it difficult for consumers to comprehend or question its suggestions.
- The low accountability score raises questions about who is responsible for the algorithm's results, particularly in times of error or harm.

Next steps for the improvement of the algorithm:

- Continue to check for prejudice and fairness concerns, particularly in circumstances where indeterminacy is high.
- Improve the algorithm's capacity to explain its decision-making process to users, particularly when the decision is not clear.
- Involve users (e.g., healthcare experts) in assessing the algorithm's explanations to ensure that transparency has significance.
- Implement organizational policies that define accountability for algorithmic judgements, especially when they may cause harm.

4. Conclusions

Since 2016, the ethics of algorithms has been a major issue of debate among academics, technology providers, and regulators. The topic also gained pace as a result of the so-called "summer of AI" and the widespread deployment of Machine Learning algorithms. One aspect that was not explicitly captured by the research debate, and which is becoming a central point of discussion in the relevant literature, is the increasing focus on the use of algorithms, AI and digital technologies more broadly, to deliver socially good outcomes [15-17].

Starting from an admittedly simplistic notion of ethics as ''the study of what we ought to do,'' our aim has been to sketch a model that could "quantify" algorithmic ethics. In this perpserctive, we propose a formal mathematical neutrosophic framework that assists stakeholders in making key decisions based on a measurable indicator, OvNEIS. The latter, which is intended to assess uncertainty in ethical judgements in algorithmic decision-making, incorporates neutrosophic logic concepts such as truth, falsehood, and indeterminacy, allowing for a more comprehensive representation of complex and ambiguous data.

Specifically, OvEIS can address and add value, by considering the key role of indeterminacy, to the following issues concerning the algorithmic ethics:

- Inconclusive and Misguided Evidence: it helps to systematically quantify and address cases where evidence is either inconclusive or misleading.
- Inscrutable Evidence: it can enhance traceability and make algorithmic processes less inaccessible, answering concerns about traceability.
- Unfair Outcomes: it can analyse fairness as one of its primary aspects, guaranteeing that algorithms are more thoroughly reviewed for bias and fairness by taking into account not just truth and falsity, but also indeterminacy in their outcomes.
- Transformative Effects: it can consider the uncertainty in their long-term impacts on society, highlighting potential ethical concerns before they manifest.
- Traceability and Accountability: by scoring algorithms on this criterion, we ensure that ethical violations are not only discovered but also quantified, supporting responsible design and implementation.

Future study should focus on extending this technique to a broader range of case studies from a variety of sectors. Validation should be enhanced and a comparative study with other fuzzy/neutrosophic MCDM methods could shed light on issues such as efficiency, complexity and robustness of the proposed method.

References

1. Floridi, L. (2012). Big data and their epistemological challenge. *Philosophy & Technology*, 25(4), 435–437.
2. de Vries, K. (2010). Identity, profiling algorithms and a world of ambient intelligence. *Ethics and Information Technology*, 12(1), 71–85.
3. Portmess, L., & Tower, S. (2014). Data barns, ambient intelligence and cloud computing: The tacit epistemology and linguistic representation of Big Data. *Ethics and Information Technology*, 17(1), 1–9.
4. Newell, S., & Marabelli, M. (2015). Strategic opportunities (and challenges) of algorithmic decision-making: A call for action on the long-term societal effects of 'datification'. *The Journal of Strategic Information Systems*, 24(1), 3–14.
5. Taddeo, M., & Floridi, L. (2015). The debate on the moral responsibilities of online service providers. *Science and Engineering Ethics*, 1–29.
6. Hill, R. K. (2016). What an algorithm is. *Philosophy & Technology*, 29, 35-59.
7. Tutt, A. (2016, March 1). An FDA for Algorithms. *69 Admin. L. Rev. 83 (2017)*, Available at SSRN: https://ssrn.com/abstract=2747994 or http://dx.doi.org/10.2139/ssrn.2747994.
8. Burrell, J. (2016). How the machine 'thinks:' Understanding opacity in machine learning algorithms. *Big Data & Society*, 3(1): 1–12.

9. Floridi, L., & Taddeo, M. (2016). What is data ethics?. Philosophical Transactions of the Royal Society A: Mathematical, *Physical and Engineering Sciences*, 374(2083), 20160360.
10. Mittelstadt, B. D., Allo, P., Taddeo, M., Wachter, S., & Floridi, L. (2016). The ethics of algorithms: Mapping the debate. *Big Data & Society*, 3(2), 2053951716679679.
11. Binns, R. (2018, January). Fairness in machine learning: Lessons from political philosophy. In *Conference on fairness, accountability and transparency (pp. 149-159)*. PMLR.
12. Crawford, K. (2016). Can an algorithm be agonistic? Ten scenes from life in calculated publics. *Science, Technology, & Human Values*, 41(1), 77-92.
13. Martin, K. (2019). Designing ethical algorithms. *MIS Quarterly Executive June*.
14. Smarandache, F. (1999). *A unifying field in Logics: Neutrosophic Logic*. In Philosophy, American Research Press, pp. 1–141.
15. Hager, G. D., Drobnis, A., Fang, F., Ghani, R., Greenwald, A., Lyons, T., & Tambe, M. (2019). Artificial intelligence for social good. *arXiv preprint arXiv:1901.05406*.
16. Paraman, P., & Anamalah, S. (2023). Ethical artificial intelligence framework for a good AI society: principles, opportunities and perils. *AI & SOCIETY*, 38(2), 595-611.
17. Cowls, J., Tsamados, A., Taddeo, M., & Floridi, L. (2021). A definition, benchmark and database of AI for social good initiatives. *Nature Machine Intelligence*, 3(2), 111-115.

Several New Types of Neutrosophic Sets

Florentin Smarandache

University of New Mexico, Mathematics Department, Gallup, NM 87301, USA; smarand@unm.edu

Smarandache, Florentin. "Several New Types of Neutrosophic Sets."
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Abstract: In the literature, new types of neutrosophic sets have been introduced in the meantime by the growing neutrosophic community. We present a few.

Keywords: Pythagorean Neutrosophic Set, Fermatean Neutrosophic Set, Generalized Fermatean Neutrosophic Set, n-power Neutrosophic Set, Cubic Spherical Neutrosophic Set, Spherical Neutrosophic Set, n-HyperSpherical Neutrosophic Set, Refined n-HyperSpherical Neutrosophic Set.

0. Introduction

We present several of the new types of neutrosophic sets have been introduced in the meantime by the growing neutrosophic community.

Let all $T, I, F \in [0,1]$, that represent the degree of Truth, degree of Indeterminacy, and degree of Falsehood respectively. Then:

1. The **Pythagorean Neutrosophic Set**:
$$0 \le T^2 + I^2 + F^2 \le 2$$
In a more general way, we may take any number strictly less than 3 as superior limit of the above sum:
$$0 \le T^2 + I^2 + F^2 \le b < 3$$
For example,
$$0 \le T^2 + I^2 + F^2 \le 1.7$$
or
$$0 \le T^2 + I^2 + F^2 \le 2.6$$
Etc.

2. The **Fermatean Neutrosophic Set** [2]
$$0 \le T^3 + I^3 + F^3 \le 2$$
where it is considered that and of course

3. The **Generalized Fermatean Neutrosophic Set**
In a more general way, we may take any number strictly less than 3 as superior limit of the above sum:

$$0 \le T^3 + I^3 + F^3 \le b < 3$$

For examples:

$$0 \le T^3 + I^3 + F^3 \le 2.5$$

or

$$0 \le T^3 + I^3 + F^3 \le 1.8$$

etc.

4. The ***n*-Power Neutrosophic Set**, $n > 1$, more general than the previous ones:

$$0 \le T^n + I^n + F^n \le b < 3$$

The varying parameters $n > 1$ and $0 < b < 3$ depend on each specific application.

For example:

$$0 \le T^7 + I^7 + F^7 \le 2.2$$

5. The **Cubic Spherical Neutrosophic Set** from the paper [1] is totally different.

Let $S = \{< a_1(T_1, I_1, F_1) >, < a_2(T_2, I_1, F_2) >, ..., < a_n(T_n, I_n, F_n) >\}$ be a single-valued neutrosophic set, where $a_1, a_2, ..., a_n \in U$ that is a universe of discourse.

One makes the averages of T's, then of I's, and of F's to get the center of the sphere.

$$T_a = \frac{1}{n}\sum_{i=1}^{n} T_i, I_a = \frac{1}{n}\sum_{i=1}^{n} I_i, F_a = \frac{1}{n}\sum_{i=1}^{n} F_i$$

The center (C) of the sphere of this neutrosophic set is:

$$C = (T_a, I_a, F_a)$$

Then the sphere's radius (r) is computing as:

$$r = \min\{\max\sqrt{[(T_1 - T_a)^2 + (I_1 - I_a)^2 + (F_1 - F_a)^2] + [(T_2 - T_a)^2 + (I_2 - I_a)^2 + (F_2 - F_a)^2] + ... + [(T_n - T_a)^2 + (I_n - I_a)^2 + (F_n - F_a)^2]}, 1\}$$

meaning the biggest distance, that does not exceed *1*, from each element (T_i, I_i, F_i) to the sphere's center (T_a, I_a, F_a).

6. **Single-Valued Spherical Neutrosophic Set**

Spherical Neutrosophic Set (SNS) was introduced by Smarandache [3] in 2017.

A Single-Valued Spherical Neutrosophic Set (SNS), of the universe of discourse U, is defined as follows:

$$A_{SNS} = \{<x, T_A(x), I_A(x), F_A(x)> \mid x \in U\},$$

where, for all $x \in U$, the functions $T_A(x), I_A(x), F_A(x) : U \to [0, \sqrt{3}]$, represent the degree of membership (truth), the degree of indeterminacy, and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \le T_A^2(x) + I_A^2(x) + F_A^2(x) \le 3.$$

The Spherical Neutrosophic Set is a generalization of Spherical Fuzzy Set, because we may restrain the SNS's components to the unit interval $T_A(x), I_A(x), F_A(x) \in [0, 1]$, and the sum of the squared components to 1, i.e. $0 \le T_A^2(x) + I_A^2(x) + F_A^2(x) \le 1$.

Further on, if replacing $I_A(x) = 0$ into the Spherical Fuzzy Set, we obtain as particular case the Pythagorean Fuzzy Set.

7. Single-Valued n-HyperSpherical Neutrosophic Set (n-HSNS)

The Single-Valued n-HyperSpherical Neutrosophic Set (n-HSNS) [3], which is a generalization of the Spherical Neutrosophic Set and of n-HyperSpherical Fuzzy Set, of the universe of discourse U, for $n \geq 1$, is defined as follows:

$$A_{n\text{-}HNS} = \{<x, T_A(x), I_A(x), F_A(x)> \mid x \in U\},$$

where, for all $x \in U$, the functions $T_A(x), I_A(x), F_A(x) : U \to [0, \sqrt[n]{3}]$, represent the degree of membership (truth), the degree of indeterminacy, and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \leq T_A^n(x) + I_A^n(x) + F_A^n(x) \leq 3.$$

8. Single-Valued Refined n-HyperSpherical Neutrosophic Set (R-n-HSNS)

The Single-Valued Refined n-HyperSpherical Neutrosophic Set (R-n-HSNS) [3], which is a generalization of the n-HyperSpherical Neutrosophic Set and of Refined n-HyperSpherical Fuzzy Set.

On the universe of discourse U, for $n \geq 1$, we define it as:

$$A_{R-n-HSNS} = \{x(T_A^1(x), T_A^2(x), ..., T_A^p(x); I_A^1(x), I_A^2(x), ..., I_A^r(x);$$
$$F_A^1(x), F_A^2(x), ..., F_A^s(x)), p + r + s \geq 4, x \in U\},$$

where p, r, s are nonzero positive integers, and for all $x \in U$, the functions

$$T_A^1(x), T_A^2(x), ..., T_A^p(x), I_A^1(x), I_A^2(x), ..., I_A^r(x), F_A^1(x), F_A^2(x), ..., F_A^s(x) : U \to [0, m^{1/n}],$$

represent the degrees of sub-membership (sub-truth) of types *1, 2, ..., p,* the degrees of sub-indeterminacy of types *1, 2, ..., r,* and degrees on sub-nonmembership (sub-falsity) of types *1, 2, ..., s* respectively, that satisfy the condition:

$$0 \leq \sum_1^p (T_A^j)^n + \sum_1^r (I_A^k)^n + \sum_1^s (F_A^l)^n \leq m, \text{ where } p + r + s = m.$$

9. Conclusion

Many types of extended or hybrid neutrosophic sets have been introduced by the neutrosophic community. We revealed only a few of them. Certainly, more in the future will follow.

Reference

[1] S. Gomathi, S. Krishnaprakash, M. Karpagadevi, Said Broumi, Cubic Spherical Neutrosophic Set, International Journal of Neutrosophic Science (IJNS) Vol. 21, No. 04, PP. 172-180, 2023.

[2] C. Antony Crispin Sweety, R. Jansi, Fermatean Neutrosophic Sets, International Journal of Advanced Research in Computer and Communication Engineering, 24-27, Vol. 10, Issue 6, June 2021.

[3] F. Smarandache, Neutrosophic Set is a Generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, and n-HyperSpherical Fuzzy Set, while Neutrosophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision (revisited), Journal of New Theory 29 (2019) 01-35,
https://fs.unm.edu/Raspunsatan.pdf
arXiv, Cornell University, New York City, NY, USA, pp. 1-50, 17-29 November 2019,
https://arxiv.org/ftp/arxiv/papers/1911/1911.07333.pdf;
University of New Mexico, Albuquerque, USA, Digital Repository, pp. 1-50,
https://digitalrepository.unm.edu/math_fsp/21,

The extended study of 2-refined neutrosophic numbers

Yaser Ahmad Alhasan
Prince Sattam bin Abdulaziz University, Alkharj, SAUDI ARABIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

Abuobida Alfahal
Prince Sattam bin Abdulaziz University, Alkharj, SAUDI ARABIA

Raja Abdullah Abdulfatah
Prince Sattam bin Abdulaziz University, Alkharj, SAUDI ARABIA

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Abstract: Numerical roots play a crucial role in real or complex equations, as well as in various mathematical models involving differentiation, integration, and other mathematical relationships. In the realm of mathematics, there is a growing trend of utilizing new formulas that incorporate non-classical numbers, such as neutrosophic and refined neutrosophic numbers. The objective of this research is to establish precise and comprehensive mathematical procedures for dealing with refined neutrosophic roots within mathematical formulas, be it equations or other mathematical constructs. This paper presents an extensive study on 2-refined neutrosophic numbers, focusing on the square root of a 2-refined neutrosophic real or complex number. Additionally, this work introduces the concept of 2-refined neutrosophic real or complex polynomials and explores the process of finding the refined neutrosophic roots to solve 2-refined neutrosophic equations. To illustrate these concepts, several examples have been provided.

Keywords: refined neutrosophic; square root; complex polynomial; real polynomial.

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1. Introduction and Preliminaries

To describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction, Smarandache suggested the neutrosophic Logic as an alternative to the current logics. Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1]

Agboola introduced the concept of refined neutrosophic algebraic structures [2]. Also, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that: [3]

$$I_1I_1 = {I_1}^2 = I_1 \quad (1)$$
$$I_2I_2 = {I_2}^2 = I_2 \quad (2)$$
$$I_1I_2 = I_2I_1 = I_1 \quad (3)$$

In addition, there are many papers presenting studies on refined neutrosophic numbers [4-5-6-7-8-12-13-14-15] and Mehmet Celik and Ahmed Hatip presented a study on the refined ah-isometry

and its applications in refined neutrosophic surfaces. Smarandache discussed neutrosophic indefinite integral (Refined Indeterminacy) [11]

Alhasan also presented several papers on calculus, in which he discussed neutrosophic definite and indefinite integrals. He also presented the most important applications of definite integrals in neutrosophic logic [9-10].

This paper dealt with several topics, in the first part of which introduction and preliminaries were presented, and in the main discussion part the 2-refined neutrosophic numbers that contain two part of indeterminacy (I_1, I_2). In the last part, a conclusion to the paper is given.

2. Main Discussion

2.1 The square root of a 2-refined neutrosophic real number

First, let's compute the square root: $\sqrt{a+bI_1+cI_2}$ whereas $a \geq 0, a+c \geq 0, a+b+c \geq 0$, while $I_1, I_2 =$ indeterminacy

$$\sqrt{a+bI_1+cI_2} = x+yI_1+zI_2$$

Where $x, y \ and \ z$ are real unknowns.

By raise both sides to the second power, we get:

$$a+bI_1+cI_2 = (x+yI_1+zI_2)^2$$

$$a+bI_1+cI_2 = (x+yI_1)^2 + 2(x+yI_1)(zI_2) + (zI_2)^2$$

$$a+bI_1+cI_2 = x^2 + 2xyI_1 + (yI_1)^2 + 2(x+yI_1)(zI_2) + (zI_2)^2$$

$$a+bI_1+cI_2 = x^2 + 2xyI_1 + y^2I_1 + 2xzI_2 + 2yzI_1 + z^2I_2$$

$$a+bI_1+cI_2 = x^2 + (y^2+2xy+2yz)I_1 + (z^2+2xz)I_2$$

Whence:

$$\begin{cases} x^2 = a \\ y^2+2xy+2yz = b \\ z^2+2xz = c \end{cases}$$

$$\Longrightarrow \begin{cases} x = \pm\sqrt{a} \\ y^2+2xy+2yz = b \quad (1) \\ z^2+2xz = c \quad (2) \end{cases}$$

Case1: $x = \sqrt{a}$ by substitution in (2)

$$z^2 + 2\sqrt{a}\ z - c = 0$$
$$\Delta = 4a + 4c$$

Then:

$$z = \frac{-2\sqrt{a}+2\sqrt{a+c}}{2} = -\sqrt{a}+\sqrt{a+c}$$
$$\left(z = \frac{-2\sqrt{a}-2\sqrt{a+c}}{2} = -\sqrt{a}-\sqrt{a+c}\right.$$

- For $z = -\sqrt{a}+\sqrt{a+c}$ we substitute in (1)

$$y^2 + 2\sqrt{a}y + 2\left(-\sqrt{a}+\sqrt{a+c}\right)y - b = 0$$

$$y^2 + 2\sqrt{a+c}\ y - b = 0$$

$$\Delta = 4(a+c) + 4b$$

Then:

$$y = \frac{-2\sqrt{a+c} + 2\sqrt{a+c+b}}{2} = -\sqrt{a+c} + \sqrt{a+c+b}$$

$$\lfloor y = \frac{-2\sqrt{a+c} - 2\sqrt{a+c+b}}{2} = -\sqrt{a+c} - \sqrt{a+c+b}$$

Hence:

$$\sqrt{a + bI_1 + cI_2} = \sqrt{a} + \left(-\sqrt{a+c} + \sqrt{a+c+b}\right)I_1 + \left(-\sqrt{a} + \sqrt{a+c}\right)I_2$$

Or:

$$= \sqrt{a} + \left(-\sqrt{a+c} - \sqrt{a+c+b}\right)I_1 + \left(-\sqrt{a} + \sqrt{a+c}\right)I_2$$

- For $z = -\sqrt{a} - \sqrt{a+c}$ we substitute in (1)

$$y^2 + 2\sqrt{a}y + 2\left(-\sqrt{a} - \sqrt{a+c}\right)y - b = 0$$

$$y^2 - 2\sqrt{a+c}\ \ y - b = 0$$

$$\Delta = 4(a+c) + 4b$$

Then:

$$y = \frac{2\sqrt{a+c} + 2\sqrt{a+c+b}}{2} = \sqrt{a+c} + \sqrt{a+c+b}$$

$$\lfloor y = \frac{2\sqrt{a+c} - 2\sqrt{a+c+b}}{2} = \sqrt{a+c} - \sqrt{a+c+b}$$

Hence:

$$\sqrt{a + bI_1 + cI_2} = \sqrt{a} + \left(\sqrt{a+c} + \sqrt{a+c+b}\right)I_1 + \left(-\sqrt{a} - \sqrt{a+c}\right)I_2$$

Or:

$$= \sqrt{a} + \left(\sqrt{a+c} - \sqrt{a+c+b}\right)I_1 + \left(-\sqrt{a} - \sqrt{a+c}\right)I_2$$

Case2: $x = -\sqrt{a}$ by substitution in (2)

$$z^2 - 2\sqrt{a}\ \ z - c = 0$$

$$\Delta = 4a + 4c$$

Then:

$$z = \frac{2\sqrt{a} + 2\sqrt{a+c}}{2} = \sqrt{a} + \sqrt{a+c}$$

$$\lfloor z = \frac{2\sqrt{a} - 2\sqrt{a+c}}{2} = \sqrt{a} - \sqrt{a+c}$$

- For $z = \sqrt{a} + \sqrt{a+c}$ we substitute in (1)

$$y^2 - 2\sqrt{a}y + 2\left(\sqrt{a} + \sqrt{a+c}\right)y - b = 0$$

$$y^2 + 2\sqrt{a+c}\ \ y - b = 0$$

$$\Delta = 4(a+c) + 4b$$

Then:

$$y = \frac{-2\sqrt{a+c} + 2\sqrt{a+c+b}}{2} = -\sqrt{a+c} + \sqrt{a+c+b}$$
$$\left(y = \frac{-2\sqrt{a+c} - 2\sqrt{a+c+b}}{2} = -\sqrt{a+c} - \sqrt{a+c+b}\right.$$

Hence:

$$\sqrt{a+bI_1+cI_2} = -\sqrt{a} + \left(-\sqrt{a+c} + \sqrt{a+c+b}\right)I_1 + \left(\sqrt{a} + \sqrt{a+c}\right)I_2$$

Or:

$$= -\sqrt{a} + \left(-\sqrt{a+c} - \sqrt{a+c+b}\right)I_1 + \left(\sqrt{a} + \sqrt{a+c}\right)I_2$$

- For $z = \sqrt{a} - \sqrt{a+c}$ we substitute in (1)

$$y^2 - 2\sqrt{a}y + 2\left(\sqrt{a} - \sqrt{a+c}\right)y - b = 0$$

$$y^2 - 2\sqrt{a+c}\ \ y - b = 0$$

$$\Delta = 4(a+c) + 4b$$

Then:

$$y = \frac{2\sqrt{a+c} + 2\sqrt{a+c+b}}{2} = \sqrt{a+c} + \sqrt{a+c+b}$$
$$\left(y = \frac{2\sqrt{a+c} - 2\sqrt{a+c+b}}{2} = \sqrt{a+c} - \sqrt{a+c+b}\right.$$

Hence:

$$\sqrt{a+bI_1+cI_2} = -\sqrt{a} + \left(\sqrt{a+c} + \sqrt{a+c+b}\right)I_1 + \left(\sqrt{a} - \sqrt{a+c}\right)I_2$$

Or:

$$= -\sqrt{a} + \left(\sqrt{a+c} - \sqrt{a+c+b}\right)I_1 + \left(\sqrt{a} - \sqrt{a+c}\right)I_2$$

The eight results are:

$(x,y,z) = \left(\sqrt{a}\ , -\sqrt{a+c} + \sqrt{a+c+b}\ , -\sqrt{a} + \sqrt{a+c}\right), \left(\sqrt{a}\ , -\sqrt{a+c} - \sqrt{a+c+b}\ , -\sqrt{a} + \sqrt{a+c}\right)$

$\left(\sqrt{a}\ , \sqrt{a+c} + \sqrt{a+c+b}\ , -\sqrt{a} - \sqrt{a+c}\right), \left(\sqrt{a}\ , \sqrt{a+c} - \sqrt{a+c+b}\ , -\sqrt{a} - \sqrt{a+c}\right)$

$\left(-\sqrt{a}\ , -\sqrt{a+c} + \sqrt{a+c+b}\ , \sqrt{a} + \sqrt{a+c}\right), \left(-\sqrt{a}\ , -\sqrt{a+c} - \sqrt{a+c+b}\ , \sqrt{a} + \sqrt{a+c}\right)$

$\left(-\sqrt{a}\ , \sqrt{a+c} + \sqrt{a+c+b}\ , \sqrt{a} - \sqrt{a+c}\right), \left(-\sqrt{a}\ , \sqrt{a+c} - \sqrt{a+c+b}\ , \sqrt{a} - \sqrt{a+c}\right)$

Because we are now calculating the square root of a 2-refined neutrosophic number (according to classical analysis), we only take the result with a positive value, hence:

$$\sqrt{a+bI_1+cI_2} = \sqrt{a} + \left(-\sqrt{a+c} + \sqrt{a+c+b}\right)I_1 + \left(-\sqrt{a} + \sqrt{a+c}\right)I_2$$

Clearly: $\sqrt{a} \geq 0\ \ , -\sqrt{a+c} + \sqrt{a+c+b} \geq 0\ \ and\ \ -\sqrt{a} + \sqrt{a+c} \geq 0$

However, when we solve the 2-refined neutrosophic equation, we take all eight results.

Example 1

Let's find: $\sqrt{9 + 9I_1 + 7I_2}$

$$\sqrt{9+9I_1+7I_2} = x + yI_1 + zI_2$$

By raise both sides to the second power, we get:

$$9 + 9I_1 + 7I_2 = (x + yI_1 + zI_2)^2$$

$$9 + 9I_1 + 7I_2 = (x + yI_1)^2 + 2(x + yI_1)(zI_2) + (zI_2)^2$$

$$9 + 9I_1 + 7I_2 = x^2 + 2xyI_1 + (yI_1)^2 + 2(x + yI_1)(zI_2) + (zI_2)^2$$

$$9 + 9I_1 + 7I_2 = x^2 + 2xyI_1 + y^2I_1 + 2xzI_2 + 2yzI_1 + z^2I_2$$

$$9 + 9I_1 + 7I_2 = x^2 + (y^2 + 2xy + 2yz)I_1 + (z^2 + 2xz)I_2$$

Whence:

$$\begin{cases} x^2 = 9 \\ y^2 + 2xy + 2yz = 9 \\ z^2 + 2xz = 7 \end{cases}$$

$$\Longrightarrow \begin{cases} x = \pm 3 \\ y^2 + 2xy + 2yz = 9 \quad (1) \\ z^2 + 2xz = 7 \quad (2) \end{cases}$$

Case1: $x = 3$ by substitution in (2)

$$z^2 + 6z - 7 = 0$$
$$(z - 1)(z + 7) = 0$$

Then:

$$\begin{cases} z = 1 \\ z = -7 \end{cases}$$

- For $z = 1$ we substitute in (1)

$$y^2 + 8y - 9 = 0$$
$$(y + 9)(y - 1) = 0$$

Then:

$$\begin{cases} y = 1 \\ y = -9 \end{cases}$$

Hence:

$$\sqrt{9+9I_1+7I_2} = 3 + I_1 + I_2 \quad \text{(Accepted)}$$

Or:

$$= 3 - 9I_1 + I_2 \quad \text{(Rejected)}$$

- For $z = -7$ we substitute in (1)

$$y^2 - 8y - 9 = 0$$
$$(y - 9)(y + 1) = 0$$

Then:

$$\begin{cases} y = 9 \\ y = -1 \end{cases}$$

Hence:

$$\sqrt{9+9I_1+7I_2} = -3 + 9I_1 - 7I_2 \text{ (Rejected)}$$

Or:

$$= -3 - I_1 - 7I_2 \quad \text{(Rejected)}$$

Case2: $x=-3$ by substitution in (2)

$$z^2-6z-7=0$$
$$(z-7)(z+1)=0$$

Then:

$$\begin{cases} z=7 \\ z=-1 \end{cases}$$

- For $z=7$ we substitute in (1)

$$y^2+8y-9=0$$

$$(y-1)(y+9)=0$$

Then:

$$\begin{cases} y=1 \\ y=-9 \end{cases}$$

Hence:

$$\sqrt{9+9I_1+7I_2}=-3+I_1+7I_2 \text{ (Rejected)}$$

Or:

$$=-3-9I_1+7I_2 \text{ (Rejected)}$$

- For $z=-1$ we substitute in (1)

$$y^2-8y-9=0$$
$$(y-9)(y+1)=0$$

$$\begin{cases} y=9 \\ y=-1 \end{cases}$$

Hence:

$$\sqrt{9+9I_1+7I_2}=-3+9I_1-I_2 \text{ (Rejected)}$$

Or:

$$=-3-I_1-I_2 \text{ (Rejected)}$$

Remarks

As a particular case we can compute $\sqrt{I_1+I_2}$

$$\sqrt{I_1+I_2}=x+yI_1+zI_2$$

By raise both sides to the second power, we get:

$$I_1+I_2=(x+yI_1+zI_2)^2$$

$$I_1+I_2=(x+yI_1)^2+2(x+yI_1)(zI_2)+(zI_2)^2$$

$$I_1+I_2=x^2+2xyI_1+(yI_1)^2+2(x+yI_1)(zI_2)+(zI_2)^2$$

$$I_1+I_2=x^2+2xyI_1+y^2I_1+2xzI_2+2yzI_1+z^2I_2$$

$$I_1+I_2=x^2+(y^2+2xy+2yz)I_1+(z^2+2xz)I_2$$

Whence:

$$\begin{cases} x^2=0 \\ y^2+2xy+2yz=1 \\ z^2+2xz=1 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \\ y^2 + 2yz = 1 \\ z^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \\ y^2 + 2yz = 1 \quad (1) \\ z = \pm 1 \end{cases}$$

- For $z = 1$ we substitute in (1)

$$y^2 + 2y - 1 = 0$$
$$\Delta = 8$$

Then:

$$y = \frac{-2 + 2\sqrt{2}}{2} = -1 + \sqrt{2}$$
$$\left(y = \frac{-2 - 2\sqrt{2}}{2} = -1 - \sqrt{2} \right.$$

Hence:

$$\sqrt{I_1 + I_2} = \left(-1 + \sqrt{2}\right) I_1 + I_2$$

Or:

$$= \left(-1 - \sqrt{2}\right) I_1 + I_2$$

- For $z = -1$ we substitute in (1)

$$y^2 - 2y - 1 = 0$$
$$\Delta = 8$$

Then:

$$y = \frac{2 + 2\sqrt{2}}{2} = 1 + \sqrt{2}$$
$$\left(y = \frac{2 - 2\sqrt{2}}{2} = 1 - \sqrt{2} \right.$$

Hence:

$$\sqrt{I_1 + I_2} = \left(1 + \sqrt{2}\right) I_1 - I_2$$

Or:

$$= \left(1 - \sqrt{2}\right) I_1 - I_2$$

The four results are:

$$(x, y, z) = \left(0 \ , -1 + \sqrt{2} \ , 1\right), \left(0 \ , -1 - \sqrt{2} \ , 1\right)$$

$$\left(0 \ , 1 + \sqrt{2} \ , -1\right), \left(0 \ , 1 - \sqrt{2} \ , -1\right)$$

Because we are now calculating the square root of a 2-refined neutrosophic number (according to classical analysis), we only take the result with a positive value, hence:

$$\sqrt{I_1 + I_2} = \left(-1 + \sqrt{2}\right) I_1 + I_2$$

However, when we solve the 2-refined neutrosophic equation, we take all four results.

2.3 2-Refined neutrosophic real or complex polynomial

Definition:

A polynomial whose coefficients (at least one of them containing I_1 or I_2 or both I_1, I_2) are 2-refined neutrosophic numbers is called 2-Refined Neutrosophic Polynomials.

Similarly, we may have 2-Refined Neutrosophic Real Polynomials if its coefficients are refined neutrosophic real numbers and 2-Refined Neutrosophic Complex Polynomials if its coefficients are 2-refined neutrosophic complex numbers.

Example 2

- $P(x) = 10x^2 + (5 - 7I_1 + 3I_2)x - 4 + 2I_2$
  Is a 2-refined refined neutrosophic real polynomial
- $Q(x) = x^4 + (2 + 3I_2 - 4i + 3I_1 i)x^3 + 6I_2 x^2 + (9 - 8I_1 + 3I_2 i)x - 4 + 2I_2 + 4I_1 i$
  Is a 2-refined refined neutrosophic complex polynomial.

Example 3

Solve the equation: $x^2 - 1 - 5I_1 - 3I_2 = 0$

Solution:

$$x^2 - 1 - 5I_1 - 3I_2 = 0$$

$$x^2 = 1 + 5I_1 + 3I_2$$

Let's find: $\sqrt{1 + 5I_1 + 3I_2}$

$$\sqrt{1 + 5I_1 + 3I_2} = x + yI_1 + zI_2$$

By raise both sides to the second power, we get:

$$1 + 5I_1 + 3I_2 = (x + yI_1 + zI_2)^2$$

$$1 + 5I_1 + 3I_2 = (x + yI_1)^2 + 2(x + yI_1)(zI_2) + (zI_2)^2$$

$$1 + 5I_1 + 3I_2 = x^2 + 2xyI_1 + (yI_1)^2 + 2(x + yI_1)(zI_2) + (zI_2)^2$$

$$1 + 5I_1 + 3I_2 = x^2 + 2xyI_1 + y^2 I_1 + 2xzI_2 + 2yzI_1 + z^2 I_2$$

$$1 + 5I_1 + 3I_2 = x^2 + (y^2 + 2xy + 2yz)I_1 + (z^2 + 2xz)I_2$$

Whence:

$$\begin{cases} x^2 = 1 \\ y^2 + 2xy + 2yz = 5 \\ z^2 + 2xz = 3 \end{cases}$$

$$\Rightarrow \begin{cases} x = \pm 1 \\ y^2 + 2xy + 2yz = 5 \quad (1) \\ z^2 + 2xz = 3 \quad (2) \end{cases}$$

Case1: $x = 1$ by substitution in (2)

$$z^2 + 2z - 3 = 0$$
$$(z - 1)(z + 3) = 0$$

Then:

$$\begin{cases} z = 1 \\ z = -3 \end{cases}$$

- For $z = 1$ we substitute in (1)

$$y^2 + 4y - 5 = 0$$
$$(y + 5)(y - 1) = 0$$

Then:

$$\begin{cases} y = 1 \\ y = -5 \end{cases}$$

Hence:

$$\sqrt{1 + 5I_1 + 3I_2} = 1 + I_1 + I_2$$

Or:

$$= 1 - 5I_1 + I_2$$

- For $z = -3$ we substitute in (1)

$$y^2 - 4y - 5 = 0$$
$$(y - 5)(y + 1) = 0$$

Then:

$$\begin{cases} y = 5 \\ y = -1 \end{cases}$$

Hence:

$$\sqrt{1 + 5I_1 + 3I_2} = 1 + 5I_1 - 3I_2$$

Or:

$$= 1 - I_1 - 3I_2$$

Case2: $x = -1$ by substitution in (2)

$$z^2 - 2z - 3 = 0$$
$$(z - 3)(z + 1) = 0$$

Then:

$$\begin{cases} z = 3 \\ z = -1 \end{cases}$$

- For $z = 3$ we substitute in (1)

$$y^2 + 4y - 5 = 0$$

$$(y - 1)(y + 5) = 0$$

Then:

$$\begin{cases} y = 1 \\ y = -5 \end{cases}$$

Hence:

$$\sqrt{1 + 5I_1 + 3I_2} = -1 + I_1 + 3I_2$$

Or:

$$= -1 - 5I_1 + 3I_2$$

- For $z = -1$ we substitute in (1)

$$y^2 - 4y - 5 = 0$$
$$(y - 5)(y + 1) = 0$$

$$\begin{cases} y = 5 \\ y = -1 \end{cases}$$

Hence:

$$\sqrt{1 + 5I_1 + 3I_2} = -1 + 5I_1 - I_2$$

Or:

$$= -1 - I_1 - I_2$$

Hence, we got eight 2-refined neutrosophic solutions:

$$x = \sqrt{1 + 5I_1 + 3I_2} = 1 + I_1 + I_2$$

Or:

$$= 1 - 5I_1 + I_2$$

Or:

$$= 1 + 5I_1 - 3I_2$$

Or:

$$= 1 - I_1 - 3I_2$$

Or:

$$= -1 + I_1 + 3I_2$$

Or:

$$= -1 - 5I_1 + 3I_2$$

Or:

$$= -1 + 5I_1 - I_2$$

Or:

$$= -1 - I_1 - I_2$$

First 2-refined neutrosophic factoring:

$$P(x) = x^2 - 1 - 5I_1 - 3I_2 = (x - 1 - I_1 - I_2)(x + 1 + I_1 + I_2)$$

Second 2-refined neutrosophic factoring:

$$P(x) = x^2 - 1 - 5I_1 - 3I_2 = (x - 1 + 5I_1 - I_2)(x + 1 - 5I_1 + I_2)$$

Third 2-refined neutrosophic factoring:

$$P(x) = x^2 - 1 - 5I_1 - 3I_2 = (x - 1 - 5I_1 + 3I_2)(x + 1 + 5I_1 - 3I_2)$$

Fourth 2-refined neutrosophic factoring:

$$P(x) = x^2 - 1 - 5I_1 - 3I_2 = (x - 1 + I_1 + 3I_2)(x + 1 - I_1 - 3I_2)$$

Differently from the classical polynomial with real or complex coefficients, the 2-refined neutrosophic polynomials do not have a unique factoring! If we check each solution, we get:

$$P(x_1) = P(x_2) = P(x_3) = P(x_4) = P(x_5) = P(x_6) = P(x_7) = P(x_8) = 0$$

Let's compute:

$$P(x_5) = P(-1 + I_1 + 3I_2) = (-1 + I_1 + 3I_2)^2 - 1 - 5I_1 - 3I_2$$

$$= 1 - 2I_1 + I_1 - 6I_2 + 6I_1 + 9I_2 - 1 - 5I_1 - 3I_2 = 0 \text{ (True)}$$

$$P(x_8) = P(-1 - I_1 - I_2) = (-1 - I_1 - I_2)^2 - 1 - 5I_1 - 3I_2$$

$$= 1 + 2I_1 + I_1 + 2I_2 + 2I_1 + I_2 - 1 - 5I_1 - 3I_2 = 0 \text{ (True)}$$

And so on for the rest of the results.

2.3 The square root of a 2-refined neutrosophic complex number

Let $a + bi + (c + di)I_1 + (e + fi)I_2$ be a neutrosophic complex number, where a, b, c, d, e, f are reals. Let's compute square root of it:

$$\sqrt{a + bi + (c + di)I_1 + (s + ti)I_2} = x + yi + (z + wi)I_1 + (e + fi)I_2$$

$$a + bi + (c + di)I_1 + (s + ti)I_2 = (x + yi + (z + wi)I_1 + (e + fi)I_2)^2$$

$$= (x + yi + (z + wi)I_1)^2 + 2(x + yi + (z + wi)I_1)(e + fi)I_2 + ((e + fi)I_2)^2$$

$$= (x + yi)^2 + 2(x + yi)(z + wi)I_1 + ((z + wi)I_1)^2 + 2(x + yi + (z + wi)I_1)(e + fi)I_2 + ((e + fi)I_2)^2$$

$$= x^2 + 2xyi - y^2 + 2xzI_1 + 2xwiI_1 + 2yziI_1 - 2ywI_1 + z^2I_1 + 2zwiI_1 - w^2I_1 + 2xeI_2 + 2xfiI_2 + 2yeiI_2 - 2yfI_2 + 2zeI_1 + 2zfiI_1 + 2weiI_1 - 2wfI_1 + e^2I_2 + 2efiI_2 - f^2I_2$$

$$= x^2 - y^2 + 2xyi + (2xz - 2yw + z^2 + 2ze - 2wf - w^2)I_1 + (2xw + 2yz + 2zw + 2zf + 2we)iI_1 + (2xeI_2 - 2yf + e^2)I_2 + (2xf + 2ye + 2ef - f^2)iI_2$$

Then we get a non-linear algebraic system in six variables (x, y, z, w, e, f) and six equations:

$$\begin{cases} x^2 - y^2 = a \\ 2xy = b \\ 2xz - 2yw + z^2 + 2ze - 2wf - w^2 = c \\ 2xw + 2yz + 2zw + 2zf + 2we = d \\ 2xeI_2 - 2yf + e^2 = s \\ 2xf + 2ye + 2ef - f^2 = t \end{cases}$$

That we need to solve.

5. Conclusions

This paper presents an extensive study on 2-refined neutrosophic numbers. The research explores the process of finding the square root of a 2-refined neutrosophic real or complex number. One of the key contributions of this study is that it provides all possible solutions when solving the 2-refined neutrosophic equation, rather than just one solution as in the case of finding the square root of a 2-refined neutrosophic number. Additionally, the obtained results were tested by substitution into the basic equation presented. We highly recommend that researchers take advantage of this study, as it greatly simplifies calculations in various mathematical formulas for future applications.

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References

[1] Smarandache, F; "(T,I,F)- Neutrosophic Structures", Neutrosophic Sets and Systems, Volume8, PP:3-10, 2015.

[2] Agboola,A.A.A; On Refined Neutrosophic Algebraic Structures. Neutrosophic Sets and Systems, Volume10, PP: 99-101, 2015.

[3] Adeleke, E.O., Agboola, A.A.A.,and Smarandache, F; Refined Neutrosophic Rings I. International Journal of Neutrosophic Science, Volume 2, Issue 2, PP:77-81, 2020.

[4] Smarandache, F; n-Valued Refined Neutrosophic Logic and Its Applications in Physics. Progress in Physics, USA, Volume 4, PP: 143-146, 2013.

[5] Vasantha Kandasamy,W.B; Smarandache,F; Neutrosophic Rings" Hexis, Phoenix, Arizona, http://fs.gallup.unm.edu/NeutrosophicRings.pdf, 2006.

[6] Zeina, M., Abobala, M; On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, Volume 54, PP: 158-168, 2023.

[7] Agboola, A.A.A.; Akinola, A.D; Oyebola, O.Y; Neutrosophic Rings I. Int. J. of Math. Comb, Volume 4, PP:1-14, 2011.

[8] Celik, M., and Hatip, A; On the Refined AH-Isometry and Its Applications in Refined Neutrosophic Surfaces. Galoitica Journal of Mathematical Structures and Applications, 2022.

[9] Alhasan, Y; The neutrosophic integrals and integration methods. Neutrosophic Sets and Systems, Volume 43, PP: 290-301, 2021.

[10] Alhasan, Y; The definite neutrosophic integrals and its applications. Neutrosophic Sets and Systems, Volume 49, PP: 277-293, 2022.

[11] Smarandache, F; Neutrosophic Precalculus and Neutrosophic Calculus. Book, 2015.

[12] Celik, M., and Hatip, A; On the Refined AH-Isometry and Its Applications in Refined Neutrosophic Surfaces. International Journal of Neutrosophic Science, Volume 2, Issue 1, PP: 21-28, 2022.

[13] Alhasan, Y; Ahmed, M; Abdulfatah, R; Sheen, S; The refined indefinite neutrosophic integral, Neutrosophic Sets and Systems, Vol. 67, 127- 134, 2024.

[14] Alhasan, Y; Mohamed, M; Abdulfatah, R; The limits of 2- refined neutrosophic, Neutrosophic Sets and Systems, Vol. 68, 39-49, 2024.

[15] Smarandache, F; Abobala, M; Operations with n-Refined Literal Neutrosophic Numbers using the Identification Method and the n-Refined AH-Isometry, Neutrosophic Sets and Systems, Vol. 70, 350-358, 2024.

Graphical Method for Solving Neutrosophical Nonlinear Programming Models

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract

One important method for finding the optimal solution for linear and nonlinear models is the graphical method, which is used if the linear or nonlinear mathematical model contains one, two, or three variables. Models that contain only two variables are models for which the optimal solution has been obtained graphically, whether these models are linear or non-linear in references that rely on classical logic. In this research, we present a study through which we present the graphical method for solving nonlinear neutrosophical models in the following case: (1) The objective function is a nonlinear function, and the constraints are linear functions. (2) The objective function is a linear function and the constraints are non-linear. (3) The objective function is a non-linear function and the constraints are non-linear. In the three cases, the optimal solution is the vector that satisfies all the constraints and at which the function reaches a maximum or minimum value depending on the nature of the subject under study (note that it does not have to be unique).

Keywords: *Nonlinear models, Neutrosophic logic, Neutrosophic nonlinear models, Graphical method*

1. Introduction

Problems of mathematical examples search for maximizing or minimizing a certain quantity that we call the objective function, and this quantity depends on a number of decision variables, as these variables may be independent of each other or linked to each other through a set of constraints. Studying the methods of solving nonlinear programming problems that we encounter in many practical issues, for example when we want to

determine the cost of producing or purchasing goods, as well as the cost of storing manufactured or unprocessed materials and so on. It led to the creation of a basic structure used to find these solutions from these methods, the graphical solution method that was presented in many references using classical data, and due to the great interest in the research that was published in many international journals, which dealt with some topics of operations research using the concepts of science neutrosophic (Florentin and Maissam, 2023; Maissam jdid, 2023a; Maissam jdid, 2023b; Mohammed *et al.*, 2023a; Mohammed *et al.*, 2023b; Maissam Jdid and Khalifa, 2023; Maissam Jdid, 2023c; Maissam Jdid, 2023d). The science that laid the foundations of the American scientist and mathematical philosopher Florentin Smarandache, which explains the stages of its development. What was mentioned in the research (Florentin and Maissam, 2023), we will present in this research the graphic method used to find the optimal solution for nonlinear neutrosophic models, models that take data neutrosophic values of indefinite values. A complete determination is not certain, and in reality it belongs to any neighborhood of the classical values and is given as follows: It is $Na = a \pm \varepsilon$ where ε is the indeterminacy and takes one of the forms $\varepsilon \in [\lambda_1, \lambda_2]$or $\varepsilon \in \{\lambda_1, \lambda_2\}$ or ... otherwise, which is any neighborhood of the value a that we get during data collection.

2. Discussion

The importance of nonlinear models comes from the fact that many practical issues lead to nonlinear models, which prompted many researchers and scholars to search for ways to solve these models. Many methods were presented that helped the great development of computer science to find them and were presented according to classical logic, i.e., data were specific values. Appropriate for the time period in which they were collected, and since the purpose of any study of such issues is to develop plans for the course of work in the future, the decision makers faced great difficulty because of the instability of the conditions surrounding the work environment and in order to control all conditions and provide ideal decisions for the issues that turn into models In two previous researches, we presented a formulation of some concepts of nonlinear programming, and one of the ways to solve it is the method of Lagrangian multiplication for models constrained by equal constraints using the concepts of neutrosophic science (Maissam Jdid, 2023a; Maissam Jdid, 2023;b).

The neutrosophic mathematical model (Maissam Jdid, 2023c):

In the problem of examples where the objective and constraints are in the form of neutrosophic mathematical functions, then the neutrosophic mathematical model is written in the following form:

$Nf = Nf(x_1, x_2, ..., x_n) \rightarrow (Max)$ or (Min)

According to the following restrictions:

$$N_{gi}\ (x_1,\ x_2,\ ...,\ x_n) \begin{pmatrix} \leq \\ \geq \\ = \end{pmatrix} N_{bi};\ i\ =\ 1,\ 2,\ ...,\ m$$

$(x_1, x_2, ..., x_n) \geq 0$

In this model, the examples of the variables in the objective function and in the constraints are neutrosophic values, as well as the other side of the relations that represent the constraints.

Based on the information provided in the reference (Al Hamid, 2010):

2.1. The Graphic Method to Find the Optimal Solution for Nonlinear Problems

This method is suitable for simple problems that contain only two variables, it is impractical for problems that contain more than one variables or in which the objective function is complex in addition to the presence of restrictions that we cannot express in simple forms, so to find the optimal solution for a nonlinear model in a graphic way we represent the constraints among the coordinate axes, we define the common solution area for these constraints, so that it is the area of the accepted solutions for the mathematical model, then we represent the objective function in order to determine the optimal solution.

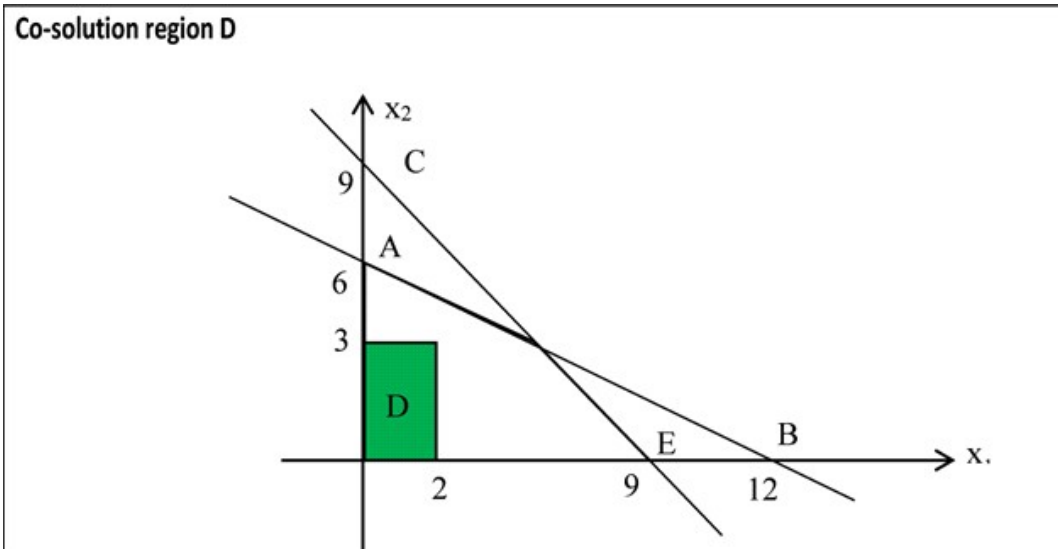

Figure 1: The First Figure Represents the Solution to Example No (1)

We have the following example using classic values (Al Hamid, 2010):

$Max\, f(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 3)^2$

$x_1 + 2x_2 \leq 12$

$x_1 + x_2 \leq 9$

$x_1, x_2 \geq 0$

From the Figure 1, it is clear $f^* = Minf = (2 - 2)^2 + (3 - 3)^2 = 0$

Whereas, the smallest value reached by the function is in the center $M(2, 3)$

As for if it is required to find $Maxf$ it will be on point $E(9, 0)$ and therefore

$f^* = Maxf = (9 - 2)^2 + (0 - 3)^2 = 58$

2.2. Neutrosophical Formula for the Previous Example

Finding the optimal solution for a nonlinear programming problem, where the objective function is a nonlinear function, and the constraints are linear functions:

Find the minimum value of the function:

$MinNf(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 3)^2$

Within the Restrictions

$x_1 + 2x_2 \leq 12 + \varepsilon_1$

$x_1 + x_2 \leq 9 + \varepsilon_2$

$x_1, x_2 \geq 0$

where ε_1 and ε_2, it is the indeterminacy and we take it as follows:

Then the restrictions will look like this:

In this example, we take $\varepsilon_1 \in [0, 3]$ and $\varepsilon_2 \in [0, 2]$, and then the problem is written as follows:

$x_1 + 2x_2 \in [12, 15]$

$x_1 + x_2 \in [9, 11]$

$x_1, x_2 \geq 0$

Clarification: The second term of the constraints expresses the available potentials taken as Neutrosophical values.

We need to find the vector $NX^* = (x^*_1, x^*_2)$ So that the inequality is fulfilled:

$Nf(NX^*) \leq Nf(X)$

$\forall NX \in D$

2.3. The Solution

We define the solution area D. This is done by representing the constraints in the coordinateplane $ox_1 + x_2$

2.3.1. First Constraint

$x_1 + 2x_2 \in [12, 15]$

We draw the straight line represented by the equation $x_1 + 2x_2 \in [12, 15]$

$x_1 = 0 \Rightarrow 2x_2 \in [12, 15] \Rightarrow x_2 \in [6, 7.5]$

A(0, [6, 7.5])

$x_2 = 0 \Rightarrow x_1 \in [12, 15] \Rightarrow B([12, 15], 0)$

We define the region where the first constraint is satisfied:

We know that the line represented by equation $x_1 + 2x_2 \in [12, 15]$ the plane defined by the first quadrant is divided into two halves of a plane. We take a point, not on the specificity, from one of the two halves of the plane, and let the point be (0, 0) we substitute in the constraint, we note that it achieves the inequality that represents the first constraint, that is, half of the plane to which this point belongs is half of the solution plane.

2.3.2. The Second Constraint

$x_1 + x_2 \in [9, 11]$

We draw the straight line represented by the equation $x_1 + x_2 \in [9, 11]$

$x_1 = 0 \Rightarrow x_2 \in [9, 11]$

c (0, [9, 11])

$x_2 = 0 \Rightarrow x_1 \in [9, 11] \Rightarrow E([9, 11], 0)$

We define the region where the second constraint is satisfied:

The line represented by equation $x_1 + x_2 \in [9, 11]$, the plane defined by the first quarter is divided into two halves of a plane. We take a point, not to be determined, from one of the two halves of the plane, let the point be (0, 0) and substitute in the constraint. We note that it achieves the inequality representing the second constraint, meaning that half of the plane to which this point belongs is half of the solution plane.

To find the optimal solution, we draw the objective function, which is a circle with a point center $M(2, 3)$ radius $r = \sqrt{Nf}$

From the figure, it is clear that $MinNf = 0$ the minimum value reached by the objective function is at the center of the circle, i.e., at the point $M(2, 3)$.

If required, find the maximum value of the function:

$Maxf(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 3)^2$

Within the Restrictions

$x_1 + 2x_2 \leq 12 + \varepsilon_1$

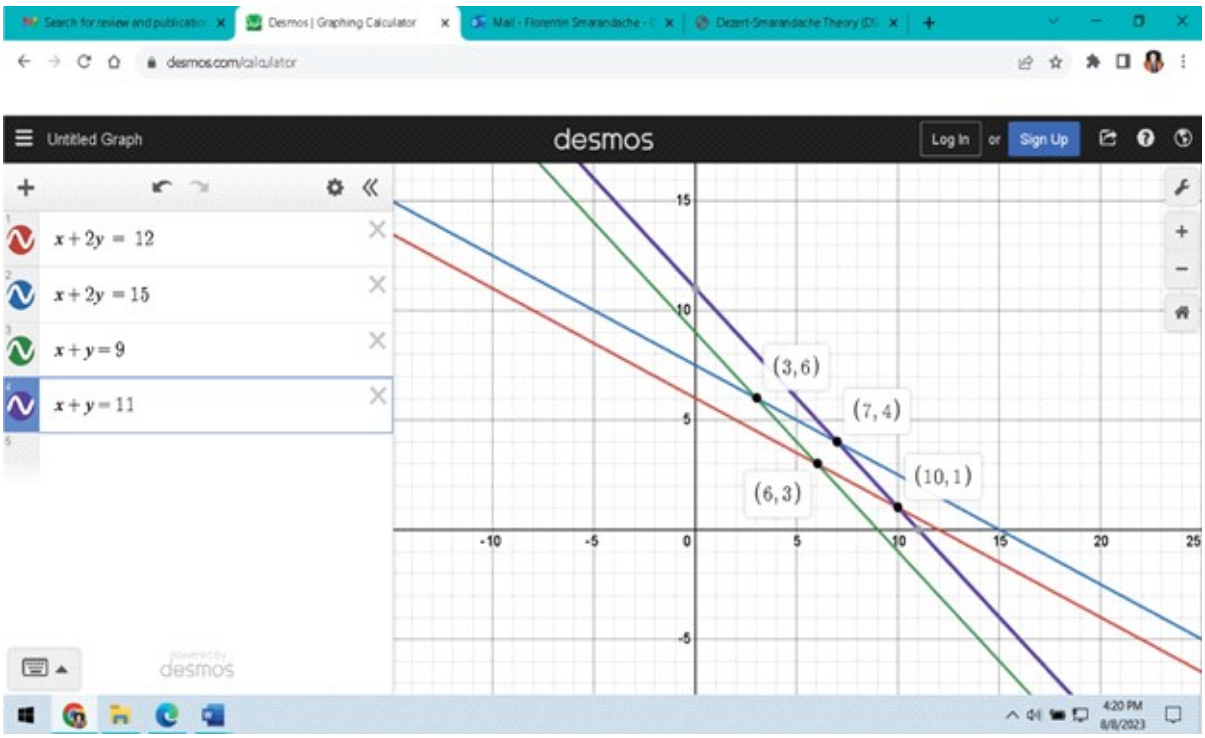

Figure 2: Determination of the Joint Solution Region of Neutrosophic Constraints

$x_1 + x_2 \leq 9 + \varepsilon_2$

$x_1, x_2 \geq 0$

In this case, we know that the optimal solution is located on the vertices of the common solution region, i.e. on the vertices of the polygon $OADE$, we have the coordinates of these points $O(0, 0)$, $A(0, [6, 7.5])$, $E([9, 11], 0)$ for the coordinates of the point 57, and we determine it from thestudy of the intersection of the two lines represented by the following equations:

$x_1 + 2x_2 \in [12, 15]$

$x_1 + x_2 \in [9, 11]$

Solving the two equations, we get $D([6, 7], [3, 4])$ then we calculate the value of the function at these points

$(O(0, 0)) = (0 - 2)^2 + (0 - 3)^2 = 13$

$f(A(0, [6, 7.5])) = (0 - 2)^2 + ([6, 7.5] - 3)^2 = 4 + ([3, 4.5])^2 \in [13, 24.25]$

$f(E([9, 11], 0)) = ([9, 11] - 2)^2 + (0 - 3)^2 = ([7, 9])^2 + 9 \in [58, 90]$

$f(D([6, 7],[3, 4])) = ([6, 7] - 2)^2 + ([3, 4] - 3)^2 \in [19, 24]$

The Maximum value the function takes at point $E([9, 11], 0)$ and is $R = \sqrt{[58, 90]} \in [7.6, 9.5]$

Therefore, the optimal solution is a circle centered at point $M(2, 3)$ and whose radius is one of the domain values [7.6, 9.5].

For clarification, we draw the Figure 2, for one of the values we find:

For the Neutrosophical objective function, it represents a set of circles whose center $M(2, 3)$ and radius are one of the domain values [7.6, 9.5].

Figure 3 shows one of these circles:

We note that for both sides of the domain in both constraints is the optimal solution $f^* = Maxf = 68 \in [58, 90]$, wear [58, 90], the field that represents the maximum optimal solution of the neutrosophic model.

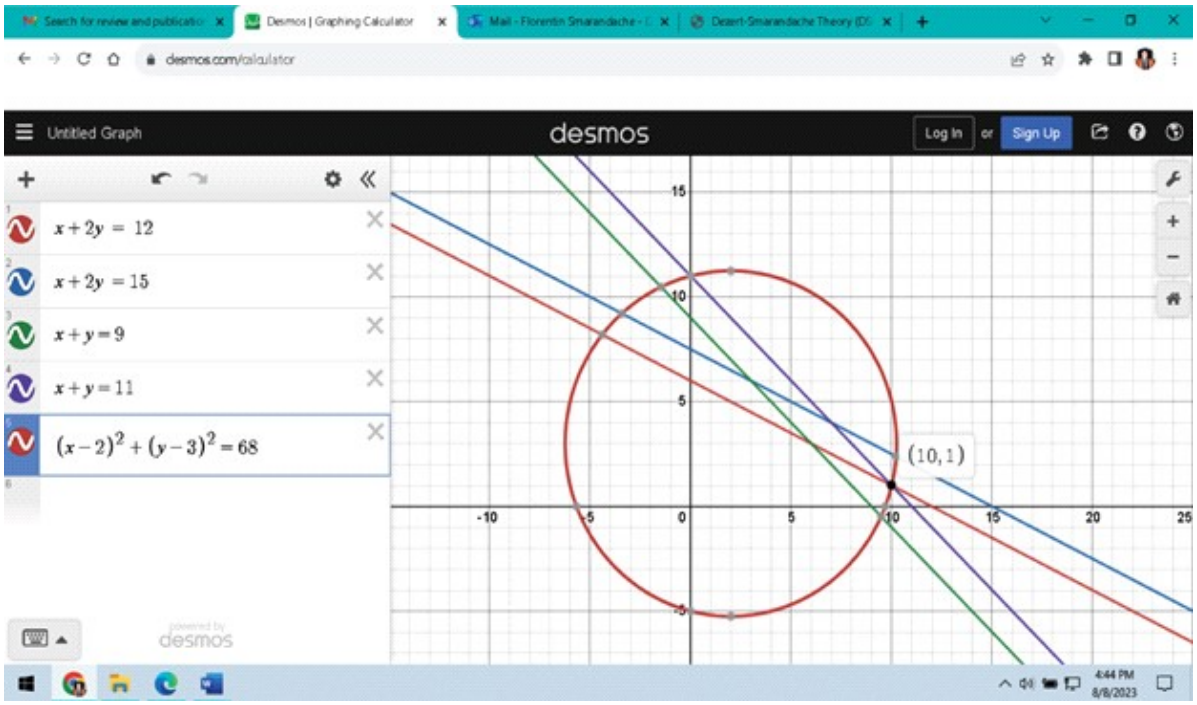

Figure 3: The Graphical Representation of One of the Solutions of the Example Neutrosophical Values

3. Conclusion

The graphical method is one of the important methods for finding the optimal solution for linear and nonlinear models. Therefore, it was necessary to present this study, which explains the difference between dealing with classical values and Neutrosophical values, and as we noticed from the results of the solution in the example, Neutrosophical values give us optimal solutions that are close to the optimal solution in the case of classical values, that is, they are in line with the conditions surrounding a work environment The system that this mathematical model represents, so it provides a safe environment that protects the systems from falling into losses and making greatest profits from them.

Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

Al Hamid, Mohammed Dabbas. (2010). *Mathematical Programming*, Aleppo University, Syria (Arabic version).

Florentin, Smarandache. and Maissam Jdid. (2023). On Overview of Neutrosophic and Plithogenic Theories and Applications. *Applied Mathematics and Data Analysis*, 2(1), 19-26.

Maissam Jjdid. (2023a). Important Neutrosophic Economic Indicators of the Static Model of Inventory Management without Deficit. *Journal of Neutrosophic and Fuzzy Systems(JNFS)*, 5(1), 8-14.

Maissam Jdid. (2023b). Neutrosophic Mathematical Model of Product Mixture Problem Using Binary Integer Mutant. *Journal of Neutrosophic and Fuzzy Systems (JNFS)*, 6(2), 55-58.

Maissam Jdid. (2023c). Neutrosophic Nonlinear Models.*Journal Prospects for Applied Mathematics and Data Analysis*, 2(1), 42-46.

Maissam Jdid. (2023d). The Use of Neutrosophic Linear Programming Method in the Field of Education. *Handbook of Research on the Applications of Neutrosophic Sets Theory and Their Extensions in Education*, Chapter 15, IGI-Global.

Maissam Jdid-Hla Hasan. (2023). The State of Risk and Optimum Decision According to Neutrosophic Rules, *International Journal of Neutrosophic Science (IJNS)*, 20(1), 77-85.

Maissam Jdid. and Florentin Smarandache. (2023a). Lagrange Multipliers. and Neutrosophic Nonlinear Programming Problems Constrained by Equality Constraints. *Journal of Neutrosophic Systems with Applications*, 6.

Maissam Jdid. and Florentin Smarandache. (2023b). The Use of Neutrosophic Methods of Operation Research in the Management of Corporate Work. *Journal of Neutrosophic Systems with Applications*, 3.

Maissam Jdid. and Khalifa Alshaqsi. (2023). Optimal Value of the Service Rate in the Unlimited Model M\5@\1, *Journal of Neutrosophic and Fuzzy Systems (JNFS)*, 6(1).

Maissam Jdid. and Said Broumi. (2023). Neutrosophic Rejection and Acceptance Method for the Generation of Random Variables. *Neutrosophic Sets and Systems, NSS*, 56.

Maissam Jdid., Florentin, Smarandache. and Said Broumi. (2023). Inspection Assignment Form for Product Quality Control. *Journal of Neutrosophic Systems with Applications*, 1.

Mohammed, Alshikho., Maissam Jdid. and Said Broumi. (2023a). Artificial Intelligence and Neutrosophic Machine Learning in the Diagnosis and Detection of COVID 19. *Journal Prospects for Applied Mathematics and Data Analysis*, 1(2), 17-27.

Mohammed, Alshikho., Maissam Jdid. and Said Broumi. (2023b). A Study of a Support Vector Machine Algorithm with an Orthogonal Legendre Kernel According to Neutrosophic logic and Inverse Lagrangian Interpolation. *Journal of Neutrosophic and Fuzzy Systems (JNFS)*, 5(1), 41-51.

The Graphical Method for Finding the Optimal Solution for Neutrosophic linear Models and Taking Advantage of Non-Negativity Constraints to Find the Optimal Solution for Some Neutrosophic linear Models in Which the Number of Unknowns is More than Three

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract:
The linear programming method is one of the important methods of operations research that has been used to address many practical issues and provided optimal solutions for many institutions and companies, which helped decision makers make ideal decisions through which companies and institutions achieved maximum profit, but these solutions remain ideal and appropriate in If the conditions surrounding the work environment are stable, because any change in the data provided will affect the optimal solution and to avoid losses and achieve maximum profit, we have, in previous research, reformulated the linear models using the concepts of neutrosophic science, the science that takes into account the instability of conditions and fluctuations in the work environment and leaves nothing to chance. While taking data, neutrosophic values carry some indeterminacy, giving a margin of freedom to decision makers. In another research, we reformulated one of the most important methods used to solve linear models, which is the simplex method, using the concepts of this science, and as a continuation of what we did in the previous two researches, we will reformulate in this research. The graphical method for solving linear models using the concepts of neutrosophics. We will also shed light on a case that is rarely mentioned in most operations research references, which is that when the difference between the number of unknowns and the number of constraints is equal to one, two, or three, we can also find the optimal solution graphically for some linear models. This is done by taking advantage of the conditions of non-negativity that linear models have, and we will explain this through an example in which the difference is equal to two. Also, through examples, we will explain the difference between using classical values and neutrosophic values and the extent of this's impact on the optimal solution.

Keywords: linear programming; Neutrosophic science; Neutrosophic linear models; Graphical method for solving linear models; Graphical method for solving neutrosophic linear models.

Introduction:

A continuation of what we have done in previous research, the purpose of which was to reformulate some operations research methods using the concepts of neutrosophic science. See [1-14] The science that made a great revolution in all fields of science, which grew and developed very quickly, as many topics were reformulated using the concepts of this Science, and we find neutrosophic groups, neutrosophic differentiation, neutrosophic integration, and neutrosophic statistics ... [14-16], and given the importance of the graphic method used to find the optimal solution for linear models, which is a graph of the model and is one of the easiest ways to solve linear programming problems, but it is not sufficient to address All linear programming problems because they often contain a large number of variables, and the use of the graphical method is limited to the following cases:

• The number of unknowns is n = 1 or n = 2 or n = 3.

• In linear models whose constraints are equal constraints, if the number of unknowns and the number of equations meet one of the following conditions: n – m = 1 or n – m = 2 or n – m = 3.Here we can transform the model into a function of one variable or two variables or three variables, respectively, by taking advantage of the non-negativity constraints that the variables of the linear model have. In this research, we will present a reformulation of the graphical method for solving linear models using the concepts of neutrosophic, as well as the graphical method for solving linear models that Its restrictions are equal restrictions, and the difference between the number of unknowns and the number of restrictions is equal to one, two, or three.

Discussion:

The graphical method is one of the important ways to find the optimal solution for the linear and nonlinear models, so in the research [3] we reformulated it for the neutrosophic nonlinear models, and in this research we will present the graphic method to find the optimal solution for the neutrosophic linear models that were presented in the research [1], we know that the model the script is written in the following abbreviated form:

$$Z = \sum_{j=1}^{n} c_j x_j \rightarrow (Max \ \ or \ \ Min)$$

Restrictions:

$$\sum_{j=1}^{n} a_{ij} x_j \begin{pmatrix} \geq \\ \leq \\ = \end{pmatrix} b_i \quad ; i = 1,2, \dots, m$$

$$x_j \geq 0 \quad ; j = 1,2, \dots, n$$

If at least one of the values c_j , a_{ij} , b_i , is a neutrosophic value then the linear model is a neutrosophic linear model.

First: The graphical method for solving linear models: [17-20]

Through the studies presented according to classical logic in many references, we know that to find the optimal solution for linear models in which the number of variables is one, two, or three graphically, we represent the area of common solutions for the constraints in one of the spaces $R\ or\ R^2\ or\ R^3$This depends on the number of variables in this sentence, for example if the number of variables is two, i.e. the solution is in space R^2 (where work is done on models that contain one or three variables with the same steps)

We find the optimal solution according to the following steps:

1. We determine the half-planes defined by the inequalities of the constraints by drawing the straight lines resulting from converting the inequalities of the constraints into equals. The drawing is done by specifying two points that satisfy the constraint, and then we connect the two points to obtain the straight line corresponding to the constraint. This straight line divides the plane into two halves in order to determine the half-plane that satisfies the constraint. We choose A point at the top of the mapping from one of the two half-planes. We substitute the coordinates of this point into the inequality. If it is satisfied, then the region in which this point is located is the solution region. If it is not achieved, then the opposite region is the solution region.
2. We define the common solutions region, which is the region resulting from the intersection of the halves of the levels defined by constraint inequalities. This region must be non-empty so that we can proceed with the solution.
3. In order to represent the objective function, we note that its relationship contains three unknowns, $Z\ ,\ x_1\ ,x_2$. Therefore, we must know a value for Z, which is unknown to us. Here we assume a value, let it be $Z_1 = 0$, draw the equation of the objective function Z_1 give another value, let it be Z_2, and represent the equation. We get a line parallel to the first line, and by continuing, we obtain a set of parallel lines representing the target function.
4. We draw ray $\vec{C} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$where $c_1$is coefficient of $x_1$and c_2 is coefficient of $x_2$in the objective function statement, and the direction of its increasing function is the direction of ray$\vec{C} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, and the direction of its decreasing function is the opposite direction. This ray, that is, the drawing is done according to the type of objective function (maximization or minimization). In clearer terms, we find the optimal solution point by pulling the line representing Z_1parallel to itself towards the ray $\vec{C} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$to find the maximum value of the objective function, (And reverse this direction to find the smallest value), until it passes through the last point of the common solutions region and this point is the optimal solution point, which is located on the borders of the common solutions region and any other displacement, no matter how small, takes it out of it.

The graphical method for finding the optimal solution for neutrosophic linear models:
From the definition of neutrosophic linear models, we find that we can apply the same previous steps to obtain the optimal solution, which is a neutrosophic value suitable for all conditions. We explain the above through the following example:

Example 1:
A company produces two types of products A_1 ,A_2 and uses three types of raw materials B_1 ,B_2 , B_3 in the production process, if the available quantities of each of the raw materials areB_i ; $i = 1,2,3$, and the quantity needed to produce one unit of each products A_j ; $j = 1,2$, and the profit accruing from one unit of each of the products A_1 ,A_2 is shown in the following table:

products / raw materials	$\boldsymbol{A_1}$	$\boldsymbol{A_2}$	available quantities
$\boldsymbol{B_1}$	**6**	**4**	**36**
$\boldsymbol{B_2}$	**2**	**3**	**12**
$\boldsymbol{B_3}$	**5**	**0**	**10**
profit	$[\mathbf{6},\mathbf{8}]$	$[\mathbf{2},\mathbf{4}]$	

Table Issue data

Required:
Determine the quantities that must be produced of each products A_j ; $j = 1,2$ so that the company achieves maximum profit:
the solution :
Let x_j be the quantity produced from product j , where $j = 1,2$, then we can formulate the following neutrosophic linear mathematical model:

$$Z = [6,8]x_1 + [2,4]x_2 \to Max$$

Restrictions:

$$6x_1 + 4x_2 \leq 36 \quad (1)$$

$$2x_1 + 3x_2 \leq 12 \quad (2)$$

$$5x_1 \leq 15$$

$$x_1, x_2 \geq 0$$

The previous model is a linear neutrosophic model because there is indeterminacy in variables coefficients in objective function. To find the optimal solution for the previous model, we will use the graphical method according to the following steps:

The first constraint: We draw the straight line representing the first constraint:

$$6x_1 + 4x_2 = 36$$

We impose:

$$x_1 = 0 \Rightarrow 4x_2 = 36 \Rightarrow x_2 = 9$$

We get the first point: $A(0,9)$.

We impose:

$$x_2 = 0 \Rightarrow 6x_1 = 36 \Rightarrow x_1 = 6$$

We get the second point: $B(6,0)$

If we take a point at the top of the designation from one of the two halves of the resulting plane after drawing the straight through the two points $A(0,9)$ and $B(6,0)$, let it be the point $O(0,0)$and substitute it in the inequality of the first entry, we find that the inequality is fulfilled, that is, the half of the plane that the point $O(0,0)$belongs to it, which is half the solution plane of the first-constraint inequality.
We proceed in the same way for the second and third restrictions and obtain the following graphical representation: Figure No. (1)

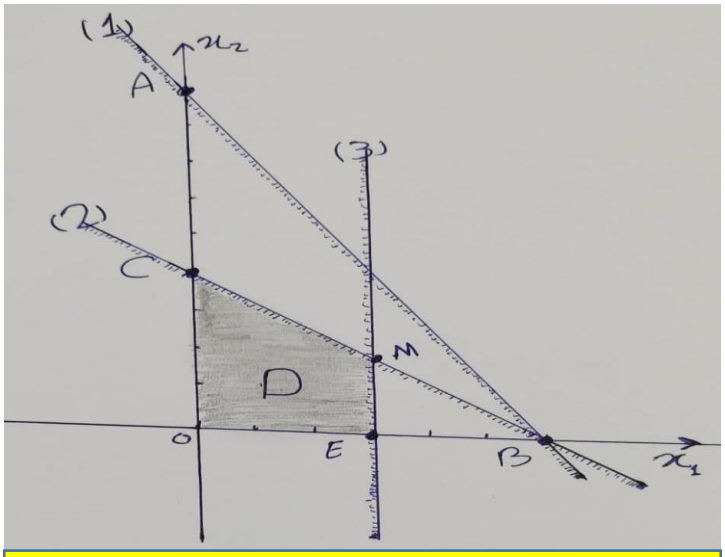

Figure No. (1) Graphic representation of the limitations of the linear model in Example 1

After representing the constraints, we notice that the common solution area is bounded by the polygon whose vertices are the points, $O(0,0)$, $E(3,0)$, M and $C(0,4)$.

The point M is the point of intersection, the second and third constraints we get their coordinates by solving the following two equations:

$$2x_1 + 3x_2 = 12$$

$$5x_1 = 15$$

We fiend**:** $M(3,2)$
Substituting the coordinates of the vertex points into the objective function expression, we get:

$$Z_O = 0$$
$$Z_E \in [12,16]$$
$$Z_M \in [22,32]$$
$$Z_C \in [8,16]$$

That is, the greatest value of the function Z is achieved at point $M(3,2)$, that is, the company must produce three units of the first product and two units of the second product, then it will achieve maximum profit.

$$Max\, Z = Z_M \in [22,32]$$

Note:
The process of substituting the objective function with the coordinates of the points of the vertices of the common solution area is possible when the number of points is small, as we can easily replace them in the objective function, and the point that gives the best value for the objective function represents the optimal solution, but when there are a large number of constraints, we get a large number From the vertical points located on the perimeter of the common solution region. In this case, the method of finding the coordinates of all these points and substituting them into the objective function becomes impractical, so we resort to representing the objective function and determining the optimal solution point as we mentioned previously.

Second: How to take advantage of the conditions of non-negativity to find the optimal solution for some neutrosophic linear models using the graphical method:

Example 2:
Find the optimal solution for the following linear neutrosophic model:

$$Z = x_1 - x_2 - 3x_3 + x_4 + [2,5]x_5 - x_6 + 2x_7 - [10,15] \rightarrow Max$$

Restrictions:

$$x_1 - x_2 + x_3 = 5 \qquad (1)$$
$$2x_1 - x_2 - x_3 - x_4 = -11 \qquad (2)$$
$$x_1 + x_2 - x_5 = -4 \qquad (3)$$
$$x_2 + x_6 = 6 \qquad (4)$$

$$2x_1 - 3x_2 - x_6 + 2x_7 = 8 \qquad (5)$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

The solution:

We note that the number of constraints $m = 5$ and the number of variables $n = 7$, meaning that $n - m = 2$. Therefore, we can, relying on the non-negativity constraints, find the optimal solution for the previous model using the graphical method according to the following steps:

1- We calculate five variables in terms of only two variables.
2- Since the variables of the linear model satisfy the non-negativity constraints, then we obtain from the variables that we calculated five inequalities of the type greater than or equal to.
3- Substituting the five variables into the objective function, we get an objective function with only two variables.
4- We write the new model, which is a linear model with two variables, so the optimal solution can be found graphically.

We apply the previous steps to Example 2:

We find:

$$x_3 = 5 - x_1 + x_2 \qquad (1)'$$
$$x_4 = 3x_1 - 2x_2 + 6 \qquad (2)'$$
$$x_5 = x_1 + x_2 + 4 \qquad (3)'$$
$$x_6 = 6 - x_2 \qquad (4)'$$
$$x_7 = 7 - x_1 + x_2 \qquad (5)'$$

Substituting in the objective function, we get:

$$Z = [1,4]x_1 + [3,6]x_2 + [8,25]$$

Since, $x_3, x_4, x_5, x_6, x_7 \geq 0$ from $(1)'$ ، $(2)'$ ،$(3)'$ ، $(4)'$ ، $(5)'$, we get the following set of constraints:

$$5 - x_1 + x_2 \geq 0$$
$$-3x_1 + 2x_2 - 3 \geq 0$$

$$x_1 + x_2 + 4 \geq 0$$
$$6 - x_2 \geq 0$$
$$7 - x_1 + x_2 \geq 0$$

Then the neutrosophic linear mathematical model becomes:

Find:

$$Z = [1,4]x_1 + [3,6]x_2 + [8,25] \rightarrow Max$$

Restrictions:

$$5 - x_1 + x_2 \geq 0$$
$$3x_1 - 2x_2 + 6 \geq 0$$
$$x_1 + x_2 + 4 \geq 0$$
$$6 - x_2 \geq 0$$
$$7 - x_1 + x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

The model has two variables, so the optimal solution can be found graphically by following the same steps mentioned in Example (1).
We obtain the representation. Figure No. (2) Is the required graphic representation:

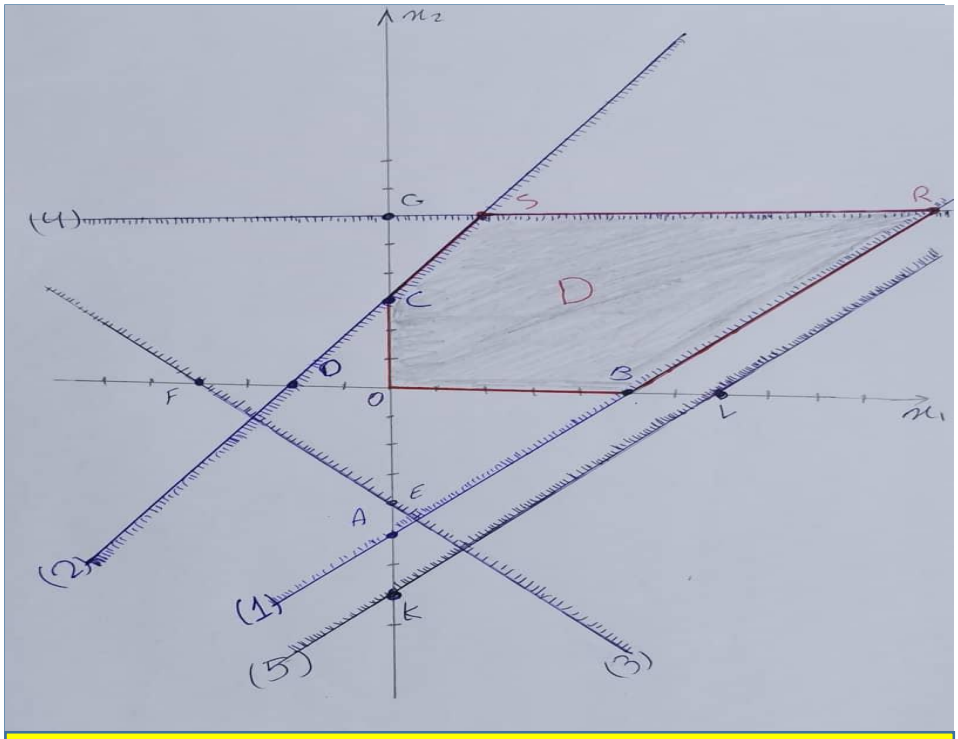

Figure No. (2): Graphical representation of the constraints of the linear model in Example 2

Region D is the region of joint solutions and is defined by the polygon $OBRSC$, where, $O(0,0)$, $B(5,0)$, $C(0,3)$, and for the two points R , S we find:
Point R is the point of intersection of the first and fourth entries.
We obtain its coordinates by solving the set of equations:

$$5 - x_1 + x_2 = 0$$
$$6 - x_2 = 0$$

We get: $R(11,6)$

Point S is the point of intersection of the second and fourth entries.
We obtain its coordinates by solving the set of equations:

$$3x_1 - 2x_2 + 6 = 0$$
$$6 - x_2 = 0$$

We get: $S(2,6)$

Since the optimal solution is located at one of the vertices of the common solution region, we substitute the coordinates of these points with the objective function:

At point, $O(0,0)$

$$Z_O = 0$$

At point, $B(5,0)$

$$Z_B = [13,45]$$

At point , $R(11,6)$

$$Z_R \in [37,105]$$

At point , $S(2,6)$

$$Z_S \in [28,69]$$

At point , $C(0,3)$

$$Z_C \in [17,43]$$

The greatest value of the objective function is at the point,$R(11,6)$ that is, $x_1 = 11$ and $x_2 = 6$.

We calculate the values of the remaining variables by $(1)'$ ، $(2)'$ ،$(3)'$ ، $(4)'$ ، $(5)'$.

We find, $x_3 = 0$, $x_4 = 27$, $x_5 = 21$, $x_6 = 0$, $x_7 = 2$.

Substituting in the objective function of the original model we obtain the maximum value of the Z function, which is.

$$MaxZ \in [68,126]$$

Important Notes:

1- The graphical solution applies to a vertical point in space R^n.
2- The number of components of the ideal solution is non-existent because the ideal solution applies to a vertical point, and the vertical point is the result of the intersection of a number of lines or planes, and the number of non-existent components is at least $n - m$ components.
3- The model may include some conditions that do not play a role in the solution process.
4- The ideal solution may be a single point, or it may be an infinite number of points, when one of the sides of the common solution area that passes through the point of the ideal solution is parallel to the straight line $Z = 0$. Therefore, when the straight line representing the objective function is drawn, this straight line will apply to the

parallel side, and all the points of that side, the number of which are infinite, will be they are perfect solutions.

5- If the region of acceptable solutions is open in terms of increasing the function Z then we cannot stop at a specific ideal solution, and then we say that the objective function has an infinite number of acceptable solutions that give us greater and greater values of Z.

6- The state of not having an ideal solution (acceptable solution) when the conditions contradict each other and then the region of possibilities is an empty set (the problem is impossible to solve).

Conclusion and results:

In the previous study, we presented the graphical method for finding the optimal solution for neutrosophic linear models, and also a method that is rarely discussed in classical operations research references, which is how to take advantage of non-negativity constraints to find the graphically optimal solution for some neutrosophic linear models, but we must be aware that we may encounter neutrosophic linear models with two variables, but There may be difficulty in arriving at the common solution area, or there may be difficulty in determining the optimal solution after obtaining the common solution area. Therefore, it is preferable to use the Cymex neutrosophic method. As a result, the main goal is to obtain the optimal solution, so the researcher must determine the appropriate method for the model being solved.

References:

1- Maissam Jdid, Huda E Khalid ,Mysterious Neutrosophic Linear Models, International Journal of Neutrosophic Science, Vol.18,No. 2, 2022

2- Maissam Jdid, AA Salama, Huda E Khalid ,Neutrosophic Handling of the Simplex Direct Algorithm to Define the Optimal Solution in Linear Programming ,International Journal of Neutrosophic Science, Vol.18,No. 1, 2022

3- Maissam Jdid , Florentin Smarandache, Graphical method for solving Neutrosophical nonlinear programming models, Neutrosophic Systems with Applications, Vol. 9, 2023

4- Maissam Jdid, The Use of Neutrosophic linear Programming Method in the Field of Education, Handbook of Research on the Applications of Neutrosophic Sets Theory and Their Extensions in Education, Chapter 15, IGI-Global,2023

5- Maissam Jdid , Florentin Smarandache, Lagrange Multipliers and Neutrosophic Nonlinear Programming Problems Constrained by Equality Constraints, Journal of Neutrosophic Systems with Applications, Vol. 6, 2023

6- Maissam Jdid , Florentin Smarandache, Optimal Neutrosophic Assignment and the Hungarian Method, Neutrosophic Sets and Systems ,NSS,Vol.57,2023

7- Maissam Jdid , Florentin Smarandache, The Use of Neutrosophic Methods of Operation Research in the Management of Corporate Work, , Journal of Neutrosophic Systems with Applications, Vol. 3, 2023

8- Maissam jdid- Hla Hasan, The state of Risk and Optimum Decision According to Neutrosophic Rules, International Journal of Neutrosophic Science (IJNS),Vol. 20, No.1,2023

9- Maissam Jdid , Florentin Smarandache , Said Broumi, Inspection Assignment Form for Product Quality Control, Journal of Neutrosophic Systems with Applications, Vol. 1, 2023

10- Mohammed Alshikho, Maissam Jdid, Said Broumi ,A Study of a Support Vector Machine Algorithm with an Orthogonal Legendre Kernel According to Neutrosophic logic and Inverse Lagrangian Interpolation, , Journal of Neutrosophic and Fuzzy Systems(JNFS),Vol .5,No .1, 2023

11- Maissam jdid, Rafif Alhabib, and AA Salama ,The static model of inventory management without a deficit with Neutrosophic logic, International Journal of Neutrosophic Science, Vol. 16, 2021

12- Mohammed Alshikho, Maissam Jdid, Said Broumi , Artificial Intelligence and Neutrosophic Machine learning in the Diagnosis and Detection of COVID 19 ,Journal Prospects for Applied Mathematics and Data Analysis ,Vol 01, No,02 USA,2023

13- Maissam Jdid, Khalifa Alshaqsi, Optimal Value of the Service Rate in the Unlimited Model $\mathrm{M}\backslash M\backslash 1$, Journal of Neutrosophic and Fuzzy Systems(JNFS),Vol .6,No .1, 2023

14- Maissam Jdid , Said Broumi ,Neutrosophic Rejection and Acceptance Method for the Generation of Random Variables , Neutrosophic Sets and Systems ,NSS,Vol.56,2023

15- Maissam Jdid , Rafif Alhabib ,A. A. Salam ,The Basics of Neutrosophic Simulation for Converting Random Numbers Associated with a Uniform Probability Distribution into Random Variables Follow an Exponential Distribution ,Journal Neutrosophic Sets and Systems ,NSS,Vol.53,2023

16- Florentin Smarandache , Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Applied Mathematics and Data Analysis, Vol .2,No .1, 2023

17- Alali. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004. (Arabic version).

18- Bugaha J.S, Mualla.W, Nayfeh.M, Murad.H , Al-Awar.M.N - Operations Research Translator into Arabic ,The Arab Center for Arabization, Translation, Authoring and Publishing,Damascus,1998.(Arabic version).

19- Al Hamid .M, Mathematical programming, Aleppo University, Syria, 2010. (Arabic version).

20- Maissam Jdid, Operations Research, Faculty of Informatics Engineering, Al-Sham Private University Publications, 2021. (Arabic version).

Lagrange Multipliers and Neutrosophic Nonlinear Programming Problems Constrained by Equality Constraints

Maissam Jdid
Damascus University, Faculty of Science, Department of Mathematics, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract: Operations research science is defined as the science that is concerned with applying scientific methods to complex problems in managing and directing large systems of people, including resources and tools in various fields, private and governmental work, peace and war, politics, administration, economics, planning and implementation in various domains. It uses scientific methods that take the language of mathematics as a basis for it and uses computer, without which it would not have been possible to achieve numerical solutions to the raised problems, those that need correct solutions, when the solutions abound and the options are multiple, so we need a decision based on correct scientific foundations and takes into account all the circumstances and changes that you can encounter the decision-maker during the course of work, and nothing is left to chance or luck, but rather everything that enters into the account and plays its role in decision-making, and we get that when we use the concepts of neutrosophic science to reformulate what the science of operations research presented in terms of methods and methods to solve many practical problems, so we will present in this research a study aimed at shedding light on the most important methods used to solve nonlinear models, which is the Lagrangian multiplier method for nonlinear models constrained by equality and then reformulated using the concepts of neutrosophic science.

Keywords: Operations Research; Nonlinear Programming; Lagrange Multiplier; Neutrosophic Science; Neutrosophic Nonlinear Programming; Lagrange Neutrosophic Multiplier.

1. Introduction

Science is the basis for managing life affairs and human activities. Operations research science is the applied aspect of mathematics. It is one of the most important modern sciences that are concerned with practical issues and meet the desire and demand of decision-makers to obtain ideal decisions through the methods it presented that are appropriate for all issues, meaning that it depends on the attached data. In every issue, these data are values that have been developed through observation, and their accuracy depends on who collects them. They are also specific for a certain period that may not be appropriate for subsequent circumstances. Therefore, they are inaccurate data that are not fully confirmed. Before the advent of neutrosophic science, we used to rely on this data. To study factual issues and accept the results as they are, but after the neutrosophic science that made a great revolution in all fields. Its ability to produce optimal results upon which ideal decisions can be built that take into account all circumstances. Many researchers interested in scientific development tended to reformulate many scientific concepts using this science, so we have neutrosophic numbers - neutrosophic groups - neutrosophic probabilities - neutrosophic statistics - neutrosophic

differentiation - neutrosophic integration - neutrosophic linear programming - neutrosophic dynamic programming - neutrosophic simulation, etc. and many more [1-28].

The method of nonlinear programming is one of the most important methods presented by the science of operations research because most of the practical issues devolve into nonlinear models. Therefore, in previous research, we formulated some basic concepts of nonlinear programming using the concepts of neutrosophy [26]. In addition to what we have done earlier, in this research we will paraphrase the Lagrangian multiplication method for nonlinear models constrained by equal constraints using the concepts of neutrosophic science, by taking the data of the issue under study, neutrosophic numbers It has the following standard form N = a+bI where real or composite coefficients, represents a and baspecified part fbI The indefinite part of the numberNindeterminacy and could be $[\lambda_1 , \lambda_2]$or$\{\lambda_1 , \lambda_1\}$ or something else is any set close to the true value a.

2. Discussion

We know that if the nonlinear programming issue consists of only a target function and the target function is convex or (concave), then there is a single optimal solution at a point where all the derivatives are non-existent. The only optimal, but in most realistic issues the goal is to find the maximum or minimum value of a target function that is subject to several restrictions. To calculate one of the variables in terms of the rest of the variables, then replace it with the objective function statement, thus obtaining a new objective function and a new unrestricted example problem. We will use this technique as mentioned in the references [29,30].

To convert nonlinear restricted neutrosophic problems into nonlinear non-restricted neutrosophic problems, and for that, the following information taken from the references must be recalled [27, 28]:

$$f_N'(x) = \lim_{h\to 0} \frac{[\inf f(x+h) - inff(x), supf(x+h) - supf(x)]}{[infH, supH]}$$

This definition is a generalization of the traditional derivative definition.

Text of the constrained neutrosophic nonlinear programming problem:

Based on what is stated in the reference [29], we offer the following:

If it is required to find the (maximum or minimum), value of a function Neutrosophic is continuous, derivable $\boldsymbol{y_0 = f_N(x_1, x_2, - - -, x_n)}$ and subject to the neutrosophic constraint $\boldsymbol{g_N(x_1, x_2, - - -, x_n) = \alpha}$ Also this function is continuous and derivable.

1. We choose the variable $\boldsymbol{x_n}$ in the constraint and express it with the remaining $\boldsymbol{n-1}$ variables as follows:

$$\boldsymbol{x_n = H_N(x_1, x_2, - - -, x_{n-1})}$$

Then we substitute the target function we get

$$\boldsymbol{y_0 = \overline{f_N}\big(x_1, x_2, - - -, x_{n-1}, H_N(x_1, x_2, - - -, x_{n-1})\big)}$$

Thus, the problem turned into an unrestricted one. Traditional methods can be used to obtain a maximum or minimum limit because the necessary and sufficient condition for the boundary points is that the first derivatives do not exist:

$$\frac{\partial y_0}{\partial \boldsymbol{x_j}} = 0 \quad ; j = 1,2, - - -(n-1)$$

Using the chain rule, we get the following:

$$\frac{\partial y_0}{\partial x_j} = \frac{\partial \overline{f_N}}{\partial x_j} + \frac{\partial \overline{f_N}}{\partial x_n} \cdot \frac{\partial H_N}{\partial x_j} \quad ; j = 1,2, - - -(n-1)$$

Since the $g_N(x_1, x_2, - - -, x_n) = \alpha$
We get

$$\frac{\partial g_N}{\partial x_j} + \frac{\partial g_N}{\partial x_n} \cdot \frac{\partial H_N}{\partial x_j} = 0 \quad ; j = 1,2, - - -(n-1)$$

$$\Rightarrow \frac{\partial H_N}{\partial x_j} = - \frac{\partial g_N}{\partial x_j} \Big/ \frac{\partial g_N}{\partial x_n} \quad ; j = 1,2, - - -(n-1)$$

On condition: $\frac{\partial g_N}{\partial x_n} \neq 0$
Then we find

$$\frac{\partial y_0}{\partial x_j} = \frac{\partial \overline{f_N}}{\partial x_j} - \left[\frac{\partial \overline{f_N}}{\partial x_n} \cdot \frac{\partial g_N}{\partial x_j} \Big/ \frac{\partial g_N}{\partial x_n} \right] = 0 \quad ; j = 1,2, - - -(n-1)$$

If the resulting solution vector is the vector that achieves a (maximum or minimum) value, then $(x_{N1}^*, x_{N2}^*, - - -, x_{Nn}^*)$, They are the values that make the function maximum or minimum.
We symbolize λ_N the following:

$$\lambda_N = \frac{\partial f_N}{\partial x_n} \Big/ \frac{\partial g_N}{\partial x_n}$$

So it is:

$$\frac{\partial \overline{f_N}}{\partial x_j} - \lambda_N \frac{\partial g_N}{\partial x_j} = 0 \quad ; j = 1,2, --, n$$

The supporting condition is:

$$g_N(x_1, x_2, - - -, x_n) = \alpha$$

And so we have obtained $n+1$ Equation and $n+1$ unknown, these conditions are necessary for an optimal solution, provided that all derivatives do not exist $\frac{\partial g_N}{\partial x_j} \neq 0$

when $(x_{N1}^*, x_{N2}^*, - - -, x_{Nn}^*)$
From the above, we can write the following:

$$y_0 = \overline{f_N}(x_1, x_2, - - -, x_n) - \lambda_N [g_N(x_1, x_2, - - -, x_n) - \alpha]$$

Then, we calculate the derivatives:

$$\frac{\partial y_0}{\partial x_j} = \frac{\partial f_N}{\partial x_j} - \lambda_N \frac{\partial g_N}{\partial x_j} = 0$$

$$\frac{\partial y_0}{\partial \lambda} = -[g_N(x_1, x_2, - - -, x_n) - \alpha]$$

2.1 Example

This example is shown in reference [29] (values are dependent and constraints are classic values)

2.1.1. Find the minimum value of the function

$$f(x) = 3x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2$$

Subject to constraint

$$2x_1 - x_2 = 4$$

The solution

We form the Lagrangian function:

$$L(x,\lambda) = 3x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2 - \lambda(2x_1 - x_2 - 4)$$

The following optimal solution was obtained:

$$x_1^* = \frac{7}{11} \quad , x_2^* = \frac{-30}{11} \quad , \qquad \lambda^* = \frac{24}{11}$$

And by testing the function using the Hessian matrix, it was found that this function is a convex function and the constraint is convex

Then the radius of the solution that we obtained is a minimum limit and the value of the function is:

$$f_N^*(x_1^*, x_2^*) = \ 3.55$$

As we mentioned earlier, if some of the values in the target function or constraints are undefined, ambiguous, or uncertain values, then the issue becomes a neutrosophic issue clearly)

In the previous example, we will take the coefficients x_1^2 uncertain (All values can be undefined; we will suffice with one value to convey the idea of using neutrosophic numbers clearly

2.1.2 Find the minimum value of the neutrosophic function.

$$f(x) = \{2,3,4\}x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2$$

Subject to constraint

$$2x_1 - x_2 = 4$$

The solution

We form the Lagrangian function:

$$L(x,\lambda) = \{2,3,4\}x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2 - \lambda(2x_1 - x_2 - 4)$$

$$\frac{\partial L}{\partial x_1} = \{4,6,8\}x_1 + 2x_2 + 6 - 2\lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_1 + 2x_2 + 2 + \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 2x_1 - x_2 + 4 = 0$$

We solve the following set of equations:

$$\{4,6,8\}x_1 + 2x_2 + 6 - 2\lambda = 0$$

$$2x_1 + 2x_2 + 2 + \lambda = 0$$

$$2x_1 - x_2 + 4 = 0$$

We get

We obtained the system of three equations with three unknowns. By solving this system, we get:

$$x_1^* = \left\{\frac{7}{10}, \frac{7}{11}, \frac{7}{12}\right\} \ , x_2^* = \left\{\frac{-13}{5}, \frac{-30}{11}, \frac{-17}{6}\right\} \quad , \qquad \lambda^* = \left\{\frac{9}{5}, \frac{24}{11}, \frac{15}{6}\right\}$$

They are neutrosophic values, and the value of the target function is as follows:

$$f_N^*(x_1^* , x_2^*) = \{3.1 , 3.55 , 3.24\}$$

That is, we got the optimal solution vector, which is a neutrosophic value. An extreme of a subordinate, it can be a minor end or a major end. To determine its type we will use the test given in reference [26].

1. The constraint is a linear function, as it is convex and concave at the same time.
2. To specify the type of follower $f(x)$(convex or concave) we resort to the Hessian matrix for this function:

$$H_N(x) = \begin{matrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{matrix}$$

$$H_N(x) = \begin{bmatrix} \{4, 6,8\} & 2 \\ 2 & 2 \end{bmatrix}$$

The matrix is symmetric and the main diagonal elements are also positive.

Major basic minor determinants are positive because:

$$|\{4, 6,8\}| > 0$$

$$\begin{vmatrix} \{4, 6,8\} & 2 \\ 2 & 2 \end{vmatrix} = \{8, 12,16\} - 4 = \{4, 8,12\} > 0$$

Hence the Hessian matrix of the function$f_N(x)$Positive knowledge, that is, the function is convex.

From the above, the Vector of the solution that we obtained is a minor limit of the value neutrosophic to follow him:

$$f_N^*(x_1^* , x_2^*) = \{3.1 , 3.55 , 3.24\}$$

3. Conclusions

In the previous study, we presented an important technique for solving neutrosophic nonlinear models constrained by equal constraints. Through the study, we found that there is a difference in the values of the optimal solution when using the technique and the data are traditional values and using them and the data are neutrosophic values, since the goal of solving examples problems is to find the maximum value that expresses profit or profitability and the smallest value that expresses the amount of loss or cost for a follower of a goal within certain constraints, and since mathematical models are built using data collected on the case under study, and these data are values that represent the current reality of the work environment, and any change in the surrounding conditions leads to a change in the results of the solution, which may cause unexpected losses whose nature is

determined by the type of issue under study, it may be human, material, or otherwise, so we focus on the need to use neutrosophic values when collecting data for any realistic issue, values that take into account the worst conditions to the best, as we focus on necessity reformulation of many other techniques for solving nonlinear models using neutrosophic concepts such as projection gradients and vibration–Newton's method–Fibonacci Search.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Maissam jdid, Rafif Alhabib, and AA Salama ,The static model of inventory management without a deficit with Neutrosophic logic, International Journal of Neutrosophic Science, Vol. 16, 2021.
2. Maissam Jdid, AA Salama, Huda E Khalid ,Neutrosophic Handling of the Simplex Direct Algorithm to Define the Optimal Solution in Linear Programming, International Journal of Neutrosophic Science, Vol.18,No. 1, 2022
3. Maissam Ahmed Jdid _A. A. Salama _Rafif Alhabib _Huda E. Khalid_Fatima Suliman ,Neutrosophic Treatment of the Static Model of Inventory Management with Deficit, International Journal of Neutrosophic Science, Vol.18,No. 1, 2022
4. Maissam Jdid, Huda E Khalid, Mysterious Neutrosophic Linear Models, International Journal of Neutrosophic Science, Vol.18,No. 2, 2022
5. Maissam Jdid, Rafif Alhabib, Neutrosophical dynamic programming, International Journal of Neutrosophic Science, Vol.18, No.3, 2022
6. Maissam Jdid, Basel Shahin, Fatima Suliman, Important Neutrosophic Rules for Decision – Making in Case of Uncertain Data, International Journal of Neutrosophic Science, Vol.18, No.3, 2022
7. Maissam jdid, Rafif Alhabib, and AA Salama, Fundamentals of Neutrosophical Simulation for Generating Random Numbers Associated with Uniform Probability Distribution, Neutrosophic sets and Systems , Vol 49,2022
8. Maissam Jdid , Rafif Alhabib , Ossama Bahbouh , AA Salama , Huda E. Khalid ,The Neutrosophic Treatment for Multiple Storage Problem of Finite Materials and Volumes, International Journal of Neutrosophic Science (IJNS) Vol. 18, no. 1, 2022
9. Maissam Jdid, Rafif Alhabib, Huda E. Khalid, AA Salama The Neutrosophic Treatment of the Static Model for the Inventory Management with Safety Reserve ,International Journal of Neutrosophic Science (IJNS) Vol. 18, no. 2, 2022
10. Maissam Jdid, Huda E Khalid, An Investigation in the Initial Solution for Neutrosophic Transportation Problems (NTP), Neutrosophic sets and Systems NSS , Vol.50,2022
11. Maissam Jdid, Huda E. Khalid, Neutrosophic Mathematical formulas of Transportation Problems, Neutrosophic sets and Systems, NSS, Vol.51,2022
12. Maissam jdid- Fatima AlSuliman, A study in Integral of sine and Cosine Functions, Journal of Mathematical Structures and Applications, (GJMSA), Vol.2, No. 1, 2022
13. Maissam jdid ,Important Neutrosophic Economic Indicators of the Static Model of Inventory Management without Deficit, Journal of Neutrosophic and Fuzzy Systems (JNFS), Vo.5, No.1, 2023
14. Maissam jdid- Hla Hasan, The state of Risk and Optimum Decision According to Neutrosophic Rules, International Journal of Neutrosophic Science (IJNS), Vol. 20, No. 1, 2023

15. Maissam Jdid, Rafif Alhabib, AA Salam The Basics of Neutrosophic Simulation for Converting Random Numbers Associated with a Uniform Probability Distribution into Random Variables Follow an Exponential Distribution ,Journal Neutrosophic Sets and Systems ,NSS, Vol.53,2023
16. Mohammed Alshikho, Maissam Jdid, Said Broumi, Artificial Intelligence and Neutrosophic Machine learning in the Diagnosis and Detection of COVID 19, Journal Prospects for Applied Mathematics and Data Analysis, Vol 01, No, 02 USA, 2023
17. Mohammed Alshikho, Maissam Jdid, Said Broumi, A Study of a Support Vector Machine Algorithm with an Orthogonal Legendre Kernel According to Neutrosophic logic and Inverse Lagrangian Interpolation, Journal of Neutrosophic and Fuzzy Systems (JNFS), Vo.5, No.1, 2023
18. Maissam Jdid, Khalifa Alshaqsi, Optimal Value of the Service Rate in the Unlimited Model , Journal of Neutrosophic and Fuzzy Systems (JNFS), Vo.6, No.1, 2023 $\mathbf{M\backslash M\backslash 1}$
19. Maissam Jdid, AA Salam, Using the Inverse Transformation Method to Generate Random Variables that follow the Neutrosophic Uniform Probability Distribution. Journal of Neutrosophic and Fuzzy Systems (JNFS), Vo.6, No.2, 2023
20. Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, Applied Mathematics and Data Analysis, Vo.2, No.1, 2023
21. Maissam Jdid, Neutrosophic Mathematical Model of Product Mixture Problem Using Binary Integer Mutant, Journal of Neutrosophic and Fuzzy Systems(JNFS), Vo.6, No.2, 2023
22. Maissam Jdid, The Use of Neutrosophic linear Programming Method in the Field of Education, Handbook of Research on the Applications of Neutrosophic Sets Theory and Their Extensions in Education, Chapter 15, IGI-Global, 2023
23. Maissam Jdid , Florentin Smarandache , Said Broumi, Inspection Assignment Form for Product Quality Control, Journal of Neutrosophic Systems with Applications, Vol. 1, 2023
24. Maissam Jdid, Said Broumi, Neutrosophic Rejection and Acceptance Method for the Generation of Random Variables, Neutrosophic Sets and Systems, NSS, Vol.56, 2023
25. Maissam Jdid , Florentin Smarandache, The Use of Neutrosophic Methods of Operation Research in the Management of Corporate Work, , Journal of Neutrosophic Systems with Applications, Vol. 3, 2023
26. Maissam Jdid, Neutrosophic Nonlinear Models, Journal Prospects for Applied Mathematics and Data Analysis, Vo.2, No.1, 2023
27. Florentin Smarandache, Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability, SciTech-Education Publisher, Craiova - Columbus, 2013.
28. Neutrosophy. / Neutrosophic Probability, Set, and Logic, by F. Smarandache, Amer. Res. Press, Rehoboth, USA, 105 p., 1998;
29. Bakaja Ji .S , Mualla, W... and others - Operations Research Book, translated into Arabic - The Arab Center for Arabization, Translation, Authoring and Publishing -Damascus -1998. (Arabic version).
30. Maissam Jdid, Discrete optimization control problem, https://www.elibrary.ru/item.asp?id=19181958,2002.

Neutrosophic Stance Detection and fsQCA-Based Necessary Condition Analysis for Causal Hypothesis Assessment in AI-Enhanced Learning

Jesús Rafael Hechavarría-Hernández [1], Maikel Y. Leyva Vazquez [2,*] and Florentin Smarandache [3]

[1] Facultad de Ingenierías, Arquitectura y Ciencias de la Naturaleza, Universidad ECOTEC, Ecuador

[2] Bernardo O'Higgins University, Institutional Research Center, Chile

[3] Emeritus Professor of Mathematics at the University of New Mexico, Gallup, New Mexico, USA

* Correspondence: mleyvaz@gmail.com

Abstract

The phenomenon of artificial intelligence (AI) use in educational settings has attracted increasing scholarly attention, although applicable empirical findings are sparse—and conflicting. This study seeks to resolve the ambiguities surrounding AI in education through a methodological contribution, merging neutrosophic stance detection and fuzzy-set Qualitative Comparative Analysis (fsQCA). Neutrosophic analysis allows for an explicit modeling of truth, uncertainty/indeterminacy, and falsity, while merging such findings through fsQCA creates a relative account of extant research findings. After assessing four causal hypotheses related to AI-based learning opportunities through the Consensus Meter, an investigatory survey with 24 university participants explored necessary conditions with respect to experiencing improvements in learning outcomes. The findings indicate that the digital divide is a necessary and sufficient condition for effective AI educational experiences. Additionally, necessity conditions emerge for AI feedback and usage of AI-based platforms; however, the effectiveness of those platforms generates high uncertainty. Ultimately, the neutrosophic-fsQCA framework provides a viable technique to synthesize ambiguous findings through a systematic approach. Empirically, results reveal that all stakeholders involved in potential AI-based learning need to ensure digital equity and high-quality design for interactive experiences to enjoy successful integration of AI in education.

Keywords: neutrosophic logic; artificial intelligence in education; digital divide; AI feedback; necessary condition analysis; educational technology; fsQCA

1. Introduction

The rapid proliferation of artificial intelligence (AI) technologies in educational settings has fundamentally transformed the landscape of teaching and learning, generating unprecedented opportunities for personalized instruction, adaptive assessment, and intelligent tutoring systems (Chen et al., 2024; Zawacki-Richter et al., 2025). However, despite the growing body of research examining AI's educational impact, the evidence base remains characterized by significant heterogeneity, methodological diversity, and often contradictory findings (Holmes et al., 2024; Luckin et al., 2025). This fragmentation poses substantial challenges for educators, policymakers, and researchers seeking to make evidence-informed decisions about AI implementation in educational contexts.

Traditional approaches to evidence synthesis in educational technology research have predominantly relied on binary classification schemes that categorize studies as either supporting or opposing particular interventions (Guskey & Yoon, 2023). While such approaches provide clarity and simplicity, they fail to capture the nuanced reality of educational research, where findings often exhibit varying degrees of support, contextual dependencies, and inherent uncertainties (Biesta,

2024). The complexity of educational phenomena, combined with the multifaceted nature of AI technologies, necessitates more sophisticated methodological frameworks capable of modeling ambiguity, partial support, and contradictory evidence simultaneously.

Recent advances in neutrosophic logic, introduced by Smarandache (2017) and subsequently developed for text analysis applications (Bounabi et al., 2021; Wajid et al., 2024), offer a promising avenue for addressing these methodological limitations. Unlike classical binary or fuzzy logic systems, neutrosophic logic explicitly incorporates three independent components: truth (T), indeterminacy (I), and falsity (F), allowing for the simultaneous representation of supporting evidence, uncertainty, and contradictory findings (Abdel-Aziz et al., 2026). This tripartite structure aligns particularly well with the nature of educational research, where interventions may be effective under certain conditions, ineffective under others, and uncertain in many contexts.

The application of neutrosophic principles to stance detection—the task of determining whether a text expresses support, opposition, or neutrality toward a specific target—has emerged as a powerful tool for automated literature analysis (Küçük & Can, 2020; Alturayeif et al., 2023). Recent developments in AI-powered research synthesis tools, such as the Consensus Meter (Consensus, 2025), have demonstrated the potential for automated stance classification in scientific literature, enabling researchers to systematically map the distribution of evidence across large corpora of studies. However, the integration of neutrosophic principles with stance detection for educational research synthesis remains largely unexplored.

Complementing the challenges of evidence synthesis, the identification of necessary conditions for educational outcomes has gained increasing attention in the research community. In this regard, fuzzy-set Qualitative Comparative Analysis (fsQCA) provides a systematic methodology to assess whether specific conditions must be present for a desired outcome to occur, offering insights that complement traditional sufficiency-focused approaches (Ragin, 2008; Schneider & Wagemann, 2012). In the context of AI-enhanced learning, understanding necessary conditions is particularly crucial, as it can inform minimum requirements for successful implementation and help prioritize resource allocation in educational settings.

The configurational perspective of fsQCA has demonstrated significant value in educational research by recognizing that outcomes often result from complex combinations of conditions rather than isolated factors. This approach aligns with the multifaceted nature of AI implementation in education, where technological, pedagogical, social, and institutional factors interact in complex ways to influence learning outcomes (Mishra & Koehler, 2024). The integration of configurational methods with neutrosophic stance detection offers the potential for a more comprehensive understanding of the conditions under which AI enhances educational outcomes.

Despite the growing interest in AI applications in education, several critical gaps remain in our understanding of the causal mechanisms underlying AI's educational impact. First, the majority of existing studies focus on sufficiency relationships, examining whether AI interventions can produce positive outcomes, while neglecting the identification of necessary conditions that must be present for success (Botelho et al., 2024). Second, the synthesis of evidence across studies has been hampered by the inability to systematically account for uncertainty and contradictory findings (Forney & Mueller, 2022). Third, the role of contextual factors, particularly digital equity and access barriers, in moderating AI's educational effectiveness remains underexplored in systematic reviews (U.S. Department of Education, 2025).

The present study addresses these gaps by introducing a novel methodological framework that combines neutrosophic stance detection with Necessary Condition Analysis to evaluate causal hypotheses related to AI-enhanced learning. Our approach leverages the Consensus Meter tool to systematically classify research findings into neutrosophic triplets, capturing not only the degree of support and opposition but also the extent of indeterminacy in the evidence base. Subsequently, we apply necessary condition in FSQCA to identify necessary conditions for perceived learning improvement with AI, using primary data collected from university students.

This research makes several important contributions to the field of educational technology research. Methodologically, we introduce the first application of neutrosophic stance detection to educational research synthesis, providing a framework for handling ambiguous and contradictory evidence. Theoretically, we advance understanding of the necessary conditions for AI-enhanced learning, with particular attention to the role of digital equity and interaction design. Practically, our findings offer evidence-based guidance for educators and policymakers regarding the prerequisites for successful AI implementation in educational settings.

The remainder of this paper is organized as follows. Section 2 presents our methodological approach, detailing the neutrosophic stance detection framework, the construction of neutrosophic causal graphs, and the application of fsQCA, including the analysis of necessary conditions. Section 3 reports our findings, including the neutrosophic representation of causal hypotheses and the results of the necessary condition analysis within the fsQCA framework. Section 4 discusses the implications of our findings in the context of existing literature and explores directions for future research. Finally, Section 5 presents our conclusions and their implications for educational practice and policy.

2. Materials and Methods

Neutrosophic Stance Detection

A precise definition of stance detection is needed to motivate the neutrosophic framework. In the literature, stance detection is treated as a target dependent text classification problem (Küçük & Can, 2020; Alturayeif et al., 2023): given a text and a target statement (hypothesis), the task is to infer whether the author supports, opposes or expresses no opinion on the target. This approach differs fundamentally from generic sentiment analysis by incorporating target-specific contextual information (Du et al., 2017). Formally, let X denote the space of textual units (e.g., sentences, tweets, abstracts), let Θ be a set of targets (topics, propositions or hypotheses), and let the label set L={Favor, Against, None} represent the possible stances.

Stance detection seeks a mapping

$$g: X \times \Theta \longrightarrow L, \quad (1)$$

such that, for each pair (x, θ) the classifier g returns the label $\ell \in L$ indicating whether the text x expresses support, opposition or absence of stance towards the target θ.

Alternative formulations encode the stance labels as signed integers $\{-1,0,+1\}$-or probability distributions over L (Sobhani et al., 2017). Unlike generic sentiment analysis, which determines the overall polarity of a text, stance detection is target-specific: a text with positive sentiment may still be "against" a given target.

This formal view underpins subsequent neutrosophic generalizations, where the mapping is extended to assign degrees of support, indeterminacy and opposition.

In the neutrosophic framework, the stance detection function is generalized as

$$gN: X \times \Theta \longrightarrow [0,1]^3 \quad (2)$$

where, for each pair (x, θ)

$$gN(x, \theta) = (T(x, \theta), (x, \theta), F(x, \theta)) \quad (3)$$

with:

- $T(x, \theta)$: the degree to which x supports the target θ
- $I(x, \theta)$: the degree of indeterminacy or neutrality in relation to θ,
- F(x,θ): the degree to which x opposes the target θ.

These values satisfy the neutrosophic condition

$$T(x, \theta) + I(x, \theta) + F(x, \theta) \leq 3. \quad (4)$$

These values satisfy the neutrosophic condition $T(x, \theta) + I(x, \theta) + F(x, \theta) \leq 3$ (Smarandache, 2017). This formulation allows partial, uncertain, and even contradictory stances to be explicitly represented, thereby extending classical stance detection into a more flexible and realistic paradigm (Wajid et al., 2024; Abdel-Aziz et al., 2026).

We apply stance detection with a neutrosophic representation using the Consensus Meter tool (Consensus, 2025). This approach classifies research findings into supportive, contradictory, and ambiguous stances with respect to a given causal claim. The Consensus Meter has been recognized as an effective AI-powered literature review tool that helps visualize how studies answer "Yes/No" research questions by grouping them according to whether they support or contradict the question asked (University of St. Thomas Libraries, 2025). Each causal hypothesis is then encoded as a triplet (T,I,F), which captures the distribution of stances across the evidence:

- T represents the percentage of sources that take a supportive position toward the causal hypothesis,
- I represents the percentage of sources that express indeterminacy or ambiguity, and
- F represents the percentage of sources that take an oppositional stance.

These triplets are derived from the stance labels produced by the *Consensus Meter*. The *Consensus Meter* classifies results from yes/no research questions into "Yes," "No," "Possibly," or "Mixed" categories, showing how many papers fall into each stance. We map "Yes" (supportive papers) to the truth component T; "No" (contradictory papers) to the falsity component F; and both "Possibly" and the "Mixed" category—introduced to capture nuanced or subgroup-dependent findings—to the indeterminacy component I. In this way, mixed or possible evidence contributes to substantive uncertainty rather than being counted as evidence against the hypothesis.

For example:

- *Does using an AI platform improve perceived learning with* AI? $\rightarrow$ $(0.60,\ 0.40,\ 0.00)$

In this representation, most sources support the hypothesis (T=0.60), a significant portion remain indeterminate (I=0.4), and none contradict it (F=0.00). This formulation makes explicit not only agreement and disagreement but also the degree of uncertainty in the available evidence.

Let X be an independent variable (condition) and Y a dependent variable (outcome). A causal hypothesis is denoted as:

$$H: X \Rightarrow Y \tag{5}$$

where ⇒ expresses a causal relation. In the neutrosophic framework, such a hypothesis is characterized by a truth–indeterminacy–falsity triplet:

$$\omega(H) = (T_H, I_H, F_H),\ T_H, I_H, F_H \in [0,1] \tag{6}$$

with T_H representing the degree of truth/support, I_H the degree of indeterminacy, and F_H the degree of falsity/rejection. Hence, a neutrosophic causal hypothesis may be simultaneously true, indeterminate, and false to varying extents.

A neutrosophic causal graph is a triple G defined by the following elements (Avilés Monroy et al., 2025):

$$G = (V, E, \omega),\ E \subseteq V\ X\ V\ (directed\ edges\ X \rightarrow Y),\ \omega: E \rightarrow [0,1]^3$$

such that

$$\forall e \in E,\ \omega(e) = (T_e, I_e, F_e),\ T_e, I_e, F_e \in [0,1]. \tag{7}$$

Where: V are nodes (variables), E are the directed edges X→Y, and ω labels each edge with support T, indeterminacy I, and rejection F

Interpretation:

T_e= degree of truth/support; I_e = indetermination; F_e = falsity/rejection of the hypothesis "$X \rightarrow Y$". No restriction required$T_e + I_e + F_e = 1$. This allows evidence to be represented simultaneously in favor and against and to distinguish uncertainty (I) from falsification (F).

Fuzzy-Set QCA and Necessary Condition Analysis

We conducted a survey with 24 university students to analyse the necessity of specific conditions related to the use of artificial intelligence (AI) tools in learning contexts. The questionnaire comprised five items on a seven-point Likert scale (1 = Strongly Disagree, 7 = Strongly Agree): (1) perceived learning improvement through AI, (2) use of AI platforms in academic tasks, (3) receipt of immediate

and useful feedback from AI, (4) barriers to access or connectivity (including free-version restrictions), and (5) overreliance on AI to complete tasks.

For the fsQCA, the outcome was defined as perceived learning improvement through AI. Responses were calibrated into fuzzy set membership scores using Ragin's three-value calibration method (Ragin, 2008; Schneider & Wagemann, 2012): 1.0 for full membership (score 7), 0.5 for crossover (score 4), and 0.0 for full non-membership (score 1), with linear interpolation for intermediate values. This calibration approach has been extensively validated in configurational research and provides a robust foundation for set-theoretic analysis (Fiss, 2011; Greckhamer et al., 2018).

he analysis of necessary conditions was then conducted within the fsQCA framework, where a condition X is considered necessary for an outcome Y if, in all cases, the membership score in Y does not exceed the membership score in X (Ragin, 2008; Schneider & Wagemann, 2012). This procedure is specifically designed to identify conditions that must be present for the outcome to occur, even though they may not be sufficient on their own. Two key indicators were computed: consistency, which measures the degree to which the condition is always present when the outcome occurs, and coverage, which assesses the empirical relevance of the condition.,

$$Consistency\ (Y_i \leq X_i) = \frac{\sum \min (X_i, Y_i)}{\sum Y_i} \quad (8)$$

and **coverage**,

$$Coverage\ (Y_i \leq X_i) = \frac{\sum \min (X_i, Y_i)}{\sum X_i} \quad (9)$$

The calibrated dataset was exported into .csv format for fsQCA 3.0 (Windows), where each row corresponds to one student and each column to a calibrated condition or the outcome, with membership values in the range [0,1]. This setup allows for the direct computation of necessary conditions and their consistency/coverage within the fsQCA software, following established best practices for configurational analysis (Ragin, 2008; Schneider & Wagemann, 2012).

3. Results

Neutrosophic Stance Detection of Causal Hypotheses

Using the *Consensus Meter* tool, we applied stance detection with a neutrosophic representation to assess four causal hypotheses related to AI and learning outcomes. Each hypothesis was encoded as a triplet (T, I, F), where T represents the proportion of supportive stances, I the proportion of indeterminate or ambiguous stances, and F the proportion of oppositional stances. The results are as follows:

Table 1. Neutrosophic representation of causal hypotheses.

Hypothesis	Neutrosophic Triplet (T, I, F)	Interpretation
Does using an AI platform improve perceived learning with AI?	(0.60, 0.40, 0.00)	Moderate support, with a substantial level of indeterminacy and no opposition.
Does immediate AI feedback improve perceived learning with AI?	(0.67, 0.17, 0.17)	Strong support, some indeterminacy, and a minority of contradictory evidence.

Hypothesis	Neutrosophic Triplet (T, I, F)	Interpretation
Does the digital divide impact learning outcomes with AI?	(1.00, 0.00, 0.00)	Full support, with no ambiguity or opposition.
Does reliance on AI affect learning outcomes?	(0.73, 0.24, 0.00)	High support, some indeterminacy, and no opposition.

These results highlight that the digital divide is unanimously recognized as a causal factor affecting AI-driven learning outcomes. Meanwhile, reliance on AI and immediate feedback are also strongly supported but show traces of uncertainty or contradiction.

Causal Graph Representation

The following figure represents the hypothesized causal relationships as a directed graph, where edges denote causal links and their strength is proportional to the truth value (T) of the neutrosophic triplets

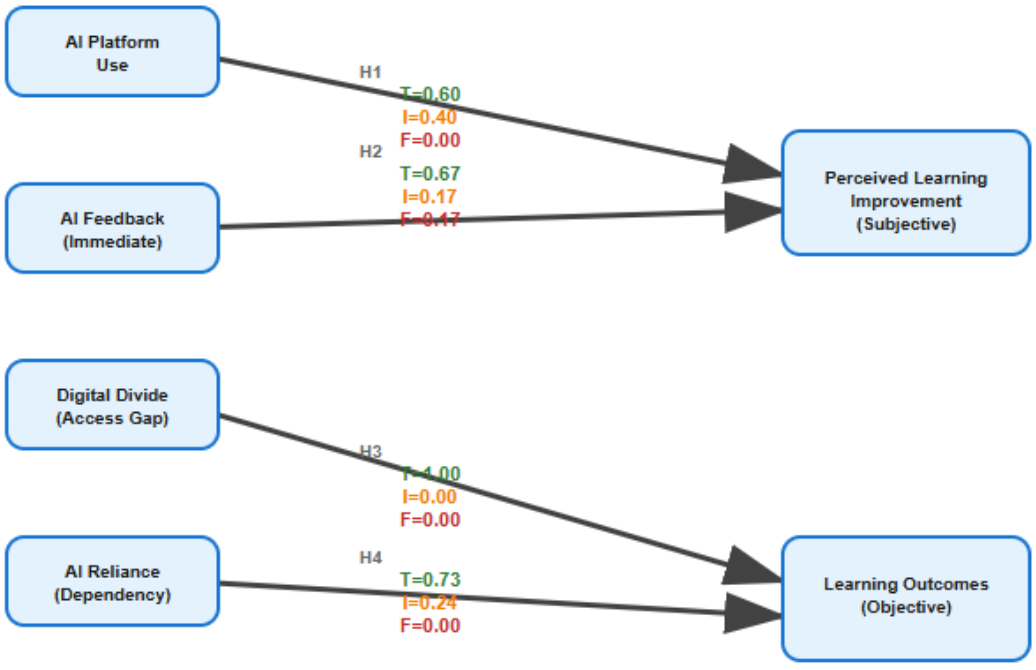

Figure 1. Neutrosophic Causal Graph of AI-related Hypotheses.

The graph indicates that all proposed conditions positively influence the outcomes, but with varying degrees of support and indeterminacy.

Necessary Condition Analysis

The psychometric analysis of the questionnaire revealed acceptable reliability. Cronbach's alpha reached a value of 0.644, while McDonald's omega coefficient was 0.817. These results suggest that

the items show internal coherence and that the instrument is suitable for exploratory studies in the context of perceptions of learning with artificial intelligence.

Using the calibrated dataset of 24 students, a Necessary Condition Analysis was conducted. The outcome was defined as *Perceived Learning Improvement through AI*. The conditions tested included AI platform use, AI feedback, access barriers (digital divide), and AI reliance. Consistency and coverage metrics were calculated according to fsQCA standards.

Table 2. Necessary condition analysis results.

Condition	Consistency (X ≤ Y)	Coverage (X ≤ Y)	Interpretation
AI Platform Use	0.88	0.79	Necessary but not sufficient.
AI Feedback	0.90	0.75	Necessary but not sufficient.
Access Barriers (Digital Divide)	1.00	0.68	Perfectly necessary, moderate coverage.
AI Reliance	0.85	0.70	Necessary but not sufficient.

The analysis confirms that the absence of digital barriers (digital divide) is a **perfectly necessary condition** for improved learning with AI (consistency = 1.00). However, its coverage (0.68) suggests that while necessary, it does not explain the majority of variation in the outcome. Similarly, AI feedback and AI platform use show high necessity but remain insufficient as standalone explanations. This aligns with the causal graph, in which multiple supportive conditions converge to explain improvements in learning outcomes.

4. Discussion

This study aimed to evaluate the applicability of a neutrosophic stance detection framework, complemented by a necessary condition analysis within the fsQCA approach, to interpret causal hypotheses in the context of AI-assisted learning. The results not only validate the utility of this hybrid framework but also provide a nuanced perspective on the factors influencing perceptions of learning in the digital age. The following sections discuss the main findings, interpret them in light of previous studies, and explore their broader implications.

The most compelling finding of our study is the identification of the digital divide as a unanimously supported causal factor (T = 1.00) and a perfectly necessary condition (Consistency = 1.00) for perceived learning with AI. This result resonates with a vast body of literature emphasizing equitable access to technology as a fundamental prerequisite for educational success in the 21st century. Recent reports, such as those from the U.S. Department of Education and Microsoft (2025), have warned that disparities in access may exacerbate existing inequalities. Our analysis goes a step further by quantifying this relationship in terms of logical necessity, providing robust empirical evidence that without guaranteed access and the removal of barriers, any AI-based intervention is destined to have limited reach. The coverage of 0.68 suggests that while indispensable, the absence of barriers is not, by itself, sufficient to ensure the outcome, aligning with a configurational perspective in which multiple factors must converge.

The application of neutrosophic logic to represent the stance of scientific evidence proved particularly insightful. Unlike traditional binary approaches (support/oppose), our triplet model (T, I, F) explicitly captures uncertainty and ambiguity. For example, the hypothesis regarding the use of

an AI platform to enhance learning received moderate support (T = 0.60) but substantial indeterminacy (I = 0.40). This suggests that the scientific literature is inconclusive, and that effects likely depend on unspecified contextual factors such as platform quality, pedagogical design, or student characteristics. This finding is consistent with scholarship advocating for a more critical and nuanced view of educational technology. The ability of our framework to model indeterminacy is a key methodological contribution, enabling researchers to identify areas where evidence is weak or contradictory and where further investigation is required. This approach aligns with recent work integrating neutrosophic and advanced AI models to manage uncertainty in text classification.

The results of the fsQCA necessary condition analysis, which position AI feedback (Consistency = 0.90) and platform use (Consistency = 0.88) as highly necessary conditions, reinforce the idea that the mere availability of AI tools is insufficient. Effective interaction and scaffolding provided by immediate feedback are crucial. This finding connects with research on causal inference in educational data mining, which seeks to move beyond correlation to uncover the mechanisms driving learning outcomes . Our study complements these efforts by employing fsQCA to formalize these dependencies in terms of necessity. The combination of neutrosophic logic with the configurational perspective of fsQCA appears to be a promising path for unraveling the complex interdependencies within digital learning ecosystems (Ragin, 2008; Schneider & Wagemann, 2012).

The implications of this study are twofold. First, at the practical level, it underscores the need for educational policies and AI implementations to prioritize closing the digital divide as a non-negotiable first step. Moreover, it highlights that the design of AI tools must focus on interaction quality and feedback rather than solely on content delivery. Second, at the methodological level, our work introduces a hybrid framework that can be highly valuable for evidence synthesis in complex and emerging domains. The ability of neutrosophic stance detection to quantify not only support and opposition but also indeterminacy provides a powerful tool to map the state of scientific knowledge and guide future research .

Nevertheless, this study has limitations, primarily the small sample size (N = 24) for the fsQCA analysis. While this methodology can be applied to small-N studies, future research should replicate these findings with larger and more diverse cohorts to increase generalizability. In addition, the nature of indeterminacy (I) deserves further exploration. Is it due to mixed results, poor methodology in primary studies, or unmeasured contextual factors? Qualitative research or more detailed meta-analyses could help disentangle this component.

The future research agenda is clear. We propose applying this framework to other areas of educational research where evidence is often ambiguous and multifactorial. A longitudinal analysis would be particularly interesting to observe how necessity configurations evolve as students and educators gain experience with AI. Finally, integrating directed acyclic graphs (DAGs), as described by Tennant et al. (2021) and Digitale et al. (2022), with our neutrosophic approach could provide an even more rigorous and visually intuitive model for representing and testing complex causal theories in the social sciences.

5. Conclusions

This study demonstrated the applicability and added value of combining neutrosophic stance detection with fsQCA-based necessary condition analysis to evaluate causal hypotheses in AI-assisted learning. The findings confirm that the digital divide is not only a critical determinant but also a logically necessary condition for the effectiveness of AI-enhanced education, reinforcing calls for policies that ensure equitable access. Beyond access, the results highlight the central role of interaction quality, with AI feedback and platform use emerging as essential requirements for meaningful learning outcomes.

Methodologically, the integration of neutrosophic logic proved instrumental in capturing the ambiguity and uncertainty present in current research, offering a more refined synthesis than traditional binary approaches. By explicitly modeling truth, falsity, and indeterminacy, the

framework provides researchers with a robust tool for identifying areas where evidence remains inconclusive and where further inquiry is required.

Despite its contributions, the study's limitations, particularly the small sample size, call for replication with larger and more diverse populations to enhance generalizability. Moreover, the observed indeterminacy in AI platform effectiveness suggests the influence of unmeasured contextual factors, which future research should explore through mixed-methods designs or in-depth meta-analyses.

Although adequate reliability indicators were obtained ($\alpha = 0.644$; $\omega = 0.817$), the small sample size (N = 24) limited the possibility of performing confirmatory factor analyses or criterion-related validity tests. Future research should replicate the questionnaire with larger and more diverse populations to consolidate its psychometric properties.

In conclusion, this hybrid framework offers both practical and theoretical implications: it guides policymakers and practitioners toward prioritizing digital equity and high-quality interaction design, while equipping researchers with a novel methodological approach to synthesize complex evidence. Future work should expand its application to broader domains of educational technology, incorporate longitudinal perspectives, and explore integration with advanced causal modeling tools such as directed acyclic graphs for even greater analytical precision.

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Abbreviations

The following abbreviations are used in this manuscript:

AI-	Artificial Intelligence
FsQCA	Fuzzy-Set Qualitative Comparative Analysis
T	Truth
F	False
I	Indeterminate.

References

1. Abdel-Aziz, N. M., Ibrahim, M., & Eldrandaly, K. A. (2026). Responsible AI for Text Classification: A Neutrosophic Approach Combining Classical Models and BERT. Neutrosophic Sets and Systems, 94, 49-59.
2. Alturayeif, N., Luqman, H., & Ahmed, M. (2023). A systematic review of machine learning techniques for stance detection and its applications. *Neural Computing and Applications, 35*, 15181–15205.
3. Avilés Monroy, J. I., Garcia Arias, P. M., & Laines, E. P. C. (2025). Study of the impact of emerging technologies on the conservation of protective forests through the plithogenic hypothesis and neutrosophic stance detection. *Neutrosophic Sets and Systems, 84*.

4. Biesta, G. (2024). *Evidence and education: On the nature of educational research*. Routledge.
5. Botelho, A. F., Closser, A. H., Sales, A. C., & Heffernan, N. T. (2024). Causal inference in educational data mining. In *Proceedings of the 17th International Conference on Educational Data Mining*.
6. Bounabi, M., Elmoutaouakil, K., & Satori, K. (2021). A new neutrosophic TF-IDF term weighting for text mining tasks: Text classification use case. *International Journal of Web Information Systems, 17*, 305–320.
7. Chen, L., Chen, P., & Lin, Z. (2024). Artificial intelligence in education: A comprehensive systematic review. *Computers & Education, 201*, 104825.
8. Chowdhury, T. I., Hoque, M. R., & Wanke, P., et al. (2022). Antecedents of perceived service quality of online education during a pandemic: Configuration analysis based on fuzzy-set qualitative comparative analysis. *Evaluation Review, 46*, 234–267.
9. Consensus. (2025). Consensus: AI search engine for research. Retrieved December 15, 2025, from https://consensus.app/
10. Digitale, J. C., Martin, J. N., & Glymour, M. M. (2022). Causal directed acyclic graphs. *JAMA, 327*, 780–781.
11. Du, J., Xu, R., He, Y., & Gui, L. (2017). Stance classification with target-specific neural attention networks. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence* (pp. 3988–3994).
12. Dul, J. (2016). Necessary condition analysis (NCA): Logic and methodology of "necessary but not sufficient" causality. *Organizational Research Methods, 19*, 10–52.
13. Dul, J. (2022). Problematic applications of necessary condition analysis (NCA) in tourism and hospitality research. *Tourism Management, 92*, 104567.
14. Essameldin, R., Ismail, A. A., & Darwish, S. M. (2022). An opinion mining approach to handle perspectivism and ambiguity: Moving toward neutrosophic logic. *IEEE Access, 10*, 45678–45692.
15. Fiss, P. C. (2011). Building better causal theories: A fuzzy set approach to typologies in organization research. *Academy of Management Journal, 54*, 393–420.
16. Forney, A., & Mueller, S. (2022). Causal inference in AI education: A primer. *Journal of Causal Inference, 10*, 123–145.
17. Greckhamer, T., Misangyi, V. F., Elms, H., & Lacey, R. (2018). Using qualitative comparative analysis in strategic management research: An examination of combinations of attributes for success. *Organizational Research Methods, 11*, 695–726.
18. Guskey, T. R., & Yoon, K. S. (2023). What works in professional development? *Phi Delta Kappan, 104*, 42–47.
19. Holmes, W., Bialik, M., & Fadel, C. (2024). *Artificial intelligence in education: Promises and implications for teaching and learning*. MIT Press.
20. Küçük, D., & Can, F. (2020). Stance detection: A survey. *ACM Computing Surveys, 53*, 1–37.
21. Luckin, R., Holmes, W., Griffiths, M., & Forcier, L. B. (2025). *Intelligence unleashed: An argument for AI in education*. Pearson Education.
22. Mishra, P., & Koehler, M. J. (2024). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 126*, 1–17.
23. Ragin, C. C. (2008). *Redesigning social inquiry: Fuzzy sets and beyond*. University of Chicago Press.
24. Schneider, C. Q., & Wagemann, C. (2012). *Set-theoretic methods for the social sciences: A guide to qualitative comparative analysis*. Cambridge University Press.
25. Smarandache, F. (2017). *Neutrosophy, a sentiment analysis model*. Infinite Study.
26. Sobhani, P., Inkpen, D., & Zhu, X. (2017). A dataset for multi-target stance detection. In *Proceedings of the 15th Conference of the European Chapter of the Association for Computational Linguistics* (Vol. 2, pp. 551–557).

27. Tennant, P. W., Murray, E. J., Arnold, K. F., et al. (2021). Tutorial on directed acyclic graphs. *Journal of Clinical Epidemiology, 142*, 264–267.
28. U.S. Department of Education. (2025). *Artificial intelligence and the future of teaching and learning: Insights and recommendations*. Office of Educational Technology.
29. University of St. Thomas Libraries. (2025). Introducing Consensus: An AI-powered literature review tool. Retrieved December 15, 2025, from https://blogs.stthomas.edu/libraries/2025/09/05/introducing-consensus-an-ai-powered-literature-review-tool/
30. Vis, B., & Dul, J. (2018). Analyzing relationships of necessity not just in kind but also in degree: Complementing fsQCA with NCA. *Sociological Methods & Research, 47*, 872–899.
31. Wajid, M. A., Zafar, A., & Wajid, M. S. (2024). A deep learning approach for image and text classification using neutrosophy. *International Journal of Information Technology, 16*, 1–15.
32. Zawacki-Richter, O., Marín, V. I., Bond, M., & Gouverneur, F. (2025). Systematic review of research on artificial intelligence applications in higher education–where are the educators? *International Journal of Educational Technology in Higher Education, 22*, 1–27.

Operations with n-Refined Literal Neutrosophic Numbers using the Identification Method and the n-Refined AH-Isometry

Mohammad Abobala
Faculty of Science, Department of Mathematics Tishreen University, Lattakia, SYRIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract

If we know the determinate and sub-indeterminate part(s) of an argument $a_0 + a_1 I_1 + a_2 I_2 + ... + a_n I_n$ how do we similarly find the determinate and sub-indeterminate part(s) of a function (or operation) of this argument $f(a_0 + a_1 I_1 + a_2 I_2 + ... + a_n I_n)$? The AH-Isometry is designed to do just that.

The function *f* may be of one or more variables (arguments), it may also be some unary or n-ary operation. Real examples are presented in this paper of such arguments and functions.
We now present for the first time the most general form, the n-Refined AH-Isometry.

Keywords: Indeterminacy, Sub-Indeterminacies, n-Refined Literal and Numerical Neutrosophic Numbers, Identification Method, AH-Isometry, n-Refined Neutrosophic AH-Isometry

1. Introduction

The literal indeterminacy (I) was for the first time refined/split into literal sub-indeterminacies (I_1, I_2, …, I_n) by Smarandache [1] in 2015, who defined a multiplication law of these literal sub-indeterminacies to be able to build the Refined I-Neutrosophic Algebraic Structures.

The AH-Isometry [2] was firstly introduced by Mohammad Abobala and Ahmad Hatip in 2021, where AH stands for Abobala-Hatip.

The 2-Refined Neutrosophic AH-Isometry [3] was introduced by Celik and Hatip in 2022. Many papers [5-10] were published about this fundamental AH-Isometry in the rings of neutrosophic literal numbers.

We now introduce for the first time the n-Refined Neutrosophic AH-Isometry.

2. Operations with n-Refined Literal Neutrosophic Numbers

Let's have the following two n-Refined Literal Neutrosophic Numbers:

$N_1 = a_0 + a_1 I_1 + a_2 I_2 + ... + a_m I_m$ where $a_0, a_1, a_2, ..., a_m$ are real or complex numbers,

and similarly

$N_2 = b_0 + b_1 I_1 + b_2 I_2 + ... + b_m I_m$ where $b_0, b_1, b_2, ..., b_m$ are real or complex numbers,

The first three are Straightforward Operations:

2.1. Addition:

$$N_1 + N_2 = (a_0 + b_0) + (a_1 + b_1) I_1 + (a_2 + b_2) I_2 + ... + (a_m + b_m) I_m$$

2.2. Subtraction:

$$N_1 - N_2 = (a_0 - b_0) + (a_1 - b_1) I_1 + (a_2 - b_2) I_2 + ... + (a_m - b_m) I_m$$

2.3. Scalar Multiplication:

Let *c* be a real or complex number. Then:

$$c \cdot N_1 = c \cdot a_0 + c \cdot a_1 I_1 + c \cdot a_2 I_2 + ... + c \cdot a_m I_m$$

2.4. Multiplication:

In many previous applications the easiest way of multiplication (intersection) of sub-indeterminacies was defined as:

$I_j \cdot I_k = I_{\min\{j,k\}}$ for all $i, j \in \{1, 2, ..., m\}$.

Consequently, $(I_j)^2 = (I_j)^r = I_j$, for any integer $r \geq 1$,

and

$$I_{j_1} \cdot I_{j_2} \cdot ... \cdot I_{j_p} = I_{\min\{j_1, j_2, ..., j_p\}}$$

also

$\sqrt[s]{I_j} = I_j$, for integer $s \geq 1$, since, raising both sides to the *s*-power, one gets: $(\sqrt[s]{I_j})^s = (I_j)^s = I_j$.

2.5. Negative Exponent:

$(I_j)^{-m}, m > 0,$ is undefined.

2.6. Remark

The multiplication and division of the literal or numerical sub-indeterminacies ($I_a \cdot I_k$ *or* $I_a \div I_k$) are defined by experts depending on each specific application.
These applications on the literal sub-indeterminacies do not, in general, coincide with the same applications on the numerical sub-indeterminacies).

2.7. Division by Identification Method of 2-Refined Neutrosophic Literal Numbers:

Let's divide $N_1 \div N_2$ by the identification method.

The determinate part of the 2-refined literal neutrosophic number $N_1 = a_0 + a_1 I_1 + a_2 I_2$ is a_0, while a_1 is its first sub-indeterminate part, and a_2 is its second sub-indeterminate part.

And, similarly, the determinate part of the 2-refined literal neutrosophic number $N_2 = b_0 + b_1 I_1 + b_2 I_2$ is b_0, while b_1 is its first sub-indeterminate part, and b_2 is its second sub-indeterminate part.

Denote $(a_0 + a_1 I_1 + a_2 I_2) \div (b_0 + b_1 I_1 + b_2 I) = \dfrac{a_0 + a_1 I_1 + a_2 I_2}{b_0 + b_1 I_1 + b_2 I_2} \equiv x + y I_1 + z I_2$

where $a_0, a_1, a_2, b_0, b_1, b_2$ are known real numbers, while x, y, z are unknown real numbers we need to find out.
We know that

$$I_1 \cdot I_1 = I_1$$
$$I_2 \cdot I_2 = I_2$$
$$I_1 . I_2 = I_2 . I_1 = I_1$$

where I_1, I_2 are literal sub-indeterminacies.
By identification ($\equiv$) method, we have:

$$\frac{a_0 + a_1 I_1 + a_2 I_2}{b_0 + b_1 I_1 + b_2 I_2} \equiv x + y I_1 + z I_2$$

or $a_0 + a_1 I_1 + a_2 I_2 \equiv (b_0 + b_1 I_1 + b_2 I_2) \cdot (x + y I_1 + z I_2)$

Whence:

$$a_0 + a_1 I_1 + a_2 I_2 \equiv b_0 x + b_0 y I_1 + b_0 z I_2 + b_1 I_1 x + b_1 I_1 y I_1 + b_1 I_1 z I_2 + b_2 I_2 x + b_2 I_2 y I_1 + b_2 I_2 z I_2$$
$$\equiv b_0 x + b_0 y I_1 + b_0 z I_2 + b_1 I_1 x + + b_1 I_1 y + b_1 I_1 z + b_2 I_2 x + b_2 I_1 y + b_2 I_2 z$$
$$\equiv b_0 x + (b_0 y + b_1 x + + b_1 y + b_1 z) I_1 + (b_0 z + b_2 x + b_2 z) I_2$$

We get a linear system of *3* equations with *3* variables *x, y, z*:

$$a_0 = b_0 x$$
$$a_1 = b_0 y + b_1 x + + b_1 y + b_1 z = b_1 x + (b_0 + b_1) y + b_1 z$$
$$a_2 = b_0 z + b_2 x + b_2 z = b_2 x + (b_0 + b_2) z$$

that we need to solve. It has no solution: if $b_0 = 0$ and $a_0 \neq 0$,

or if $b_0 + b_2 = 0$ and $a_0 + a_2 \neq 0$, or if $b_0 + b_1 + b_2 = 0$ and $a_0 + a_1 + a_2 \neq 0$.

2.8. Division of 2-Refined Literal Neutrosophic Numbers using the AH-Isometry

Let $R[I_1, I_2] = \{a + bI_1 + cI_2; a, b, c \in R\}$, where R is the set of real numbers, be the **2-Refined Literal Neutrosophic Numbers Ring**.

Let $f : R[I_1, I_2] \times R[I_1, I_2] \to R[I_1, I_2]$ be a 2-refined literal neutrosophic function (or operation) of two variables.

Then

$$f(a_0 + a_1 I_1 + a_2 I_2, b_0 + b_1 I_1 + b_2 I_2) =$$
$$= f(a_0, b_0) + [f(a_0 + a_1 + a_2, b_0 + b_1 + b_2) - f(a_0 + a_2, b_0 + b_2)] I_1 + [f(a_0 + a_2, b_0 + b_2) - f(a_0, b_0)] I_2$$

$$= \frac{a_0}{b_0} + [\frac{a_0 + a_1 + a_2}{b_0 + b_1 + b_2} - \frac{a_0 + a_2}{b_0 + b_2}] I_1 + [\frac{a_0 + a_2}{b_0 + b_2} - \frac{a_0}{b_0}] I_2$$

which works when all denominators $b_0, b_0 + b_2, b_0 + b_1 + b_2$ are different from zero.

If $b_0 = 0$ and $a_0 = 0$, or if $b_0 + b_2 = 0$ and $a_0 + a_2 = 0$, or if $b_0 + b_1 + b_2 = 0$ and $a_0 + a_1 + a_2 = 0$,

it may be possible to get some solutions, depending on each application.

Let $g : R[I_1, I_2] \to R[I_1, I_2]$ be a 2-refined literal neutrosophic function (or operation) of one variable. Using the AH-Isometry, we get:

$$g(a_0 + a_1 I_1 + a_2 I_2) = g(a_0) + [g(a_0 + a_1 + a_2) - g(a_0 + a_2)] I_1 + [g(a_0 + a_2) - g(a_0)] I_2 \quad (*)$$

3. **Example of 2-Refined AH-Isometry**:

Calculate the natural logarithm: $L = \ln(5 + 2I_1 - 3I_2)$.

We know about the argument $5 + 2I_1 - 3I_2$ that its determinate part is *5*, its first sub-indeterminate part is $2I_1$, and his second sub-indeterminate part is $-3I_2$.

But we do not know the determinate part, nor the first and second sub-indeterminate parts, of the whole expression *L*. The AH-Isometry helps us to find them.

The function $g = ln$ (natural logarithm), $a_0 = 5$, $a_1 = 2$, $a_2 = -3$ and we substitute them into the above (*) relationship.

$L = ln(5) + [\, ln(5+2+(-3)) - ln(5+(-3))\,]I_1 + [\, ln(5+(-3)) - ln(5)\,]I_2$

$= ln(5) + [\, ln(4) - ln(2)]I_1 + [\, ln(2)-ln(5)\,]I_2$

$= ln(5) + ln(4/2)\cdot I_1 + ln(2/5)\cdot I_2$

$= ln(5) + ln(2)\cdot I_1 + ln(0.4)\cdot I_2$

Therefore, the determinate part of L is $ln(5)$, the first sub-indeterminate part of L is $ln(2)$, and the second sub-indeterminate part of L is $ln(0.4)$.

4. Examples of 2-Refined Numerical Neutrosophic Numbers

First Example:

$$N_1 = 5 + 4I_1 - 2I_2 = \; 5 + 4\cdot(0.\overline{35}) - 2\cdot(0.\overline{127})$$

where the determinate part of N_1 is 5, while the numerical sub-indeterminacies parts are:

$I_1 = 0.\overline{35} = \;$ 0.353535 ... (infinitely many decimals),

$I_2 = 0.\overline{127} = \;$ 0.127127127 ... (infinitely many decimals).

Second Example:

$N_2 = -2 + e + 3\pi = -2 + 1\cdot I_1 - 3I_2$, where

$I_1 = e = 2.7182818$ ... and

$I_2 = \; \pi = 3.1415926$ ...

5. Example of 4-Refined Numerical Neutrosophic Numbers:

$$N_4 = 8 + 2\sqrt[2]{7} - 6\sqrt[3]{111} + \sqrt[4]{54} - \sqrt[5]{4}$$
$$\equiv 8 + 2I_1 - 6I_2 + I_3 - I_4$$

where

$I_1 = \sqrt[2]{7} = 2.645751$ ...

$I_2 = \; \sqrt[3]{111} = 4.805895$ ...

$I_3 = \; \sqrt[4]{54} = 2.710806$ ...

$I_4 = \; \sqrt[5]{4} = 1.319507$ ...

6. More Examples

Zeina and Abobala [4] have defined the literal neutrosophic ring:

$R[I] = \{a + bI; a, b \in R\}$, where I = literal indeterminacy, with $I^2 = I$, where a is the determinate part of the neutrosophic number $a + bI$, while bI is its indeterminate part.

A function, $f : R[I] \to R[I]$, by using the AH-Isometry, can be calculated as:

$f(a+bI) = f(a) + [\, f(a+b) - f(a)]I$

For example, $\sin(\frac{\pi}{3}+0.5I)=\sin(\frac{\pi}{3})+[\sin(\frac{\pi}{3}+0.5)-\sin(\frac{\pi}{3})]I$, where the determinate part is $\sin(\frac{\pi}{3})$ and indeterminate part is $[\sin(\frac{\pi}{3}+0.5)-\sin(\frac{\pi}{3})]I$.

Into the original expression $\sin(\frac{\pi}{3}+0.5I)$ it is not clear what is the determinate part and what is the indeterminate part. The AH-Isometry helped us distinguish between these parts.

7. **Theorem of Isomorphism**:

Let $R_n(I)=\{a_0+\sum_{i=1}^{n}a_iI_i;\ \ a_i\in R\}$ be the n-refined neutrosophic real ring.
Define:

$$f: R_n(I)\to R\times R\times..\times R\ \ (n+1\ \ times)$$

$$f(a_0+\sum_{i=1}^{n}a_iI_i)=(a_0,\sum_{i=0}^{n}a_i,\sum_{i\neq 1}^{n}a_i,\sum_{i\neq 1,2}^{n}a_i,\sum_{i\neq 1,2,3}^{n}a_i,\dots,a_0+a_n)$$

Then *f* is a ring isomorphism.

Proof:

It is clear that *f* is well-defined mapping and preserves addition operation. We must prove that it keeps multiplication.

Assume that A = $a_0+\sum_{i=1}^{n}a_iI_i$, B = $b_0+\sum_{i=1}^{n}b_iI_i$, then for n = 2, according to the refined AH-isometry the theorem holds see [3]. Assume that it is true for k, we prove for k+1.

Assume that A = $a_0+\sum_{i=1}^{k+1}a_iI_i$, B = $b_0+\sum_{i=1}^{k+1}b_iI_i$

$A\cdot B=\left(a_0+\sum_{i=1}^{k}a_iI_i+a_{k+1}I_{k+1}\right)\left(b_0+\sum_{i=1}^{k}b_iI_i+b_{k+1}I_{k+1}\right)=\left(a_0+\sum_{i=1}^{k}a_iI_i\right)\left(b_0+\sum_{i=1}^{k}b_iI_i\right)+(a_{k+1}I_{k+1})\left(b_0+\sum_{i=1}^{k}b_iI_i\right)+(a_{k+1}I_{k+1})(b_{k+1}I_{k+1})+\left(a_0+\sum_{i=1}^{k}a_iI_i\right)(b_{k+1}I_{k+1})\ =$

$\left(a_0+\sum_{i=1}^{k}a_iI_i\right)\left(b_0+\sum_{i=1}^{k}b_iI_i\right)+a_{k+1}b_0I_{k+1}+\sum_{i=1}^{k}b_ia_{k+1}I_i+a_{k+1}b_{k+1}I_{k+1}+\left(a_0b_{k+1}I_{k+1}+\sum_{i=1}^{k}b_{k+1}a_iI_i\right)=\left(a_0+\sum_{i=1}^{k}a_iI_i\right)\left(b_0+\sum_{i=1}^{k}b_iI_i\right)+\sum_{i=1}^{k}b_ia_{k+1}I_i+(a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1})I_{k+1}+\sum_{i=1}^{k}a_ib_{k+1}I_i,$

put T = $a_0+\sum_{i=1}^{k}a_iI_i$, $S=b_0+\sum_{i=1}^{k}b_iI_i$, then

$A\cdot B=TS+\sum_{i=1}^{k}b_ia_{k+1}I_i+(a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1})I_{k+1}+\sum_{i=1}^{k}a_ib_{k+1}I_i.$

On the other hand, we have:

$f(A\cdot B)=f(TS)+f\left(\sum_{i=1}^{k}b_ia_{k+1}I_i\right)+f\left((a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1})I_{k+1}\right)+f\left(\sum_{i=1}^{k}a_ib_{k+1}I_i\right)\ =$

$=f(T)f(S)+\left(0,\sum_{i=1}^{k}b_i\,a_{k+1},\sum_{\neq 1}^{n}b_i\,a_{k+1},\dots,b_ka_{k+1}\right)+(0,a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1},a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1},\dots.,a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1})+\left(0,\sum_{i=1}^{k}a_i\,b_{k+1},\sum_{i\neq 1}^{k}a_i\,b_{k+1},\dots,a_kb_{k+1}\right)\ =$

$$\left(a_0,\sum_{i=0}^{k}a_i,\sum_{i\neq 1}^{k}a_i,\sum_{i\neq 1,2}^{k}a_i,\sum_{i\neq 1,2,3}^{k}a_i,\dots,a_0+a_k\right).\left(b_0,\sum_{i=0}^{k}b_i,\sum_{i\neq 1}^{k}b_i,\sum_{i\neq 1,2}^{k}b_i,\sum_{i\neq 1,2,3}^{k}b_i,\dots,b_0+b_k\right)+$$

$(0, \ a_{k+1}, \dots, \ a_{k+1}) \cdot \left(0, \sum_{i=1}^{k} b_i, \sum_{i\neq 0,1}^{k} b_i, \sum_{i\neq 0,1,2}^{k} b_i, \sum_{i\neq 0,1,2,3}^{k} b_i, \dots, b_k\right) + (0, a_{k+1}b_0 + a_{k+1}b_{k+1} + a_0 b_{k+1}, a_{k+1}b_0 + a_{k+1}b_{k+1} + a_0 b_{k+1}, \dots, a_{k+1}b_0 + a_{k+1}b_{k+1} + a_0 b_{k+1}) + (0, b_{k+1}, \dots, \ b_{k+1}) \cdot \left(0, \sum_{i=1}^{k} a_i, \sum_{i\neq 0,1}^{k} a_i, \sum_{i\neq 0,1,2}^{k} a_i, \sum_{i\neq 0,1,2,3}^{k} a_i, \dots, a_k\right) =$

$$\left(a_0, \sum_{i=0}^{k} a_i, \sum_{i\neq 1}^{k} a_i, \sum_{i\neq 1,2}^{k} a_i, \sum_{i\neq 1,2,3}^{k} a_i, \dots, a_0 + a_k\right) \cdot \left(b_0, \sum_{i=0}^{k} b_i, \sum_{i\neq 1}^{k} b_i, \sum_{i\neq 1,2}^{k} b_i, \sum_{i\neq 1,2,3}^{k} b_i, \dots, b_0 + b_k\right) +$$

$(0, \ a_{k+1}, \dots, \ a_{k+1}) \cdot (0, \sum_{i=1}^{k} b_i + b_0 + b_{k+1}, \sum_{i\neq 0,1}^{k} b_i + b_0 + b_{k+1}, \sum_{i\neq 0,1,2}^{k} b_i + b_0 + b_{k+1}, \dots b_0 + b_k + b_{k+1}) + (0, \ a_0, \dots, \ a_0) \cdot (0, \ b_{k+1}, \dots, \ b_{k+1}) =$
$\left(a_0, \sum_{i=0}^{k+1} a_i, \sum_{i\neq 1}^{k+1} a_i, \sum_{i\neq 1,2}^{k+1} a_i, \sum_{i\neq 1,2,3}^{k+1} a_i, \dots, a_0 + a_k\right) \cdot$
$\left(b_0, \sum_{i=0}^{k+1} b_i, \sum_{i\neq 1}^{k+1} b_i, \sum_{i\neq 1,2}^{k+1} b_i, \sum_{i\neq 1,2,3}^{k+1} b_i, \dots, b_0 + b_k\right)) = f(A) \cdot f(B).$

Thus, f is a ring isomorphism, and it is called the ***n-Refined Neutrosophic AH-Isometry****.*

8. Examples of 3- and 4-Refined Neutrosophic Isomorphisms

For *n = 3*, the *3*-refined neutrosophic case, the formula of *f* is
$f(a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3) = (a_0, a_0 + a_1 + a_2 + a_3, a_0 + a_2 + a_3, a_0 + a_3).$

For *n = 4*, the *4*-refined neutrosophic case, the formula of *f* is
$f(a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3 + a_4 I_4)$
$= (a_0, a_0 + a_1 + a_2 + a_3 + a_4, a_0 + a_2 + a_3 + a_4, a_0 + a_3 + a_4, a_0 + a_4).$

9. Inverse AH-Isomorphism

The formula of inverse isomorphism is:
$f^{-1}: R \times R \times \dots \times R \ \ (n+1 \ \ times) \to R_n(I)$ such that*:*
$f^{-1}(a_0, a_1, a_2, a_3 \dots a_n) = a_0 + I_1(a_1 - a_2) + I_2(a_2 - a_3) + I_3(a_3 - a_4) + \cdots + \ I_n(a_n - a_0).$

Application:
Find the inverse of the following *3*-refined neutrosophic real number $x = 1 + I_1 + I_2 + I_3$.
Now, we will see how the n-refined AH-isometry is very powerful in simplifying hard computations.

First, we compute: $f(x) = (1,4,3,2)$, $[f(x)]^{-1} = (1, \frac{1}{4}, \frac{1}{3}, \frac{1}{2})$, hence:

$$x^{-1} = f^{-1}[[f(x)]^{-1}] = 1 + \left(\frac{1}{4} - \frac{1}{3}\right) I_1 + \left(\frac{1}{3} - \frac{1}{2}\right) I_2 + \left(\frac{1}{2} - 1\right) I_3 = 1 + \left(-\frac{1}{12}\right) I_1 + \left(-\frac{1}{6}\right) I_2 + \left(-\frac{1}{2}\right) I_3.$$

By an easy computation, we can see:

$$x \cdot x^{-1} = 1 + \left(-\frac{1}{12} - \frac{1}{12} - \frac{1}{12} - \frac{1}{12} - \frac{1}{6} - \frac{1}{2} + 1\right) I_1 + \left(-\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{2} + 1\right) I_2 + \left(-\frac{1}{2} - \frac{1}{2} + 1\right) I_3 = 1.$$

10. Conclusion

We extended for the first time the Neutrosophic AH-Isometry to n-Refined Neutrosophic AH-Isometry, to find the determinate part of a function, as well as its multiple sub-indeterminate parts.

Many real examples of n-refined neutrosophic numbers, that have a determinate part and many types of sub-indeterminacies, were presented.

The AH-Isometry is designed to find the determinate part and sub-indeterminate parts of a n-refined neutrosophic function of the form $f(a_0 + a_1 I_1 + a_2 I_2 + ... + a_n I_n)$.

References

[1] Florentin Smarandache, Refined Literal Indeterminacy and the Multiplication Law of Subindeterminacies, Ch. 5, pp. 133- 160, in his book Symbolic Neutrosophic Theory, EuropaNova asbl, Clos du Parnasse, 3E 1000, Bruxelles, Belgium, https://fs.unm.edu/SymbolicNeutrosophicTheory.pdf

[2] Mohammad Abobala, Ahmad Hatip, An Algebraic to Neutrosophic Euclidean Geometry, Neutrosophic Sets and Systems, Vol. 43, 114-123, 2021, https://digitalrepository.unm.edu/cgi/viewcontent.cgi?article=1836&context=nss_journal

[3] Mehmet Celik, Ahmed Hatip, On the Refined AH-Isometry and Its Applications in Refined Neutrosophic Surfaces, Galoitica Journal Of Mathematical Structures And Applications (GJMSA), Vol. 02, No. 01, PP. 21-28, 2022, https://ia600506.us.archive.org/9/items/on-the-refined-ahisometry/OnTheRefinedAHIsometry.pdf

[4] M. B. Zeina and M. Abobala, "A Novel Approach of Neutrosophic Continuous Probability Distributions using AH-Isometry used in Medical Applications," in Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics, Elsevier, 2023.

[5] Hasan Sankari and Mohammad Abobala, AH-Homomorphisms in Neutrosophic Rings and Refined Neutrosophic Rings, Neutrosophic Sets and Systems, vol. 38, 2020, pp. 524-536. DOI: 10.5281/zenodo.4300590

[6] Mohammad Abobala: AH-Subspaces in Neutrosophic Vector Spaces. International Journal of Neutrosophic Science, 2020, 6, 80-86, 8 p.; DOI: 10.5281/zenodo.3841628

[7] Abuobida M. Ahmed Alfahal, Sara Sawalmeh, Raja Abdullah Abdulfatah, Yaser Ahmad Alhasan, On Some n-Refined Neutrosophic Groups For $3 \leq n \leq 5$, International Journal of Neutrosophic Science (IJN) Vol. 22, No. 03, PP. 119-127, 2023.

[8] Yaser Ahmad Alhasan, Ahmad Abdullah Almekhlef, Mohamed Elghazali Ali Mohieldin Mohamed, and Raja Abdullah Abdulfatah, The indefinite refined neutrosophic integrals by parts, Neutrosophic Sets and Systems, Vol. 68, 154-164, 2024.

[9] Yaser Ahmad Alhasan, Mohammed Mustafa Ahmed, Raja Abdullah Abdulfatah3 and Suliman Sheen, The refined indefinite neutrosophic integral, Neutrosophic Sets and Systems, Vol. 67, 127-134, 2024.
[10] Yaser Ahmad Alhasan, Mohamed Elghazali Ali Mohieldin Mohamed, and Raja Abdullah Abdulfatah, The limits of 2- refined neutrosophic, Neutrosophic Sets and Systems, Vol. 68, 39-49, 2024.

Neutrosophic One-Round Zero-Knowledge Proof

Ranulfo Paiva Barbosa
Independent Researcher, Web3 Blockchain Entrepreneur, COSTA RICA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

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Abstract

Zero-Knowledge Proofs (ZKPs) are cryptographic tools that enables one party, known as the prover, to prove to another party, the verifier, that a certain statement is true without revealing any information beyond the validity of the statement itself. We introduce the Neutrosophic One-Round Zero-Knowledge Proof protocol (N-1-R) ZKP, which is an extension of the One-Round (1-R) ZKP in the realm of Neutrosophic numbers. The N-1-R ZKP is the first Neutrosophic ZK protocol.

Keywords: Zero-Knowledge Proofs; Neutrosophic One-Round Zero-Knowledge Proof Protocol; Neutrosophic Zero-Knowledge Protocol.

1 | Introduction

Since Smarandache introduced Neutrosophy to study the basis, nature, and range of neutralities as well as their contact with ideational spectra in the 1990s [1], we have seen the emergence of neutrosophic algebraic structures such as neutrosophic groups and rings [2], neutrosophic numbers [3], single-valued neutrosophic sets (SVNSs) [4], Neutrosophic number theory [5], and several applications[6]. Kandasamy and Smarandache [2] defined Neutrosophic algebraic structures and inserted the algebraic symbol indeterminacy (I) with the logical property I2 = I.

The inception of neutrosophic number theory occurred in 2020 [5]. This nascent field witnessed the exploration of fundamental concepts including neutrosophic greatest common divisor (GCD) [7], neutrosophic Diophantine equations [8], neutrosophic Euler's function, and neutrosophic congruence [9]. Merkepci et al. suggested for the first time the idea of using neutrosophic numbers in cryptography [10]. In recent years, researchers have been actively developing neutrosophic versions of well-known cryptographic systems with the potential to enhance security, such as RSA [11], El Gamal [12], and Diffie-Hellman key exchange [13]. The Neutrosophic community has made significant progress, yet there remains substantial potential for advancing neutrosophic cryptography, especially considering the growing threats to traditional cryptographic systems and protocols.

Traditional cryptographic systems, such as those based on prime factorization (RSA) [14], discrete logarithm problem (DLP) [15], and elliptic curve cryptography (ECC) [16, 17], including TLS protocols [18], are susceptible to quantum attacks [19]. In this context, organizations like NIST (National Institute of Standards and Technology) proposed new algorithms that are potentially quantum resistant [20]. Unfortunately, the threats posed to cryptographic protocols by the emerging computing paradigm, MemComputing [21, 22], are

not widely recognized yet. Unlike quantum computers, digital MemComputing machines (DMMs) can be constructed using standard electronic components [23].

None of the mentioned cryptographic systems were developed with Neutrosophic numbers. Since neutrosophic number theory emerged in 2020, it should be devised to start exploring ZKP protocols in the realm of Neutrosophic number theory.

In this paper, we explore a simple and efficient Zero-Knowledge Proof protocol (ZKP), the One-Round ZKP developed by Almuhammadi and Neuman [24], which basis resides in the Discrete Logarithm Problem (DLP) [24]. Historically, the DLP was used by Diffie-Hellman key exchange [15] marking the beginning of asymmetric cryptography in the 1970s. And considering this historic algorithm we chose it to start exploring the use of Neutrosophic numbers in the realm of ZKP. We intend to explore more modern approaches as lattices among others in the future.

The next section presents an overview of Neutrosophic number theory and section 3 presents a brief introduction to ZKP and the One-Round ZKP. In section 4, we introduce the Neutrosophic One-Round ZKP.

2 | Some Elements of Neutrosophic Number Theory

Definition 2.1. [25] Let Z be the ring of integers, we say that (I) = {a + bI; a b ∈Z} is the neutrosophic ring of integers.

Definition 2.2. [25]

a) let a +bI, and c + dI be two neutrosophic integers, then:

   a + bI ≤ c + dI if and only if a ≤ a + b ≤ c + d.

b) a + bI is called positive neutrosophic integer if a > 0 and a + b >0.

Example 2.3. 5 + 2I is a positive neutrosophic integer, that is because 5 >0, 5 + 2 = 7 >0.

Definition 2.4. [25] (Addition) Let a + bI, c + dI: (a + bI) + (c + dI) = (a + c) + (b + d)I (1)

Definition 2.5. [25] (Multiplication) Let a + bI, c + dI: (a + bI) * (c + dI) = ac + I(ad + bc + bd) (2)

Scalar Multiplication; Let x be a scalar (real number). Then:

α*(a + bI) = αa + αbI (3)

The AH-Isometry (Abobala-Hatip): [38] Let a + bI, the AH-Isometry relation is given by: f(a + bI) = f(a) + I [f(a + b) - f(a)]

Definition 2.6. [25] (Neutrosophic Exponentiation); Let a + bI, c + dI: $(a+bI)^{(c+dI)} = a^c + I[\,(a+b)^{c+d} - a^c\,]$ (4)

Definition 2.7. [25] (Division)

Let a + bI, c + dI: $\frac{a+bI}{c+dI} = \frac{a}{c} + I[\,\frac{a+b}{c+d} - \frac{a}{c}\,]$ (5)

Definition 2.8. [25] (Root Index n) Let a + bI, its Root index n is given by:

$\sqrt[n]{a+bI} = \sqrt[n]{a} + I\,[\,\sqrt[n]{a+bI} - \sqrt[n]{a}\,]$ (6)

3 | Zero-Knowledge Proofs Protocols

Zero-knowledge proofs (ZKPs) are cryptographic tools that enable one party, known as the prover, to prove to another party, the verifier, that a certain statement is true without revealing any information beyond the validity of the statement itself [26].

Since Goldwasser, Micali, and Rackoff laid the groundwork for ZKPs in the 1980s [26, 27, 28], other important properties as succinctness [29], non-interactive [30] were incorporated in the realm of ZKP paving the way to the emergence of several cryptographic primitives as ZK-SNARKS [31], Bulletproofs [32], ZK-STARKS [33] among others described on ZKP surveys [34-36].

The cornerstone upon which ZKPs applications are built is the verifiable computation, i.e., the ability to prove that an external computation was performed correctly without revealing the inputs or the computation process. This foundational attribute of ZKPs serves as a gateway to their two practical value propositions: succinctness and privacy [36].

Succinctness in ZKPs allows for the quick verification of the correctness of a computation without the extensive resources typically required for direct computation execution. Privacy, the second major value proposition, emerges from the intrinsic nature of ZKPs to prove the correctness of information without revealing the information itself. This characteristic is particularly transformative in scenarios where sensitive or confidential data is involved [36].

The ZKPs became important cryptographic methods in multiple areas, specifically, blockchain [35, 37] and non-blockchain applications [36].

3.1 | One-Round Zero-Knowledge Proof

Almuhammadi and Neuman [24] developed the One-Round ZKP (1-Round ZKP) for the Discrete Logarithm (DL) problem. That is, given a prime p, a generator g for the multiplicative group Zp, and b ∈ Zp, Peggy wants to prove in zero-knowledge that she knows x such that, $g^x \equiv b \ (mod\ p)$. She proves to Victor that her claim is true without revealing the 'x' value through the steps described in Table 1.

Table 1. One-round ZKP of DL problem.

Step		Peggy(P)	Victor(V)
0	Setup	g, p, b, x	g, p, b
1	V generates a random y		y
2	V sends $c \equiv g^y (mod\ p)$ to P	c	$c \equiv g^y (mod\ p)$
3	P sends $r \equiv c^x (mod\ p)$ to V	$r \equiv c^x (mod\ p)$	r
4	V verifies that $r \equiv b^y (mod\ p)$		

This is a one-round proof based on the framework. All parameters are set up at Step 0. There are no more auxiliary messages needed for this protocol [24]. The authors provide the proof of correctness:

Assuming Peggy knows the secret x, she just computes $r \equiv c^x (mod\ p)$ and sends r to Victor. Since Victor knows y, he can verify that:

$r \equiv b^y (mod\ p) \equiv gxy \equiv (gx)y \equiv b^y (mod\ p)$.

If Peggy does not know x, Victor can verify easily that her claim is false [24].

The one-round protocol definitively establishes Peggy's knowledge of 'x' without divulging any information about 'x' to Victor. Almuhammadi and Neuman have demonstrated that non-interactive ZKPs offer superior efficiency in terms of computational resources and communication overhead. By eliminating the need for multiple rounds of interaction, these protocols accelerate execution and minimize latency [24].

4 | Neutrosophic One-Round ZKP

The Neutrosophic 1-Round ZKP to the Discrete Logarithm problem is an extension of the previous protocol in the realm of Neutrosophic numbers. Given the following neutrosophic numbers (in bold), **p**, **g**, **b**, **x**, where:

- **p** is a neutrosophic prime number, $\mathbf{p} = p_1 + p_2 I > 0$, i.e., $p_1, p_1 + p_2 I > 0$

- **g** is a generator, $\mathbf{g} = g_1 + g_2I$, i.e., $g_1, g_1 + g_2I > 0$
- **b** is given by, $\mathbf{b} = b_1 + b_2I$
- **x** is given by, $\mathbf{x} = x_1 + x_2I$

Peggy claims knowing $\mathbf{x} = x_1 + x_2I$, such that: $\boldsymbol{g^x \equiv b\ (mod\ p)}$. From the Neutrosophic exponentiation [25], it follows that $\boldsymbol{g^x \equiv b\ (mod\ p)}$, is given by:

$$(g_1 + g_2I)^{(x_1+x_2I)} = {g_1}^{x_1}(mod\ p_1) + I[\ (g_1 + g_2)^{x_1+x_2}(mod\ p_1 + p_2) - {g_1}^{x_1}(mod\ p_1)\ ]$$

She proves to Victor that her claim is true without revealing the '**x**' value through the steps described in Table 2.

Table 2. Neutrosophic One-Round ZKP of DL problem.

Step		Peggy(P)	Victor(V)
0	Setup	**g, p, b, x**	**g, p, b**
1	V generates a random **y**		**y**
2	V sends $\boldsymbol{c \equiv g^y(mod\ p)}$ to P	$\boldsymbol{c}$	$\boldsymbol{c \equiv g^y(mod\ p)}$
3	P sends $\boldsymbol{r \equiv c^x(mod\ p)}$ to V	$\boldsymbol{r \equiv c^x(mod\ p)}$	**r**
4	V verifies that $\boldsymbol{r \equiv b^y(mod\ p)}$		

Step 1. Given the neutrosophic numbers (described previously): **p**, **g**, **b**, **x**.

Step 2. Victor chooses a neutrosophic number $\mathbf{y} = y_1 + y_2I$.

Step 3. Victor computes $\boldsymbol{c \equiv g^y(mod\ p)}$, and sends **c** to Peggy. **c** is given by:

$$(g_1 + g_2I)^{(y_1+y_2I)} = {g_1}^{y_1}(mod\ p_1) + I[\ (g_1 + g_2)^{y_1+y_2}(mod\ p_1 + p_2) - {g_1}^{y_1}(mod\ p_1)\ ]$$

Step 4. Peggy sends to Victor, **r**, $\boldsymbol{r \equiv c^x(mod\ p)}$

Step 5. Victor verifies that $\boldsymbol{r \equiv b^y(mod\ p)}$

$$\boldsymbol{r \equiv c^x(mod\ p) \equiv g^{yx}(mod\ p) \equiv g^{xy}(mod\ p) \equiv b^y(mod\ p)}$$

$$\boldsymbol{r} \equiv {g_1}^{y_1x_1}(mod\ p_1) + I[\ (g_1 + g_2)^{(y_1+y_2)(x_1+x_2)}(mod\ p_1 + p_2) - {g_1}^{y_1x_1}(mod\ p_1)\ ]$$

$$\boldsymbol{r} \equiv {g_1}^{x_1y_1}(mod\ p_1) + I[\ (g_1 + g_2)^{(x_1+x_2)(y_1+y_2)}(mod\ p_1 + p_2) - {g_1}^{x_1y_1}(mod\ p_1)\ ]$$

$$\boldsymbol{r \equiv b^y(mod\ p)}$$

The Neutrosophic One-Round ZKP is the first ZKP using Neutrosophic number theory, and it opens a new research area to evaluate the potentiality of Neutrosophic cryptographic ZKP schemes against quantum-computer attacks, as well as, MemComputing digital machines attacks.

5 | Conclusion

We introduced the first Neutrosophic ZK protocol, the Neutrosophic One-Round Zero Knowledge Proof considering the Discrete Logarithm Problem. The use of Neutrosophy in the field of ZKP can, potentially, help improve the privacy and security of communications under insecure channels.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] Smarandache, F. A unifying field in Logics: Neutrosophic Logic. In Philosophy. American Research Press, 1-141, 1992.
[2] Kandasamy, W.B.V, and Smarandache, F. Basic Neutrosophic algebraic structures and their applications to fuzzy and Neutrosophic models. Hexis, 2004.
[3] Smarandache, F. Introduction to neutrosophic statistics. 2014.
[4] Wang, H.; Smarandache, F.; Zhang, Y.; and Sunderraman, R. 2010. Single valued SVNS sets, Multisp Multistruct 4: 410–413.
[5] Ceven, Y.; Tekin, S., Some Properties of Neutrosophic Integers. Kırklareli University Journal of Engineering and Science, Vol. 6, pp.50-59, 2020.
[6] El-Hefenaway, N., Metwally, M.A., Ahmed, Z.M., El-Henawy, I.M. A review on the applications of neutrosophic sets. J. Comp. Theor. Nanoscience, 13, 936-944, 2016.
[7] Ceven Y, Cetin, O. The greatest common divisors and the least common multiples in Neutrosophic integers. J. Natural & Applied Sciences, 27(3), 411-416, 2023.
[8] Sankari H, Abobala M. Neutrosophic linear Diophantine equations with two variables. Neutrosophic Sets and Systems, 38: 399-408, 2020.
[9] Abobada M. Foundations of Neutrosophic number theory. Neutrosophic Sets and Systems, 39: 120-132, 2021.
[10] Merkepci M, and Sarkis, M, "An application of Pythagorean circles In cryptography and some ideas for future non-classical systems", Galoitica Journal Of Mathematical Structures and Applications, 2022.
[11] Merkepci M, Abobala M, Allouf A. The applications of fusion neutrosophic number theory in public key cryptography and the improvement of RSA algorithm. Fusion: Practice and Applications (FPA). 2023; 10(2):69-74.
[12] Merkepci M, Abobala M. Security model for encrypting uncertain based on refined Neutrosophic integers algorithm. Fusion: practice and applications (FPA), v. 10(2):34-41, 2023.
[13] Allouf A, 'An Era of Cryptography Based on Neutrosophic Number Theory', the Role of Cybersecurity in the Industry 5.0 Era [Working Title]. IntechOpen, Jul. 04, 2024. Doi: 10.5772/intechopen.114975.
[14] Rivest R, Shamir A, Adleman L. A method for obtaining digital signatures and public-key cryptosystems. Communications of the ACM, 21(2): 120-126.
[15] Diffie W, Hellman M, New directions in cryptography. IEEE Transactions on Information Theory, 22(6): 644–654, 1976.
[16] Miller V, Use of Elliptic Curves in Cryptography. Advances in Cryptology — CRYPTO '85 Proceedings, 1986, 85: 417–426.
[17] Koblitz N, Elliptic curve cryptosystems. Mathematics of Computation. 1987, 48(177): 203–209.
[18] Duplys, P, Schmitz, R. TLS cryptography in-depth. Packt Publishing 2023.
[19] Petrenko A, Petrenko S. Applied quantum cryptoanalysis. River Publishers. 2023.
[20] https://csrc.nist.gov/Projects/post-quantum-cryptography/post-quantum-cryptography-standardization/round-3-submissions
[21] Di Ventra M. Memcomputing: fundamentals and applications. Oxford University Press, USA, 2022.
[22] Sharp TA, Khare R, Pederson E and Traversa FL, "A Memcomputing approach to prime factorization," 2023 IEEE International Conference on Rebooting Computing (ICRC), San Diego, CA, USA, 2023, pp. 1-10.

[23] Zhang Y-H, Di Ventra M. Implementation of digital MemComputing standard electronic components. 2023. https://arxiv.org/pdf/2309.12437v1.
[24] Almuhammadi S and Neuman C. Security and privacy using one-round zero-knowledge proofs, Seventh IEEE International Conference on E-Commerce Technology (CEC'05), Munich, Germany, 2005, pp. 435-438.
[25] Abobala M. Partial foundation of neutrosophic number theory. Neutrosophic Sets and Systems. 2021; 39: 120-132.
[26] Goldreich O, Micali S, Rackoff C, The knowledge complexity of interactive proof-systems, in Proc. of the Seventeenth Annual ACM Symposium on Theory of Computing, STOC '85. New York, NY, USA: Association for Computing Machinery, 1985, p. 291–304
[27] Goldreich O, Micali S, Wigderson A, Proofs that yield nothing but their validity and a methodology of cryptographic protocol design, in 27th Annual Symposium on Foundations of Computer Science, 1986, p.174–187.
[28] Ben-Or M, Goldreich O, Goldwasser S, Hastad J, Kilian J, Micali S, Rogaway P, Everything provable is provable in zero-knowledge, in Advances in Cryptology—CRYPTO'88: Proceedings 8. Springer, 1990, p. 37–56.
[29] Bitansky N, Canetti R, Chiesa A, Tromer E, From extractable collision resistance to succinct non-interactive arguments of knowledge, and back again. IACR Cryptology ePrint Archive, vol. 2011, p. 443.
[30] Groth J, Short pairing-based non-interactive zero-knowledge arguments, in Advances in Cryptology ASIACRYPT 2010, M. Abe, Ed. Berlin, Heidelberg: Springer Berlin Heidelberg, 2010, pp. 321–340.
[31] Bitansky N, Canetti R, Chiesa A, Tromer E, From extractable collision resistance to succinct non-interactive arguments of knowledge, and back again. Proc. 3rd Innovations in Theoretical Comp. Science Conf. on - ITCS '2012. ACM. pp. 326–349.
[32] Bünz B; Bootle J, Boneh D, Poelstra A, Wuille P, Maxwell G, Bulletproofs: Short Proofs for Confidential Transactions and More. 2018 IEEE Symposium on Security and Privacy (SP). pp. 315–334.
[33] Ben-Sasson E, Bentov I, Horesh Y, Riabzev M, Scalable, transparent, and post-quantum secure computational integrity, Cryptology ePrint Archive, Paper 2018/046, https://eprint.iacr.org/2018/046.
[34] Morais E, Koens T, Van Wijk C, Koren A, A survey on zero knowledge range proofs and applications, SN Applied Sciences, 1:1–17, 2019.
[35] Sun X, Yu FR, Zhang P, Sun Z, Xie W, Peng X, A survey on zero-knowledge proof in blockchain, IEEE network, 35 (4): 198–205, 2021.
[36] Lavin R, Liu X, Mohanty H, Norman L, Zaarour G, Krishnamachari B. A survey on the applications of Zero-Knowledge Proofs. 2024, https://arxiv.org/html/2408.00243v1.
[37] Zhan R, Xue R, Liu L, Security and privacy on blockchain, ACM Computing Surveys (CSUR), 52(3):1-34, 2019 https://arxiv.org/abs/1903.07602.
[38] Abobala M, Hatip A. An algebraic approach to neutrosophic Euclidean geometry. Neutrosophic Sets and Systems, 43:114-123.

Medical Diagnosis via Refined Neutrosophic Fuzzy Logic: Detection of Illness using Neutrosophic Sets

K. Hemabala
Department of Mathematics, Mohan Babu University (Erstwhile Sree Vidyanikethan Engineering College), Tirupathi, Andhra Pradesh, INDIA

B. Srinivasa Kumar
Department of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Andhra Pradesh, INDIA

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

Hemabala, K., et al. "Medical Diagnosis via Refined Neutrosophic Fuzzy Logic: Detection of Illness using Neutrosophic Sets." *Journal of Advanced Zoology*, Vol. 44, Issue 1, 2023, pp. 15-24.

Abstract

The objective of the paper is to implement and validate diagnosis in the medical field via refined neutrosophic fuzzy logic (RNFL). As such, we have proposed a Max-Min composition (MMC) method in RNFL. This method deals with the diagnosis under certain constraints like uncertainty and indeterminacy. Further, we have considered the diagnosis problems to validate the sensitivity analysis of the novel multi attribute decision-making technique. Finally, we gave the graphical representations and compared the obtained results with other existing measures in refined neutrosophic fuzzy sets.

Keywords: *Fuzzy set, Multi Fuzzy set, Refined Neutrosophic Fuzzy set, Max-Min composition.*

1. Introduction

Several authors established many concepts of decision-making in fuzzy sets. Fuzzy set theory is very successful in handling uncertainties arising from the vagueness of an element in a set. As such, the concept of interval-valued fuzzy sets came into force to capture uncertain situations in 1975, (L.A, Zadeh 1965). Generalization of the fuzzy set which is known as intuitionistic fuzzy sets (IFs) was made by (K. Atanassov, 1986). In IFs, instead of one membership grade, there is also a non-membership grade attached to each element. The conception of IFs can be viewed as an alternative approach in case where available information is not sufficient to define the impreciseness by the single value fuzzy set. From a philosophical standpoint, (F. Smarandache, 2002; F. Smarandache, 2005) proposed the neutrosophic set (NS) to deal with imprecise uncertain, inconsistent and partial information that exists in reality. This theory is very important in many applications areas since indeterminacy is quantified explicitly. Later, (F.Smarandache, 2013) introduced n-valued neutrosophic logic. More on Fuzzy Neutrosophic sets and Fuzzy Neutrosophic Topological spaces are

discussed by (Arockiarani et al., 2014; S. Sabu & Ramakrishnan, 2011; S. Sabu & Ramakrishnan, 2010), discussed about Multi fuzzy topology. Modeling diffusion transport mechanisms by utilizing the polynomials is proposed by (Rao & Chakraverty, 2018). Further, I.Deli et.al. (2015), proposed neutrosophic multisets and their application in medical diagnosis. Fuzzy Decision-Making Using Max-Min Method and Minimization of Regret Method given by Muhammad Siddique (2009). M.Z. Ragab & Emam (1995), also discussed the min-max composition of fuzzy matrices, fuzzy sets and systems. Fuzzy Max-Min Composition Technique in Medical Diagnosis analyzed by Edward Samuel & M. Balamurugan (2012). Wang et al. (2010), analyzed various aspects related to single valued neutrosophic sets. Entropy-based grey relational analysis method for multi-attribute decision making under single valued neutrosophic assessments given by Biswas et al. (2014). Further, Zeshui (2010) proposed method based on distance measure for interval-valued intuitionistic fuzzy group decision-making. Multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment is given by (Ye J, 2015; Broumi et al., 2022) discussed about the medical diagnosis problems based on neutrosophic sets and their hybrid structures. This article has five sections the first section deals with the introductory part and the next with the preliminaries then the third section gives a brief report on Max-Min Composition in Refined Neutrosophic fuzzy Sets, the fourth will discuss about the diagnosis problems of the proposed concept. Further, the fifth section gives the validation and sensitivity analysis and the final section will deliver the conclusion of this study.

Preliminaries

Refined Neutrosophic fuzzy relation in different product space can be combined with each other by the operation called "composition". They are many composition methods in use, e.g., max – product method, max- average method and max-min method and distance measures used in decision-making problems. But the max-min composition method is best known in refined neutrosophic fuzzy logic applications. Before illustrating the max min composition method in refined neutrosophic fuzzy sets. Here we give some basic definitions of fuzzy, multi-fuzzy, neutrosophic fuzzy and refined neutrosophic fuzzy sets. Generally fuzzy sets handling the concept of partial truth- truth values between complete truth and complete false whereas neutrosophic set is an extension of the intuitionistic fuzzy set which includes indeterminacy.

Fuzzy Set

Let X is an Universe of discourse and A be a fuzzy set defined on X[1]. A fuzzy set can be defined by $A=\{(x,\mu_A(x))/x\in X,\mu_A(x)\in[0,1]\}$

Ex: Let $X=\{1,2,3,4,5\}$ and $A=\{(1,0.1),(2,0.2),(3,0.3),(4,0.4),(5,0.5)\}$.

Multi Fuzzy Set

A multi $fuzzy$ set A in X is defined by way of a sequence of well-ordered set[6,7]

$A=\{(x,\mu_{A_1}(x),\mu_{A_2}(x),....\mu_{A_n}(x))/x\in X,\mu_{A_i}(x)\in[0,1]\}\ where\ i=1\ to\ n$

Ex: Let $X=(a,b,c)$ then $A=\{(a,0.1,0.2,0.5),(b,0.2,0.2,0.3),(c,0.2,0.3,0.4)\}$.

Properties

Let us consider two multi $fuzzy$ sets A and B in X[6,7]

Then we have for each $i=1\ to\ n$ where n is a positive integer

$A\subseteq B\ if\ and\ only\ if\ \mu_{A_i}(x)\le\mu_{B_i}(x)$

$A=B\ if\ and\ only\ if\ \mu_{A_i}(x)=\mu_{B_i}(x)$

$A\cup B=\{x,\max(\mu_{A_i}(x),\mu_{B_i}(x))\}$

$$A \cap B = \left\{ x, \min\left(\mu_{A_i}(x), \mu_{B_i}(x)\right) \right\}$$

$$C(A) = \left\{ (x, C\mu_{A_i}(x)), C\mu_{A_i}(x) = 1 - \mu_{A_i}(x) \right\}$$

Neutrosophic Fuzzy set

Consider a neutrosophic $fuzzy$ set[3,4] $\mathcal{N}$ in X(nonempty)is normally characterized by three proposes namely T_N, I_N, F_N. Then neutrosophic fuzzy set is defined by

$$N = \left\{ < x, T_N(x), I_N(x), F_N(x) > : x \in X, T_N(x), I_N(x), F_N(x) \in [0,1] \right\}$$

$$0^- \le T_N(x) + I_N(x) + F_N(x) \le 3^+$$

T_N = Truth membership function

I_N = Indeterminacy membership function

F_N = Falsity membership function

The n-Numerical Valued Refined Neutrosophic Fuzzy Logic

In general [18] T_N can be split many types of truths like $T_N^1, T_N^2, ...T_N^p$, I_N can be split many types of indeterminacies like $I_N^1, I_N^2, ...I_N^r$ and F_N can be split many types of falsities like $F_N^1, F_N^2, ...F_N^s$. All subcomponents T_N^i, I_N^j, F_N^k are subsets of [0,1], for all $i \in \{1,2,...p\}, j \in \{1,2,...r\}\ and\ k \in \{1,2,...s\}$. Then it can be defined X as follows

$$N = \left\{ < x, \left(T_N^1(x), T_N^2(x), ...T_N^p(x)\right), \left(I_N^1(x), I_N^2(x), ...I_N^r(x)\right), \left(F_N^1(x), F_N^2(x), ...F_N^s(x)\right) > x \in X \right\}$$

If all sources of information that separately provide neutrosophic values for a specific subcomponent are independent sources, then in the general case we consider that each of the subcomponents T_N^i, I_N^j, F_N^k is independent with respect to the others and it is in the non-standard set $\left]^-0,1^+\right[$.

$$^-0 \le \sum_{i=1}^{p} T_N^i + \sum_{j=1}^{r} I_N^j + \sum_{k=1}^{s} F_N^k \le n^+ \quad \text{where } n = p + r + s.$$

Properties

Let A and B are two n-numerical valued refined neutrosophic fuzzy **sets** in X and x be the generic element in X then the following properties hold :

$$A \subseteq B\ if\ and\ only\ if\ T_A^i(x) \le T_B^i(x), I_A^j(x) \ge I_B^j(x), F_A^k(x) \ge F_B^k(x)$$

$$A = B\ if\ and\ only\ if\ T_A^i(x) = T_B^i(x), I_A^j(x) = I_B^j(x), F_A^k(x) = F_B^k(x)$$

$$A \cup B = \left\{ x, \max\left(T_A^i(x), T_B^i(x)\right), \min\left(I_A^j(x), I_B^j(x)\right), \min\left(F_A^k(x), F_B^k(x)\right) \right\}$$

$$A \cap B = \left\{ x, \min\left(T_A^i(x), T_B^i(x)\right), \max\left(I_A^j(x), I_B^j(x)\right), \max\left(F_A^k(x), F_B^k(x)\right) \right\}$$

Max-Min Composition in refined neutrosophic Fuzzy Sets

Many authors developed uncertain problems in decision making by using max min composition method in fuzzy sets (Muhammad, 2009; M.Z. Ragab & E.G. Emam, 1995; A. Edward Samuel & M. Balamurugan, 2012). But this method has some limitations when the data includes indeterminacy. Since the fuzzy set takes only one value between 0 and 1. This may affect the decision drawn from the information. Using the max-min method on a single set of data may not give accurate results. As such, we get the minimum error by collecting the data two or more times to apply the max min method. To

overcome these situations, we are proposing max min method in the refined neutrosophic fuzzy set, where we can see the data includes indeterminacy and information also includes multi fuzzy. The indeterminacy function does not exist in fuzzy sets, implicitly exist in intuitionistic fuzzy sets and explicitly exist in neutrosophic fuzzy sets. Here, we present the max-min composition method to handle uncertainty in multiple constraints and plays a vital role to handling multiple data sets.

Let X be the universe of discourse and A and B are two refined neutrosophic fuzzy set defined on X. Then the data set in A and B can be represented in the form of matrices say Z and Y is given by:

$Z = [a_{lm}]_{LxM}$ where $a_{lm} = \left(T_A^i(x_l / y_m), I_A^j(x_l / y_m), F_A^k(x_l / y_m)\right)$, $l = 1 to L, m = 1 to M$.

$Y = [b_{mn}]_{MxN}$ where $b_{mn} = \left(T_B^i(y_m / z_n), I_B^j(y_m / z_n), F_B^k(y_m / z_n)\right)$, $m = 1 to M, n = 1 to N$.

Then the Max-Min composition of the two matrices Z and Y is defined by: $Z\,o\,Y = C = \lfloor c_{\ln} \rfloor_{LxN}$ where:

$$c_{l\,n} = \begin{pmatrix} \max\left(\max_l\left(\min_m\left(T_A^i(x_l / y_m), T_B^i(y_m / z_n)\right)\right)\right), \min\left(\min_l\left(\max_m\left(I_A^j(x_l / y_m), I_B^j(y_m / z_n)\right)\right)\right), \\ \min\left(\min_l\left(\max_m\left(F_A^k(x_l / y_m), F_B^k(y_m / z_n)\right)\right)\right) \end{pmatrix}$$

For better result get the data for the proposed method via refined neutrosophic fuzzy sets at two or more different intervals.

Propositions of Max-Min composition

Max min method can exhibit various useful properties, a few of which are given here for effectiveness and efficiency of the proposed method.

Let Z,Y,D are three refined neutrosophic fuzzy matrices defined on X then $maxmin$ composition on Z,Y is denoted by Z o Y and it is easily verify the following algebraic properties.

3.1.1 Associative

Z o (Y o D) = (Z o Y) o D for all Z, Y, D

3.1.2 Commutative

Z o Y = Y o Z for all Z, Y

3.1.3 Transitive

Z o Y = Y o D then Z o D for all Z, Y, D

Algorithm

Making choices between future, and uncertain possibilities is defined as decision making. It is a choice between numerous methods for achieving a goal. In business, finance, and economics, as well as engineering, social, physical, and medical sciences, decision-making is critical. It should be highlighted that all decisions are made with the future in mind. There can be no decision-making when there are no choices. Due to issues such as insufficient and imprecise information, vagueness, and the uncertainty of the situation, it is a tough process. These characteristics demonstrate that judgments are made in a fuzzy logic environment. The act of making a decision is characterized by the selection of an option from a set of options. Stipulated goals must be met while keeping obstacles in mind during this procedure. Consider a simple goal-and-constraints decision-making paradigm.

Let X be a Universal set and A be the refined neutrosophic fuzzy sets,

Consider $x_1, x_2, \ldots . x_m$ be the set of m elements and $y_1, y_2, \ldots . y_n$ be the set of n elements then we have the following algorithm for the max min composition method in refined neutrosophic fuzzy sets. Based on this algorithm we can draw a decision for the given data.

Algorithm for the proposed method

Step 1: Let $x_1, x_2, \ldots . x_L$ be the symptoms characterstics of every patient. The data must include percentage of truth values indeterminacy values and falsity values (T, I, F). The data may be quantitative or qualitative in nature.

Step 2: Observe the symptoms at different times.

Step 3: Convert data obtained in Step 2 into membership values by using triangular membership function and form a matrix Z of order LxM.

Step 4: Next collect the data of symptoms characteristics for the diagnosis from expert which we compare with the patient symptoms. The data must include truth values indeterminacy values and falsity values (T, I, F). The data may be quantitative or qualitative in nature. This data is already existed and standard data in nature.

Step 5: Convert the data obtained in Step 4 into membership values by using triangular membership function and form a matrix Y of order MxN.

Step 6: Apply max min composition between A and B by using the formula

$$c_{l\,\mathrm{n}} = \left(\begin{array}{c} \max\left(\max_l \left(\min_m \left(T_A^i\left(x_l / y_m\right), T_B^i\left(y_m / z_n\right)\right)\right)\right), \min\left(\min_l \left(\max_m \left(I_A^j\left(x_l / y_m\right), I_B^j\left(y_m / z_n\right)\right)\right)\right), \\ \min\left(\min_l \left(\max_m \left(F_A^k\left(x_l / y_m\right), F_B^k\left(y_m / z_n\right)\right)\right)\right) \end{array} \right)$$

Step 7: The highest value of T in the entries of the matrix C gives the decision.

Step 8: If T is same then look into Falsity value in the relation matrix C. The least value of F in the matrix C gives the decision provided that Indeterminacy value is less than or equal to Falsity membership value. If indeterminacy membership value is greater than the Falsity value then conclusion cannot be drawn and information is bias.

Diagnosis problems by proposed method

Diagnosis problem 1:

When a person is affected by a particular disease that a person will have a more than one symptom such as Temperature, cough, throat infection, headache, sneezing etc. Also, each viral disease will have more than one symptom. For instance, malaria, typhoid, chicken gunya will have various symptoms like temperature, body pain, cough etc. A person who is affected by corona virus also will have the symptoms of temperature, cough, sneezing, body pain etc. Now to find out the core symptom of corona virus we will use max min composition via refined neutrosophic multi fuzzy sets. We can find the maximum value in T can conclude the kind of sickness affecting the patient suffering from.

Let us take 8 patients for our study i.e $P = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$. Each patient is experiencing more than one symptom say S = {Temperature, Headache, body pain, cough, sneezing}. Now using the neutrosophic data we want to find out the kind of disease affecting the person from the common prevalent diseases, say d = {Viral fever, malaria, typhoid, chicken gunya, corona virus}.

The data given in the form of matrix Z, Y.

The following matrix represents the patient's vs symptoms.

$$Z=\begin{bmatrix}((0.6,0.7),(0.3,0.2),(0.3,0.1)) & ((0.5,0.4),(0.2,0.1),(0.4,0.3)) & ((0.3,0.5),(0.5,0.4),(0.2,0.1)) & ((0.4,0.3),(0.4,0.2),(0.4,0.2)) & ((0.3,0.5),(0.4,0.3),(0.5,0.4))\\ ((0.1,0.2),(0.6,0.7),(0.4,0.3)) & ((0.4,0.3),(0.6,0.5),(0.3,0.2)) & ((0.3,0.2),(0.5,0.5),(0.4,0.4)) & ((0.3,0.3),(0.5,0.4),(0.4,0.4)) & ((0.3,0.2),(0.6,0.7),(0.7,0.5))\\ ((0.6,0.4),(0.3,0.2),(0.4,0.2)) & ((0.6,0.6),(0.2,0.1),(0.4,0.3)) & ((0.4,0.4),(0.5,0.5),(0.5,0.4)) & ((0.2,0.2),(0.5,0.3),(0.5,0.3)) & ((0.2,0.2),(0.4,0.2),(0.3,0.3))\\ ((0.4,0.3),(0.3,0.2),(0.2,0.2)) & ((0.4,0.4),(0.4,0.4),(0.4,0.3)) & ((0.2,0.1),(0.4,0.4),(0.5,0.4)) & ((0.5,0.4),(0.2,0.1),(0.4,0.2)) & ((0.4,0.3),(0.3,0.3),(0.4,0.3))\\ ((0.2,0.1),(0.4,0.3),(0.6,0.5)) & ((0.2,0.1),(0.4,0.4),(0.0,0.0)) & ((0.7,0.6),(0.6,0.5),(0.1,0.1)) & ((0.2,0.1),(0.4,0.4),(0.7,0.6)) & ((0.3,0.2),(0.2,0.1),(0.7,0.5))\\ ((0.3,0.3),(0.4,0.2),(0.5,0.4)) & ((0.6,0.4),(0.4,0.3),(0.3,0.1)) & ((0.6,0.6),(0.3,0.3),(0.1,0.0)) & ((0.5,0.5),(0.4,0.3),(0.7,0.5)) & ((0.5,0.4),(0.4,0.3),(0.6,0.7))\\ ((0.4,0.7),(0.5,0.4),(0.3,0.2)) & ((0.6,0.5),(0.5,0.5),(0.1,0.1)) & ((0.5,0.5),(0.4,0.4),(0.4,0.3)) & ((0.5,0.4),(0.3,0.2),(0.4,0.3)) & ((0.4,0.4),(0.5,0.5),(0.4,0.3))\\ ((0.6,0.6),(0.3,0.3),(0.7,0.6)) & ((0.6,0.4),(0.2,0.2),(0.3,0.2)) & ((0.6,0.6),(0.3,0.3),(0.6,0.6)) & ((0.4,0.3),(0.3,0.2),(0.4,0.2)) & ((0.7,0.7),(0.1,0.0),(0.2,0.2))\end{bmatrix}$$

For each patient the type of symptoms usually found, so that we can obtain symptom disease relation. The following matrix represents the symptoms vs diseases.

$$Y=\begin{bmatrix}(0.6,0.3,0.3) & (0.2,0.5,0.3) & (0.2,0.6,0.4) & (0.1,0.6,0.6) & (0.1,0.6,0.4)\\ (0.4,0.5,0.3) & (0.2,0.6,0.4) & (0.1,0.5,0.4) & (0.2,0.4,0.6) & (0.1,0.6,0.4)\\ (0.1,0.6,0.3) & (0.0,0.6,0.4) & (0.2,0.5,0.5) & (0.8,0.2,0.2) & (0.1,0.7,0.1)\\ (0.4,0.4,0.4) & (0.4,0.1,0.5) & (0.2,0.5,0.5) & (0.1,0.7,0.4) & (0.4,0.5,0.4)\\ (0.1,0.7,0.4) & (0.1,0.6,0.3) & (0.1,0.6,0.4) & (0.1,0.7,0.4) & (0.8,0.2,0.2)\end{bmatrix}$$

The resultant matrix is patient's vs diseases.

$$C=\begin{bmatrix}(0.6,0.3,0.3) & (0.3,0.2,0.3) & (0.2,0.5,0.4) & (0.5,0.4,0.2) & (0.3,0.3,0.1)\\ (0.4,0.4,0.3) & (0.3,0.4,0.3) & (0.2,0.5,0.4) & (0.2,0.5,0.4) & (0.3,0.5,0.4)\\ (0.6,0.3,0.3) & (0.2,0.3,0.3) & (0.2,0.5,0.4) & (0.4,0.4,0.4) & (0.2,0.2,0.3)\\ (0.4,0.3,0.3) & (0.4,0.1,0.3) & (0.2,0.4,0.4) & (0.2,0.4,0.4) & (0.4,0.3,0.3)\\ (0.2,0.3,0.3) & (0.2,0.4,0.4) & (0.2,0.5,0.4) & (0.7,0.4,0.2) & (0.3,0.2,0.1)\\ (0.4,0.4,0.3) & (0.4,0.3,0.4) & (0.2,0.5,0.4) & (0.6,0.3,0.2) & (0.5,0.3,0.1)\\ (0.6,0.4,0.3) & (0.4,0.2,0.3) & (0.2,0.5,0.4) & (0.5,0.4,0.3) & (0.4,0.5,0.3)\\ (0.6,0.3,0.3) & (0.4,0.2,0.3) & (0.2,0.5,0.4) & (0.6,0.3,0.4) & (0.7,0.2,0.2)\end{bmatrix}$$

We can conclude that P_1, P_2, P_3, P_7 suffers with Viral fever, P_4 suffers with Malaria, P_5, P_6 suffers with Chicken gunya, P_8 suffers with Corona virus.

Diagnosis Problem 2:

Three patients came with symptoms of three viruses namely Herpes simplex, Hepatitis, Influenza. Doctor suggested three tests to three patients namely Elisa test, Blood test and Viral culture.

Let $P=\{P_1,P_2,P_3\}$ be the universal set where P_1,P_2,P_3 represents the set of patients.

$V=\{V_1,V_2,V_3\}$ be the set of viruses and $T=\{T_1,T_2,T_3\}$ be the set of tests. They have taken two samples and the report has given in terms of the relation matrix. The following matrix is the relation matrix of patients and their tested values (The matrices A and B given in this paper are directly converted raw data into membership values)

$$A=\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\begin{array}{c} \begin{array}{ccc} T_1 & T_2 & T_3 \end{array} \\ \begin{bmatrix} ((0.6,0.5),(0.2,0.2),(0.2,0.2)) & ((0.8,0.8),(0.1,0.1),(0.1,0.1)) & ((0.4,0.3),(0.2,0.2),(0.4,0.5)) \\ ((0.8,0.7),(0.1,0.2),(0.1,0.1)) & ((0.4,0.3),(0.2,0.3),(0.4,0.4)) & ((0.6,0.5),(0.2,0.1),(0.3,0.3)) \\ ((0.4,0.4),((0.1,0.1),(0.5,0.5)) & ((0.3,0.3),(0.1,0.2),(0.6,0.5)) & ((0,8,0.8),(0.1,0.1).(0.1,0.1)) \end{bmatrix} \end{array}$$

Doctor is having some standard values for the tests based on the viruses. These values may help for the junior doctor for the better treatment. The following matrix is the relation matrix of test values based on viruses.

The relation matrix B is given by

$$B=\begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array}\begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{bmatrix} ((0.8,0.8),(0.1,0.1),(0.1,0.1)) & ((0.5,0.5),(0.1,0.1),(0.4,0.4)) & ((0.4,0.4),(0.1,0.1),(0.5,0.5)) \\ ((0.4,0.4),(0.1,0.1),(0.5,0.5)) & ((0.8,0.8),(0.1,0.1),(0.1,0.1)) & ((0.4,0.4),(0.1,0.1),(0.5,0.5)) \\ ((0.5,0.5),(0.1,0.1),(0.4,0.4)) & ((0.4,0.4),(0.1,0.1),(0.5,0.5)) & ((0.8,0.8),(0.1,0.1),(0.1,0.1)) \end{bmatrix} \end{array}$$

The resultant matrix

$$D=\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{bmatrix} (0.6,0.1,0.2) & (0.8,0.1,0.1) & (0.4,0.1,0.5) \\ (0.8,0.1,0.1) & (0.5,0.1,0.4) & (0.6,0.1,0.3) \\ (0.5,0.1,0.4) & (0.4,0.1,0.5) & (0.8,0.1,0.1) \end{bmatrix} \end{array}$$

From the resultant matrix it is easily observe that P_1 has highest truth membership value at V_2, P_2 has highest truth membership value at V_1 and P_3 has highest truth membership value at V_3 .We conclude that P_1 suffers with V_2, P_2 suffers with V_1 and P_3 suffers with V_3.

Validation and Sensitivity Analysis

Validity and Sensitivity analysis of the proposed methodology regarding max-min composition of RNFS is done with various distance measures like Normalized Hamming distance measure, Normalized Euclidean distance measure, Hausdroff metric for the same data and the comparison results are shown in the below tables and graphs. As such, various distance measures are listed below (Tongjuan, 2021).

(a) Normalized Hamming distance measure

$$D_{Ham}^{N}(A,B)=\frac{1}{3n}\sum_{i=1}^{n}\left(\left|T_A(x_i)-T_B(x_i)\right|+\left|I_A(x_i)-I_B(x_i)\right|+\left|F_A(x_i)-F_B(x_i)\right|\right)$$

(b) Normalized Euclidean distance measure

$$D_{Euc}^{N}(A,B)=\sqrt{\frac{1}{3n}\sum_{i=1}^{n}\left(\left(T_A(x_i)-T_B(x_i)\right)^2+\left(I_A(x_i)-I_B(x_i)\right)^2+\left(F_A(x_i)-F_B(x_i)\right)^2\right)}$$

(c) Hausdroff measure

$$D_{Haus}^{N}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\max\left(\left|T_A(x_i)-T_B(x_i)\right|,\left|I_A(x_i)-I_B(x_i)\right|,\left|F_A(x_i)-F_B(x_i)\right|\right)$$

Due to large calculations, I compared proposed method with the distance measures by taking diagnosis problem 2.

	MMC	NHDM	NEDM	HM
(P1, V1)	(0.6,0.1,0.2)	0.15	0.1353	0.25
(P1, V2)	(0.8,0.1,0.1)	0.2	0.2449	0.3
(P1, V3)	(0.4.0.1,0.5)	0.0667	0.0818	0.1
(P2, V1)	(0.8,0.1,0.1)	0.2667	0.3109	0.4
(P2, V2)	(0.5,0.1,0.4)	0.3	0.3266	0.45
(P2, V3)	(0.6,0.1,0.3)	0.1333	0.1526	0.2
(P3, V1)	(0.5,0.1,0.4)	0.0667	0.0816	0.1
(P3, V2)	(0.4,0.1,0.5)	0.0667	0.0816	0.1
(P3, V3)	(0.8,0.1,0.1)	0	0	0

The following figure shows that the graphical representation between proposed method and distance measures.

The graphical representation between proposed method and other measure

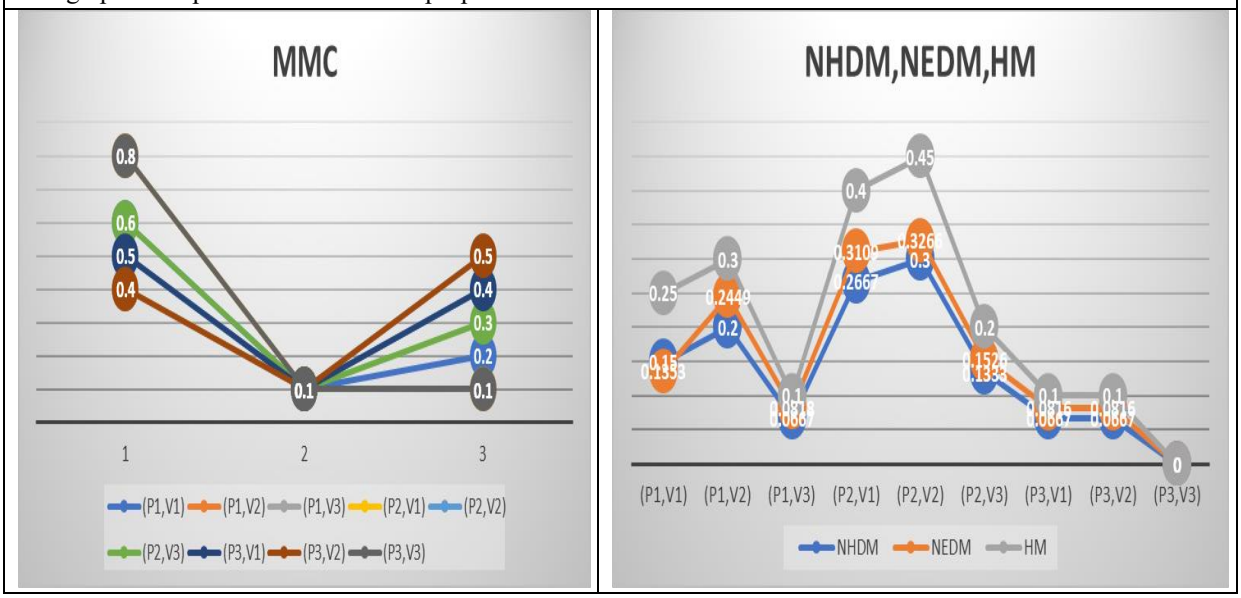

Figure (1): Graph of MMC with NHDM, NEDM & HM

Table 1 depicts the comparison values between the proposed method other method for the considered problem 2. In distance methods usually, the minimum value gives the decision. When you collect the neutrosophic data these methods fail. This we can see in the table 1. Whereas our proposed method gives the correct decision. In Fig.1, we can easily observe that the graph of proposed method shows the three values such as Truth membership value, Falsity membership values and Inderminacy value whereas the graph of other measures like NHDM, NEDM & HM shows only a single crisp value.

2. Conclusion

The discussion in this paper is related to the analysis and validation of the Max-Min Composition method. In this method the decision is based on truth membership value. Further, the decision depends on the indeterminacy membership value ($F < I$ & $F > I$). As such, in this paper, Max-Min composition method is developed and implemented to examine its feasibility for selecting a diagnosis problem. The main concluding remarks of the proposed method are accurate decision by taking into account of truth membership value (T), indeterminacy membership values (I) and falsity membership values (F). We compared this method with various measures where the decision depends on the crisp set. In these measures the authors merged all three membership values into a crisp set. In the first illustration, the proposed method and measuring methods exhibit the same result but in the second

Illustration the proposed method has given the correct decision whereas the existing methods do not. In the realm of medical diagnosis, the utilisation of neutrosophic tests presents an innovative and alternative method for detecting paten illnesses. This study explores the application of neutrosophic sets as an unconventional approach to healthcare diagnosis aiming to enhance the accuracy and reliability of disease detection. By incorporating the concept of neutrosophy, which deals with indeterminacy, uncertainty, and vagueness, this research offers a fresh perspective on how medical professionals can make informed decisions when faced with complex and ambiguous clinical data. Through the examination of real-world case studies and the development of novel algorithms, this study investigates the potential benefits of neutrosophic sets in improving patient care and medical decision making. The findings shed light on the feasibility and effectiveness of this alternative methodology in the context of diagnosing various medical conditions, ultimately contributing to the ongoing evolution of healthcare practices.

Neutrosophic set based diagnosis heavily relies on data. Incomplete, inaccurate or biased data can lead to erroneous results, limited access to high quality medical data may restrict the effectiveness of this approach. The use of unconventional diagnostic methods like neutrosophic sets may raise ethical and legal questions, particularly in case of misdiagnosis or patient harm. Neutrosophic sets should not replace the expertise of healthcare professionals. They should be considered as complementary tools rather than standalone diagnostic methods.

References:

A. Edward Samuel and M. Balamurugan (2012), Fuzzy Max-Min Composition Technique in Medical Diagnosis, Applied Mathematical Sciences, vol. 6, no. 35, 1741 – 1746. http://www.m-hikari.com/ams/ams-2012/ams-33-36-2012/samuelAMS33-36-2012.pdf

Arockiarani, J. Martina Jency (2014), More on Fuzzy Neutrosophic sets and Fuzzy Neutrosophic Topological spaces, International journal of innovative research and studies, vol. 3, no. 5, 643-652. https://fs.unm.edu/SN/Neutro-MoreFuzzyNeutroSets.pdf

Broumi, S., Dhar, M., Bakhouyi, A., Bakali, A., & Talea, M. (2022). Medical diagnosis problems based on neutrosophic sets and their hybrid structures: A survey. *Neutrosophic Sets and Systems*, *49*(1), 1-18. https://digitalrepository.unm.edu/cgi/viewcontent.cgi?article=2036&context=nss_journal

F. Smarandache (2002), Neutrosophy,a new branch of Philosophy logic in multiple-valued logic,An international journal, vol. 8, no. 3,297-384. https://fs.unm.edu/Neutrosophy-A-New-Branch-of-Philosophy.pdf

F.Smarandache (2005), Neutrosophic set –a generalization of the Intuitionistic fuzzy sets, J.Pure Appl.Math., vol. 24, 287-297. https://www.ijpam.eu/contents/2005-24-3/1/1.pdf

F.Smarandache (2013) n-Valued Refined Neutrosophic Logic and Its Applications to Physics, Progress in physics, vol. 4, October,143-146. https://doi.org/10.48550/arXiv.1407.1041

I.Deli, S.Broumi, F. Smarandache (2015), On neutrosophic multisets and its application in medical diagnosis. J. New Theory, vol. 6, 88–98. https://doi.org/10.5281/zenodo.49112

K. Atanassov (1986), Intuitionistic fuzzy sets, Fuzzy Sets and Systems, vol. 20, 87- 96. https://doi.org/10.1016/S0165-0114(86)80034-3

L.A. Zadeh (1965) Fuzzy sets, Information control, vol.8, 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

M.Z. Ragab, E.G. Emam (1995), On the min-max composition of fuzzy matrices, fuzzy sets and systems, vol. 75, no. 1, 13 October, 83-92. https://doi.org/10.1016/0165-0114(94)00332-2

Muhammad Siddique (2009), Fuzzy Decision Making Using Max-Min Method and Minimization of Regret Method(MMR), (720301-P254), thesis. https://www.diva-portal.org/smash/get/diva2:830339/FULLTEXT01.pdf

P.Biswas, S. Pramanik, and B.C.Giri (2014) Entropy based grey relational analysis method for multi attribute decision making under single valued neutrosophic assessments. Neutrosophic Sets and Systems, 2, 102-110. http://dx.doi.org/10.5281/zenodo.22459

Rao, T. D., & Chakraverty, S. (2018). Modeling radon diffusion equation by using fuzzy polynomials in Galerkin's method. *Recent Advances in Applications of Computational and Fuzzy Mathematics*, 75-93. http://dx.doi.org/10.1007/978-981-13-1153-6_4

S. Sabu and T.V. Ramakrishnan (2010), Multi fuzzy sets, International mathematical Forum, vol. 50,2471-2476. https://www.researchgate.net/publication/284899193_Multi-Fuzzy_Sets

S. Sabu and T.V. Ramakrishnan (2011), Multi fuzzy topology, International journal of Applied Mathematics. Volume 24 No. 1 , 117-129. https://www.researchgate.net/publication/268160837_Multi-fuzzy_topology

Tongjuan Wang (2021), Distance of Single-Valued Neutrosophic Set and Its Application in Pattern Recognition, Journal of Physics, 2025, 012019. http://dx.doi.org/10.1088/1742-6596/2025/1/012019

Wang, H., F. Smarandache. Zhang, Y., & Sunderraman, R.(2010),Single valued neutrosophic sets. Multispace and Multistructure 4,410-413. https://www.researchgate.net/publication/262047656_Single_valued_neutrosophic_sets

Ye, J. (2015). Multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment. *Journal of Intelligent Systems*, *24*(1), 23-36. https://doi.org/10.1515/jisys-2014-0001

Zeshui Xu. (2010) A method based on distance measure for interval-valued intuitionistic fuzzy group decision making, Information Sciences, 180, 181-190. https://doi.org/10.1016/j.ins.2009.09.005

Three Decades of Neutrosophic and Plithogenic Theories with their Applications (1995 - 2024), and seven decades of life of Florentin Smarandache (1954 - 2024)

Florentin Smarandache
Department of Mathematics, Physics and Natural Sciences Division, University of New Mexico, Gallup, NM, UNITED STATES OF AMERICA

Smarandache, Florentin. "Three Decades of Neutrosophic and Plithogenic Theories with their Applications (1995 - 2024)." *Neutrosophic Sets and Systems,* vol. 73, 2024, pp. 1-10.

Zadeh introduced the **degree of membership/truth** (T) in 1965 and defined the fuzzy set. *Atanassov* introduced the **degree of nonmembership/falsehood** (F) in 1986 and defined the intuitionistic fuzzy set. *Smarandache* introduced the **degree of indeterminacy/neutrality** (I) as independent component in 1995 (published in 1998) and he defined the neutrosophic set on three components: (T, I, F) = (Truth, Indeterminacy, Falsehood), where in general T, I, F are subsets of the interval [0, 1]; in particular T, I, F may be intervals, hesitant sets, single-values, etc.; *Indeterminacy* (or *Neutrality*), as independent component from the truth and from the falsehood, is the main distinction between Neutrosophic Theories and other classical and fuzzy theory or fuzzy extension theories:
https://fs.unm.edu/Indeterminacy.pdf
See F. Smarandache, Neutrosophy / Neutrosophic probability, set, and logic", Proquest, Michigan, USA, 1998.
https://arxiv.org/ftp/math/papers/0101/0101228.pdf
https://fs.unm.edu/eBook-Neutrosophics6.pdf;
reviewed in Zentralblatt für Mathematik (Berlin, Germany):
https://zbmath.org/?q=an:01273000
And cited by Denis Howe in The Free Online Dictionary of Computing, England, 1999. Neutrosophic Set and Logic are generalizations of classical, fuzzy, and intuitionist fuzzy set and logic:
https://arxiv.org/ftp/math/papers/0404/0404520.pdf
https://arxiv.org/ftp/math/papers/0303/0303009.pdf

Etymology

The words "neutrosophy" and "neutrosophic" were coined/invented by F. Smarandache in his 1998 book. Neutrosophy: A branch of philosophy, introduced by F. Smarandache in 1980, which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy considers a proposition, theory, event, concept, or entity <A> in relation to its opposite <antiA>, and with their neutral <neutA>. Neutrosophy (as dynamic of opposites and their neutrals) is an extension of the Dialectics and Yin Yang (which are the dynamic of opposites only).
Neutrosophy is the basis of neutrosophic set, neutrosophic logic, neutrosophic measure, neutrosophic probability, neutrosophic statistics etc.
https://arxiv.org/ftp/math/papers/0010/0010099.pdf

Neutrosophic Set is a Generalization of Intuitionist Fuzzy Set, Inconsistent Intuitionist Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionist Fuzzy Set of second type), q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, and n-HyperSpherical Fuzzy Set, while Neutrosophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision.

https://arxiv.org/ftp/arxiv/papers/1911/1911.07333.pdf
https://fs.unm.edu/Raspunsatan.pdf

Neutrosophic Logic is a general framework for unification of many existing logics, such as fuzzy logic (especially intuitionistic fuzzy logic), paraconsistent logic, intuitionist logic, etc. The main idea of NL is to characterize each logical statement in a 3D-Neutrosophic Space, where each dimension of the space represents respectively the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of $]^{-}0, 1^{+}[$ with not necessarily any connection between them.
For all engineering, technical, administrative and other practical applications the classical unit interval [0, 1] should be used.
While Neutrosophic Probability and Statistics are generalizations of classical and imprecise probability and classical statistics respectively.

The Most Important Books and Papers on the Advancement of Neutrosophics

1980s - Foundation of **Paradoxism** that is an international movement in science and culture based on excessive use of contradictions, antitheses, oxymoron, and paradoxes [Smarandache]. During three decades (1980-2020) hundreds of authors from tens of countries around the globe contributed papers to 15 international paradoxist anthologies: https://fs.unm.edu/a/paradoxism.htm

1995-1998 – Smarandache extended the paradoxism (*based on opposites*) to a new branch of philosophy called **Neutrosophy** (*based on opposites and their neutral/indeterminacies*), that gave birth to many scientific branches, such as: neutrosophic logic, neutrosophic set, neutrosophic probability and statistics, neutrosophic algebraic structures, and so on with multiple applications in all fields.

Neutrosophy is also an extension of the *Dialectics*, the *Yin-Yang* ancient Chinese philosophy, the *Manichaeism*, and in general of the *Dualism.*
https://fs.unm.edu/Neutrosophy-A-New-Branch-of-Philosophy.pdf

Introduced the neutrosophic set/logic/probability/statistics; introduces the single-valued neutrosophic set (pp. 7-8);

https://arxiv.org/ftp/math/papers/0101/0101228.pdf (fourth edition)
https://fs.unm.edu/eBook-Neutrosophics6.pdf (online sixth edition)

Single Valued Neutrosophic Sets

https://fs.unm.edu/SingleValuedNeutrosophicSets.pdf

I**ndeterminacy** in Neutrosophic Theories and their Applications.
https://fs.unm.edu/Indeterminacy.pdf

1998, 2019 - Extended **Nonstandard Neutrosophic Logic, Set, Probability based on NonStandard Analysis**

https://arxiv.org/ftp/arxiv/papers/1903/1903.04558.pdf https://fs.unm.edu/AdvancesOfStandardAndNonstandard.pdf

Improved Definition of NonStandard Neutrosophic Logic and Introduction to **Neutrosophic Hyperreals** (Third version), arXiv, Cornell University, New York City, USA, **https://arxiv.org/ftp/arxiv/papers/1812/1812.02534.pdf,**
https://fs.unm.edu/NonStandardAnalysis-Imamura-proven-wrong.pdf

2002 – Introduction of **corner cases of sets / probabilities / statistics / logics**, such as:
- Neutrosophic intuitionistic set (different from intuitionist fuzzy set), neutrosophic paraconsistent set, neutrosophic faillibilist set, neutrosophic paradoxist set, neutrosophic pseudo-paradoxist set, neutrosophic tautological set, neutrosophic nihilist set, neutrosophic dialetheist set, neutrosophic trivialist set;
- Neutrosophic intuitionistic probability and statistics, neutrosophic paraconsistent probability and statistics, neutrosophic faillibilist probability and statistics, neutrosophic paradoxist probability and statistics, neutrosophic pseudo-paradoxist probability and statistics, neutrosophic tautological probability and statistics, neutrosophic nihilist probability and statistics, neutrosophic dialetheist probability and statistics,neutrosophic trivialist probability and statistics;
- Neutrosophic paradoxist logic (or paradoxism), neutrosophic pseudo-paradoxist logic (or neutrosophic pseudo-paradoxism), neutrosophic tautological logic (or neutrosophic tautologism):
https://arxiv.org/ftp/math/papers/0301/0301340.pdf
https://fs.unm.edu/DefinitionsDerivedFromNeutrosophics.pdf

2003 – Introduction by Kandasamy and Smarandache of **Neutrosophic Numbers** (a+bI, where I = *literal indeterminacy*, $I^2 = I$, which is different from the *numerical indeterminacy* I = real set), I-Neutrosophic Algebraic Structures and **Neutrosophic Cognitive Maps**

https://arxiv.org/ftp/math/papers/0311/0311063.pdf
https://fs.unm.edu/NCMs.pdf

2005 - Introduction of **Interval Neutrosophic Set/Logic**

https://arxiv.org/pdf/cs/0505014.pdf
https://fs.unm.edu/INSL.pdf

2006 – Introduction of **Degree of Dependence and Degree of Independence between the Neutrosophic Components** T, I, F.
For single valued neutrosophic logic, the sum of the components is:
$0 \leq t+i+f \leq 3$ when all three components are independent;
$0 \leq t+i+f \leq 2$ when two components are dependent, while the third one is independent from them;
$0 \leq t+i+f \leq 1$ when all three components are dependent.
When three or two of the components T, I, F are independent, one leaves room for background incomplete information (sum < 1), paraconsistent and contradictory information (sum > 1), or complete information (sum = 1).
If all three components T, I, F are dependent, then similarly one leaves room for incomplete information (sum < 1), or complete information (sum = 1).
In general, the sum of two components x and y that vary in the unitary interval [0, 1] is:
$0 \leq x + y \leq 2 - d°(x, y)$, where $d°(x, y)$ is the degree of dependence between x and y, while $d°(x, y)$ is the degree of independence between x and y.
Degrees of Dependence and Independence between Neutrosophic Components T, I, F are independent components, leaving room for *incomplete information* (when their superior sum < 1), *paraconsistent and contradictory information* (when the superior sum > 1), or *complete information* (sum of components = 1).
For software engineering proposals the classical unit interval [0, 1] is used.
https://doi.org/10.5281/zenodo.571359
https://fs.unm.edu/eBook-Neutrosophics6.pdf (p. 92)
https://fs.unm.edu/NSS/DegreeOfDependenceAndIndependence.pdf

2007 – The Neutrosophic Set was extended [Smarandache, 2007] to **Neutrosophic Overset** (when some neutrosophic component is > 1), since he observed that, for example, an employee working overtime deserves a degree of membership > 1, with respect to an employee that only works regular full-time and whose degree of membership = 1;
and to **Neutrosophic Underset** (when some neutrosophic component is < 0), since, for example, an employee making more damage than benefit to his company deserves a degree of membership < 0, with respect to an employee that produces benefit to the company and has the degree of membership > 0;
and to and to **Neutrosophic Offset** (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and some neutrosophic component < 0).
Then, similarly, the Neutrosophic Logic/Measure/Probability/Statistics etc. were extended to respectively **Neutrosophic Over-/Under-/Off- Logic / Measure / Probability / Statistics** etc.
https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf
https://fs.unm.edu/NeutrosophicOversetUndersetOffset.pdf
https://fs.unm.edu/SVNeutrosophicOverset-JMI.pdf
https://fs.unm.edu/IV-Neutrosophic-Overset-Underset-Offset.pdf
https://fs.unm.edu/NSS/DegreesOf-Over-Under-Off-Membership.pdf

2007 – Smarandache introduced the **Neutrosophic Tripolar Set** and **Neutrosophic Multipolar Set** and consequently the **Neutrosophic Tripolar Graph** and **Neutrosophic Multipolar Graph**
https://fs.unm.edu/eBook-Neutrosophics6.pdf (p. 93)
https://fs.unm.edu/IFS-generalized.pdf

2009 – Introduction of **N-norm and N-conorm**
https://arxiv.org/ftp/arxiv/papers/0901/0901.1289.pdf
https://fs.unm.edu/N-normN-conorm.pdf

2013 - Development of **Neutrosophic Measure** and **Neutrosophic Probability** (*chance that an event occurs, indeterminate chance of occurrence, chance that the event does not occur*)
https://arxiv.org/ftp/arxiv/papers/1311/1311.7139.pdf
https://fs.unm.edu/NeutrosophicMeasureIntegralProbability.pdf

2013 – Smarandache **Refined / Split the Neutrosophic Components** (T, I, F) into Neutrosophic SubComponents (T_1, T_2, ...; I_1, I_2, ...; F_1, F_2, ...):
https://arxiv.org/ftp/arxiv/papers/1407/1407.1041.pdf
https://fs.unm.edu/n-ValuedNeutrosophicLogic-PiP.pdf

2014 – Introduction of the **Law of Included Multiple-Middle** (as extension of the Law of Included Middle)
(<A>; <neutA_1>, <neutA_2>, …, <neutA_n>; <antiA>)

https://fs.unm.edu/LawIncludedMultiple-Middle.pdf

and the **Law of Included Infinitely-Many-Middles** (2023)

https://fs.unm.edu/NSS/LawIncludedInfinitely1.pdf
(<A>; <neutA_1>, <neutA_2>, …, <neut$A_{infinity}$>; <antiA>)

2014 - Development of **Neutrosophic Statistics** (*indeterminacy* is introduced into classical statistics with respect to any data regarding the sample / population, probability distributions / laws / graphs / charts etc., with respect to the individuals that only partially belong to a sample / population, and so on):

https://fs.unm.edu/NS/NeutrosophicStatistics.htm

Neutrosophic Numbers used in Neutrosophic Statistics

https://fs.unm.edu/NS/AppurtenanceInclusionEquations-revised.pdf

2015 - Extension of the Analytical Hierarchy Process (AHP) to **α-Discounting Method for Multi-Criteria Decision Making (α-D MCDC)**
https://fs.unm.edu/ScArt/AlphaDiscountingMethod.pdf
https://fs.unm.edu/ScArt/CP-IntervalAlphaDiscounting.pdf
https://fs.unm.edu/ScArt/ThreeNonLinearAlpha.pdf
https://fs.unm.edu/alpha-DiscountingMCDM-book.pdf

2015 - Introduction of **Neutrosophic Precalculus** and **Neutrosophic Calculus**

https://arxiv.org/ftp/arxiv/papers/1509/1509.07723.pdf
https://fs.unm.edu/NeutrosophicPrecalculusCalculus.pdf

2015 – **Refined Neutrosophic Numbers** $(a+ b_1I_1 + b_2I_2 + \dots + b_nI_n)$, where $I_1, I_2, \dots, I_n$ are SubIndeterminacies of Indeterminacy I.

2015 – **(t,i,f)-Neutrosophic Graphs**.

2015 - **Thesis-AntiThesis-NeutroThesis**, and NeutroSynthesis, Neutrosophic Axiomatic System, neutrosophic dynamic systems, symbolic neutrosophic logic, (t, i, f)-Neutrosophic Structures, I-Neutrosophic Structures, Refined Literal Indeterminacy, Quadruple Neutrosophic Algebraic Structures, Multiplication Law of SubIndeterminacies, and Neutrosophic Quadruple Numbers of the form a + bT + cI + dF, where T, I, F are literal neutrosophic components, and a, b, c, d are real or complex numbers:

https://arxiv.org/ftp/arxiv/papers/1512/1512.00047.pdf
https://fs.unm.edu/SymbolicNeutrosophicTheory.pdf

2015 – Introduction of the **SubIndeterminacies** of the form $I_0^k = \frac{k}{0}$, for $k \in \{0, 1, 2, \dots, n-1\}$, into the ring of modulo integers Z_n - called natural neutrosophic indeterminacies (Vasantha-Smarandache)
https://fs.unm.edu/MODNeutrosophicNumbers.pdf

2015 – Introduction of **Neutrosophic Crisp Set and Topology** (Salama & Smarandache)
https://fs.unm.edu/NeutrosophicCrispSetTheory.pdf

2016 - **Addition, Multiplication, Scalar Multiplication, Power, Subtraction, and Division of Neutrosophic Triplets (T, I, F)**

https://fs.unm.edu/CR/SubstractionAndDivision.pdf

2016 – Introduction of **Neutrosophic Multisets** (as generalization of classical multisets)

https://fs.unm.edu/NeutrosophicMultisets.htm

2016 – Introduction of **Neutrosophic Triplet Structures** and m-valued refined neutrosophic triplet structures [Smarandache - Ali] **https://fs.unm.edu/NeutrosophicTriplets.htm**

2016 – Introduction of **Neutrosophic Duplet Structures**

https://fs.unm.edu/NeutrosophicDuplets.htm

2017 - 2020 - **Neutrosophic Score, Accuracy, and Certainty Functions** form a total order relationship on the set of (single-valued, interval-valued, and in general subset-valued) neutrosophic triplets (T, I, F); and these functions are used in MCDM (Multi-Criteria Decision Making): **https://fs.unm.edu/NSS/TheScoreAccuracyAndCertainty1.pdf**

2017 - In biology Smarandache introduced the **Theory of Neutrosophic Evolution: Degrees of Evolution, Indeterminacy or Neutrality, and Involution** (as extension of *Darwin's Theory of Evolution*): **https://fs.unm.edu/neutrosophic-evolution-PP-49-13.pdf**

https://fs.unm.edu/V/NeutrosophicEvolution.mp4

https://fs.unm.edu/NeutrosophicEvolution.pdf

2017 - Introduction by F. Smarandache of **Plithogeny** (as generalization of Yin-Yang, Manichaeism, Dialectics, Dualism, and Neutrosophy), and **Plithogenic Set / Plithogenic Logic as generalization of MultiVariate Logic / Plithogenic Probability and Plithogenic Statistics as generalizations of MultiVariate Probability and Statistics** (as generalization of fuzzy, intuitionistic fuzzy, neutrosophic set/logic/probability/statistics): **https://arxiv.org/ftp/arxiv/papers/1808/1808.03948.pdf**
https://fs.unm.edu/Plithogeny.pdf

2017 - Enunciation of the Law that: **It Is Easier to Break from Inside than from Outside a Neutrosophic Dynamic System** (Smarandache - Vatuiu):

https://fs.unm.edu/EasierMaiUsor.pdf

2018 - 2023 - Introduction of new types of soft sets: **HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, SuperHyperSoft Set, TreeSoft Set**:

https://fs.unm.edu/TSS/NewTypesSoftSets-Improved.pdf

https://fs.unm.edu/TSS/SuperHyperSoftSet.pdf

https://fs.unm.edu/NSS/IndetermSoftIndetermHyperSoft38.pdf

(with *IndetermSoft Operators* acting on *IndetermSoft Algebra*)

https://fs.unm.edu/TSS/

2018 – Introduction to **Neutrosophic Psychology** (*Neutropsyche, Refined Neutrosophic Memory: conscious, aconscious, unconscious, Neutropsychic Personality, Eros / Aoristos / Thanatos, Neutropsychic Crisp Personality*):

https://fs.unm.edu/NeutropsychicPersonality-ed3.pdf

2019 - **Theory of Spiral Neutrosophic Human Evolution** (Smarandache - Vatuiu): **https://fs.unm.edu/SpiralNeutrosophicEvolution.pdf**

2019 - Introduction to **Neutrosophic Sociology** (*NeutroSociology*) [neutrosophic concept, or (T, I, F)-concept, is a concept that is T% true, I% indeterminate, and F% false]: **https://fs.unm.edu/Neutrosociology.pdf**

2019 - **Refined Neutrosophic Crisp Set**

https://fs.unm.edu/RefinedNeutrosophicCrispSet.pdf

2019-2024 - Introduction of sixteen new types of topologies: **NonStandard Topology, Largest Extended NonStandard Real Topology, Neutrosophic Triplet Weak/Strong Topologies, Neutrosophic Extended Triplet Weak/Strong Topologies, Neutrosophic Duplet Topology, Neutrosophic Extended Duplet Topology, Neutrosophic MultiSet Topology, NonStandard Neutrosophic Topology, NeutroTopology, AntiTopology, Refined Neutrosophic Topology, Refined Neutrosophic Crisp Topology, SuperHyperTopology,** and **Neutrosophic SuperHyperTopology**:

https://fs.unm.edu/TT/RevolutionaryTopologies.pdf

https://fs.unm.edu/TT/

2019 - Generalization of the classical Algebraic Structures to NeutroAlgebraic Structures (or **NeutroAlgebras**) {whose **operations and axioms are partially true, partially indeterminate, and partially false} as extensions of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebras) {whose operations and axioms are totally false}.**

https://fs.unm.edu/NA/NeutroAlgebra.htm

https://fs.unm.edu/NA/NeutroAlgebra.pdf

And, in general, he extended any classical Structure, in no matter what field of knowledge, to a NeutroStructure and an AntiStructure:

https://fs.unm.edu/NA/NeutroStructure.pdf

As alternatives and generalizations of the **Non-Euclidean Geometries** he introduced in 2021 the **NeutroGeometry & AntiGeometry.** While the Non-Euclidean Geometries resulted from the total negation of only one specific axiom (Euclid's Fifth **Postulate), the AntiGeometry results from the total negation of any axiom and even of more axioms from any geometric axiomatic system (Euclid's, Hilbert's, etc.), and the NeutroGeometry results from the partial negation of one or more axioms [and no** total negation of no axiom] from any geometric axiomatic system.

https://fs.unm.edu/NSS/NeutroGeometryAntiGeometry31.pdf

https://fs.unm.edu/NG/

2019-2022 - Extension of HyperGraph to **SuperHyperGraph and Neutrosophic SuperHyperGraph**

https://fs.unm.edu/NSS/n-SuperHyperGraph.pdf

2020 - Introduction to **Neutrosophic Genetics:** **https://fs.unm.edu/NeutrosophicGenetics.pdf**
2021 - Introduction to Neutrosophic Number Theory (Abobala)

https://fs.unm.edu/NSS/FoundationsOfNeutrosophicNumberTheory10.pdf

2021 - As alternatives and generalizations of the Non-Euclidean Geometries, Smarandache introduced in 2021 the NeutroGeometry & AntiGeometry. While the Non-Euclidean Geometries resulted from the total negation of only one specific axiom (Euclid's Fifth Postulate), the AntiGeometry results from the total negation of any axiom and even of more axioms from any geometric axiomatic system (Euclid's, Hilbert's, etc.), and the NeutroGeometry results from the partial negation of one or more axioms [and no total negation of no axiom] from any geometric axiomatic system:

https://fs.unm.edu/NSS/NeutroGeometryAntiGeometry31.pdf

Real Examples of NeutroGeometry and AntiGeometry:

https://fs.unm.edu/NSS/ExamplesNeutroGeometryAntiGeometry35.pdf

2021 - Introduction of Plithogenic Logic as a generalization of MultiVariate Logic

https://fs.unm.edu/NSS/IntroductionPlithogenicLogic1.pdf

2021 - Introduction of Plithogenic Probability and Statistics as generalizations of MultiVariate Probability and Statistics respectively

https://fs.unm.edu/NSS/PlithogenicProbabilityStatistics20.pdf

2021 - Introduction of the AH-Isometry f(x+yI) = f(x) + I[f(x+y) - f(x)] and foundation of the Neutrosophic Euclidean Geometry (by Abobala & Hatip).

https://fs.unm.edu/NSS/AlgebraicNeutrosophicEuclideanGeometry10.pdf

and extension to n-Refined AH-Isometry (Smarandache & Abobala, 2024)

https://fs.unm.edu/NSS/RefinedLiteral21.pdf

2016 - 2022 SuperHyperAlgebra & Neutrosophic SuperHyperAlgebra

https://fs.unm.edu/SuperHyperAlgebra.pdf

2022 - SuperHyperFunction, SuperHyperTopology

https://fs.unm.edu/NSS/SuperHyperFunction37.pdf

2022 - 2023 Neutrosophic Operational Research (Smarandache - Jdid)

https://fs.unm.edu/NeutrosophicOperationsResearch.pdf

2023 - Symbolic Plithogenic Algebraic Structures built on the set of Symbolic Plithogenic Numbers of the form $a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n$ where the multiplication $P_i \cdot P_j$ is based on the prevalence order and absorbance law **https://fs.unm.edu/NSS/SymbolicPlithogenicAlgebraic39.pdf**

2023 - Foundation of Neutrosophic Cryptology (Merkepci-Abobala-Allouf)

https://fs.unm.edu/NeutrosophicCryptography1.pdf
https://fs.unm.edu/NeutrosophicCryptography2.pdf
https://fs.unm.edu/NSS/2OnANovelSecurityScheme.pdf

2023 - The MultiNeutrosophic Set (a neutrosophic set whose elements' degrees T, I, F are evaluated by multiple sources):

https://fs.unm.edu/NSS/MultiNeutrosophicSet.pdf

2023 - The MultiAlist System of Thought (an open dynamic system of many opposites, with their neutralities or indeterminacies, formed by elements from many systems):

https://fs.unm.edu/NSS/MultiAlistSystemOfThought.pdf

2023 - Appurtenance Equation, Inclusion Equation, & Neutrosophic Numbers used in Neutrosophic Statistics.

https://fs.unm.edu/NS/AppurtenanceInclusionEquations-revised.pdf

2024 - SuperHyperStructure and Neutrosophic SuperHyperStructure

https://fs.unm.edu/SHS/

2024 - Zarathustra & Neutrosophy

https://fs.unm.edu/Zoroastrianism.pdf

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Neutrosophic Researchers:

There are about 7,500 neutrosophic researchers, within 90 countries around the globe, that have produced about 4,000 articles and books, and over 70 PhD and MSc theses, within more than three decades. Many neutrosophic researchers got specialized into various fields of neutrosophics, plithogenics, NeutroAlgebra and AntiAlgebra, NeutroGeometry and AntiGeometry, new types of topologies, new types of soft sets, SuperHyperStructures, etc.

References

University of New Mexico *(USA)* web sites:

https://fs.unm.edu/neutrosophy.htm
https://fs.unm.edu/NSS/Articles.htm
https://fs.unm.edu/CR/CR-Articles.htm
https://fs.unm.edu/NCML/Articles.htm ***(****Spanish)*
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Other journals:

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